ALGEBRAIC INDEPENDENCE OF A CERTAIN SERIES AND ITS SUBSERIES WITH SUBSCRIPTS IN A GEOMETRIC PROGRESSION

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ABSTRACT. The main results of this paper, Theorems 4 and 5, assert the algebraic independence of a certain subseries $\sum_{n=1}^{\infty} a_n$ and its subseries $\sum_{n=1}^{\infty} a_{d^n}$ with d an integer greater than 1, where $\{a_n\}_{n\geq 1}$ is given, in Example 1 of Theorem 4, by $[\log_d n]/(F_nF_{n+2k})$ with $\{F_n\}_{n\geq 0}$ the sequence of Fibonacci numbers. These results are proved by using Mahler's method for algebraic independence with Theorems 1, 2 and Corollaries 1, 2 stating key formulas of this paper, which are deduced by using the crucial Lemma 1. For instance in the case of Example 1 of Theorem 4, Corollary 2 gives linear relations over $\mathbb{Q}$ between the numbers $\sum_{n=1}^{\infty}[\log_d n]/(F_nF_{n+2k})$, $\sum_{n=1}^{\infty} 1/(F_{d^n}F_{d^n+k})$ $(k \in \mathbb{N})$.

1. INTRODUCTION

Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0). \tag{1}$$

Brousseau [2] proved that for every $k \in \mathbb{N}$

$$\sigma_k = \sum_{n=1}^{\infty} \frac{(-1)^n}{F_nF_{n+k}} = \frac{1}{F_k}\left(\frac{k(1-\sqrt{5})}{2} + \sum_{n=1}^{k}\frac{F_{n-1}}{F_n}\right).$$

Rabinowitz [7] proved that for every $k \in \mathbb{N}$

$$\sigma_k^* = \sum_{n=1}^{\infty} \frac{1}{F_nF_{n+2k}} = \frac{1}{F_{2k}}\sum_{n=1}^{k}\frac{1}{F_{2n-1}F_{2n}}.$$

The author [8, 9] considered the arithmetic nature of the sums of similarly constructed series such as

$$\sum_{n=1}^{\infty} \frac{(-1)^n[\log_d n]}{F_nF_{n+k}} \quad (d \in \mathbb{N}\setminus\{1\},\ k \in \mathbb{N})$$

and

$$\sum_{n=1}^{\infty} \frac{[\log_d n]}{F_nF_{n+2k}} \quad (d \in \mathbb{N}\setminus\{1\},\ k \in \mathbb{N}),$$

where $[x]$ denotes the largest integer not exceeding the real number x. These sums are not only transcendental but also algebraically independent in contrast with the sums σ_k and σ_k^* which are algebraic numbers.

In what follows, let $\{R_n\}_{n\geq 0}$ be the linear recurrence defined by

$$R_{n+2} = A_1 R_{n+1} + A_2 R_n \quad (n \geq 0),$$

where R_0, R_1, A_1, and A_2 are real algebraic numbers with $A_1 A_2 \neq 0$, $A_1^2 + 4A_2 > 0$, and R_0, R_1 are not both zero. Then

$$R_n = a\alpha^n + b\beta^n \quad (n \geq 0),$$

where α, β ($|\alpha| \geq |\beta|$) are the roots of $\Phi(X) = X^2 - A_1 X - A_2$ and a, b are real algebraic numbers. It is easily seen that $|\alpha| > |\beta| > 0$. Assume in addition that $|a| \geq |b| > 0$. Then $\{R_n\}_{n\geq 0}$ is not a geometric progression and $R_n \neq 0$ for any $n \geq 1$. The sequence $\{F_n\}_{n\geq 0}$ of the Fibonacci numbers is an example of $\{R_n\}_{n\geq 0}$, since we have

