Anticyclotomic main conjecture for modular forms and integral Perrin-Riou twists

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Abstract.

We prove a one-sided divisibility relation for the anticyclotomic Iwasawa main conjecture for modular forms in terms of the p-adic L-function constructed by Bertolini-Darmon-Prasanna and Brakočević. The divisibility relation is known by Castella if p is ordinary and by Castella-Wan for the elliptic curve case. Here we prove the higher weight non-ordinary case with a treatment that works uniformly for both ordinary and non-ordinary cases. In the proof, we establish a theory of integral Perrin-Riou twist. It enables us not only to twist systems of generalized Heegner cycles (which are not norm-compatible) by any continuous p-adic anticyclotomic characters but also to investigate the denominators of resulting systems explicitly.

§1. Introduction

1.1. Our setting of the main conjecture

The aim of this paper is to prove a one-sided divisibility relation for the anticyclotomic Iwasawa main conjecture for modular forms. The anticyclotomic Iwasawa theory has a long history and there are many works in different settings. Here we clarify our setting comparing with others.

Let f be a normalized (elliptic) eigen newform of weight k for $\Gamma_0(N)$. Let K be an imaginary quadratic field. We consider a factorization $N = N^+N^-$ where a prime factor of N divides N^+ (resp. N^-) if and

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only if it is split (resp. inert or ramified) in K. The arithmetic of f over K depends significantly on the condition on N^- and the Iwasawa theory also depends on the condition on a fixed prime p (ordinary or not for f, split or not in K). For example, the classical Heegner hypothesis is that $N^- = 1$. Then the sign of the functional equation of the L-function of f over K is -1 and we have special cycles called Heegner cycles. The Gross-Zagier formula relates the derivative of the L-function at the central point to the height of the Heegner cycle of conductor 1. The existence of such special cycles gives a special flavor in the anticyclotomic Iwasawa theory. (We recall it in §1.2 below.) On the other hand, if N^- is a square-free product of the *odd* number of inert primes, the sign of the functional equation is $+1$. Special cycles do not exist directly, however, Bertolini-Darmon [4] studied the Iwasawa theory in the ordinary case of weight 2 by considering congruences from Heegner points on various Shimura curves. Chida-Hsieh [14] generalized their work to the higher weight ordinary case. If f has complex multiplication by our fixed imaginary quadratic field K (hence N^- contains ramified primes and it is not square-free), there are works by Agboola-Howard ([1], [2]).

In this paper, we assume the classical Heegner hypothesis $N^- = 1$ and our prime p splits in K and $p \nmid N$ but there is no condition on the ordinarity for f. We think our method also works when N^- is a square-free product of the *even* number of inert primes.

1.2. Background

Here we recall the background of our work. The origin goes back to the study of the behavior of the Mordell-Weil rank of elliptic curves in $\mathbb{Z}_p$-extensions by Mazur (cf. [26]). The behavior in the anticyclotomic $\mathbb{Z}_p$-extension (with the classical Heegner hypothesis) is special because of the existence of a system of Heegner points of higher conductors. Then Perrin-Riou [29] formulated an Iwasawa theoretic conjecture on Heegner points, which is considered as the Iwasawa main conjecture in this context. Since our result is intrinsically related to her conjecture, we recall it briefly.

Let E be an elliptic curve defined over $\mathbb{Q}$ with conductor N. Let K be an imaginary quadratic field satisfying the classical Heegner hypothesis. Let p be a good *ordinary* prime for E. Let K_∞/K be the anticyclotomic $\mathbb{Z}_p$-extension and put $\Lambda := \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$. Let $\mathcal{X}_\infty$ be the Pontryagin dual of the discrete p-Selmer group of E over K_∞. We also consider the projective limit of compact Selmer groups $\mathcal{S}_\infty := \varprojlim_n \mathrm{Sel}(K_n, T_pE)$, which naturally contains the limit of the Mordell-Weil groups $\varprojlim_n E(K_n) \otimes \mathbb{Z}_p$ (In fact, we have $\mathcal{S}_\infty = \varprojlim_n E(K_n) \otimes \mathbb{Z}_p$ if the Tate-Shafarevich groups are finite). We take a modular prametrization $\pi : X_0(N) \to E$ and then the system of Heegner points defines a Λ-submodule $\mathcal{H}_\infty$ in $\varprojlim_n E(K_n) \otimes \mathbb{Z}_p$. We remark that the natural system of Heegner points of p-power conductors is not norm compatible and here we need the ordinarity condition on p for E to stabilize the system to obtain the module $\mathcal{H}_\infty$. We come back to this point later when we consider the non-ordinary case.

Conjecture 1.1. (*Perrin-Riou*)

(1) *$\mathcal{S}_\infty$ is a free Λ-module of rank 1.*
(2) *There exists a finitely generated torsion Λ-module $\mathcal{M}$ such that $\mathcal{X}_\infty$ is pseudo-isomorphic to $\Lambda \oplus \mathcal{M} \oplus \mathcal{M}$.*
(3) *We have the equation of characteristic ideals*

$$\mathrm{Char}(\mathcal{M}) = c_\pi^{-1} u_K^{-1} \mathrm{Char}(\mathcal{S}_\infty / \mathcal{H}_\infty)$$

where c_π is the Manin constant of π and $u_K := (\sharp \mathcal{O}_K^\times)/2$.

This conjecture is a Λ-adic version of Kolyvagin's result relating the square root of the order of the Tate-Shafarevich group to the index of the Heegner point in the Mordell-Weil group (if the Heegner point is non-trivial). By Cornut, Vatsal and Howard ([15], [39], [18]), now we know (1), (2) and a one-sided divisibility for (3) under mild assumptions. Recently, there is progress on the other divisibility for (3) (See [8], [40].) There is also a generalization of the Perrin-Riou conjecture to modular forms of higher weight by using (generalized) Heegner cycles or a Hida theoretic deformation of Heegner points. See Castella [10] and Longo-Vigni [25].

In the Perrin-Riou conjecture, the ordinary condition on p for E is essential, otherwise, $\mathcal{H}_\infty = 0$ since there is no universal norm (or norm compatible system of rational points) for E in the supersingular case. However, the system of Heegner points of p-power conductors satisfies a certain norm relation associated to the p-Euler factor of E. (See [29] or [20].) One remedy is to use the idea in [19] to decompose the system to obtain a norm compatible system non-canonically, that is, to consider the plus/minus version of the Perrin-Riou conjecture. In fact, if $a_p(E) := 1 + p - \sharp E(\mathbb{F}_p) = 0$, Castella-Wan [13] formulated the plus/minus Perrin-Riou conjecture and obtained similar results known in the ordinary case. It seems to be possible to generalize their method to more general settings by using the idea by Sprung in [38]. However, in this paper, we do not pursue this direction, instead, we use another formulation of the conjecture that works for the both ordinary and non-ordinary cases equally well.

In general, we have (at least) two types of the formulation of the Iwasawa main conjecture. One uses a system of "zeta elements" and the other uses a p-adic L-function, and both formulations are important. For example, in the classical Iwasawa theory of cyclotomic fields, cyclotomic units are used as zeta elements and played the key role to prove the one-sided divisibility via the Euler system in the main conjecture. On the other hand, the Kubota-Leopoldt p-adic L-function is used in the other formulation where the divisibility in the opposite direction is shown by using congruences of modular forms. In the Perrin-Riou conjecture, Heegner points are considered as zeta elements and form an Euler system. It is natural to expect the formulation of the main conjecture in terms of p-adic L-function equivalent to the Perrin-Riou conjecture.

Though the Perrin-Riou conjecture was formulated in 1980's, the corresponding p-adic L-function was found almost 30 years later by Bertolini-Darmon-Prasanna [5] and Brakočević [7]. (For simplicity, we call it BDP-B's p-adic L-function.) Its square interpolates the special values of the L-function of f over K twisted by anticyclotomic characters with appropriate infinity type. See §2.2 for details. One can then formulate an Iwasawa main conjecture in terms of BDP-B's p-adic L-function following the philosophy or recipe of Bloch-Kato or Greenberg. However, it is surprising (at least for the authors) that BDP-B's p-adic L-function lives in the Iwasawa algebra Λ without having huge denominators even in the non-ordinary case (though the condition p splits in K is still very important), and the formulation of the main conjecture via BDP-B's p-adic L-function works for the both ordinary and non-ordinary cases equally well without any modification. This phenomenon is explained by the fact that BDP-B's case (twists of the Galois representation of f by anticyclotomic characters with split p) satisfies the Panchishkin condition even when p is non-ordinary for f. Castella [10, 11] showed that at ordinary primes, this main conjecture is equivalent to the Perrin-Riou conjecture (and its generalization to the higher weight case). For elliptic curves with $a_p(E) = 0$, Castella-Wan [13] showed the equivalence to the plus/minus Perrin-Riou conjecture. In this paper, though we do not formulate the Perrin-Riou type conjecture in terms of generalized Heegner cycles in the non-ordinary case, we prove a one-sided divisibility in the main conjecture via BDP-B's p-adic L-function in the higher weight non-ordinary case (cf. Theorem 1.5).

1.3. Difficulties in the non-ordinary higher weight case

Concerning the dependence on the ordinarity condition on p for f, it is interesting to compare with the cyclotomic Iwasawa theory for modular forms. In the cyclotomic case, there is a formulation of the main conjecture by Kato using his zeta elements, which works for the both ordinary and non-ordinary primes (even for bad primes) equally well. On the other hand, the cyclotomic p-adic L-function (by Amice-Vélu-Vishik) has huge denominators if p is non-ordinary and it is not straightforward to formulate the main conjecture in terms of the p-adic L-function. (One needs Perrin-Riou's local theory or plus/minus theory, etc.) We explained that the anticyclotomic case was the contrary. In fact, in the following sense, the anticyclotomic case is more subtle.

In the cyclotomic case, the obstruction of denominators comes p-locally. That is, the p-adic L-function is the image of the Perrin-Riou (or Coleman) map of zeta elements and the denominator comes from the Perrin-Riou map, which is p-locally defined. Kato zeta elements are integral and norm compatible in the p-power direction. The norm compatibility plays an important role when we consider the Soulé twist. In fact, Kato first constructed the system related to non-critical values of the L-function and used the Soulé twist and the explicit reciprocity law to obtain the system related to critical values.

In the anticyclotomic case, the obstruction of the denominator comes globally. Since the defining interpolation range of BDP-B's p-adic L-function is different from that of (generalized) Heegner cycles, we need a Soulé type twist and an explicit reciprocity law to relate these. However, the system of generalized Heegner cycles is not norm compatible in the p-power direction, and if p is non-ordinary, we need denominators for the stabilization. These denominators break not only the principle of the Soulé twist but also the argument of the Euler system itself. The problem of the Soulé twist was solved in [21] for twists of *algebraic* anticyclotomic characters but it was not sufficient for proving the divisibility in the main conjecture. In this paper, we consider the integral Perrin-Riou twist generalizing the twist of [21], and it can be applied for *all* continuous anticyclotomic p-adic characters. (Precicely speaking, our twisting method is not completely identical to that in [21].)

Readers may think that the language of the analytic distribution seems more natural to describe our integral Perrin-Riou twist. In fact, such approach is taken in [23] to twist Beilinson-Flach elements. However, in our case, we needed more subtle calculations of denominators to check the axiom of the Euler system at $l \neq p$ (cf. Lemma 4.5 (2)) in cohomology groups with huge torsion subgroups.

1.4. The conjecture and our main result

In order to state our main result precisely, we fix notation. Let $f \in S_{2r}(\Gamma_0(N))$ be a normalized eigen newform of weight $k = 2r \geq 2$ for

$\Gamma_0(N)$. Let K be an imaginary quadratic field with discriminant $-D_K$ satisfying the classical Heegner hypothesis, that is, the rational primes dividing N split in K. Let p be an odd prime, and fix embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. We assume that p splits in K and write $(p) = \mathfrak{p}\overline{\mathfrak{p}}$, where $\mathfrak{p}$ denotes the prime in K above p compatible with ι_p. Let F_f be the minimal field in $\mathbb{C}_p$ containing $\mathbb{Q}_p$ and all the Fourier coefficients a_n of f, and let $\mathscr{O}_f$ be its ring of integers. Let $V_f \cong F_f^{\oplus 2}$ be the Galois representation attached to f and T_f an $\mathscr{O}_f$-lattice of V_f stable under the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. Let K_∞ be the anticyclotomic $\mathbb{Z}_p$-extension of K and put $\Gamma = \mathrm{Gal}(K_\infty/K)$. We put $\Lambda = \mathscr{O}_f[[\Gamma]]$ and $\Lambda^{\mathrm{ur}} = \hat{\mathscr{O}}_f^{\mathrm{ur}}[[\Gamma]]$, where $\hat{\mathscr{O}}_f^{\mathrm{ur}}$ denotes the ring of integers in the p-adic completion of the maximal unramified extension of F_f. We denote BDP-B's p-adic L-function by $\mathscr{L}_\mathfrak{p}(f) \in \Lambda^{\mathrm{ur}}$, whose square interpolates algebraic parts of $L(f,\chi,r)$ for anticyclotomic Hecke characters χ of infinity type $(j,-j)$ with $j \geq r$. We note that $\mathscr{L}_\mathfrak{p}(f)$ depends on the choices of periods, and we refer the reader to Subsection 2.2 for details. We put $W = (V_f/T_f)(r)$, where (r) denotes the r-th Tate twist, and define

(1.1)
$$H^1_{\emptyset,0}(K_\infty, W) = \mathrm{Ker}\left(H^1(K_\infty, W) \to \prod_{w\mid\overline{\mathfrak{p}}} H^1(K_{\infty,w}, W) \times \prod_{w'\nmid p} H^1(K^{\mathrm{ur}}_{\infty,w'}, W)\right)$$

where w (resp. w') runs over all places of K_∞ dividing $\overline{\mathfrak{p}}$ (resp. not dividing p). Let $X_{\emptyset,0}(K_\infty, W)$ be the Pontryagin dual of $H^1_{\emptyset,0}(K_\infty, W)$. It is a finitely generated Λ-module for the canonical Λ-action.

