On the functions having ‘perfect’ algebraic independence property at algebraic numbers

Taka-aki Tanaka

Keio University

Transcendental number theory

In transcendental number theory, we are interested in the transcendental numbers given by the values of analytic functions at algebraic numbers. In other words, its research objects are transcendental numbers represented as explicit power series of algebraic numbers.

Transcendental number theory

In transcendental number theory, we are interested in the transcendental numbers given by the values of analytic functions at algebraic numbers. In other words, its research objects are transcendental numbers represented as explicit power series of algebraic numbers.

$$\text{Algebraic numbers} \xrightarrow{\text{analytic function}} \text{Transcendental numbers}$$

Transcendental number theory

In transcendental number theory, we are interested in the transcendental numbers given by the values of analytic functions at algebraic numbers. In other words, its research objects are transcendental numbers represented as explicit power series of algebraic numbers.

$$\text{Algebraic numbers} \xrightarrow{\text{analytic function}} \text{Transcendental numbers}$$

One of the main purpose in transcendental number theory is to obtain a necessary and sufficient condition for the values of a given analytic function at algebraic numbers to be algebraically independent.

Transcendental number theory

$$\text{Algebraic numbers} \xrightarrow{\text{analytic function}} \text{Transcendental numbers}$$

One of the main purpose in transcendental number theory is to obtain a necessary and sufficient condition for the values of a given analytic function at algebraic numbers to be algebraically independent.

Remarkable case

$$\text{Algebraic numbers} \xrightarrow{\text{analytic function}} \text{Transcendental numbers}$$

One of the main purpose in transcendental number theory is to obtain a necessary and sufficient condition for the values of a given analytic function at algebraic numbers to be algebraically independent.

Remarkable case

$$\text{Algebraic numbers} \xrightarrow{\text{analytic function}} \text{Transcendental numbers}$$

One of the main purpose in transcendental number theory is to obtain a necessary and sufficient condition for the values of a given analytic function at algebraic numbers to be algebraically independent.

The remarkable case is that the values of a given analytic function at any distinct algebraic numbers are algebraically independent.

Remarkable case

The remarkable case is that the values of a given analytic function at any distinct algebraic numbers are algebraically independent.

How about well-known functions

The remarkable case is that the values of a given analytic function at any distinct algebraic numbers are algebraically independent.

How about well-known functions

The remarkable case is that the values of a given analytic function at any distinct algebraic numbers are algebraically independent.

Well-known functions do not have this property.

How about well-known functions

The remarkable case is that the values of a given analytic function at any distinct algebraic numbers are algebraically independent.

Well-known functions do not have this property.

For example, the famous Lindemann-Weierstrass theorem asserts that the values $e^{\alpha_1}, \dots, e^{\alpha_s}$ of exponential function at algebraic numbers $\alpha_1, \dots, \alpha_s$ are algebraically dependent if and only if $\alpha_1, \dots, \alpha_s$ are linearly dependent over $\mathbb{Q}$.

How about well-known functions

Well-known functions do not have this property.

For example, the famous Lindemann-Weierstrass theorem asserts that the values $e^{\alpha_1}, \ldots, e^{\alpha_s}$ of exponential function at algebraic numbers $\alpha_1, \ldots, \alpha_s$ are algebraically dependent if and only if $\alpha_1, \ldots, \alpha_s$ are linearly dependent over $\mathbb{Q}$.

How about well-known functions

Well-known functions do not have this property.

For example, the famous Lindemann-Weierstrass theorem asserts that the values $e^{\alpha_1}, \ldots, e^{\alpha_s}$ of exponential function at algebraic numbers $\alpha_1, \ldots, \alpha_s$ are algebraically dependent if and only if $\alpha_1, \ldots, \alpha_s$ are linearly dependent over $\mathbb{Q}$.

How about well-known functions

Well-known functions do not have this property.

For example, the famous Lindemann-Weierstrass theorem asserts that the values $e^{\alpha_1}, \ldots, e^{\alpha_s}$ of exponential function at algebraic numbers $\alpha_1, \ldots, \alpha_s$ are algebraically dependent if and only if $\alpha_1, \ldots, \alpha_s$ are linearly dependent over $\mathbb{Q}$.

Moreover, the Gamma function $\Gamma(z)$ does not have this property, since $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ for any $\alpha \in \overline{\mathbb{Q}} \setminus \{0, -1, -2, -3, \ldots\}$.

How about well-known functions

Well-known functions do not have this property.

Moreover, the Gamma function $\Gamma(z)$ does not have this property, since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ for any $\alpha \in \overline{\mathbb{Q}} \setminus \{0, -1, -2, -3, \ldots\}$.