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (n \geq 0).$$

Let $f(x)$ be a real-valued function on $x \geq 0$ such that $f'(x) > 0$ for any $x > 0$ and $f(\mathbb{N}) \subset \mathbb{N}$. Let $f^{-1}(x)$ be the inverse function of $f(x)$. For any $k \in \mathbb{N}$ we put

$$S_k = \sum_{n=f(1)}^{\infty} \frac{(-A_2)^n [f^{-1}(n)]}{R_n R_{n+k}}, \quad S_k^* = \sum_{n=f(1)}^{\infty} \frac{A_2^n [f^{-1}(n)]}{R_n R_{n+k}},$$

$$T_k = \sum_{n=f(1)}^{\infty} \frac{(-A_2)^n [f^{-1}(n)]}{R_{n+k-1} R_{n+k}},$$

and

$$U_k = \sum_{n=1}^{\infty} \frac{(-A_2)^{f(n)}}{R_{f(n)} R_{f(n)+k}}.$$

Let $\{F_n^*\}_{n\geq 0}$ be the Fibonacci type sequence defined by

$$F_0^* = 0, \quad F_1^* = 1, \quad F_{n+2}^* = A_1 F_{n+1}^* + A_2 F_n^* \quad (n \geq 0).$$

Theorem 1. *For any $k \in \mathbb{N}$*

$$S_k = \frac{1}{F_k^*} \sum_{l=1}^{k} (-A_2)^{l-1} T_l$$

and

$$U_k = \frac{1}{F_k^*} \left(T_1 - (-A_2)^k T_{k+1} \right).$$

Corollary 1. *For any $k \in \mathbb{N}$*

$$S_k = \frac{1}{F_k^*} \left(k S_1 - \sum_{l=1}^{k-1} F_l^* U_l \right).$$

Theorem 2. *If $f(n) \equiv f(1) \pmod 2$ for any $n \geq 1$, then*

$$S_{2k}^* = \frac{(-1)^{f(1)}}{F_{2k}^*} \sum_{l=1}^{2k} A_2^{l-1} T_l$$

for any $k \in \mathbb{N}$.

By Theorems 1 and 2 we have

Corollary 2. *If $f(n) \equiv f(1) \pmod 2$ for any $n \geq 1$, then*

$$S_{2k}^* = \frac{(-1)^{f(1)}}{F_{2k}^*} \sum_{l=1}^{2k-1} (-1)^{l+1} F_l^* U_l$$

for any $k \in \mathbb{N}$.

In what follows, let d be an integer greater than 1. Letting $f(x) = d^x$, we have the following algebraic independence result.

Theorem 3. *The numbers*

$$\sum_{n=1}^{\infty} \frac{n^l \xi^n (-A_2)^{d^n}}{R_{d^n} R_{d^n+k}} \quad (\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N}) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+1}}$$

are algebraically independent.

As a special case of Theorem 3 we have the following:

Corollary 3. *The numbers*

$$\sum_{n=1}^{\infty} \frac{(-A_2)^{d^n}}{R_{d^n} R_{d^n+k}}, \quad \sum_{n=1}^{\infty} \frac{n(-A_2)^{d^n}}{R_{d^n} R_{d^n+k}} \quad (k \in \mathbb{N}), \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+1}}$$

are algebraically independent.

Using Corollaries 2 and 3, we have the following:

Theorem 4. *The numbers*

$$\sum_{n=1}^{\infty} \frac{[\log_d n] A_2^n}{R_n R_{n+2k}}, \quad \sum_{n=1}^{\infty} \frac{n A_2^{d^n}}{R_{d^n} R_{d^n+2k}} \quad (k \in \mathbb{N})$$

are algebraically independent.

Proof. Letting $f(x) = d^x$, we see that

$$S_{2k}^* = \sum_{n=1}^{\infty} \frac{[\log_d n] A_2^n}{R_n R_{n+2k}} \quad \text{and} \quad U_k = \sum_{n=1}^{\infty} \frac{(-A_2)^{d^n}}{R_{d^n} R_{d^n+k}}.$$

By Corollary 2 the numbers $\{S_{2l}^* \mid 1 \le l \le k\}$ are expressed as linearly independent linear combinations over $\mathbb{Q}$ of the numbers $\{U_l \mid 1 \le l \le 2k-1\}$. Noting that $(-1)^{d^n} = (-1)^d$ for any $n \ge 1$, we have

$$\sum_{n=1}^{\infty} \frac{nA_2^{d^n}}{R_{d^n}R_{d^n+2k}} = (-1)^d \sum_{n=1}^{\infty} \frac{n(-A_2)^{d^n}}{R_{d^n}R_{d^n+2k}}.$$

Hence by Corollary 3, the proof of the theorem is completed. □

Example 1. Let $\{F_n\}_{n\ge0}$ be the sequence of the Fibonacci numbers defined by (1). Then the numbers

$$\sum_{n=1}^{\infty} \frac{[\log_d n]}{F_nF_{n+2k}}, \quad \sum_{n=1}^{\infty} \frac{n}{F_{d^n}F_{d^n+2k}} \quad (k \in \mathbb{N})$$

are algebraically independent.