Castella formulated the anticyclotomic Iwasawa main conjecture in terms of BDP-B's p-adic L-function at ordinary primes ([10, 11].) Since our representation satisfies the Panchishkin condition, it is natural to expect the anticyclotomic Iwasawa main conjecture precisely in the same form as for the ordinary primes even for non-ordinary primes:

Conjecture 1.2. (1) *The Λ-module $X_{\emptyset,0}(K_\infty, W)$ is torsion.*
(2) *We have*
$$\mathrm{Char}(X_{\emptyset,0}(K_\infty, W)) \otimes_\Lambda \Lambda^{\mathrm{ur}} = \mathscr{L}_\mathfrak{p}(f)^2 \Lambda^{\mathrm{ur}}$$
where Char *denotes the characteristic ideal of finitely generated torsion Λ-modules.*

Twisting by Hecke characters, the above conjecture is equivalent to the Iwasawa main conjecture of the following form that relates the p-adic L-function to the Bloch-Kato Selmer group.

Conjecture 1.3. *Let ψ be an anticyclotomic Hecke character of infinity type $(j,-j)$ such that $j \geq r$ and the p-adic avatar $\hat{\psi}$ factors* *thorough* Γ. *Let* $H^1_{\mathrm{f}}(K_\infty, W(\psi^{-1}))$ *be the Bloch-Kato Selmer group associated to the twist* $W(\psi^{-1})$ *of* W *by* $\hat{\psi}^{-1}$ *(see Remark 2.6 and (2.6) for details). Then, its Pontryagin dual* $H^1_{\mathrm{f}}(K_\infty, W(\psi^{-1}))^\vee$ *is a finitely generated torsion* Λ*-module, and*

$$\mathrm{Char}(H^1_{\mathrm{f}}(K_\infty, W(\psi^{-1}))^\vee) \otimes_\Lambda \Lambda^{\mathrm{ur}} = \mathrm{Tw}_\psi(\mathscr{L}_{\mathfrak{p}}(f)^2)\Lambda^{\mathrm{ur}}$$

where $\mathrm{Tw}_\psi : \Lambda^{\mathrm{ur}} \to \Lambda^{\mathrm{ur}}$ *denotes the twist by* $\hat{\psi}$.

Remark 1.4. (1) By definition, $\mathrm{Tw}_\psi(\mathscr{L}_{\mathfrak{p}}(f)^2)$ interpolates the algebraic parts of $L(f, \chi^{-1}\psi^{-1}, r)$ for anticyclotomic Hecke characters χ such that the infinity type of $\chi\psi$ is of the form $(i, -i)$ with $i \geq r$.

(2) Our Selmer group is pseudo-isomorphic to Greenberg's Selmer group introduced in [17] (see Proposition 2.8 for details).

(3) Conjecture 1.2 (2) is independent of the choice of the lattice T_f assuming Conjecture 1.2 (1). (cf. Proposition 2.9.)

The following is our main result, which gives an evidence of the above conjecture.

Theorem 1.5. *Suppose that* $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ *and* D_K *is odd or divisible by* 8. *Assume that* $p \nmid 6h_K$ *where* h_K *denotes the class number of* K. *Assume also that the image of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}_{\mathscr{O}_f}(T_f) \cong \mathrm{GL}_2(\mathscr{O}_f)$ *contains the subgroup consisting of the elements* $g \in \mathrm{GL}_2(\mathbb{Z}_p)$ *such that* $\det(g) \in (\mathbb{Z}_p^\times)^{2r-1}$. *Then, Conjecture 1.2 (1) holds, and we have*

$$\mathscr{L}_{\mathfrak{p}}(f)^2 \in \mathrm{Char}(X_{\emptyset,0}(K_\infty, W)) \otimes_\Lambda \Lambda^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \tag{1.2}$$

We note that by twisting (1.2), we also have a similar result for Conjecture 1.3 as follows.

Theorem 1.6. *Let* ψ *be an anticyclotomic character as in Conjecture 1.3. Under the same assumption as in Theorem 1.5, the finitely generated* Λ*-module* $H^1_{\mathrm{f}}(K_\infty, W(\psi^{-1}))^\vee$ *is torsion, and we have*

$$\mathrm{Tw}_\psi(\mathscr{L}_{\mathfrak{p}}(f)^2) \in \mathrm{Char}(H^1_{\mathrm{f}}(K_\infty, W(\psi^{-1}))^\vee) \otimes_\Lambda \Lambda^{\mathrm{ur}}.$$

Recently, a similar result is obtained by K. Büyükboduk and Lei in a slightly different setting by a completely different method using Beilinson-Flach elements ([9]).

1.5. Plan of our proof

Our strategy is as follows. First, for sufficiently many continuous characters $\chi : \Gamma \to \overline{\mathbb{Q}}_p^\times$, we compare the specialization of both sides of Conjecture 1.2 (2). Here we use the Euler system obtained by our integral Perrin-Riou twist of generalized Heegner cycles by χ. Next, by using a certain control theorem, we glue up the comparisons between specializations to prove our main result. The key point is that in terms of generalized Heegner cycles, a naive gluing argument does not work in the non-ordinary case but it works in the formulation in terms of BDP-B's p-adic L-function.

More precisely, our proof consists of the following three steps.

Step 1: For almost all height-one prime ideals $\mathfrak{P} \neq p\Lambda$ of Λ, we compare the following two quantities:

- The p-adic valuation of $\chi_{\mathfrak{P}}(\mathscr{L}) \in \overline{\mathbb{Q}}_p$, where $\mathscr{L} \in \Lambda$ is a generator of the ideal $\mathscr{L}_{\mathfrak{p}}(f)\Lambda^{\mathrm{ur}} \cap \Lambda$, and $\chi_{\mathfrak{P}} : \Gamma \to \overline{\mathbb{Q}}_p^{\times}$ is the continuous character induced by choosing an embedding $\Lambda/\mathfrak{P} \hookrightarrow \overline{\mathbb{Q}}_p$.
- The length of the Selmer group $H^1_{\emptyset,0}(K, W(\chi_{\mathfrak{P}}^{-1}))$.

By our integral Perrin-Riou twist, we obtain an Euler system related to $\chi_{\mathfrak{P}}(\mathscr{L})$ from generalized Heegner cycles. Then, by similar arguments to those in the ordinary case (cf. [10]), we complete Step 1 (cf. Proposition 5.4).

Step 2: We prove a control theorem which says that the length of $H^1_{\emptyset,0}(K, W(\chi_{\mathfrak{P}}^{-1}))$ and that of $H^1_{\emptyset,0}(K_\infty, W)[\mathfrak{P}^\iota]$ (the part killed by $\mathfrak{P}^\iota$) are the same up to a constant when $\mathfrak{P}$ varies. Here, $\iota : \Lambda \to \Lambda$ is the involution sending $g \in \Gamma$ to g^{-1}, and $\mathfrak{P}^\iota := \iota(\mathfrak{P})$. See Proposition 5.7 for the details.

Step 3: Let $\mathfrak{Q} \neq p\Lambda$ be a height-one prime ideal of Λ, and we take a certain sequence $\{\mathfrak{P}^{(m)}\}_{m\geq 0}$ of height-one prime ideals such that $\mathrm{Im}(\chi_{\mathfrak{Q}}\chi_{\mathfrak{P}^{(m)}}^{-1}) \subseteq 1 + p^m\overline{\mathbb{Z}}_p$, where $\overline{\mathbb{Z}}_p$ denotes the ring of integers in $\overline{\mathbb{Q}}_p$. Since

$$X_{\emptyset,0}(K_\infty, W)/\mathfrak{P}^{(m)} = \mathrm{Hom}(H^1_{\emptyset,0}(K_\infty, W)[\mathfrak{P}^{(m),\iota}], \mathbb{Q}_p/\mathbb{Z}_p),$$

the control theorem above relates the asymptotic behavior of the length of $H^1_{\emptyset,0}(K, W(\chi_{\mathfrak{P}^{(m)}}^{-1}))$ (as m varies) to $\mathrm{ord}_{\mathfrak{Q}}(\mathrm{Char}(X_{\emptyset,0}(K_\infty, W)))$. On the other hand, the asymptotic behavior of the p-adic valuation of the evaluation $\chi_{\mathfrak{P}^{(m)}}(\mathscr{L})$ is related to $\mathrm{ord}_{\mathfrak{Q}}(\mathscr{L})$. Then, by using the first step, we conclude that

$$2\,\mathrm{ord}_{\mathfrak{Q}}(\mathscr{L}) \geq \mathrm{ord}_{\mathfrak{Q}}(\mathrm{Char}(X_{\emptyset,0}(K_\infty, W))).$$

Step 3 is completed in Subsection 5.3.

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§2. The p-adic L-function and the Selmer group

In this section, we recall the p-adic L-function of Bertolini-Darmon-Prasanna and Brakočević and basic properties of our Selmer group.

2.1. Notation and settings

Here we fix our notation and settings that are used throughout this paper. For a number field or valuation field L, we denote by $\mathscr{O}_L$ the ring of integers in L. For a number field L and a place of v, we write its completion at v by L_v, its adèle ring by $\mathbb{A}_L$, and the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$ by G_L. For an algebraic extension L of $\mathbb{Q}$ and a place v of L, we denote by L_v the union of the completion at v of finite extensions in L. For a local field L, we denote the maximal unramified extension of L in the algebraic closure $\overline{L}$ by L^{ur} and its completion by $\widehat{L^{\mathrm{ur}}}$.

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a normalized eigen newform of weight $k = 2r$ for $\Gamma_0(N)$. Let K be an imaginary quadratic field with discriminant $-D_K$ satisfying the classical Heegner hypothesis for f, and let $\mathfrak{N}$ be an ideal of $\mathscr{O}_K$ such that $\mathscr{O}_K/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$. We also assume that $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ and D_K is odd or divisible by 8.

For a natural number c, let $K[c]$ be the ring class field of K of conductor c. We put $K[cp^\infty] := \cup_{n=1}^{\infty} K[cp^n]$. Let K_∞ be the anticyclotomic $\mathbb{Z}_p$-extension of K, that is, the unique extension of K in $K[p^\infty]$ whose Galois group $\Gamma = \mathrm{Gal}(K_\infty/K)$ is isomorphic to $\mathbb{Z}_p$. For an integer $m \geq 0$, let K_m be the unique subfield of K_∞ with $[K_m : K] = p^m$.

Let $p \nmid 2N$ be a prime which splits in K and write $p\mathscr{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$, where $\mathfrak{p}$ is the prime ideal of $\mathscr{O}_K$ above p compatible with a fixed embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. We also fix an embedding $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and denote by $\tau \in G_{\mathbb{Q}}$ the complex conjugation induced by ι_∞. Let F_f be the minimal field extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$ containing all the Fourier coefficients of f via ι_∞ and ι_p. We denote by $V_f \cong F_f^{\oplus 2}$ the Galois representation of Deligne attached to f. Let $\mathscr{O}_f$ be the ring of integers in F_f and T_f an $\mathscr{O}_f$-lattice of V_f which is stable under the action of $G_{\mathbb{Q}}$.

Let $\mathscr{O}$ be the ring of integers in a finite extension F of F_f containing the roots of $X^2 - a_p X + p^{2r-1}$. In the rest of this paper, by abuse of notation, we denote by T_f (resp. V_f) the scalar extension $T_f \otimes_{\mathscr{O}_f} \mathscr{O}$ (resp.

$V_f \otimes \mathscr{O}$). We note that the scalar extension is harmless to prove the main result (Theorem 1.5).

2.2. Hecke characters and p-adic L-functions

Let $I(D_K)$ be the group of fractional ideals of K relatively prime to D_K. Then, by [34] (see also [41]) there exists a Hecke character $\varphi_K : I(D_K) \to \mathbb{C}^\times$ such that

- the conductor $\mathfrak{f}$ of φ_K is divisible only by primes dividing D_K,
- for $\mathfrak{a} \in I(D_K)$, we have $\varphi_K(\overline{\mathfrak{a}}) = \overline{\varphi_K(\mathfrak{a})}$,
- for $\alpha \in K^\times$ relatively prime to $\mathfrak{f}$, we have $\varphi_K(\alpha\mathscr{O}_K) = \pm\alpha$.

(Note that $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$.) Let H be the Hilbert class field of K and A an elliptic curve over H with complex multiplication by $\mathscr{O}_K$ such that $j(A) = j(\mathscr{O}_K)$ and its Serre-Tate character ψ_A is given by $\psi_A = \varphi_K \circ N_{H/K}$, where $N_{H/K}$ is the norm map.