How about well-known functions

Well-known functions do not have this property.

Moreover, the Gamma function $\Gamma(z)$ does not have this property, since $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$ for any $\alpha \in \overline{\mathbb{Q}} \setminus \{0,-1,-2,-3,\ldots\}$.

Furthermore, the Riemann zeta function $\zeta(s)=\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^s}$ does not have this property, since $\zeta(2k) \in \mathbb{Q}^{\times}\pi^{2k}$ for any $k \in \mathbb{Z}_{>0}$ and so $\zeta(2k)^{\ell}\zeta(2\ell)^{-k} \in \mathbb{Q}^{\times}$ for any distinct $k,\ell \in \mathbb{Z}_{>0}$.

Perfect algebraic independence property

In this talk,
an analytic function $f(z)$ is said to have the perfect algebraic independence property
if the values of $f(z)$ at any nonzero algebraic numbers within the natural boundary of $f(z)$ are algebraically independent, namely
the infinite set

$$\left\{ f(\alpha) \;\middle|\; \alpha \in \overline{\mathbb{Q}}^{\times} \cap D_f \right\}$$

is algebraically independent, where D_f denotes the domain of existence of the analytic function $f(z)$.

Differential perfect algebraic independence property

In this talk,
an analytic function $g(z)$ is said to have the
differential perfect algebraic independence property
if the values of $g(z)$ as well as the derivatives of
$g(z)$ of any order at any nonzero algebraic numbers
within the natural boundary of $g(z)$ are algebraically
independent, namely
the infinite set

$$\left\{ g^{(l)}(\alpha) \;\middle|\; l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times} \cap D_g \right\}$$

is algebraically independent, where D_g denotes the
domain of existence of the analytic function $g(z)$.

Ingredients of this talk

The speaker will introduce 5 types of functions having the (differential) perfect algebraic independence property:

1. Complex entire functions having the differential perfect algebraic independence property,
2. Complex functions with natural boundary $|z| = 1$ having the (differential) perfect algebraic independence property,
3. Complex entire functions represented as infinite products and having the differential perfect algebraic independence property without their zeroes,
4. Complex functions of three variables having the perfect algebraic independence property, and
5. Functions defined over function fields of positive characteristic, and having the differential perfect algebraic independence property.

Complex entire functions having the differential perfect algebraic independence property

The following complex entire function $g(z)$ has the differential perfect algebraic independence property, namely the infinite set $\{g^{(l)}(\alpha) \mid l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times}\}$ is algebraically independent:

Complex entire functions having the differential perfect algebraic independence property

The following complex entire function $g(z)$ has the differential perfect algebraic independence property, namely the infinite set $\{g^{(l)}(\alpha) \mid l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times}\}$ is algebraically independent: Fix $\beta \in \overline{\mathbb{Q}}$ with $0 < |\beta| < 1$ and $d \in \mathbb{Z}$ with $d \geq 2$.

Complex entire functions having the differential perfect algebraic independence property

The following complex entire function $g(z)$ has the differential perfect algebraic independence property, namely the infinite set $\{g^{(l)}(\alpha) \mid l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times}\}$ is algebraically independent:

Fix $\beta \in \overline{\mathbb{Q}}$ with $0 < |\beta| < 1$ and $d \in \mathbb{Z}$ with $d \geq 2$.

Nishioka (1986): $g(z) = \sum_{k=0}^{\infty} \beta^{k!} z^k$.

Complex entire functions having the differential perfect algebraic independence property

The following complex entire function $g(z)$ has the differential perfect algebraic independence property, namely the infinite set $\{g^{(l)}(\alpha) \mid l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times}\}$ is algebraically independent: Fix $\beta \in \overline{\mathbb{Q}}$ with $0 < |\beta| < 1$ and $d \in \mathbb{Z}$ with $d \geq 2$.

Nishioka (1986): $g(z) = \sum_{k=0}^{\infty} \beta^{k!} z^k$.

Nishioka (1996): $g(z) = \sum_{k=0}^{\infty} \beta^{d^k} z^k$.

Complex entire functions having the differential perfect algebraic independence property

The following complex entire function $g(z)$ has the differential perfect algebraic independence property, namely the infinite set $\{g^{(l)}(\alpha) \mid l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times}\}$ is algebraically independent:

Fix $\beta \in \overline{\mathbb{Q}}$ with $0 < |\beta| < 1$ and $d \in \mathbb{Z}$ with $d \geq 2$.

Nishioka (1986): $g(z) = \displaystyle\sum_{k=0}^{\infty} \beta^{k!} z^k$.

Nishioka (1996): $g(z) = \displaystyle\sum_{k=0}^{\infty} \beta^{d^k} z^k$.