Example 2. Let $\{L_n\}_{n\ge0}$ be the sequence of Lucas numbers defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \quad (n \ge 0). \tag{2}$$

Then the numbers

$$\sum_{n=1}^{\infty} \frac{[\log_d n]}{L_nL_{n+2k}}, \quad \sum_{n=1}^{\infty} \frac{n}{L_{d^n}L_{d^n+2k}} \quad (k \in \mathbb{N})$$

are algebraically independent.

It is interesting that the second series of Theorem 4 is regarded as a subseries of the first one obtained by replacing n by d^n. It seems difficult to find in literature the results which assert the algebraic independence of $\sum_{n=1}^{\infty} a_n$ and its subseries $\sum_{n=1}^{\infty} a_{d^n}$, where $\{a_n\}_{n\ge1}$ is a sequence of rational numbers such that $\sum_{n=1}^{\infty} a_n$ absolutely converges. For example, the algebraic independency of the numbers $\sum_{n=1}^{\infty} 1/F_n$ and $\sum_{n=1}^{\infty} 1/F_{d^n}$ $(d \ge 3)$ is open. On the other hand, Lucas [3] showed that $\sum_{n=1}^{\infty} 1/F_{2^n} = (5-\sqrt{5})/2$. André-Jeannin [1] proved the irrationality of $\sum_{n=1}^{\infty} 1/F_n$, while its transcendency is open. Nishioka, Tanaka, and Toshimitsu [6] proved that the numbers $\sum_{n=1}^{\infty} 1/F_{d^n}$ $(d \ge 3)$ are algebraically independent.

Combining Corollary 1 with $f(x) = d^x$ and Corollary 3, we immediately have the following:

Theorem 5. *The numbers*

$$\sum_{n=1}^{\infty} \frac{(-A_2)^n[\log_d n]}{R_nR_{n+k}}, \quad \sum_{n=1}^{\infty} \frac{n(-A_2)^{d^n}}{R_{d^n}R_{d^n+k}} \quad (k \in \mathbb{N})$$

are algebraically independent.

Example 3. Let $\{F_n\}_{n\ge0}$ be the sequence of the Fibonacci numbers defined by (1). Since $a_n = F_{2n}$ satisfies $a_{n+2} = 3a_{n+1} - a_n$ $(n \ge 0)$, the numbers

$$\sum_{n=1}^{\infty} \frac{[\log_d n]}{F_{2n}F_{2n+2k}}, \quad \sum_{n=1}^{\infty} \frac{n}{F_{2d^n}F_{2d^n+2k}} \quad (k \in \mathbb{N})$$

are algebraically independent.

Example 4. Let $\{L_n\}_{n\ge0}$ be the sequence of the Lucas numbers defined by (2). Since $b_n = L_{2n}$ satisfies $b_{n+2} = 3b_{n+1} - b_n$ $(n \ge 0)$, the numbers

$$\sum_{n=1}^{\infty} \frac{[\log_d n]}{L_{2n}L_{2n+2k}}, \quad \sum_{n=1}^{\infty} \frac{n}{L_{2d^n}L_{2d^n+2k}} \quad (k \in \mathbb{N})$$

are algebraically independent.

Example 5. Let $\{R_n\}_{n\ge0}$ be the linear recurrence defined by

$$R_0 = 2, \quad R_1 = \sqrt{5}, \quad R_{n+2} = \sqrt{5}R_{n+1} - R_n \quad (n \ge 0).$$

Then $R_nR_{n+2k-1} = \sqrt{5}(F_{2n+2k-1} + F_{2k-1})$ and $R_nR_{n+2k} = L_{2n+2k} + L_{2k}$, where $\{F_n\}_{n\ge0}$ and $\{L_n\}_{n\ge0}$ are the sequences of the Fibonacci numbers and the Lucas numbers defined respectively by (1) and (2), and so the numbers

$$\sum_{n=1}^{\infty} \frac{[\log_d n]}{F_{2n+2k-1}+F_{2k-1}}, \quad \sum_{n=1}^{\infty} \frac{[\log_d n]}{L_{2n+2k}+L_{2k}}, \quad \sum_{n=1}^{\infty} \frac{n}{F_{2d^n+2k-1}+F_{2k-1}}, \quad \sum_{n=1}^{\infty} \frac{n}{L_{2d^n+2k}+L_{2k}} \quad (k \in \mathbb{N})$$

are algebraically independent.