For a place v of $\mathbb{Q}$, let $N_v : (H \otimes \mathbb{Q}_v)^\times \to (K \otimes \mathbb{Q}_v)^\times$ be the norm map. We denote by $\tilde{\psi}_A : \mathbb{A}_H^\times \to K^\times$ the idelic Serre-Tate character associated to ψ_A. (cf. [36, Theorem 10].) We define

$$\psi_{A,v} : \mathbb{A}_H^\times \to (K \otimes \mathbb{Q}_v)^\times, \qquad a \mapsto \tilde{\psi}_A(a)N_v(a_v)^{-1},$$

where a_v denotes the component of a in $(H \otimes \mathbb{Q}_v)^\times$, and then $\psi_{A,v}$ factors through $\mathbb{A}_H^\times/H^\times$. By class field theory, $\psi_{A,v}$ factors through the Artin map $\mathrm{art}_H : \mathbb{A}_H^\times/H^\times \to \mathrm{Gal}(H^{\mathrm{ab}}/H)$, and induces a map $G_H \to (K \otimes \mathbb{Q}_v)^\times$, which is also denoted by $\psi_{A,v}$. For $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, we denote by $\psi_\mathfrak{q} : G_H \to K_\mathfrak{q}^\times$ the composite

$$G_H \xrightarrow{\psi_{A,p}} (K \otimes \mathbb{Q}_p)^\times \to K_\mathfrak{q}^\times,$$

where the last map is the projection induced by $K \otimes \mathbb{Q}_p = K_\mathfrak{p} \oplus K_{\overline{\mathfrak{p}}}$. We note that G_H acts on $A[\mathfrak{p}^n]$ and $A[\overline{\mathfrak{p}}^n]$ by $\psi_\mathfrak{p}$ and $\psi_{\overline{\mathfrak{p}}}$, respectively. By identifying $K \otimes \mathbb{R}$ with $\mathbb{C}$ via the fixed embedding $\iota_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$, we obtain a Hecke character $\psi_{A,\infty} : \mathbb{A}_H^\times/H^\times \to \mathbb{C}^\times$.

We denote by $\mathfrak{p}_0$ the prime ideal of H above $\mathfrak{p}$ compatible with ι_p. We fix a minimal Weierstrass model of A over the localization of $\mathscr{O}_H$ at $\mathfrak{p}_0$ and write the Néron differential by ω_A. We fix an isomorphism $t_{p^\infty} : \hat{\mathbb{G}}_m \cong \hat{A}$ of formal groups over $\widehat{\mathbb{Z}_p^{\mathrm{ur}}}$ and a complex uniformization $t_\infty : \mathbb{C}/\mathscr{O}_K \cong A(\mathbb{C})$. Then, there exists $(\Omega_K, \Omega_p) \in \mathbb{C}^\times \times \widehat{\mathbb{Z}_p^{\mathrm{ur}}}^\times$ such that

$$\Omega_K 2\pi i dz = t_\infty^*(\omega_A), \quad \Omega_p \frac{dT}{1+T} = t_{p^\infty}^*(\omega_A),$$

where z and T denote the standard coordinate of $\mathbb{C}/\mathscr{O}_K$ and $\hat{\mathbb{G}}_m$, respectively.

Now we recall the p-adic L-function constructed in [5, 7]. We say that a Hecke character $\chi : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ is of infinity type (m,n) if $\chi(a) = a^{-m}\overline{a}^{-n}$ for $a \in (K \otimes \mathbb{R})^\times$, where $\overline{a}$ denotes the complex conjugation of a, and we identify $(K \otimes \mathbb{R})^\times$ with $\mathbb{C}^\times$ by ι_∞ and put it in the archimedean component in $\mathbb{A}_K^\times$. We note that the sign of our definition of infinity type is the same as that in [5] and opposed to that in [12]. We denote by Σ the set of anticyclotomic Hecke characters whose infinity type is of the form $(j,-j)$. Then $s = r$ is a critical point of $L(f,\chi^{-1},s)$ for $\chi \in \Sigma$ in the sense of Deligne. We consider the decomposition $\Sigma = \Sigma^{(1)} \cup \Sigma^{(2)} \cup \Sigma^{(2')}$, where

$$\begin{aligned}\Sigma^{(1)} &= \{\, \chi \text{ is of infinity type } (j,-j) \mid -r < j < r \,\},\\ \Sigma^{(2)} &= \{\, \chi \text{ is of infinity type } (j,-j) \mid j \geq r \,\},\\ \Sigma^{(2')} &= \{\, \chi \text{ is of infinity type } (j,-j) \mid j \leq -r \,\}.\end{aligned}$$

Then a character $\chi \in \Sigma$ lies in $\Sigma^{(1)}$ if and only if the sign of the functional equation of $L(f,\chi^{-1},s)$ is equal to -1 (cf. [5, §4.1]).

We put $\widehat{K} = K \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, where $\widehat{\mathbb{Z}} = \prod_{l:\text{primes}} \mathbb{Z}_l$. For a Hecke character $\varphi : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ of infinity type (m,n), we define the p-adic avatar $\hat{\varphi} : \widehat{K}^\times/K^\times \to \mathbb{C}_p^\times$ of φ by

$$\hat{\varphi}(a) = \iota_p \circ \iota_\infty^{-1}(\varphi(a)) a_{\mathfrak{p}}^{-m} a_{\overline{\mathfrak{p}}}^{-n}.$$

Here, $a_{\mathfrak{p}}$ and $a_{\overline{\mathfrak{p}}}$ denote the $\mathfrak{p}$-component and $\overline{\mathfrak{p}}$-component of a, respectively. We note that the image of $\widehat{K}^\times$ under φ lies in a finite extension of K. By class field theory, we obtain $\hat{\varphi} : G_K \to \overline{\mathbb{Q}}_p^\times$.

Let $\mathcal{G}_{p^\infty} = \mathrm{Gal}(K[p^\infty]/K)$ be the Galois group of the tower of the ring class fields of p-power conductor over K. Let $\mathscr{L}_{\mathfrak{p}}{}^{\mathrm{BDP-B}}(f) \in \hat{\mathscr{O}}^{\mathrm{ur}}[[\mathcal{G}_{p^\infty}]]$ be the p-adic L-function constructed in [5, 7] characterized by the following interpolation property: for every $\chi \in \Sigma^{(2)}$ of infinity type $(j,-j)$ such that $\hat{\chi}$ factors through $\mathcal{G}_{p^\infty}$, we have

$$\left(\frac{\hat{\chi}(\mathscr{L}_{\mathfrak{p}}{}^{\mathrm{BDP-B}}(f))}{\Omega_p^{2j}}\right)^2 = L^{\mathrm{alg}}(f,\chi^{-1},r) e_{\mathfrak{p}}(f,\chi^{-1}) \chi(\mathfrak{N}) 2^3 \varepsilon(f) u_K^2 \sqrt{D_K}, \tag{2.3}$$

where $L^{\mathrm{alg}}(f,\chi^{-1},r) \in \overline{\mathbb{Q}}$ is equal to $L(f,\chi^{-1},r)/\Omega_K^{4j}$ up to explicit factors. We refer the reader to [12, Proposition 3.8] for details (the sign of infinity type in *loc. cit.* is opposed to ours). We note that the twist of $\iota(\mathscr{L}_{\mathfrak{p}}{}^{\mathrm{BDP-B}}(f))$ by $\psi \in \Sigma^{(2')}$ coincides with $\mathscr{L}_{\mathfrak{p},\psi}(f)$ of [12, Definition 3.7], where $\iota : \hat{\mathscr{O}}^{\mathrm{ur}}[[\mathcal{G}_{p^\infty}]] \to \hat{\mathscr{O}}^{\mathrm{ur}}[[\mathcal{G}_{p^\infty}]]$ is induced by the involution $\mathcal{G}_{p^\infty} \to \mathcal{G}_{p^\infty}$ sending g to g^{-1}.

2.3. Selmer groups

In this subsection, we fix general terminologies on Selmer groups.

Let V be a finite dimensional F-vector space with continuous G_K-action. We assume that for almost all primes v of K, the representation V is unramified at v. Let T be an $\mathscr{O}_F$-lattice of V which is stable under the G_K-action. We put $W = V/T$.

Given a Selmer structure $\mathscr{F}$ on V (cf. [18, Definition 1.1.10]), by abuse of notation, we denote by the same symbol $\mathscr{F}$ the Selmer structures on T and W defined as follows. Let L be a finite extension of K. For a prime v of L, we let $H^1_{\mathscr{F}}(L_v, T)$ (resp. $H^1_{\mathscr{F}}(L_v, W)$) be the inverse image (resp. the image) of $H^1_{\mathscr{F}}(L_v, V)$ under the natural map $H^1(L_v, T) \to H^1(L_v, V)$ (resp. $H^1(L_v, V) \to H^1(L_v, W)$). For the maximal ideal $\mathfrak{m}$ of $\mathscr{O}$, we also define Selmer structures on $T/\mathfrak{m}^n$ and $W[\mathfrak{m}^n]$ by letting $H^1_{\mathscr{F}}(L_v, T/\mathfrak{m}^n)$ (resp. $H^1_{\mathscr{F}}(L_v, W[\mathfrak{m}^n])$) be the image (resp. the inverse image) of $H^1_{\mathscr{F}}(L_v, T)$ (resp. $H^1_{\mathscr{F}}(L_v, W)$) under $H^1(L_v, T) \to H^1(L_v, T/\mathfrak{m}^n)$ (resp. $H^1(L_v, W[\mathfrak{m}^n]) \to H^1(L_v, W)$).

The Bloch-Kato local condition is given by

$$H^1_{\mathrm{f}}(L_v, V) = \begin{cases} \mathrm{Ker}\left(H^1(L_v, V) \to H^1(L, V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})\right) & \text{if } v \mid p, \\ H^1_{\mathrm{ur}}(L_v, V) := \mathrm{Ker}\left(H^1(L_v, V) \to H^1(L_v^{\mathrm{ur}}, V)\right) & \text{if } v \nmid p. \end{cases} \tag{2.4}$$

Definition 2.1. Let $\mathscr{F}, \mathscr{G}$ be Selmer structures on V such that for every prime v of L not dividing p, we have $H^1_{\mathscr{F}}(L_v, V) = H^1_{\mathscr{G}}(L_v, V) = H^1_{\mathrm{f}}(L_v, V)$. Let M be one of the representations $V, T, T/\mathfrak{m}^n, W$ and $W[\mathfrak{m}^n]$. Then we define $H^1_{\mathscr{F},\mathscr{G}}(L, M)$ as the kernel of

$$H^1(L, M) \to \prod_{v \mid \mathfrak{p}} \frac{H^1(L_v, M)}{H^1_{\mathscr{F}}(L_v, M)} \times \prod_{v \mid \overline{\mathfrak{p}}} \frac{H^1(L_v, M)}{H^1_{\mathscr{G}}(L_v, M)} \times \prod_{v \nmid p} \frac{H^1(L_v, M)}{H^1_{\mathrm{f}}(L_v, M)}.$$

If $\mathscr{F}$ satisfies $H^1_{\mathscr{F}}(L_v, V) = \{0\}$ (resp. $H^1_{\mathscr{F}}(L_v, V) = H^1(L_v, V)$) for $v \mid \mathfrak{p}$ then we denote the Selmer group $H^1_{\mathscr{F},\mathscr{G}}(L, M)$ by $H^1_{0,\mathscr{G}}(L, M)$ (resp. $H^1_{\emptyset,\mathscr{G}}(L, M)$). We similarly define $H^1_{\mathscr{F},0}(L, M)$, $H^1_{\mathscr{F},\emptyset}(L, M)$, $H^1_{0,\emptyset}(L, M)$ and $H^1_{\emptyset,0}(L, M)$. We put

$$X_{\mathscr{F}}(L, W) = \left(H^1_{\mathscr{F}}(L, W)\right)^{\vee}, \quad X_{\mathscr{F},\mathscr{G}}(L, W) = \left(H^1_{\mathscr{F},\mathscr{G}}(L, W)\right)^{\vee},$$

where $(\quad)^{\vee}$ denotes the Pontryagin dual.

For a $\mathbb{Z}_p$-extension K_∞ of K, we define the Selmer groups over K_∞ as $H^1_{\mathscr{F},\mathscr{G}}(K_\infty, M) = \varinjlim_n H^1_{\mathscr{F},\mathscr{G}}(K_n, M)$ where $\mathscr{F}$, $\mathscr{G}$ are one of 0, $\emptyset$ and the Bloch-Kato local condition. Then its Pontryagin dual $X_{\mathscr{F},\mathscr{G}}(K_\infty, W)$ is a finitely generated $\mathcal{O}_F[[\mathrm{Gal}(K_\infty/K)]]$-module.

Now we define Greenberg's Selmer groups. Suppose that for each $v|p$ a subrepresentation F_v^+V of $V|_{G_{K_v}}$ is given. Then F_v^+M is defined similarly and we put

$$H^1_{\mathrm{Gr}}(L_v, M) := \operatorname{Ker}\left(H^1(L_v, M) \to H^1(K_v^{\mathrm{ur}}, M/F_v^+M)\right).$$

Greenberg's Selmer group $H^1_{\mathrm{Gr}}(L, M)$ is defined to be $H^1_{\mathrm{Gr},\mathrm{Gr}}(L, M)$. The group $H^1_{\mathrm{Gr}}(K_\infty, M)$ is also defined similarly.

2.4. The main theorem

Let $\Lambda^{\mathrm{ur}} = \hat{\mathscr{O}}^{\mathrm{ur}}[[\Gamma]]$, and we denote by $\mathscr{L}_{\mathfrak{p}}(f) \in \Lambda^{\mathrm{ur}}$ the image of $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP-B}}(f)$ under the natural projection $\hat{\mathscr{O}}^{\mathrm{ur}}[[\mathcal{G}_{p^\infty}]] \to \Lambda^{\mathrm{ur}}$. We put $\Lambda = \mathscr{O}[[\Gamma]]$ and $W = (V_f/T_f)(r)$. We now state the anticyclotomic Iwasawa main conjecture.