T (1996): $g(z) = \displaystyle\sum_{k=0}^{\infty} \beta^{R_k} z^k$, where $\{R_k\}_{k\geq 0}$ belongs to a certain class of linear recurrences, which will be explained later and includes the sequence $\{F_k\}_{k\geq 0}$ of Fibonacci numbers.

Complex functions with natural boundary $|z| = 1$ having the perfect algebraic independence property

The following complex function $f(z)$ has the natural boundary $|z| = 1$ and the perfect algebraic independence property, namely the infinite set

$$\left\{ f(\alpha) \;\middle|\; \alpha \in \overline{\mathbb{Q}}^{\times} \cap D \right\} = \{ f(\alpha) \mid \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1 \}$$

is algebraically independent, where

$$D = \{ z \in \mathbb{C} \ : \ |z| < 1 \}.$$

Complex functions with natural boundary $|z| = 1$ having the perfect algebraic independence property

The following complex function $f(z)$ has the natural boundary $|z| = 1$ and the perfect algebraic independence property, namely the infinite set $\{f(\alpha) \mid \alpha \in \overline{\mathbb{Q}}^{\times} \cap D\}$ is algebraically independent:

Complex functions with natural boundary $|z| = 1$ having the perfect algebraic independence property

The following complex function $f(z)$ has the natural boundary $|z| = 1$ and the perfect algebraic independence property, namely the infinite set $\{f(\alpha) \mid \alpha \in \overline{\mathbb{Q}}^{\times} \cap D\}$ is algebraically independent:

Nishioka (1987): $f(z) = \sum_{k=0}^{\infty} z^{k!+k}$.

Complex functions with natural boundary $|z| = 1$ having the perfect algebraic independence property

The following complex function $f(z)$ has the natural boundary $|z| = 1$ and the perfect algebraic independence property, namely the infinite set $\{f(\alpha) \mid \alpha \in \overline{\mathbb{Q}}^{\times} \cap D\}$ is algebraically independent:

Nishioka (1987): $f(z) = \displaystyle\sum_{k=0}^{\infty} z^{k!+k}$.

Masser (1999): $f(z) = \displaystyle\sum_{k=1}^{\infty} [k\omega] z^k$, the Hecke-Mahler series,

where $\omega \in \mathbb{R}$ is a quadratic irrational number and $[x]$ denotes the largest integer not exceeding the real number x.

Complex functions with natural boundary $|z| = 1$ having the perfect algebraic independence property

The following complex function $f(z)$ has the natural boundary $|z| = 1$ and the perfect algebraic independence property, namely the infinite set $\{f(\alpha) \mid \alpha \in \overline{\mathbb{Q}}^{\times} \cap D\}$ is algebraically independent:

Nishioka (1987): $f(z) = \displaystyle\sum_{k=0}^{\infty} z^{k!+k}$.

Masser (1999): $f(z) = \displaystyle\sum_{k=1}^{\infty} [k\omega] z^k$, the Hecke-Mahler series,

where $\omega \in \mathbb{R}$ is a quadratic irrational number and $[x]$ denotes the largest integer not exceeding the real number x.

T (unpublished): $f(z) = \displaystyle\sum_{k=0}^{\infty} z^{d^k+k}$, where $d \in \mathbb{Z}$ with $d \geq 2$.

Complex functions with natural boundary $|z| = 1$ having the differential perfect algebraic independence property

The following complex function $g(z)$ has the natural boundary $|z| = 1$ and the differential perfect algebraic independence property, namely the infinite set

$$\left\{ g^{(l)}(\alpha) \;\middle|\; l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times} \cap D \right\}$$

is algebraically independent, where

$$D = \{z \in \mathbb{C} \ :\ |z| < 1\}.$$

Complex functions with natural boundary $|z| = 1$ having the differential perfect algebraic independence property

The following complex function $g(z)$ has the natural boundary $|z| = 1$ and the differential perfect algebraic independence property, namely the infinite set $\{g^{(l)}(\alpha) \mid l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times} \cap D\}$ is algebraically independent:

Complex functions with natural boundary $|z| = 1$ having the differential perfect algebraic independence property

The following complex function $g(z)$ has the natural boundary $|z| = 1$ and the differential perfect algebraic independence property, namely the infinite set $\{g^{(l)}(\alpha) \mid l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times} \cap D\}$ is algebraically independent:

Nishioka (1987): $g(z) = \displaystyle\sum_{k=0}^{\infty} z^{k!+k}$.