2. Lemmas

The following lemma plays an essential role in the proof of Theorems 1 and 2.

Lemma 1. *Let $f(x)$ be a real-valued function on $x \ge 0$ such that $f'(x) > 0$ for any $x > 0$ and $f(\mathbb{N}) \subset \mathbb{N}$. Let $f^{-1}(x)$ be the inverse function of $f(x)$. Let K be any field of characteristic 0 endowed with an absolute value $|\ \ |_v$. Let $\{a_n\}_{n\ge1}$ be a sequence in K with $|a_n|_v = o(1/f^{-1}(n))$. Suppose the sum $\sum_{n=1}^{\infty} |a_n|_v$ converges in $\mathbb{R}$. Then in the completion K_v of K we have*

$$\sum_{n=f(1)}^{\infty} [f^{-1}(n)](a_n - a_{n+1}) = \sum_{h=1}^{\infty} a_{f(h)}. \tag{3}$$

Proof. Let $h \in \mathbb{N}$ and $n \in \mathbb{N}$. Since $f'(x) > 0$ for any $x > 0$, $(f^{-1}(x))' > 0$ for any $x \geq f(1)$. Hence, if $f(h) \leq n < f(h+1)$, then $h \leq f^{-1}(n) < h+1$ and so $[f^{-1}(n)] = h$. Therefore, letting

$$\chi(n) = \begin{cases} 1 & (n = f(h)) \\ 0 & (\text{otherwise}) \end{cases} \quad \text{and} \quad s_n = \sum_{k=1}^{n} \chi(k),$$

we see that $s_n = [f^{-1}(n)]$ for $n \geq f(1)$. Then, letting $H \in \mathbb{N}$ and $N = f(H)$, we have

$$\begin{aligned} \sum_{h=1}^{H} a_{f(h)} &= \sum_{n=f(1)}^{N} \chi(n) a_n \\ &= \sum_{n=f(1)}^{N-1} s_n(a_n - a_{n+1}) + s_N a_N \\ &= \sum_{n=f(1)}^{N-1} [f^{-1}(n)](a_n - a_{n+1}) + [f^{-1}(N)] a_N. \end{aligned} \tag{4}$$

Since $|a_n|_v = o(1/f^{-1}(n))$, $[f^{-1}(N)]a_N$ tends to 0 as $N \to \infty$. Since $\sum_{n=1}^{\infty} |a_n|_v$ converges in $\mathbb{R}$, the sum of the subseries $\sum_{h=1}^{\infty} a_{f(h)}$ also converges in K_v. Letting $H \to \infty$ in (4), we have (3). □

Remark 1. The condition $|a_n|_v = o(1/f^{-1}(n))$ of Lemma 1 is satisfied if

$$|a_n|_v = o(n^{-1}), \tag{5}$$

since we have $[f^{-1}(n)] = s_n \leq n$. We shall use the condition (5) instead in the proof of Theorems 1 and 2.

Lemma 2 (A special case of Theorem 3.3.2 in Nishioka [4]). *Let K be an algebraic number field and d an integer greater than 1. Suppose that $f_{ij}(z) \in K[[z]]$ $(i = 1, \dots, m,\ j = 1, \dots, n(i))$ converge in $|z| < r$. Assume that, for every i, $f_{i1}(z), \dots, f_{in(i)}(z)$ satisfy the system of functional equations*

$$\begin{pmatrix} f_{i1}(z) \\ \vdots \\ \vdots \\ f_{in(i)}(z) \end{pmatrix} = \begin{pmatrix} a_i & 0 & \cdots & 0 \\ a_{21}^{(i)} & a_i & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ a_{n(i)1}^{(i)} & \cdots & a_{n(i)\,n(i)-1}^{(i)} & a_i \end{pmatrix} \begin{pmatrix} f_{i1}(z^d) \\ \vdots \\ \vdots \\ f_{in(i)}(z^d) \end{pmatrix} + \begin{pmatrix} b_{i1}(z) \\ \vdots \\ \vdots \\ b_{in(i)}(z) \end{pmatrix}, \tag{6}$$

where $a_i, a_{st}^{(i)} \in K$ and $b_{ij}(z) \in K(z)$. Let α be an algebraic number with $0 < |\alpha| < \min\{1, r\}$. If $f_{ij}(z)$ $(i = 1, \dots, m,\ j = 1, \dots, n(i))$ are algebraically independent over $K(z)$, then the values $f_{ij}(\alpha)$ $(i = 1, \dots, m,\ j = 1, \dots, n(i))$ are algebraically independent.