Conjecture 2.2. (1) *The Λ-module $X_{\emptyset,0}(K_\infty, W)$ is torsion.*
(2) *We have* $\mathscr{L}_{\mathfrak{p}}(f)^2\Lambda^{\mathrm{ur}} = \mathrm{Char}(X_{\emptyset,0}(K_\infty, W)) \otimes_\Lambda \Lambda^{\mathrm{ur}}$.

We assume the following.

Condition 2.3. (1) $p \nmid 6h_K$. Fixing an $\mathscr{O}_f$-basis of T_f, the image of $G_{\mathbb{Q}} \to \mathrm{Aut}_{\mathscr{O}}(T_f) \cong \mathrm{GL}_2(\mathscr{O}_f)$ contains

$$\{g \in \mathrm{GL}_2(\mathbb{Z}_p) \mid \det(g) \in (\mathbb{Z}_p^\times)^{2r-1}\}.$$

Remark 2.4. If f is non-CM, then Condition 2.3 is satisfied for almost all primes p ([33]). The condition $p \nmid h_K$ is assumed in [18], and we use it only when we use results in in [18].

Our main theorem is the following.

Theorem 2.5. *We assume Condition 2.3. Then*

(1) *Conjecture 2.2 (1) holds.*
(2) *We have*

$$\mathscr{L}_{\mathfrak{p}}(f)^2 \in \mathrm{Char}(X_{\emptyset,0}(K_\infty, W)) \otimes_\Lambda \Lambda^{\mathrm{ur}} \otimes \mathbb{Q}_p. \tag{2.5}$$

Remark 2.6. For $\psi \in \Sigma^{(2)}$, the Hodge-Tate weights of $V(\psi^{-1})|_{G_{K_{\mathfrak{p}}}}$ are all positive, and those of $V(\psi^{-1})|_{G_{K_{\overline{\mathfrak{p}}}}}$ are less than or equal to zero, where $V(\psi^{-1})$ denotes $V(\hat{\psi}^{-1})$, and in our convention the Hodge-Tate weight of $\mathbb{Q}_p(1)$ is 1. In particular, for $n \geq 0$ we have

$$H^1_{\mathrm{f}}(K_{n,\mathfrak{p}}, V(\psi^{-1})) = H^1(K_{n,\mathfrak{p}}, V(\psi^{-1})), \quad H^1_{\mathrm{f}}(K_{n,\bar{\mathfrak{p}}}, V(\psi^{-1})) = \{0\},$$

where $K_{n,v}$ ($v \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$) denotes the completion of K_n at any prime above v. Hence the Bloch-Kato Selmer group $H^1_{\mathrm{f}}(K_\infty, W(\psi^{-1}))$ is explicitly written as

$$\mathrm{Ker}\left(H^1_{\emptyset,\emptyset}(K_\infty, W(\psi^{-1})) \to \prod_{w|\overline{\mathfrak{p}}} H^1(K_{\infty,w}, W(\psi^{-1})) \right), \tag{2.6}$$

and twisting (2.5) by $\hat{\psi}$ yields a similar result for Conjecture 1.3. This also explains why the Selmer structure on $X_{\emptyset,0}(K_\infty, W)$ is natural.

In [17], for Galois representations satisfying the Panchishkin condition, Greenberg has formulated Iwasawa main conjectures (without precise formulation of p-adic L-functions). We note that by Remark 2.6, for $\psi \in \Sigma^{(2)}$, the Galois representation $V(\psi^{-1})$ satisfies the Panchishkin condition. In the rest of this subsection, we compare the Bloch-Kato Selmer group with Greenberg's Selmer group with $F_v^+ V(\psi^{-1}) = V(\psi^{-1})$ for $v|\mathfrak{p}$ and $F_v^+ V(\psi^{-1}) = \{0\}$ for $v|\overline{\mathfrak{p}}$. Explicitly, our Greenberg's Selmer group is given by

$$H^1_{\mathrm{Gr}}(K_\infty, W(\psi^{-1})) = \mathrm{Ker}\left(H^1_{\emptyset,\emptyset}(K_\infty, W(\psi^{-1})) \to \prod_{w|\overline{\mathfrak{p}}} H^1(K^{\mathrm{ur}}_{\infty,w}, W(\psi^{-1})) \right).$$

We also define $H^1_{\mathrm{Gr}}(K_\infty, W)$ by twisting the above.

Lemma 2.7. *For a place w of K_∞ dividing p, $H^0(K_{\infty,w}, V) = \{0\}$, and $H^0(K_{\infty,w}, W)$ is finite.*

Proof. Assume that $V_0 := H^0(K_{\infty,w}, V) \neq \{0\}$. Then, by the existence of a $G_{\mathbb{Q}}$-equivariant, non-degenerate paring $V \times V \to F(1)$, we have $\dim_F(V_0) = 1$. Let E be an elliptic curve with complex multiplication by $\mathscr{O}_K$ defined over a finite field $\mathbb{F}_q$ of characteristic p. We denote by α, β the roots of the p-Euler factor of E. Since V_0 is a one-dimensional crystalline representation of $\mathrm{Gal}(K_{\infty,w}/K_v)$, it is a power of the anticyclotomic character up to a finite character. Hence, if we denote by $D_{\mathrm{crys}}(V_0)$ the filtered φ-module associated to V_0, then a non-zero power of the eigenvalue λ_{V_0} of the Frobenius on $D_{\mathrm{crys}}(V_0)$ is equal to $(\alpha/\beta)^m$ for some $m \in \mathbb{Z}$. By the Weil conjecture for E, α and β are q-Weil numbers. Hence, λ_{V_0} is a 1-Weil number. However, the Weil conjecture for f implies that λ_{V_0} must be a p-Weil number, which is a contradiction. Hence, $H^0(K_{\infty,w}, V) = \{0\}$.

We next prove that $H^0(K_{\infty,w}, W)$ is finite. We denote by $G_{\infty,w}$ the absolute Galois group of $K_{\infty,w}$. Assume that $H^0(K_{\infty,w}, W)$ is infinite. Then, for $m \geq 1$, by identifying $W[p^m] = T/p^m$, the group $(T/p^m)^{G_{\infty,w}}$

has an element of order p^m. Hence, for $n \geq 1$

$$\mathfrak{T}_n := \varprojlim_{k \leq n} \left(T/p^k \setminus \{0\}\right)^{G_{\infty,w}} \times \prod_{m \geq n} T/p^m$$

is non-empty, closed subset of $\prod_{m\geq 1} T/p^m$. Since $\mathfrak{T}_n \supseteq \mathfrak{T}_{n+1}$, the family $\{\mathfrak{T}_n\}_{n\geq 1}$ has the finite intersection property. Hence, by the compactness of $\prod_{m\geq 1} T/p^m$, we have $\cap_{n\geq 1}\mathfrak{T}_n \neq \emptyset$, and then

$$T^{G_{\infty,w}} \setminus \{0\} = \varprojlim_m (T/p^m)^{G_{\infty,w}} \setminus \{0\} \neq \emptyset,$$

which contradicts that $H^0(K_{\infty,w}, V) = \{0\}$. Q.E.D.

Proposition 2.8. *The following assertions hold.*

(1) *The Λ-modules $X_{\emptyset,0}(K_\infty, W)$ and $H^1_{\mathrm{Gr}}(K_\infty, W)^\vee$ are pseudo-isomorphic.*

(2) *The Λ-modules $H^1_{\mathrm{f}}(K_\infty, W(\psi^{-1}))^\vee$ and $H^1_{\mathrm{Gr}}(K_\infty, W(\psi^{-1}))^\vee$ are pseudo-isomorphic.*

Proof. Since

$$H^1_{\emptyset,0}(K_\infty, W)(\psi^{-1}) \cong H^1_{\mathrm{f}}(K_\infty, W(\psi^{-1})),$$
$$H^1_{\mathrm{Gr}}(K_\infty, W)(\psi^{-1}) \cong H^1_{\mathrm{Gr}}(K_\infty, W(\psi^{-1}))$$

as Λ-modules, the assertion (2) follows from (1). We prove (1). By definition, we have an exact sequence

$$0 \to H^1_{\emptyset,0}(K_\infty, W) \to H^1_{\mathrm{Gr}}(K_\infty, W) \to \oplus_{w|\overline{\mathfrak{p}}} H^1(K^{\mathrm{ur}}_{\infty,w}/K_{\infty,w}, W^{I_w}),$$

where I_w denotes the inertia subgroup of G_w. Hence, it suffices to show that for each $w \mid \overline{\mathfrak{p}}$ the module $H^1(K^{\mathrm{ur}}_{\infty,w}/K_{\infty,w}, W^{I_w})$ is finite.

We denote by $\mathrm{Fr}_w \in G_w/I_w$ the Frobenius map of w, which topologically generates $G_w/I_w \cong \widehat{\mathbb{Z}}$. Then, there exists an exact sequence

$$0 \to H^0(K_{\infty,w}, W) \to W^{I_w} \to W^{I_w} \to H^1(K^{\mathrm{ur}}_{\infty,w}/K_{\infty,w}, W^{I_w}) \to 0,$$

where the map $W^{I_w} \to W^{I_w}$ is given by $\mathrm{Fr}_w - 1$. Hence, by Lemma 2.7, the group $H^1(K^{\mathrm{ur}}_{\infty,w}/K_{\infty,w}, W^{I_w})$ is finite. Q.E.D.

Proposition 2.9. *Assume that Conjecture 1.2 (1) is true. Then Conjecture 1.2 (2) is independent of the choice of the lattice T_f.*

Proof. First, by construction, the p-adic L-function $\mathscr{L}_{\mathfrak{p}}(f)$ is independent of T_f. (The complex period related to the interpolation property is the CM period of K and not the period of f.) For the characteristic ideal, the choice of the lattice affects only the μ-invariant. Since our Selmer group is pseudo-isomorphic to Greenberg's one, we can use Perrin-Riou's formula in [30, §1] to calculate the difference. Suppose that T_1 and T_2 are lattices of $V_f(r)$ stable under the Galois action such that $T_1 \subseteq T_2$. We denote by μ_i the μ-invariant of the Selmer group associated to $V_f(r)/T_i$. Then the theorem in [30, §1] implies that

$$\mu_2 - \mu_1 = \mathrm{ord}_p(\#H^0(\mathbb{C}, C)) - \sum_{v|p} \mathrm{ord}_p(\#(F_v C)),$$

where $C := T_2/T_1$, v ranges over all primes dividing p, and the filtrations $F_v C$ on C are given by $F_{\mathfrak{p}} C = C$ and $F_{\overline{\mathfrak{p}}} C = \{0\}$. Hence, $\mu_2 - \mu_1$ is equal to $\mathrm{ord}_p(\#C) - \mathrm{ord}_p(\#C) = 0$, and hence the μ-invariant is independent of the choice of the lattice. The case where T_1 is not contained in T_2 is reduced to the above case by considering $T_1 \cap T_2 \subseteq T_i$. Precisely speaking, the Galois representation in [30] is assumed to be ordinary and the ordinary filtration is used to define the Greenberg Selmer group. However, we checked that the proof therein works for any local filtration over p if the associated Greenberg Selmer group and its contragredient are Λ-cotortion. In particular, since our V is self-dual, if we assume Conjecture 1.2 (1), Perrin-Riou's formula works for our filtration. Q.E.D.

§3. Integral Perrin-Riou twists

In this section, we give a generalization of the Perrin-Riou twist with integral coefficients which works also for global Galois representations (cf. Proposition 3.7). Although a global generalization of Perrin-Riou twist is given by Loeffler-Zerbes ([23]), our advantage is to construct twisted elements with explicit expression, which enables us to investigate the denominators of twists precisely. This expression is crucial to study the images of generalized Heegner classes in the local cohomology groups with huge torsions.

Since this section is independent of the other sections of this paper and we consider general cases, we start with fixing notation.

3.1. Setup

Let p be a prime. Let L be a finite extension of $\mathbb{Q}$ or of $\mathbb{Q}_l$ for some prime l, which is allowed to be p. Let L_∞/L be a $\mathbb{Z}_p$-extension and L_n the n-th layer. We put $\Gamma = \mathrm{Gal}(L_\infty/L)$ and fix a topological generator $\gamma \in \Gamma$. Let F be a finite extension of $\mathbb{Q}_p$ with the integer ring

$\mathscr{O}$, and let V be a finite-dimensional F-vector space with continuous G_L-action. Let $\rho : \Gamma \to \mathscr{O}^\times$ be a non-trivial, continuous character such that $\mathrm{Im}(\rho) \subseteq 1+2p\mathscr{O}$. Let T be an $\mathscr{O}$-lattice stable under the G_L-action.

For $m \geq 0$ and $h \geq 1$ we put

$$\omega_{m,h,\rho}(X) = \prod_{0 \leq i \leq h-1} (\rho(\gamma)^{-ip^m}(1+X)^{p^m} - 1).$$

When there is no fear of confusion, to simplify the notation we write $\omega_{m,h} = \omega_{m,h,\rho}$. For a positive integer i and an element a of a commutative $\mathbb{Q}$-algebra, we put

$$\binom{a}{i} = \frac{a(a-1)(a-2)\cdots(a-i+1)}{i!}.$$

We also put $\binom{a}{0} = 1$.

For a continuous character $\chi : \Gamma \to \mathbb{C}_p^\times$, we denote by $\mathscr{O}(\chi)$ the representation of G_L whose underlying space is the completion of the ring of integers in $F(\mathrm{Im}(\chi))$ and whose G_L-action is given by χ. We denote by e_χ the basis of $\mathscr{O}(\chi)$ corresponding to $1 \in F(\mathrm{Im}(\chi))$. For a topological $\mathscr{O}$-module M with continuous G_L-action, we put $M(\chi) = M \otimes_{\mathscr{O}} \mathscr{O}(\chi)$, and we note that there is a canonical isomorphism of Γ-modules

$$H^1(L_\infty, M(\chi)) \cong H^1(L_\infty, M)(\chi).$$

By this isomorphism, we often identify the two Γ-modules.