Complex functions with natural boundary $|z| = 1$ having the differential perfect algebraic independence property

The following complex function $g(z)$ has the natural boundary $|z| = 1$ and the differential perfect algebraic independence property, namely the infinite set
$\{g^{(l)}(\alpha) \mid l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times} \cap D\}$ is algebraically independent:

Nishioka (1987): $g(z) = \displaystyle\sum_{k=0}^{\infty} z^{k!+k}$.

Recently Tanuma-T proved that

$g(z) = \displaystyle\sum_{k=1}^{\infty} [k\omega] z^k$, where $\omega \in \mathbb{R}$ is a quadratic irrational number with $|\omega - \omega'| > 2$ (ω': the conjugate of ω).

Complex functions with natural boundary $|z| = 1$ having the (differential) perfect algebraic independence property

All the (differential) perfect algebraic independence results within the unit circle are proved by using Vandermonde determinant whose entries are distinct roots of unity.

Complex functions with natural boundary $|z| = 1$ having the (differential) perfect algebraic independence property

All the (differential) perfect algebraic independence results within the unit circle are proved by using Vandermonde determinant whose entries are distinct roots of unity. Presuming from $f(z) = \sum_{k=0}^{\infty} z^{d^k+k}$ and $g(z) = \sum_{k=0}^{\infty} z^{k!+k}$, we raise the following

Problem

Complex functions with natural boundary $|z| = 1$ having the (differential) perfect algebraic independence property

All the (differential) perfect algebraic independence results within the unit circle are proved by using Vandermonde determinant whose entries are distinct roots of unity.
Presuming from $f(z) = \sum_{k=0}^{\infty} z^{d^k+k}$ and $g(z) = \sum_{k=0}^{\infty} z^{k!+k}$, we raise the following

Problem

Let $f(z) = \sum_{k=0}^{\infty} z^{e_k} \in \{0,1\}[[z]]$. Does $f(z)$ have the (differential) perfect algebraic independence property within the unit circle if the following 2 conditions are both satisfied?

- The exponents satisfy $\liminf_{k\to\infty} e_{k+1}/e_k > 1$.
- For any $N \in \mathbb{Z}_{>0}$ and for any $a \in \{0, 1, \ldots, N-1\}$, there are infinitely many k such that $e_k \equiv a \pmod{N}$.

Complex entire functions represented as infinite products and having the differential perfect algebraic independence property without their zeroes

Let

$$g_d(z) = \prod_{k=0}^{\infty} \left(1 - \frac{z}{3^{d^k} - 2^{d^k}}\right) \qquad (d = 2, 3, 4, \ldots).$$

Then the infinite set

$$\left\{ g_d^{(l)}(\alpha) \;\middle|\; d \geq 2,\ l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times} \setminus \{3^{d^k} - 2^{d^k}\}_{k \geq 0} \right\}$$

is algebraically independent.

Complex entire functions represented as infinite products and having the differential perfect algebraic independence property without their zeroes

Let

$$g_d(z) = \prod_{k=0}^{\infty} \left(1 - \frac{z}{3^{d^k} - 2^{d^k}}\right) \qquad (d = 2, 3, 4, \ldots).$$

Then the infinite set

$$\left\{ g_d^{(l)}(\alpha) \;\middle|\; d \geq 2,\ l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times} \setminus \{3^{d^k} - 2^{d^k}\}_{k \geq 0} \right\}$$

is algebraically independent.

Complex entire functions represented as infinite products and having the differential perfect algebraic independence property without their zeroes

Let

$$g_d(z) = \prod_{k=0}^{\infty}\left(1 - \frac{z}{3^{d^k} - 2^{d^k}}\right) \qquad (d = 2, 3, 4, \ldots).$$

Then the infinite set

$$\left\{ g_d^{(l)}(\alpha) \;\middle|\; d \geq 2,\ l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times} \setminus \{3^{d^k} - 2^{d^k}\}_{k \geq 0} \right\}$$

is algebraically independent.

This is an example of the following result:

Complex entire functions represented as infinite products and having the differential perfect algebraic independence property without their zeroes

A special case of Theorem of Kurosawa-Tachiya-T (2014)

Let

$$g_d(z) = \prod_{k=0}^{\infty}\left(1 - \frac{z}{c_1\rho_1^{d^k} + c_2\rho_2^{d^k}}\right) \qquad (d = 2, 3, 4, \ldots),$$

where $c_1, c_2, \rho_1, \rho_2 \in \overline{\mathbb{Q}}^{\times}$ *and* ρ_1, ρ_2 *are multiplicatively independent and satisfy* $\rho_1 > \max\{1,\ \rho_2\}$. *Then the infinite set*

$$\left\{ g_d^{(l)}(\alpha) \;\middle|\; d \geq 2,\ l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times} \setminus \{c_1\rho_1^{d^k} + c_2\rho_2^{d^k}\}_{k \geq 0} \right\}$$

is algebraically independent.