Remark 2. It is not necessary in Lemma 2 to assume that $b_{ij}(\alpha^{d^k})$ $(i = 1, \dots, m,\ j = 1, \dots, n(i))$ are defined for all $k \geq 0$, which is satisfied by (6) and the fact that $f_{ij}(\alpha^{d^k})$ $(i = 1, \dots, m,\ j = 1, \dots, n(i))$ are defined for all $k \geq 0$ since $|\alpha^{d^k}| < r$.

Lemma 3 (Theorem 3.2.1 in Nishioka [4]). *Let C be a field of characteristic 0. Suppose that $f_{ij}(z) \in C[[z]]$ $(i = 1, \dots, m,\ j = 1. \dots, n(i))$ satisfy the functional equations of the form (6) with $a_i, a_{st}^{(i)} \in C, a_i \neq 0, a_{s\,s-1}^{(i)} \neq 0$ $(2 \leq s \leq n(i))$, and $b_{ij}(z) \in C(z)$. If $f_{ij}(z)$ $(i = 1, \dots, m,\ j = 1. \dots, n(i))$ are algebraically dependent over $C(z)$, then there exists a nonempty subset $\{i_1, \dots, i_r\}$ of $\{1, \dots, m\}$ with $a_{i_1} = \cdots = a_{i_r}$ such that $f_{i_1 1}, \dots, f_{i_r 1}$ are linearly dependent over C modulo $C(z)$, that is, there exist $c_1, \dots, c_r \in C$, not all zero, such that*

$$c_1 f_{i_1 1} + \cdots + c_r f_{i_r 1} \in C(z).$$

Lemma 4 (Lemmas 2, 3, and 6 in Nishioka [5]). *Let ξ be a nonzero complex number and $a_1, \dots, a_n$ nonzero complex numbers satisfying $|a_i| \neq 1$, $|a_i| \neq |a_j|$ $(i \neq j)$. Let $f_i(z) \in \mathbb{C}[[z]]$ $(0 \leq i \leq n)$ satisfy the functional equations*

$$\begin{aligned} f_0(z) &= \xi f_0(z^d) + \frac{z^r}{1 + \varepsilon z^r}, \\ f_i(z) &= \xi f_i(z^d) + \frac{z^r}{1 + a_i z^r} \quad (1 \leq i \leq n), \end{aligned}$$

where $r \in \mathbb{N}$ and $\varepsilon = \pm 1$. If $d = \xi = 2$ and $\varepsilon = 1$, then $f_i(z)$ $(1 \leq i \leq n)$ are linearly independent over $\mathbb{C}$ modulo $\mathbb{C}(z)$, otherwise so are $f_i(z)$ $(0 \leq i \leq n)$.

Remark 3. If $d = \xi = 2$ and $\varepsilon = 1$, then

$$f_0(z) = \sum_{h=0}^{\infty} \frac{2^h z^{r2^h}}{1 + z^{r2^h}} = \frac{z^r}{1 - z^r} \in \mathbb{C}(z).$$

3. Proof of Theorems 1 and 2

Before stating the proof of Theorems 1 and 2, we recall that $\{R_n\}_{n \geq 0}$ is expressed as

$$R_n = a\alpha^n + b\beta^n \quad (n \geq 0),$$

where α, β are the roots of $\Phi(X) = X^2 - A_1 X - A_2$ such that $|\alpha| > |\beta| > 0$ and a, b are real algebraic numbers satisfying $|a| \geq |b| > 0$. Using the same α and β, we can express the sequence $\{F_n^*\}_{n \geq 0}$ defined before Theorem 1 by

$$F_n^* = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (n \geq 0).$$

Proof of Theorem 1. Since $R_n = a\alpha^n + b\beta^n$ $(n \geq 0)$ and $-A_2 = \alpha\beta$, we have

$$\begin{aligned}\frac{(-A_2)^n}{R_nR_{n+k}} &= \frac{1}{a(\alpha^k-\beta^k)}\left(\frac{\beta^n}{a\alpha^n+b\beta^n} - \frac{\beta^{n+k}}{a\alpha^{n+k}+b\beta^{n+k}}\right)\\ &= \frac{1}{a(\alpha^k-\beta^k)}\left(\frac{\beta^n}{R_n} - \frac{\beta^{n+k}}{R_{n+k}}\right).\end{aligned} \tag{7}$$