For $h \geq 1$ and a subfield $H \subseteq \mathbb{C}_p$, we put

$$\mathscr{H}_{h,H}(\Gamma) = \left\{ \sum_{n \geq 0} a_n(\gamma-1)^n \in H[[\gamma-1]] \;\middle|\; \lim_{n\to\infty} n^{-h}|a_n|_p \to 0 \right\} \tag{3.7}$$

where $|\cdot|_p$ denotes the multiplicative valuation on $\mathbb{C}_p$ normalized by $|p|_p = p^{-1}$. We put $\mathscr{H}_{\infty,H}(\Gamma) = \cup_{m\geq 1}\mathscr{H}_{m,H}(\Gamma)$ and

$$H^1_{\mathrm{Iw}}(L,T) = \varprojlim_m H^1(L_m,T), \quad H^1_{\mathrm{Iw}}(L,V) = H^1_{\mathrm{Iw}}(L,T) \otimes \mathbb{Q}_p.$$

For a continuous character $\chi : \Gamma \to \mathbb{C}_p^\times$, we define

$$\mathrm{pr}_{n,\chi} : H^1_{\mathrm{Iw}}(L,T) \to H^1(L_n, T(\chi))$$

as the composite

$$H^1_{\mathrm{Iw}}(L,T) \to H^1_{\mathrm{Iw}}(L,T)(\chi) \xrightarrow{\cong} H^1_{\mathrm{Iw}}(L,T(\chi)) \to H^1(L_n,T(\chi)),$$

where the first map is $x \mapsto x \otimes e_\chi$, the second map is due to [35, Proposition 6.2.1], and the third map is the projection. By abuse of notation, we denote by

$$(3.8) \qquad \mathrm{pr}_{n,\chi} : H^1_{\mathrm{Iw}}(L,T) \otimes_{\mathscr{O}[[\Gamma]]} \mathscr{H}_{\infty,F}(\Gamma) \to H^1(L_n, V(\chi))$$

the scalar extension.

Now, we briefly explain the idea of the Perrin-Riou twist. Given a system as in Proposition 3.7 below, one can construct an element of $H^1_{\mathrm{Iw}}(L,T) \otimes_{\mathscr{O}[[\Gamma]]} \mathscr{H}_{\infty,F}(\Gamma)$ (see Remark 3.13), and then its image under $\mathrm{pr}_{n,\chi}$ is the twist of the given system by χ. The main result (Proposition 3.7) of this section gives the twist without constructing elements of $H^1_{\mathrm{Iw}}(L,T) \otimes_{\mathscr{O}[[\Gamma]]} \mathscr{H}_{\infty,F}(\Gamma)$.

3.2. The integral twist

Lemma 3.1. *Let $\chi : \Gamma \to \mathbb{C}_p^\times$ be a continuous character and k an integer such that $\chi(\gamma)^{p^k} \in 1 + p\mathscr{O}_{\mathbb{C}_p}$. Then, for $n \geq 1$, we have*

$$\mathrm{ord}_p \left(\frac{(\chi(\gamma)-1)^n}{n} \right) \geq \frac{n}{p^k} - \log_p n,$$

where $\mathrm{ord}_p : \mathbb{C}_p^\times \to \mathbb{Q}$ denotes the additive valuation such that $\mathrm{ord}_p(p) = 1$, and $\log_p$ denotes the real logarithmic function with base p.

Proof. Since $\chi(\gamma)^{p^k} \in 1 + p\mathscr{O}_{\mathbb{C}_p}$, we have $(\chi(\gamma)-1)^{p^k} \in p\mathscr{O}_{\mathbb{C}_p}$, and hence $\mathrm{ord}_p(\chi(\gamma)-1) \geq 1/p^k$. Combining this with $\mathrm{ord}_p(n) \leq \log_p n$, we obtain the inequality. Q.E.D.

Lemma 3.2. *For $m \geq 0$, $h \geq 1$ and a continuous character $\chi : \Gamma \to \mathbb{C}_p^\times$,*

$$\omega_{m,h,\rho}(\chi(\gamma)-1) \equiv 0 \mod p^{h(m+B(\chi))}\mathscr{O}(\chi),$$

where we define $B(\chi)$ as the maximal integer m such that for every $0 \leq i \leq h-1$ and $n \geq 1$, $m \leq \mathrm{ord}_p\left((\rho(\gamma)^{-i}\chi(\gamma)-1)^n/n\right)$.

Proof. The lemma is proven by using the congruence

$$\begin{aligned}\rho(\gamma)^{-ip^m}\chi(\gamma)^{p^m}-1 &= (\rho(\gamma)^{-i}\chi(\gamma)-1+1)^{p^m}-1\\ &= \sum_{j=1}^{p^m}\binom{p^m}{j}(\rho(\gamma)^{-i}\chi(\gamma)-1)^j\\ &= \sum_{j=1}^{p^m}p^m\binom{p^m-1}{j-1}\frac{(\rho(\gamma)^{-i}\chi(\gamma)-1)^j}{j}\\ &\equiv 0 \mod p^{m+B(\chi)}\mathscr{O}(\chi).\end{aligned}$$

Q.E.D.

For $\mu\in\mathbb{R}^{>0}$ and $g=\sum_{n\geq 0}a_n(g)X^n\in\mathbb{C}_p[[X]]$, we put

$$v(g,\mu)=\inf_{n\geq 0}\{\mathrm{ord}_p(a_n(g))+n\mu\}\in\mathbb{R}\cup\{\pm\infty\}$$

with the convention that $\mathrm{ord}_p(0)=+\infty$. For a subfield H of $\mathbb{C}_p$, we define

$$\mathrm{L}_H[\mu,+\infty]=\left\{\sum_{n\geq 0}a_nX^n\in H[[X]]\ \middle|\ \lim_{n\to+\infty}(\mathrm{ord}_p(a_n)+n\mu)=+\infty\right\}.$$

Lemma 3.3. *Let $\mu>0$ be a real number and $\ell(X)=\sum_{n\geq 0}a_n(\ell)X^n$ an element of $\mathrm{L}_H[\mu,+\infty]$. Let $\omega=\sum_{n=0}^{s}a_n(\omega)X^n\in H[X]$ be a polynomial of degree s which is μ-dominant, that is,*

$$v(\omega,\mu)=\mathrm{ord}_p(a_s(\omega))+\mu s.$$

Then, there exist a power series $g\in L_H[\mu,+\infty]$ and a polynomial $P=\sum_{n=0}^{s-1}a_n(P)X^n\in H[X]$ such that $\ell=\omega g+P$. Moreover, P and g are uniquely determined by these properties, and we have

$$v(P,\mu)\geq v(\ell,\mu), \tag{3.9}$$

$$v(g,\mu)\geq v(\ell,\mu)-v(\omega,\mu). \tag{3.10}$$

Proof. This is [22, Lemme 1]. Q.E.D.

For $j\in\mathbb{Z}$, we put

$$\ell_{\rho,j}(1+X)=\frac{\log(1+X)}{\log(\rho(\gamma))}-j=\frac{1}{\log(\rho(\gamma))}\sum_{n\geq 1}\frac{(-1)^{n-1}X^n}{n}-j. \tag{3.11}$$

Lemma 3.4. *For an integer $m \geq 0$, there exist a polynomial $P_m \in \mathbb{Q}_p(\mathrm{Im}(\rho))[X]$ and a power series $g_m = \sum_{n\geq 0} a_n(g)X^n \in \mathbb{Q}_p(\mathrm{Im}(\rho))[[X]]$ such that*

(1) $\deg(P_m) < hp^m$,
(2) $\ell_{\rho,0}(1+X) = \omega_{m,h}(X)g_m(X) + P_m(X)$,
(3) *for $\mu > 0$, we have $g_m \in \mathrm{L}_{\mathbb{Q}_p(\mathrm{Im}(\rho))}[\mu, +\infty]$.*

Moreover, P_m and g_m are uniquely determined by the properties above, and the following holds.

(4) *If we write $P_m = \sum_{n=1}^{hp^m-1} a_n(P_m)X^n \in \mathbb{Q}_p(\mathrm{Im}(\rho))[X]$, then for $n \geq 1$ we have*

$$\mathrm{ord}_p(a_n(P_m)) > -\mathrm{ord}_p(\log(\rho(\gamma))) - \log_p(h) - m, \tag{3.12}$$

$$\mathrm{ord}_p(a_n(g_m)) \geq -\mathrm{ord}_p(\log(\rho(\gamma))) - \log_p(h) - m - \log_p(n+1). \tag{3.13}$$

Remark 3.5. If $\mathrm{Im}(\rho) \subseteq \mathbb{Z}_p^\times$, then $P_m \in \mathbb{Q}_p[X]$ and

$$\mathrm{ord}_p(a_n(P_m)) \geq -\mathrm{ord}_p(\log(\rho(\gamma))) - \log_p(h) - m + 1.$$

Proof. We first verify the assumptions in Lemma 3.3. We note that for all $\mu > 0$, we have $\ell_{\rho,0}(1+X) \in \mathrm{L}_F[\mu, +\infty]$. We also note that the roots of $\omega_{m,h}(X)$ are of the form $\rho(\gamma)^j\zeta_{p^m}^a - 1$, where ζ_{p^m} is a p^m-th root of unity, $0 \leq j \leq h-1$ and $a \in \mathbb{Z}$. Hence, by [22, (2.7')], if $0 < \mu \leq 1/(p^{m-1}(p-1))$, then the polynomial $\omega_{m,h}(X)$ is μ-dominant. For such μ, Lemma 3.3 implies that there exists a unique pair (P_m, g) such that we have (1), (2) and that $g \in \mathrm{L}_F[\mu, +\infty]$. By the uniqueness, P_m and g are independent of μ. Hence, the assertion (3) also holds.

It remains to prove the assertion (4). We note that for $\mu > 0$, if we define a real-valued function $\nu(x) = \mu x - \log_p x$, then

$$\min_{x>0}\{\nu(x)\} = \nu\left(\frac{1}{\mu\ln(p)}\right) = \frac{1}{\ln(p)} + \log_p(\mu\ln(p)), \tag{3.14}$$

where ln denotes the natural logarithm function. Since

$$\log(1+X) = \omega_{m,h}(X)\log(\rho(\gamma))g + \log(\rho(\gamma))P_m,$$

if $0 < \mu \leq 1/(p^{m-1}(p-1))$ and $0 \leq n \leq hp^m - 1$, then by (3.9)

$$\begin{aligned}\mathrm{ord}_p(\log(\rho(\gamma))a_n(P_m)) + \mu n &\geq \inf_{k\geq 0}\{\mathrm{ord}_p(1/k) + \mu k\} \\ &\geq \inf_{k\geq 0}\{-\log_p(k) + \mu k\} \\ &\geq \ln(p)^{-1} + \log_p(\mu\ln(p)),\end{aligned} \tag{3.15}$$

where the last equality follows from (3.14). Let μ be $h^{-1}p^{-m}\ln(p)^{-1}$, which is less than $1/(p^{m-1}(p-1))$. Then, by (3.15) we have

$$\begin{aligned}\mathrm{ord}_p(\log(\rho(\gamma))a_n(P_m)) &\geq -h^{-1}p^{-m}\ln(p)^{-1}n + \ln(p)^{-1} - \log_p(hp^m)\\ &> -\log_p(h) - m,\end{aligned}$$

where the last inequality follows from $n < hp^m$.

We next consider g_m. By (3.10), for a non-negative integer n and $0 < \mu \leq 1/(p^{m-1}(p-1))$, we have

$$\begin{aligned}\mathrm{ord}_p(\log(\rho(\gamma))a_n(g_m)) + n\mu &\geq \ln(p)^{-1} + \log_p(\mu\ln(p)) - v(\omega_{m,h}(X),\mu)\\ &= \ln(p)^{-1} + \log_p(\mu\ln(p)) - hp^m\mu,\end{aligned}$$

where the equality follows from the fact that $\omega_{m,h}(X)$ is μ-dominant. If we take $\mu = (n+1)^{-1}h^{-1}p^{-m}\ln(p)^{-1}$, then we have

$$\begin{aligned}\mathrm{ord}_p(\log(\rho(\gamma))a_n(g_m)) &= \frac{n(hp^m-1)}{(n+1)hp^m\ln(p)} - \log_p(n+1) - \log_p(h) - m\\ &\geq -\log_p(n+1) - \log_p(h) - m,\end{aligned}$$

which implies (3.13). Q.E.D.

For $m \geq 1$, we denote by φ_m the isomorphism of F-algebras

$$\varphi_m : F[\gamma-1]/\omega_{m,h}(\gamma-1) \cong \oplus_{i=0}^{h-1} F[\Gamma_m](\rho^i)$$

that sends $\gamma - 1$ to $((\rho(\gamma)^i\gamma - 1)\otimes e_{\rho^i})_i$.

Lemma 3.6. *Let M be an $\mathscr{O}$-module and $b_0, b_1, \ldots, b_{h-1}$ elements in M. We put*

$$d = \sum_{j=0}^{h-1}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k} b_k \otimes \binom{P_m(\gamma-1)}{j} \in M\otimes F[\gamma-1].$$

Then, we have $\varphi_m(d) = (b_i \otimes e_{\rho^i})_i$, where by abuse of notation, we denote by the same symbol φ_m the homomorphism $M\otimes F[\gamma-1] \to \oplus_{i=0}^{h-1} M\otimes F[\Gamma_m](\rho^i)$ induced by φ_m.