Complex functions of three variables having the (quasi) perfect algebraic independence property

Let $\{G_k\}_{k\geq 0}$ be the generalized Fibonacci numbers defined by

$$G_0 = 0, \quad G_1 = 1, \quad G_{k+2} = bG_{k+1} + G_k \quad (k \geq 0),$$

where b is a positive integer. Then the infinite set

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{G_1+G_2+\cdots+G_k}}{(1-aq^{G_1})(1-aq^{G_2})\cdots(1-aq^{G_k})} \;\middle|\; \begin{array}{c} x, a, q \in \overline{\mathbb{Q}} \setminus \{0\}, \\ |a| \leq 1, \ |q| < 1 \end{array} \right\}$$

is algebraically independent.

Complex functions of three variables having the (quasi) perfect algebraic independence property

Let $\{G_k\}_{k\geq 0}$ be the generalized Fibonacci numbers defined by

$$G_0 = 0, \quad G_1 = 1, \quad G_{k+2} = bG_{k+1} + G_k \quad (k \geq 0),$$

where b is a positive integer. Then the infinite set

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{G_1+G_2+\cdots+G_k}}{(1-aq^{G_1})(1-aq^{G_2})\cdots(1-aq^{G_k})} \;\middle|\; \begin{array}{c} x, a, q \in \overline{\mathbb{Q}} \setminus \{0\}, \\ |a| \leq 1, \ |q| < 1 \end{array} \right\}$$

is algebraically independent.

This is an example of the following result:

Complex functions of three variables having the (quasi) perfect algebraic independence property

Let $\{R_k\}_{k\geq 1}$ be a linear recurrence of positive integers satisfying

$$R_{k+n} = c_1 R_{k+n-1} + \cdots + c_n R_k \quad (k \geq 1),$$

where $n \geq 2$ and $c_1, \ldots, c_n \in \mathbb{Z}_{\geq 0}$ with $c_n \neq 0$.

Complex functions of three variables having the (quasi) perfect algebraic independence property

Let $\{R_k\}_{k\geq 1}$ be a linear recurrence of positive integers satisfying

$$R_{k+n} = c_1 R_{k+n-1} + \cdots + c_n R_k \quad (k \geq 1),$$

where $n \geq 2$ and $c_1, \ldots, c_n \in \mathbb{Z}_{\geq 0}$ with $c_n \neq 0$. Define

$$\begin{aligned}\Theta(x,a,q) &= \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\cdots+R_k}}{(1-aq^{R_1})(1-aq^{R_2})\cdots(1-aq^{R_k})} \\ &= \sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{xq^{R_l}}{1-aq^{R_l}}.\end{aligned}$$

Complex functions of three variables having the (quasi) perfect algebraic independence property

In what follows, let

$$U = \left\{ (x, a, q) \;\middle|\; x, a, q \in \overline{\mathbb{Q}} \setminus \{0\},\ |a| \leq 1,\ |q| < 1 \right\}.$$

Then $\Theta(x, a, q) = \displaystyle\sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{xq^{R_l}}{1 - aq^{R_l}}$ converges at any point in U. Let $(x_1, a_1, q_1), (x_2, a_2, q_2) \in U$. We write $(x_1, a_1, q_1) \sim (x_2, a_2, q_2)$ if $x_1/a_1 = x_2/a_2$ and if $a_1 q_1^{R_k} = a_2 q_2^{R_k}$ for all sufficiently large k. Then $\sim$ is an equivalence relation.

Complex functions of three variables having the (quasi) perfect algebraic independence property

Theorem (T, 2009)

Let $\{R_k\}_{k\geq 1}$ be a linear recurrence of positive integers defined above. Suppose $\{R_k\}_{k\geq 1}$ is not a geometric progression. Let $\Phi(X) = X^n - c_1X^{n-1} - \cdots - c_n$. Assume that $\Phi(\pm 1) \neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Then the values

$$\Theta(x, a, q) = \sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{xq^{R_l}}{1 - aq^{R_l}} \quad ((x, a, q) \in U)$$

are algebraically dependent if and only if there exist distinct $(x_1, a_1, q_1), (x_2, a_2, q_2) \in U$ such that $(x_1, a_1, q_1) \sim (x_2, a_2, q_2)$.