Hence, noting that $n|\beta^n/R_n| \to 0$ as $n \to \infty$, we have by Lemma 1 with Remark 1

$$\begin{aligned}S_k &= \frac{1}{a(\alpha^k-\beta^k)}\sum_{n=f(1)}^{\infty}[f^{-1}(n)]\left(\sum_{l=0}^{k-1}\frac{\beta^{n+l}}{R_{n+l}} - \sum_{l=0}^{k-1}\frac{\beta^{n+l+1}}{R_{n+l+1}}\right)\\ &= \frac{1}{a(\alpha^k-\beta^k)}\sum_{h=1}^{\infty}\sum_{l=0}^{k-1}\frac{\beta^{f(h)+l}}{R_{f(h)+l}}.\end{aligned} \tag{8}$$

Letting $k = 1$ and replacing n by $n + l - 1$ in (7), we have

$$\frac{(-A_2)^{n+l-1}}{R_{n+l-1}R_{n+l}} = \frac{1}{a(\alpha-\beta)}\left(\frac{\beta^{n+l-1}}{R_{n+l-1}} - \frac{\beta^{n+l}}{R_{n+l}}\right).$$

Hence by Lemma 1 with Remark 1

$$\begin{aligned}T_l &= \frac{(-A_2)^{1-l}}{a(\alpha-\beta)}\sum_{n=f(1)}^{\infty}[f^{-1}(n)]\left(\frac{\beta^{n+l-1}}{R_{n+l-1}} - \frac{\beta^{n+l}}{R_{n+l}}\right)\\ &= \frac{(-A_2)^{1-l}}{a(\alpha-\beta)}\sum_{h=1}^{\infty}\frac{\beta^{f(h)+l-1}}{R_{f(h)+l-1}}.\end{aligned} \tag{9}$$

Therefore by (8) and (9) we have

$$S_k = \frac{1}{F_k^*}\sum_{l=1}^{k}(-A_2)^{l-1}T_l.$$

Replacing n by $f(h)$ in (7), we have

$$\frac{(-A_2)^{f(h)}}{R_{f(h)}R_{f(h)+k}} = \frac{1}{a(\alpha^k-\beta^k)}\left(\frac{\beta^{f(h)}}{R_{f(h)}} - \frac{\beta^{f(h)+k}}{R_{f(h)+k}}\right).$$

Hence

$$U_k = \frac{1}{a(\alpha^k-\beta^k)}\sum_{h=1}^{\infty}\left(\frac{\beta^{f(h)}}{R_{f(h)}} - \frac{\beta^{f(h)+k}}{R_{f(h)+k}}\right)$$

and so

$$U_k = \frac{1}{F_k^*}\left(T_1 - (-A_2)^kT_{k+1}\right),$$

which completes the proof of the theorem. □

Proof of Theorem 2. Replacing k by $2k$ in (7) and multiplying its both sides by $(-1)^n = (-1)^{n+2k}$, we have

$$\begin{aligned}\frac{A_2^n}{R_nR_{n+2k}} &= \frac{1}{a(\alpha^{2k}-\beta^{2k})}\left(\frac{(-\beta)^n}{R_n} - \frac{(-\beta)^{n+2k}}{R_{n+2k}}\right)\\ &= \frac{1}{a(\alpha^{2k}-\beta^{2k})}\left(\sum_{l=0}^{2k-1}\frac{(-\beta)^{n+l}}{R_{n+l}} - \sum_{l=0}^{2k-1}\frac{(-\beta)^{n+l+1}}{R_{n+l+1}}\right).\end{aligned}$$