Proof. First we claim that $\varphi_m(P_m(\gamma-1)) = (i\otimes e_{\rho^i})_i$. We note that for $0 \leq i \leq h-1$ there exists an injective homomorphism of F-algebras $F[\Gamma_m](\rho^i) \hookrightarrow \prod_{j=0}^{p^m-1}\mathbb{C}_p(\rho^i)$ sending γ to $(\zeta_{p^m}^j \otimes e_{\rho^j})_j$, where ζ_{p^m} is a primitive p^m-th root of unity. We note that the image of $\varphi_m(P_m(\gamma-1))$ in $\prod_{j=0}^{p^m-1}\mathbb{C}_p(\rho^i)$ is given by

$$(P_m(\rho(\gamma)^i\zeta_{p^m}^j - 1)\otimes e_{\rho^i})_j.$$

By letting $X = \rho(\gamma)^i \zeta_{p^m}^j - 1$ in Lemma 3.4 (2), for $j \in \mathbb{Z}$, we have

$$P_m(\rho(\gamma)^i \zeta_{p^m}^j - 1) = i,$$

which implies that the i-th component of the element $\varphi_m(P_m(\gamma - 1))$ in $\prod_{i=0}^{h-1} F[\Gamma_m](\rho^i)$ is $i \otimes e_{\rho^i}$. Then, the claim follows.

By the claim, the image of $\varphi_m(d)$ in $M \otimes F[\Gamma_m](\rho^i)$ is

$$\sum_{j=0}^{h-1} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} b_k \otimes \binom{i}{j} e_{\rho^i} = \left(\sum_{\substack{k,l,m \geq 0, \\ k+l+m=i}} (-1)^{\ell} \frac{i!}{k!\ell!m!} b_k \right) \otimes e_{\rho^i}.$$

This is the coefficient of $t^i/i!$ of the product of power series $e^t \cdot e^{-t} \cdot \sum_{k=0}^{\infty} b_k \frac{t^k}{k!}$, that is, b_i. Q.E.D.

For $m \geq n$, we put

$$N_{m/n} = \sum_{j=0}^{p^{m-n}-1} \gamma^{jp^n} \in \mathbb{Z}[\Gamma].$$

For $n \geq 0$ or $n = \infty$ and for a finitely generated $\mathscr{O}$-module M with continuous G_{L_n}-action, we put

$$\tilde{H}^1(L_n, M) = \mathrm{Im}\left(H^1(L_n, M) \to H^1(L_n, M \otimes F)\right).$$

We note that if $H^1(L_n, M)$ is torsion-free (e.g. $H^0(L_n, M \otimes F/\mathscr{O}) = \{0\}$), then we may identify $\tilde{H}^1(L_n, M)$ and $H^1(L_n, M)$.

Proposition 3.7. *Let $h \geq 1$ and $\alpha \in F^{\times}$ an element such that $|p^h/\alpha|_p < 1$. Suppose that for $0 \leq i \leq h-1$, we have a system $(c_{n,i})_n \in \prod_{n \geq 0} \tilde{H}^1(L_n, T(\rho^i))$ satisfying the following conditions:*

(a) $\mathrm{Cor}_{n+1/n} c_{n+1,i} = \alpha c_{n,i}$ *for $n \geq 0$, where $\mathrm{Cor}_{n+1/n}$ denotes the corestriction map relative to L_{n+1}/L_n,*

(b) *for $0 \leq i \leq h-1$, we have*

$$\sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \mathrm{Res}_{\infty}(c_{n,k}) \otimes e_{\rho}^{\otimes -k} \equiv 0 \mod p^{in},$$

where Res_{∞} denotes the restriction maps $\tilde{H}^1(L_n, T(\rho^k)) \to \tilde{H}^1(L_{\infty}, T(\rho^k))$.

Let $P_m(X) \in F[X]$ be the polynomial as in Lemma 3.4. Then, for a continuous character $\chi : \Gamma \to \mathbb{C}_p^\times$ and $n \geq 0$, the sequence

$$\frac{1}{\alpha^m} N_{n+m/n} \left(\sum_{i=0}^{h-1} \binom{P_{n+m}(\chi(\gamma)-1)}{i} \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \mathrm{Res}_\infty(c_{n+m,k}) \otimes e_{\rho^{-k}\chi} \right)$$

$(m = 1, 2, \dots)$ in $H^1(L_\infty, V(\chi))$ converges to an element c_n^χ that satisfies the following conditions.

(1) *For $n \geq 0$, we have $c_n^\chi \in H^0\left(\Gamma^{p^n}, H^1(L_\infty, V(\chi))\right)$. In particular, if $H^0(G_{L_\infty}, T) = \{0\}$, then c_n^χ is the restriction of an element in $H^1(L_n, V(\chi))$, which we denote by the same symbol c_n^χ.*

(2) *For $n \geq 0$, $N_{n+1/n} c_{n+1}^\chi = \alpha c_n^\chi$.*

(3) *For $n \geq 0$, we have*

$$(h-1)! p^{(h-1)\delta(h)} (\log^{h-1} \rho(\gamma)) \alpha^{\max\{0, 1-B(\chi)\}} c_n^\chi \in \tilde{H}^1(L_\infty, T(\chi)),$$

where we put $\delta(h) = \min\left\{ a \in \mathbb{Z}_{\geq 0} \;\middle|\; a \geq \log_p(h) \right\}$. Moreover, if $n \geq 1 - B(\chi)$, then we have

$$(h-1)! p^{(h-1)\delta(h)} (\log^{h-1} \rho(\gamma)) c_n^\chi \in \tilde{H}^1(L_\infty, T(\chi)).$$

Remark 3.8. If $\mathrm{Im}(\rho) \subseteq \mathbb{Z}_p^\times$, then for $n \geq 0$ we have

$$(h-1)! p^{(h-1)(\delta(h)-1)} (\log^{h-1} \rho(\gamma)) \alpha^{\max\{0, 1-B(\chi)\}} c_n^\chi \in \tilde{H}^1(L_\infty, T(\chi))$$

(see Remark 3.9 below for details). In particular, if p is odd and $\mathrm{Im}(\rho) = 1 + p\mathbb{Z}_p$, then

$$(h-1)! p^{(h-1)\delta(h)} \alpha^{\max\{0, 1-B(\chi)\}} c_n^\chi \in \tilde{H}^1(L_\infty, T(\chi)).$$

Proof. By Lemma 3.4 (4), for $0 \leq i < h$ we have

$$\binom{P_m(X)}{i} \in \frac{1}{i!(p^{\delta(h)} \log(\rho(\gamma)) p^m)^i} \mathscr{O}[X]. \tag{3.16}$$

For $m \geq n$, if we put

$$d_m = \sum_{i=0}^{h-1} \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \mathrm{Res}_\infty(c_{m,k}) \otimes e_{\rho^{-k}} \otimes_{\mathscr{O}} \binom{P_m(\gamma - 1)}{i},$$

then by (3.16) and the assumption (b), we have

$$(h-1)! C_\rho^{h-1} d_m \in \tilde{H}^1(L_\infty, T) \otimes_{\mathscr{O}} \mathscr{O}[\gamma - 1], \tag{3.17}$$

where $C_\rho := p^{\delta(h)}\log(\rho(\gamma))$. Under the isomorphism of Γ-modules

$$\varphi_m : H^1(L_\infty, T)\otimes_{\mathscr{O}} F[\gamma-1]/\omega_{m,h}(\gamma-1) \cong \prod_{0\le i\le h-1} H^1(L_\infty, V)\otimes_F F[\Gamma_m](\rho^i),$$

by Lemma 3.6 we have

(3.18)

$$\varphi_m(d_m) = (\mathrm{Res}_\infty(c_{m,i}))_i \in \prod_{0\le i\le h-1} H^1(L_\infty, V(\rho^i))^{\Gamma^{p^m}} \subseteq \prod_{0\le i\le h-1}\left(H^1(L_\infty, V)\otimes_F F[\Gamma_m](\rho^i)\right)^{\Gamma^{p^m}}.$$

Here, we identify $H^1(L_\infty, V(\rho^i))^{\Gamma^{p^m}}$ with a subspace of the vector space $\left(H^1(L_\infty, V)\otimes_F F[\Gamma_m](\rho^i)\right)^{\Gamma^{p^m}}$ by the map induced by the natural inclusion $F\to F[\Gamma_m]$. Hence, since $\omega_{m,h}(X)\in\mathscr{O}[X]$ is monic up to a unit in $\mathscr{O}^\times$, we have

$$(\gamma^{p^m}-1)d_m \in (h-1)!^{-1}C_\rho^{1-h}\tilde{H}^1(L_\infty, T)\otimes J_m. \tag{3.19}$$

Here we put $J_m = \omega_{m,h}(\gamma-1)\mathscr{O}[\gamma-1]$. For simplicity, we put $M_m = (h-1)!^{-1}C_\rho^{1-h}\tilde{H}^1(L_\infty, T)\otimes J_m$. By (3.19), we have

$$N_{m+n+1/n}d_{m+n+1} \equiv N_{n+m/n}N_{n+m+1/n+m}d_{n+m+1} \mod M_{n+m+1}.$$

By the assumption (a), we also have $\varphi_m\left(N_{m+1/m}d_{m+1}\right) = \varphi_m(\alpha d_m)$, and hence

$$N_{m+1/m}d_{m+1} - \alpha d_m \in M_m. \tag{3.20}$$

Therefore we also have

$$N_{n+m/n}N_{n+m+1/n+m}d_{n+m+1} \equiv \alpha N_{n+m/n}d_{n+m} \mod M_{n+m}.$$

Hence, we have

$$N_{m+n+1/n}d_{m+n+1} \equiv \alpha N_{n+m/n}d_{n+m} \mod M_{n+m}. \tag{3.21}$$

Let

$$\mathrm{ev}_\chi : \tilde{H}^1(L_\infty, T)\otimes\mathscr{O}[\gamma-1] \to \tilde{H}^1(L_\infty, T)(\chi)$$

be the evaluation map by $\mathscr{O}$-algebra $\mathscr{O}[\gamma-1]\to\mathscr{O}(\chi)$ sending γ to $\chi(\gamma)$, which is Γ-linear. Then, by (3.21) and Lemma 3.2, we obtain

(3.22)

$$\mathrm{ev}_\chi\Big(\frac{1}{\alpha^{m+1}}N_{m+n+1/n}d_{m+n+1}\Big)\equiv\mathrm{ev}_\chi\left(\frac{1}{\alpha^m}N_{n+m/n}d_{n+m}\right) \mod \frac{p^{h(n+B(\chi))}}{(h-1)!C_\rho^{h-1}\alpha}\left(\frac{p^h}{\alpha}\right)^m\tilde{H}^1(L_\infty,T)(\chi).$$

Hence, by $|p^h/\alpha|_p<1$ the sequence

$$\left\{\mathrm{ev}_\chi\left(\frac{1}{\alpha^m}N_{n+m/n}d_{n+m}\right)\right\}_m$$

converges to an element c_n^χ of $H^1(L_\infty,V(\chi))$.

We next prove that

$$c_n^\chi\in H^0(\Gamma^{p^n},H^1(L_\infty,V(\chi))). \tag{3.23}$$

Since $(\gamma^{p^n}-1)N_{n+m/n}=\gamma^{p^{m+n}}-1$, by (3.19) we have

$$(\gamma^{p^n}-1)N_{n+m/n}d_{n+m}\in M_{m+n}.$$

Combining this with Lemma 3.2 implies (3.23).

It remains to verify the properties (2) and (3). By (3.19) and (3.20), we have

$$\begin{aligned}&N_{n+1/n}N_{m+n+1/n+1}d_{m+n+1}-\alpha N_{m+n/n}d_{m+n}\\ \equiv&N_{m+n/n}\left(N_{m+n+1/m+n}d_{m+n+1}-\alpha d_{m+n}\right)\equiv 0 \mod M_{m+n}.\end{aligned}$$

By combining this with Lemma 3.2 and taking the limit as $m\to\infty$, we conclude (2).

We next prove (3). For $m\geq 0$ and $0\leq i<h$, by (3.16) we have

$$p^{i(m+n)}\binom{P_{n+m}(\chi(\gamma)-1)}{i}\in(i!)^{-1}C_\rho^{-i}\mathscr{O}(\chi),$$

and hence $\mathrm{ev}_\chi(d_{n+m})\in(h-1)!^{-1}C_\rho^{1-h}\tilde{H}^1(L_\infty,T(\chi))$. Then, the congruence (3.22) implies that for $n\geq 1-B(\chi)$

$$c_n^\chi\in(h-1)!^{-1}C_\rho^{1-h}\tilde{H}^1(L_\infty,T(\chi)). \tag{3.24}$$

This implies the assertion (3). Q.E.D.

Remark 3.9. If $\mathrm{Im}(\rho) \subseteq \mathbb{Z}_p^\times$, then by Remark 3.5, instead of (3.16) we obtain

$$\binom{P_m(X)}{i} \in \frac{1}{i!(p^{\delta(h)}\log(\rho(\gamma))p^{m-1})^i}\mathbb{Z}_p[X].$$

Then, the rest of the argument in the proof works even if we replace C_ρ above by $p^{\delta(h)-1}\log(\rho(\gamma))$, and we have the assertion in Remark 3.8.