Complex functions of three variables having the (quasi) perfect algebraic independence property

Recall that $(x_1, a_1, q_1) \sim (x_2, a_2, q_2)$ if $x_1/a_1 = x_2/a_2$ and if $a_1 q_1^{R_k} = a_2 q_2^{R_k}$ for all sufficiently large k. We have

Complex functions of three variables having the (quasi) perfect algebraic independence property

Recall that $(x_1,a_1,q_1) \sim (x_2,a_2,q_2)$ if $x_1/a_1 = x_2/a_2$ and if $a_1 q_1^{R_k} = a_2 q_2^{R_k}$ for all sufficiently large k. We have

Corollary (T, 2009)

Suppose that $\{R_k\}_{k\geq 1}$ satisfies $R_{k+n} = c_1 R_{k+n-1} + \cdots + c_{n-1} R_{k+1} + R_k$ $(k \geq 1)$. Let $N^* = \mathrm{g.c.d.}(R_2 - R_1,\ R_3 - R_2, \ldots, R_{n+1} - R_n)$. Let ζ be a primitive N^*-th root of unity and $G = \langle(\zeta^{R_1}, \zeta^{R_1}, \zeta^{-1})\rangle$ a cyclic group generated by $(\zeta^{R_1}, \zeta^{R_1}, \zeta^{-1})$ with componentwise multiplication. Then the values $\Theta(x,a,q) = \sum_{k=1}^{\infty}\prod_{l=1}^{k} \frac{xq^{R_l}}{1-aq^{R_l}}$ $((x,a,q) \in U)$ are algebraically dependent if and only if there exist distinct $(x_1,a_1,q_1),(x_2,a_2,q_2) \in U$ such that $(x_1/x_2, a_1/a_2, q_1/q_2) \in G$.

Complex functions of three variables having the (quasi) perfect algebraic independence property

Corollary (T, 2009)

Suppose in addition $\{R_k\}_{k\geq 1}$ *satisfies* $R_{k+n} = c_1 R_{k+n-1} + \cdots + c_{n-1}R_{k+1} + R_k \ (k \geq 1)$. *Let* $N^* = \mathrm{g.c.d.}(R_2 - R_1,\ R_3 - R_2, \ldots, R_{n+1} - R_n)$. *Let* ζ *be a primitive* N^**-th root of unity and* $G = \langle(\zeta^{R_1}, \zeta^{R_1}, \zeta^{-1})\rangle$ *a cyclic group generated by* $(\zeta^{R_1}, \zeta^{R_1}, \zeta^{-1})$ *with componentwise multiplication. Then the values* $\Theta(x, a, q) = \sum_{k=1}^{\infty}\prod_{l=1}^{k}\frac{xq^{R_l}}{1 - aq^{R_l}}$ $((x, a, q) \in U)$ *are algebraically dependent if and only if there exist distinct* $(x_1, a_1, q_1), (x_2, a_2, q_2) \in U$ *such that* $(x_1/x_2, a_1/a_2, q_1/q_2) \in G$.

Complex functions of three variables having the (quasi) perfect algebraic independence property

Corollary (T, 2009)

Suppose in addition $\{R_k\}_{k\geq 1}$ *satisfies* $R_{k+n} = c_1 R_{k+n-1} + \cdots + c_{n-1} R_{k+1} + R_k \ (k \geq 1)$. *Let* $N^* = \mathrm{g.c.d.}(R_2 - R_1,\ R_3 - R_2, \ldots, R_{n+1} - R_n)$. *Let* ζ *be a primitive* N^**-th root of unity and* $G = \langle(\zeta^{R_1}, \zeta^{R_1}, \zeta^{-1})\rangle$ *a cyclic group generated by* $(\zeta^{R_1}, \zeta^{R_1}, \zeta^{-1})$ *with componentwise multiplication. Then the values* $\Theta(x, a, q) = \displaystyle\sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{xq^{R_l}}{1 - aq^{R_l}}$ $((x, a, q) \in U)$ *are algebraically dependent if and only if there exist distinct* $(x_1, a_1, q_1), (x_2, a_2, q_2) \in U$ *such that* $(x_1/x_2, a_1/a_2, q_1/q_2) \in G$.

By this corollary we have the example stated above.

Complex functions of three variables having the (quasi) perfect algebraic independence property

By this corollary we have the example stated above,

Complex functions of three variables having the (quasi) perfect algebraic independence property

By this corollary we have the example stated above, since $N^* = \mathrm{g.c.d.}(G_2 - G_1,\ G_3 - G_2) = \mathrm{g.c.d.}(b-1, b^2 - b + 1) = 1$.

Complex functions of three variables having the (quasi) perfect algebraic independence property

By this corollary we have the example stated above, since $N^* = \text{g.c.d.}(G_2 - G_1,\ G_3 - G_2) = \text{g.c.d.}(b-1, b^2-b+1) = 1$.