Hence, noting that $n|(-\beta)^n/R_n| \to 0$ as $n \to \infty$, we have by Lemma 1 with Remark 1

$$\begin{aligned}S_{2k}^* &= \frac{1}{a(\alpha^{2k}-\beta^{2k})}\sum_{n=f(1)}^{\infty}[f^{-1}(n)]\left(\sum_{l=0}^{2k-1}\frac{(-\beta)^{n+l}}{R_{n+l}} - \sum_{l=0}^{2k-1}\frac{(-\beta)^{n+l+1}}{R_{n+l+1}}\right)\\ &= \frac{1}{a(\alpha^{2k}-\beta^{2k})}\sum_{h=1}^{\infty}\sum_{l=0}^{2k-1}\frac{(-\beta)^{f(h)+l}}{R_{f(h)+l}}\\ &= \frac{1}{a(\alpha^{2k}-\beta^{2k})}\sum_{l=0}^{2k-1}(-1)^{l+f(1)}\sum_{h=1}^{\infty}\frac{\beta^{f(h)+l}}{R_{f(h)+l}},\end{aligned}$$

since $f(h) \equiv f(1) \pmod 2$ for any $h \geq 1$. Therefore we have by (9)

$$S_{2k}^* = \frac{(-1)^{f(1)}}{F_{2k}^*}\sum_{l=1}^{2k}A_2^{l-1}T_l,$$

which completes the proof of the theorem. □

4. Proof of Theorem 3

Remark 4. For $Q(z) \in \mathbb{C}(z)$ with $Q(0) = 0$, we define

$$f(x,z) = \sum_{n=1}^{\infty}x^nQ(z^{d^n}),$$

where x is a variable and d is an integer greater than 1. Letting $D = x\partial/\partial x$, we see that

$$f_l(x,z) := D^lf(x,z) = \sum_{n=1}^{\infty}n^lx^nQ(z^{d^n}) \qquad (l \geq 0)$$

satisfy

$$\begin{aligned}f_0(x,z) &= xf_0(x,z^d) + xQ(z^d),\\ f_1(x,z) &= xf_1(x,z^d) + xf_0(x,z^d) + xQ(z^d),\\ &\vdots\\ f_m(x,z) &= \sum_{l=0}^{m}\binom{m}{l}xf_l(x,z^d) + xQ(z^d).\end{aligned}$$

Hence for a complex number x, the functions $f_0(x,z),\dots,f_m(x,z)$ satisfy a system of functional equations of the form (6).

Proof of Theorem 3. Let $c=a^{-1}b$, $\gamma=\alpha^{-1}\beta$, and

$$f_{\xi lk}(z)=\sum_{n=1}^{\infty}n^l\xi^n\left(\frac{z^{d^n}}{1+cz^{d^n}}-\frac{\gamma^k z^{d^n}}{1+c\gamma^k z^{d^n}}\right)\quad(\xi\in\overline{\mathbb{Q}}^\times,\ l\geq 0,\ k\in\mathbb{N}).$$

Then

$$f_{\xi lk}(\gamma)=a^2(\alpha^k-\beta^k)\sum_{n=1}^{\infty}\frac{n^l\xi^n(-A_2)^{d^n}}{R_{d^n}R_{d^n+k}}. \tag{10}$$

Using (8) in the proof of Theorem 1 and letting $k=1$, $f(x)=d^x$, and $g(z)=\sum_{n=1}^{\infty}z^{d^n}/(1+cz^{d^n})$, we have

$$\sum_{n=1}^{\infty}\frac{(-A_2)^n[\log_d n]}{R_nR_{n+1}}=\frac{1}{a(\alpha-\beta)}\sum_{n=1}^{\infty}\frac{\beta^{d^n}}{R_{d^n}}=\frac{g(\gamma)}{a^2(\alpha-\beta)}. \tag{11}$$

Therefore it is enough by (10) and (11) to prove the algebraic independence of the values $f_{\xi lk}(\gamma)$ $(\xi\in\overline{\mathbb{Q}}^\times,\ l\geq 0,\ k\in\mathbb{N})$ and $g(\gamma)$. We see that each $f_{\xi 0k}(z)$ $(\xi\in\overline{\mathbb{Q}}^\times,\ k\in\mathbb{N})$ satisfies the functional equation

$$f_{\xi 0k}(z)=\xi f_{\xi 0k}(z^d)+\xi\left(\frac{z^d}{1+cz^d}-\frac{\gamma^k z^d}{1+c\gamma^k z^d}\right)$$