Lemma 3.10. *For elements a, b of a commutative $\mathbb{Q}$-algebra and an integer i, we have $\binom{a+b}{i} = \sum_{k=0}^{i}\binom{a}{k}\binom{b}{i-k}$.*

Proof. We are reduced to the case where $a = X, b = Y \in \mathbb{Q}[X, Y]$. Since $\binom{X+Y}{i}$ and $\sum_{k=0}^{i}\binom{X}{k}\binom{Y}{i-k}$ are polynomials in $\mathbb{Q}[X, Y]$, it suffices to prove that for all positive integers m, n, $\binom{m+n}{i} = \sum_{k=0}^{i}\binom{m}{k}\binom{n}{i-k}$, which follows from considering the coefficient of $t^i s^{m+n-i}$ in $(t+s)^{m+n} = (t+s)^m(t+s)^n \in \mathbb{Z}[t, s]$. Q.E.D.

Proposition 3.11. *With same notation as in Proposition 3.7, the sequence*

$$\frac{1}{\alpha^m}N_{n+m/n}\left(\sum_{i=0}^{h-1}\binom{\ell_{\rho,0}(\chi(\gamma))}{i}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\mathrm{Res}_\infty(c_{n+m,k}) \otimes e_{\rho^{-k}\chi}\right)$$

$(m = 1, 2, \dots)$ in $H^1(L_\infty, V(\chi))$ converges to the element c_n^χ in Proposition 3.7.

Proof. For $g(X) \in F[[X]]$, we write $\chi(g) = g(\chi(\gamma) - 1)$ if it converges. For $m \geq 1$, by Lemma 3.4 and Lemma 3.10, we have

$$\begin{aligned}\binom{\ell_{\rho,0}(\chi(\gamma))}{i} &= \binom{\chi(g_{n+m})\chi(\omega_{m+n}) + \chi(P_{n+m})}{i} \\ &= \sum_{j=0}^{i}\binom{\chi(g_{n+m})\chi(\omega_{m+n})}{j}\binom{\chi(P_{n+m})}{i-j}.\end{aligned}$$

By Lemma 3.4 (4), we have

$$\mathrm{ord}_p\left(\binom{\chi(P_{n+m})}{i-j}\right) \geq -m(i-j) + c_0$$

for some constant c_0 independent of m. Lemma 3.2 implies that

$$\mathrm{ord}_p\left(\binom{\chi(g_{n+m})\chi(\omega_{m+n})}{j}\right) \geq (h-1)jm + c_1$$

for some constant c_1 independent of m. Therefore, we have

$$\text{(3.25)}\quad \mathrm{ord}_p\left(\binom{\chi(g_{n+m})\chi(\omega_{m+n})}{j}\binom{\chi(P_{n+m})}{i-j}\right) \geq (hj-i)m+c_0+c_1.$$

Hence, by the assumption (b) of Proposition 3.7, we have

$$\frac{1}{\alpha^m}\binom{\chi(g_{n+m})\chi(\omega_{m+n})}{j}\binom{\chi(P_{n+m})}{i-j}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\mathrm{Res}_\infty(c_{n+m,k})\otimes e_{\rho^{-k}\chi}$$
$$\in \frac{p^{jhm+c_0+c_1}}{\alpha^m}\in \tilde{H}^1(L_\infty, T(\chi)),$$

which converges to $0\in \tilde{H}^1(L_\infty, T(\chi))$ as $m\to\infty$ unless $j=0$. Our assertion follows from this. Q.E.D.

We show that our c_n^χ is compatible with $\mathrm{pr}_{n,\chi}$. We write $\mathrm{pr}_{n,i} = \mathrm{pr}_{n,\rho^i}$.

Proposition 3.12. *With the same notation and assumption as in Proposition 3.7, suppose that c_∞ is an element of $H^1_{\mathrm{Iw}}(L,T)\otimes_{\mathscr{O}[[\Gamma]]}\mathscr{H}_{h,F}(\Gamma)$ such that for $0\leq i\leq h-1$ and $m\geq 0$, we have $\mathrm{pr}_{m,i}(c_\infty) = \frac{1}{\alpha^m}c_{m,i}$. Assume that $H^0(G_{L_\infty},T)=\{0\}$. Then, for a continuous character $\chi:\Gamma\to\mathbb{C}_p^\times$ and $n\geq 0$ we have $\mathrm{pr}_{n,\chi}(c_\infty)=\frac{1}{\alpha^n}c_n^\chi$.*

Remark 3.13. In the case where L is a finite extension of $\mathbb{Q}_p$, [32, §1.8] implies the existence of c_∞ as above. See [23, Propositions 2.3.3 and 2.4.5] for the case where L is a number field.

Proof. We denote by

$$\mathrm{Res}'_\infty : H^1_{\mathrm{Iw}}(L,T)\otimes_{\mathscr{O}[[\Gamma]]}\mathscr{H}_{h,F}(\Gamma)\to H^1(L_\infty, T\otimes\mathscr{H}_{h,F}(\Gamma))^\Gamma$$

the composite

$$\begin{aligned}\mathrm{Res}'_\infty : H^1_{\mathrm{Iw}}(L,T)\otimes_{\mathscr{O}[[\Gamma]]}\mathscr{H}_{h,F}(\Gamma) &= H^1(L, T\otimes\mathscr{O}[[\Gamma]])\otimes_{\mathscr{O}[[\Gamma]]}\mathscr{H}_{h,F}(\Gamma)\\ &\to H^1(L, T\otimes_{\mathscr{O}}\mathscr{H}_{h,F}(\Gamma))\\ &\to H^1(L_\infty, T\otimes\mathscr{H}_{h,F}(\Gamma))^\Gamma\\ &= (H^1(L_\infty,T)\otimes\mathscr{H}_{h,F}(\Gamma))^\Gamma,\end{aligned}$$

where the first equality is due to Shapiro's lemma, and the second arrow is induced by the restriction map. Let $\chi:\Gamma\to\mathbb{C}_p^\times$ be a continuous character and F_χ the completion of the field $F(\mathrm{Im}(\chi))$. By [22, (2.7')], the polynomial $\chi(\gamma)^{-p^m}(1+X)^{p^m}-1$ is μ-dominant for sufficiently small $\mu>0$, and by Lemma 3.3 we have an isomorphism

$$\mathscr{H}_{F_\chi}(\Gamma)/(\chi(\gamma)^{-p^m}\gamma^{p^m}-1)\cong F_\chi[\gamma-1]/(\chi(\gamma)^{-p^m}\gamma^{p^m}-1)$$

where

$$\mathscr{H}_{F_\chi}(\Gamma) = \left\{ g(\gamma-1) \in F_\chi[[\gamma-1]] \mid g(X) \in \cap_{\mu>0} \mathrm{L}_{F_\chi}[\mu, +\infty] \right\}.$$

Then we denote by $\widetilde{\mathrm{pr}}_{m,\chi}$ the composite

$$\begin{aligned}\widetilde{\mathrm{pr}}_{m,\chi} : (H^1(L_\infty, T) \otimes \mathscr{H}_{h,F}(\Gamma))^\Gamma & \\ &\to \Big(H^1(L_\infty, T) \otimes (F_\chi[\gamma-1]/(\chi(\gamma)^{-p^m}\gamma^{p^m}-1))\Big)^\Gamma \\ &\cong \big(H^1(L_\infty, T) \otimes F[\Gamma_m](\chi)\big)^\Gamma \\ &= H^1(L_\infty, T(\chi) \otimes F[\Gamma_m])^\Gamma \\ &= H^1(L, T(\chi) \otimes F[\Gamma_m]).\end{aligned}$$

Here, the isomorphism is induced by $\mathrm{Tw}_\chi : F_\chi[\Gamma] \to F_\chi[\Gamma]$ sending every element $g(\gamma-1)$ to $g(\chi(\gamma)\gamma-1)$, and the last equality is due to the assumption that $H^0(G_{L_\infty}, T) = \{0\}$. If we denote by

$$\mathrm{sha}_m : H^1(L,\ T(\chi) \otimes F[\Gamma_m]) \to H^1(L_m, T(\chi))$$

the isomorphism induced by Shapiro's theorem, then

$$\mathrm{pr}_{m,\chi} = \mathrm{sha}_m \circ \widetilde{\mathrm{pr}}_{m,\chi} \circ \mathrm{Res}'_\infty. \tag{3.26}$$

We first consider the case where $n = 0$. We write

$$\mathrm{Res}'_\infty(c_\infty) = \sum_{j=1}^{l} x_j \otimes f_j(\gamma-1) \in (H^1(L_\infty, T) \otimes \mathscr{H}_{h,F}(\Gamma))^\Gamma.$$

By (3.26) with $m = 0$, if we denote by $\mathrm{Res}_\infty(\mathrm{pr}_{0,\chi}(c_\infty))$ the image of $\mathrm{pr}_{0,\chi}(c_\infty)$ in $H^1(L_\infty, V(\chi))^\Gamma = \big(H^1(L_\infty, T) \otimes F(\chi)\big)^\Gamma$, then

$$\mathrm{Res}_\infty(\mathrm{pr}_{0,\chi}(c_\infty)) = \sum_j x_j \otimes f_j(\chi(\gamma)-1)e_\chi. \tag{3.27}$$

As is proved in the proof of Lemma 3.3, for $0 < \mu \le 1/(p^{m-1}(p-1))$, the polynomial $\omega_{m,h}(X)$ is μ-invariant. Then, by using Lemma 3.3, we denote by $R_{j,m} \in F[X]$ the polynomial such that $\deg(R_{j,m}) < hp^m$ and

$$f_j(\gamma-1) \equiv R_{j,m}(\gamma-1) \bmod \omega_{m,h}(\gamma-1)\mathscr{H}_F(\Gamma). \tag{3.28}$$

Since $\mathrm{pr}_{m,i}(c_\infty) = \alpha^{-m}c_{m,i}$, (3.26) and (3.28) imply that

$$\sum_j x_j \otimes R_{j,m}(\gamma-1) \bmod (\rho(\gamma)^{-ip^m}\gamma^{p^m}-1) = \alpha^{-m}\mathrm{Res}_\infty(\mathrm{sha}_m^{-1}(c_{m,i}))$$

as an equality in $H^1(L_\infty, T) \otimes F[\Gamma_m](\rho^i)$. Hence, by Lemma 3.6 and by noting that $\mathrm{Res}_\infty(\mathrm{sha}_m^{-1}(c_{m,i})) = \sum_{b=0}^{m-1} \gamma^{p^b}\mathrm{Res}_\infty(c_{m,i}) \otimes \gamma^{p^b}$, we have an equality in $H^1(L_\infty, T) \otimes F[\gamma-1]/(\omega_{m,h}(\gamma-1))$

$$\begin{aligned}&\alpha^m \sum_j x_j \otimes R_{j,m}(\gamma-1)\\ &= \sum_{b=0}^{m-1}\sum_{i=0}^{h-1}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\gamma^{p^b}(\mathrm{Res}_\infty(c_{m,k}) \otimes e_{\rho^{-k}}) \otimes \gamma^{p^b}\binom{P_m(\gamma-1)}{i}.\end{aligned}$$

Since $\omega_{m,h}(X)$ is monic up to unit and $\deg(R_{j,m}) < \deg(\omega_{m,h}(X))$, (3.17) implies that

$$\begin{aligned}&\alpha^m \sum_j x_j \otimes R_{j,m}(\gamma-1)\\ &\equiv \sum_{b=0}^{m-1}\sum_{i=0}^{h-1}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\gamma^{p^b}(\mathrm{Res}_\infty(c_{m,k}) \otimes e_{\rho^{-k}}) \otimes \gamma^{p^b}\binom{P_m(\gamma-1)}{i}\\ &\qquad \mod (h-1)!^{-1}C_\rho^{1-h}\tilde{H}^1(L_\infty, T) \otimes \omega_{m,h}(\gamma-1)\mathscr{O}[\gamma-1].\end{aligned}$$

Hence, by considering $H^1(L_\infty, V) \otimes F[\gamma-1] \to H^1(L_\infty, V) \otimes F(\chi)$, we have

$$\begin{aligned}&\sum_j x_j \otimes R_{j,m}(\chi(\gamma)-1)\\ &\equiv \frac{1}{\alpha^m}N_{m/0}\sum_{i=0}^{h-1}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\mathrm{Res}_\infty(c_{m,k}) \otimes e_{\rho^{-k}\chi}\binom{P_m(\chi(\gamma)-1)}{i}\\ &\qquad \mod \frac{\omega_{m,h}(\chi(\gamma)-1)}{\alpha^m(h-1)!C_\rho^{h-1}}\tilde{H}^1(L_\infty, T(\chi)).\end{aligned}$$

By Proposition 3.7, the right hand side converges to c_0^χ as $m \to \infty$. We also note that each $R_{j,m}(\chi(\gamma)-1)$ converges to $f_j(\chi(\gamma)-1)$ as $m \to \infty$ (cf. [3, Proposition IV. 1]). Hence, by Lemma 3.2 and (3.27), letting $m \to \infty$, we have

$$\mathrm{pr}_{0,\chi}(c_\infty) = c_0^\chi. \tag{3.29}$$

We next consider general n. Since the natural map from $F[\Gamma_n]$ to $\prod_{\phi:\Gamma_n \to \overline{\mathbb{Q}}_p^\times} F(\phi)$ is injective, by the assumption that $H^0(G_{L_\infty}, T) = \{0\}$, the composite

(3.30)

$$H^1(L_n, V(\chi)) = H^1(L, V(\chi) \otimes F[\Gamma_n]) \to H^1(L_\infty, V(\chi) \otimes F[\Gamma_n])^\Gamma \\ \to \prod_{\phi\colon \Gamma_n \to \overline{\mathbb{Q}}_p^\times} H^1(L_\infty, V(\chi\phi))^\Gamma = \prod_{\phi\colon \Gamma_n \to \overline{\mathbb{Q}}_p^\times} H^1(L, V(\chi\phi))$$

is injective. For $\phi : \Gamma_n \to \overline{\mathbb{Q}}_p^\times$, the map

$$\pi_\phi : H^1(L_n, V(\chi)) \to H^1(L, V(\chi\phi))$$

coming from (3.30) sends each x to $\mathrm{Cor}_{n/0}(x \otimes e_\phi)$. Hence, $\pi_\phi \circ \mathrm{pr}_{n,\chi} = \mathrm{pr}_{0,\chi\phi}$. We also note that by the explicit expression in Proposition 3.7 of c_n^χ, we have $\pi_\phi(c_n^\chi) = \alpha^n c_0^{\chi\phi}$. By (3.29) and replacing χ by $\chi\phi$, we have $\mathrm{pr}_{0,\chi\phi}(c_\infty) = c_0^{\chi\phi}$, and hence $\pi_\phi(\mathrm{pr}_{n,\chi}(c_\infty)) = \pi_\phi(\alpha^{-n} c_n^\chi)$. By the injectivity of the composite (3.30), we complete the proof. Q.E.D.