Let $\{G_k\}_{k\geq 0}$ be the generalized Fibonacci numbers defined by

$$G_0 = 0, \quad G_1 = 1, \quad G_{k+2} = bG_{k+1} + G_k \quad (k \geq 0),$$

where b is a positive integer. Then the infinite set

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{G_1+G_2+\cdots+G_k}}{(1-aq^{G_1})(1-aq^{G_2})\cdots(1-aq^{G_k})} \;\middle|\; \begin{array}{c} x, a, q \in \overline{\mathbb{Q}} \setminus \{0\}, \\ |a| \leq 1,\ |q| < 1 \end{array} \right\}$$

is algebraically independent.

Complex functions of three variables having the (quasi) perfect algebraic independence property

Theorem (T, 2009)

Let $\{R_k\}_{k\geq 1}$ be a linear recurrence of positive integers defined above. Suppose $\{R_k\}_{k\geq 1}$ is not a geometric progression. Let $\Phi(X) = X^n - c_1X^{n-1} - \cdots - c_n$. Assume that $\Phi(\pm 1) \neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Then the values

$$\Theta(x, a, q) = \sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{xq^{R_l}}{1 - aq^{R_l}} \quad ((x, a, q) \in U)$$

are algebraically dependent if and only if there exist distinct $(x_1, a_1, q_1), (x_2, a_2, q_2) \in U$ such that $(x_1, a_1, q_1) \sim (x_2, a_2, q_2)$.

Function field in positive characteristic

Let p be a prime and $\mathbb{F}_q$ the finite field of q elements with $q = p^e$.

Function field in positive characteristic

Let p be a prime and $\mathbb{F}_q$ the finite field of q elements with $q = p^e$.

Correspondence between the notions in function field over $\mathbb{F}_q$ and those of $\mathbb{Q}$ is as follows:
$A := \mathbb{F}_q[\theta] \longleftrightarrow \mathbb{Z}, \quad K := \mathbb{F}_q(\theta) \longleftrightarrow \mathbb{Q}$,
monic irreducible polynomial $\in A \longleftrightarrow$ prime $\in \mathbb{Z}$.
For any maximal ideal $P \subset A$, which is generated by a monic irreducible polynomial of A, we can construct the "P-adic" completion K_P of K in a similar way to construct the p-adic number field $\mathbb{Q}_p$.
For example, $K_{(\theta)} = \mathbb{F}_q((\theta))$.

Function field in positive characteristic

On the other hand, for $a = b/c \in K^\times$ with $b, c \in A\backslash\{0\}$, define $|a|_\infty := q^{\deg_\theta(a)}$, where $\deg_\theta(a) := \deg_\theta(b) - \deg_\theta(c)$.

Function field in positive characteristic

On the other hand, for $a = b/c \in K^\times$ with $b, c \in A\backslash\{0\}$, define $|a|_\infty := q^{\deg_\theta(a)}$, where $\deg_\theta(a) := \deg_\theta(b) - \deg_\theta(c)$.

Let $K_\infty = \mathbb{F}_q((1/\theta))$, which is the completion of K with respect to $|\cdot|_\infty$. Let $K_\infty^{\text{alg.}}$ be the algebraic closure of K_∞ and C_∞ the completion of $K_\infty^{\text{alg.}}$, which is algebraically closed.
We treat not only C_v with $v = \infty$ but also with $v = P$, the maximal ideal of A generated by a monic irreducible polynomial in A. For example, $C_{(\theta)}$ is the completion of $K_{(\theta)}^{\text{alg.}}$ with $K_{(\theta)} = \mathbb{F}_q((\theta))$.

The result

We give a positive characteristic analogue of the following result in all the complete, algebraically closed field C_v for any nontrivial absolute value $|\cdot|_v$ on K.

Theorem (Nishioka, 1996)

For an integer $d \geq 2$ and for $\beta \in \overline{\mathbb{Q}}$ with $0 < |\beta| < 1$, define $g(z) := \sum_{k=0}^{\infty} \beta^{d^k} z^k$. Then, the infinite set

$$\left\{ g^{(j)}(\alpha) \;\middle|\; j \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times} \right\} \subset \mathbb{C}$$

is algebraically independent over $\mathbb{Q}$.

Complex entire functions having the differential perfect algebraic independence property

The following complex entire function $g(z)$ has the differential perfect algebraic independence property, namely the infinite set $\{g^{(l)}(\alpha) \mid l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times}\}$ is algebraically independent:
Fix $\beta \in \overline{\mathbb{Q}}$ with $0 < |\beta| < 1$ and $d \in \mathbb{Z}$ with $d \geq 2$.

Nishioka (1986): $g(z) = \displaystyle\sum_{k=0}^{\infty} \beta^{k!} z^k$.

Nishioka (1996): $g(z) = \displaystyle\sum_{k=0}^{\infty} \beta^{d^k} z^k$.