and $f_{\xi lk}(z)$ $(l\geq 0)$ satisfy a system of functional equations of the form (6) for every fixed ξ and k by Remark 4. We see also that $g(z)$ satisfies the functional equation

$$g(z)=g(z^d)+\frac{z^d}{1+cz^d}.$$

Hence by Lemma 2 the values $f_{\xi lk}(\gamma)$ $(\xi\in\overline{\mathbb{Q}}^\times,\ l\geq 0,\ k\in\mathbb{N})$ and $g(\gamma)$ are algebraically independent if the functions $f_{\xi lk}(z)$ $(\xi\in\overline{\mathbb{Q}}^\times,\ l\geq 0,\ k\in\mathbb{N})$ and $g(z)$ are algebraically independent over $\mathbb{C}(z)$.

We assert that for every fixed $\xi\neq 1$ the functions $f_{\xi 0k}(z)$ $(k\in\mathbb{N})$ are linearly independent over $\mathbb{C}$ modulo $\mathbb{C}(z)$ and so are the functions $f_{10k}(z)$ $(k\in\mathbb{N})$ with $g(z)$, which implies by Lemma 3 that the functions $f_{\xi lk}(z)$ $(\xi\in\overline{\mathbb{Q}}^\times,\ l\geq 0,\ k\in\mathbb{N})$ and $g(z)$ are algebraically independent over $\mathbb{C}(z)$. Let

$$h_{\xi k}(z)=\sum_{n=1}^{\infty}\frac{\gamma^k\xi^n z^{d^n}}{1+c\gamma^k z^{d^n}}\quad(\xi\in\overline{\mathbb{Q}}^\times,\ k\geq 0).$$

Then

$$f_{\xi 0k}(z)=h_{\xi 0}(z)-h_{\xi k}(z)$$

for every fixed $\xi\in\overline{\mathbb{Q}}^\times$ and $k\in\mathbb{N}$ and each $h_{\xi k}(z)$ $(\xi\in\overline{\mathbb{Q}}^\times,\ k\geq 0)$ satisfies the functional equation

$$h_{\xi k}(z)=\xi h_{\xi k}(z^d)+\frac{\xi\gamma^k z^d}{1+c\gamma^k z^d}.$$

Suppose there exists a $\xi\neq 1$ such that $f_{\xi 01}(z),\dots,f_{\xi 0k}(z)$ are linearly dependent over $\mathbb{C}$ modulo $\mathbb{C}(z)$ for some k. If $d=\xi=2$ and $c=1$, we see by Remark 3 that $h_{20}(z)=2z^2/(1-z^2)\in\mathbb{C}(z)$ and so $h_{21}(z),\dots,h_{2k}(z)$ are linearly dependent over $\mathbb{C}$ modulo $\mathbb{C}(z)$; otherwise, so are $h_{\xi 0}(z),h_{\xi 1}(z),\dots,h_{\xi k}(z)$, which contradicts Lemma 4, since $H_{\xi k}(z):=\xi^{-1}\gamma^{-k}h_{\xi k}(z)$ satisfies the functional equation

$$H_{\xi k}(z)=\xi H_{\xi k}(z^d)+\frac{z^d}{1+c\gamma^k z^d}.$$

Therefore, if $f_{\xi lk}(z)$ $(\xi\in\overline{\mathbb{Q}}^\times,\ l\geq 0,\ k\in\mathbb{N})$ and $g(z)=h_{10}(z)$ are algebraically dependent over $\mathbb{C}(z)$, then $h_{10}(z),f_{101}(z),\dots,f_{10k}(z)$ are linearly dependent over $\mathbb{C}$ modulo $\mathbb{C}(z)$ for some k, and hence so are $h_{10}(z),h_{11}(z),\dots,h_{1k}(z)$, which contradicts Lemma 4. Therefore the functions $f_{\xi lk}(z)$ $(\xi\in\overline{\mathbb{Q}}^\times,\ l\geq 0,\ k\in\mathbb{N})$ and $g(z)$ are algebraically independent over $\mathbb{C}(z)$ and so the values $f_{\xi lk}(\gamma)$ $(\xi\in\overline{\mathbb{Q}}^\times,\ l\geq 0,\ k\in\mathbb{N})$ and $g(\gamma)$ are algebraically independent, which completes the proof of the theorem. □

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