T (1996): $g(z) = \displaystyle\sum_{k=0}^{\infty} \beta^{R_k} z^k$, where $\{R_k\}_{k\geq 0}$ belongs to a certain class of linear recurrences, which includes the sequence $\{F_k\}_{k\geq 0}$ of Fibonacci numbers.

The result

We give a positive characteristic analogue of the following result in all the complete, algebraically closed field C_v for any nontrivial absolute value $|\cdot|_v$ on K.

Theorem (Nishioka, 1996)

For an integer $d \geq 2$ and for $\beta \in \overline{\mathbb{Q}}$ with $0 < |\beta| < 1$, define $g(z) := \sum_{k=0}^{\infty} \beta^{d^k} z^k$. Then, the infinite set

$$\left\{ g^{(j)}(\alpha) \;\middle|\; j \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times} \right\} \subset \mathbb{C}$$

is algebraically independent over $\mathbb{Q}$.

Hasse-Teichmüller derivatives

Since the derivative $g^{(p)}(x)$ of order p vanishes over the field of characteristic $p > 0$, we consider Hasse-Teichmüller derivatives defined as follows instead of the usual derivatives:

Hasse-Teichmüller derivatives

Since the derivative $g^{(p)}(x)$ of order p vanishes over the field of characteristic $p > 0$, we consider Hasse-Teichmüller derivatives defined as follows instead of the usual derivatives: For any Laurent series $\sum_{k=m}^{\infty} c_k x^k \in R((x))$ with coefficients in any ring R and for any nonnegative integer j we define the Hasse-Teichmüller derivative $H^{(j)}$ of order j by

$$H^{(j)}\left(\sum_{k=m}^{\infty} c_k x^k\right) = \sum_{k=m}^{\infty} c_k \binom{k}{j} x^{k-j}.$$

The Hasse-Teichmüller derivatives satisfy the product rule, the quotient rule, and the chain rule.

Functions defined over function fields of positive characteristic, and having the differential perfect algebraic independence property

Theorem (Goto-T, submitted)

For an integer $d \geq 2$ not divisible by the characteristic p and for $\beta \in K^{\mathrm{alg.}}$ with $0 < |\beta|_v < 1$, define $g(z) := \sum_{k=0}^{\infty} \beta^{d^k} z^k$. Then, the infinite set

$$\left\{ H^{(j)} g(\alpha) \;\middle|\; j \in \mathbb{Z}_{\geq 0},\ \alpha \in \left(K^{\mathrm{alg.}}\right)^{\times} \right\} \subset C_v$$

is algebraically independent over K.

The result

For $g(z) := \sum_{k=0}^{\infty} \beta^{p^k} z^k$ we have

$$\alpha^p g(\alpha)^p = \sum_{k=0}^{\infty} \beta^{p^{k+1}} \alpha^{p(k+1)} = g(\alpha^p) - \beta.$$

The result

Not only in C_v with $v = \infty$ but also in C_v with $v = P$, the maximal ideal of A generated by a monic irreducible polynomial in A, we gave a positive characteristic analogue of the following

Theorem (Nishioka, 1996)

For an integer $d \geq 2$ and for $\beta \in \overline{\mathbb{Q}}$ with $0 < |\beta| < 1$, define $g(z) := \sum_{k=0}^{\infty} \beta^{d^k} z^k$. Then, the infinite set

$$\left\{ g^{(j)}(\alpha) \;\middle|\; j \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times} \right\} \subset \mathbb{C}$$

is algebraically independent over $\mathbb{Q}$.

Functions defined over function fields of positive characteristic, and having the differential perfect algebraic independence property

Theorem (Nishioka, 1996)

For an integer $d \geq 2$ and for $\beta \in \overline{\mathbb{Q}}$ with $0 < |\beta| < 1$, define $g(z) := \sum_{k=0}^{\infty} \beta^{d^k} z^k$. Then, the infinite set

$$\{g^{(j)}(\alpha) \mid j \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}}^{\times}\} \subset \mathbb{C}$$

is algebraically independent over $\mathbb{Q}$.

Theorem (Goto-T, submitted)

For an integer $d \geq 2$ not divisible by the characteristic p and for $\beta \in K^{\mathrm{alg.}}$ with $0 < |\beta|_v < 1$, define $g(z) := \sum_{k=0}^{\infty} \beta^{d^k} z^k$. Then, the infinite set

$$\{H^{(j)} g(\alpha) \mid j \in \mathbb{Z}_{\geq 0},\ \alpha \in \left(K^{\mathrm{alg.}}\right)^{\times}\} \subset C_v$$

is algebraically independent over K.

Thank you very much for your attention!