§4. Twists of generalized Heegner classes and Selmer groups

In this section, we apply the integral Perrin-Riou twist to generalized Heegner classes, and by following arguments as in [18], we bound the twisted Selmer group by the resulting classes.

Unlike the previous section, we follow the notation fixed in Subsection 2.1. We also assume Condition 2.3 here. Let $\mathscr{P}$ be the set of (rational) primes inert in K. We denote by $\mathscr{N}$ the set of square-free products of primes in $\mathscr{P}$. By abuse of notation, for $l \in \mathscr{P}$ we denote by the same symbol l the prime ideal of K above l.

In the following, we write

$$V = V_f(r), \quad T = T_f(r), \quad W = V/T.$$

4.1. Generalized Heegner classes

We fix some notation on generalized Heegner classes, which are the images of generalized Heegner cycles (introduced in [5]) under the p-adic Abel-Jacobi map.

For a natural number c, we put $\mathscr{O}_c = \mathbb{Z} + c\mathscr{O}_K$ and we denote by A_c the elliptic curve given by the Weierstrass model associated to the lattice $\Omega_K \mathscr{O}_c$. By CM theory, A_c is defined over $K[c]$, and the isogeny $\mathbb{C}/\Omega_K\mathscr{O}_K \to \mathbb{C}/\Omega_K\mathscr{O}_c$ sending z to cz induces an isogeny $\pi_c : A \to A_c$ over $K[c]$. We fix an element $t_A \in A[\mathfrak{N}]$ of order N, which gives an $H(A[\mathfrak{N}])$-rational point of $X_1(N)$ represented by (A, t_A). Then, for $c \geq 1$ with $(c, N) = 1$, we have the generalized Heegner cycle Δ_{π_c}

associated to π_c (cf. [5, §2.3]), which is defined over $\tilde{K}[c] := K[c](A[\mathfrak{N}])$. As in [12, §4.2], its image under the p-adic Abel-Jacobi map gives an element of $H^1(\tilde{K}[c], V_f \otimes \mathrm{Sym}^{2r-2} H^1_{\mathrm{et}}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)(2r-1))$. We denote by $z[c] \in H^1(K[c], V_f \otimes \mathrm{Sym}^{2r-2} H^1_{\mathrm{et}}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)(2r-1))$ its image under the corestriction map. We note that

$$H^1(K[c], V_f \otimes \mathrm{Sym}^{2r-2} H^1_{\mathrm{et}}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)(2r-1)) \cong \prod_{1 \le i \le 2r-1} H^1(K[c], V_f(\psi_{\mathfrak{p}}^i \psi_{\overline{\mathfrak{p}}}^{2r-i})).$$

Here the isomorphism above is given by choosing an appropriate basis u, v of T_pA (cf. Section 7 of [21].) For $1 \le i \le 2r-1$, we denote by $z^{(i)}[c] \in H^1(K[c], V_f(\psi_{\mathfrak{p}}^i \psi_{\overline{\mathfrak{p}}}^{2r-i}))$ the i-th component of $z[c]$ under the decomposition. We note that the denominators of $z^{(i)}[c]$ are bounded as c and i vary.

Let $\alpha \in \mathscr{O}$ be a root of $X^2 - a_pX + p^{2r-1}$ such that $|p^{2r-1}/\alpha|_p < 1$. For a positive integer c relatively prime to pN, a natural number n and $1 \le i \le 2r-1$, we put

$$z^{(i)}[cp^n]_\alpha = \alpha z^{(i)}[cp^n] - p^{2r-2} \mathrm{Res}_{K[cp^{n-1}], K[cp^n]}(z^{(i)}[cp^{n-1}])$$

in $H^1(K[cp^n], V_f(\psi_{\mathfrak{p}}^i \psi_{\overline{\mathfrak{p}}}^{2r-i}))$. Then, [12, Proposition 4.4] implies that for $n \ge 1$,

$$\mathrm{Cor}_{n+1/n} z^{(i)}[cp^{n+1}]_\alpha = \alpha z^{(i)}[cp^n]_\alpha, \tag{4.31}$$

where $\mathrm{Cor}_{n+1/n}$ denotes the corestriction map relative to the extension $K[cp^{n+1}]/K[cp^n]$.

4.2. Generalized Heegner classes and the p-adic L-function

First, we briefly recall the explicit reciprocity law relating generalized Heegner classes with the p-adic L-function (cf. [5, 12, 21]). We mostly follow [21].

Let c be a positive integer relatively prime to pN. For $0 \le n \le \infty$, we denote by $\hat{H}_{cp^n}$ the completion of $K[cp^n]$ at the prime $\mathfrak{p}_{cp^n}$ above $\mathfrak{p}$ induced by ι_p, and we let $\mathscr{W}_{cp^n}$ be its ring of integers. We put $\hat{\mathcal{G}}_{cp^n} = \mathrm{Gal}(\hat{H}_{cp^n}/K_{\mathfrak{p}})$. For $i \in \mathbb{Z}$, we put

$$V\langle i \rangle = V(\psi_{\mathfrak{p}}^i \psi_{\overline{\mathfrak{p}}}^{-i}) = V_f(\psi_{\mathfrak{p}}^{i+r} \psi_{\overline{\mathfrak{p}}}^{r-i}), \quad T\langle i \rangle = T(\psi_{\mathfrak{p}}^i \psi_{\overline{\mathfrak{p}}}^{-i}).$$

We also put $\hat{\Lambda}_c = \mathscr{O}[[\hat{\mathcal{G}}_{cp^\infty}]]$, $\mathscr{H}_{h,F}(\mathcal{G}_{cp^\infty}) = \mathscr{H}_{h,F}(\Gamma)[G'_c]$ (cf. (3.7)), where G'_c is a finite abelian group such that $\mathcal{G}_{cp^\infty} := \mathrm{Gal}(K[cp^\infty]/K) =$

$\Gamma \times G'_c$. Following [31, §3.6.1], if we put $\mathscr{H}_\infty(\mathcal{G}_{cp^\infty}) = \cup_{h\geq 1}\mathscr{H}_{h,F}(\mathcal{G}_{cp^\infty})$, then we have a natural pairing $\langle -,-\rangle_{T,c}$:

$$H^1_{\mathrm{Iw}}(\hat{H}_c, T)\otimes_{\hat{\Lambda}_c}\mathscr{H}_\infty(\mathcal{G}_{cp^\infty}) \times H^1_{\mathrm{Iw}}(\hat{H}_c, T)\otimes_{\hat{\Lambda}_c}\mathscr{H}_\infty(\mathcal{G}_{cp^\infty}) \to \mathscr{H}_\infty(\mathcal{G}_{cp^\infty}),$$

where $H^1_{\mathrm{Iw}}(\hat{H}_c, T) := \varprojlim_n H^1(\hat{H}_{cp^n}, T)$. We note that since we have fixed a prime of $K[cp^n]$ above $\mathfrak{p}$ via ι_p, [35, Corollary B.5.2] implies that $\mathscr{O}[\mathcal{G}_{cp^n}]\otimes_{\mathscr{O}[\hat{\mathcal{G}}_{cp^n}]} H^1(\hat{H}_{cp^n}, T\langle i\rangle)$ is canonically isomorphic to the direct sum $\oplus_{v|\mathfrak{p}} H^1(K[cp^n]_v, T\langle i\rangle)$, where v ranges over all primes of $K[cp^n]$ above $\mathfrak{p}$. For $n\geq 0$, $i\in\mathbb{Z}$, we denote by

$$\mathrm{pr}_{n,i} : H^1_{\mathrm{Iw}}(\hat{H}_c, T)\otimes_{\hat{\Lambda}_c}\mathscr{H}_\infty(\mathcal{G}_{cp^\infty}) \to \oplus_{v|\mathfrak{p}} H^1(K[cp^n]_v, V\langle i\rangle)$$

the twisted natural projection (cf. (3.8)).

Let k_c be the residue field of $\hat{H}_c$ and $\overline{A_c}$ the reduction modulo $\mathfrak{p}_c$ of the elliptic curve A_c over $K[c]$ introduced in §4.1. We put $\tilde{\mathfrak{R}}_c = \prod_{\tau\in\mathcal{G}_c}\mathfrak{R}^\tau_c$, where $\mathfrak{R}^\tau_c$ denotes the universal deformation ring for the reduction $\overline{A^\tau_c}$ of A^τ_c. Then, $\tilde{\mathfrak{R}}_c$ is a $\mathscr{W}_c[[\mathcal{G}_{cp^\infty}]]$-algebra. For $h\geq r$, we denote by

$$\tilde{\Omega}^\epsilon_{V,h,c} : \tilde{\mathfrak{R}}^{\psi=0}_c \otimes_{\mathbb{Z}_p} D_{\mathrm{crys}}(V|_{G_{K_\mathfrak{p}}}) \to H^1_{\mathrm{Iw}}(\hat{H}_c, T)\otimes_{\hat{\Lambda}_c}\mathscr{H}_\infty(\mathcal{G}_{cp^\infty})$$

the semi-local Perrin-Riou exponential map (see [21, §5] for the details), where ψ is the σ^{-1}-semilinear map on $\tilde{\mathfrak{R}}_c$ introduced in [21, §4.2] (σ is the Frobenius on $\mathscr{W}_c$), and it is regarded as an analogue of ψ in [31, 1.1.3]. We note that $\tilde{\mathfrak{R}}^{\psi=0}_c$ is a free $\mathscr{W}_c[[\mathcal{G}_{cp^\infty}]]$-module of rank one (cf. [21, Proposition 5.4]).

We denote by $D_{\mathrm{crys}}(V_f|_{G_{K_\mathfrak{p}}})^{\varphi=\alpha}$ the one-dimensional subspace of the F-vector space $D_{\mathrm{crys}}(V_f|_{G_{K_\mathfrak{p}}})$ on which φ acts by α. We define $D_{\mathrm{crys}}(V_f|_{G_{K_\mathfrak{p}}})^{\varphi=\alpha}(r)$ as $D_{\mathrm{crys}}(V_f|_{G_{K_\mathfrak{p}}})^{\varphi=\alpha}\otimes D_{\mathrm{crys}}(\mathbb{Q}_p(r))$. By [21, Theorem 4.15], there exists $\boldsymbol{g}_c \in \tilde{\mathfrak{R}}^{\psi=0}_c\otimes_{\mathbb{Z}_p} D_{\mathrm{crys}}(V_f|_{G_{K_\mathfrak{p}}})^{\varphi=\alpha}(r)$ such that if we put $\boldsymbol{z}[c] = \tilde{\Omega}^\epsilon_{V,r,c}(\boldsymbol{g}_c) \in \mathscr{H}_\infty(\mathcal{G}_{cp^\infty})\otimes_{\hat{\Lambda}_c} H^1_{\mathrm{Iw}}(\hat{H}_c, T)$, then for $n\geq 1$ and $-r < i < r$, we have

$$\mathrm{pr}_{n,i}(\boldsymbol{z}[c]) = \mathrm{loc}_\mathfrak{p}(\alpha^{-n} z^{(i+r)}[cp^n]_\alpha). \tag{4.32}$$

Here, $\mathrm{loc}_\mathfrak{p} : H^1(K[cp^n], -) \to \oplus_{v|\mathfrak{p}} H^1(K[cp^n]_v, -)$ denotes the sum of localization maps. We write as $\boldsymbol{g}_c = \mathscr{I}_c\otimes\eta$ where η is a basis of $D_{\mathrm{crys}}(V_f|_{G_{K_\mathfrak{p}}})^{\varphi=\alpha}(r)$ and $\mathscr{I}_c\in\tilde{\mathfrak{R}}^{\psi=0}_c$. Let $\boldsymbol{e}$ be a basis of $\tilde{\mathfrak{R}}^{\psi=0}_1$ as a $\mathscr{W}_1[[\mathcal{G}_{p^\infty}]]$-module and $\omega\in D_{\mathrm{crys}}(V)$ be an element such that $[\eta,\omega]_{D_{\mathrm{crys}}(V)} = 1$, where $[-,-]_{D_{\mathrm{crys}}(V_f|_{G_{K_\mathfrak{p}}})(r)}$ denotes the de Rham pairing. We denote by $\mathscr{K}_\infty(\mathcal{G}_{cp^\infty})$ the total quotient ring of $\mathscr{H}_\infty(\mathcal{G}_{cp^\infty})$. We fix a topological generator γ of Γ and put

$$\tilde{\Omega}^{\epsilon}_{V,1-r,c} = \left(\prod_{j=1-r}^{r-1} \ell_{\psi_{\mathfrak{p}}\psi_{\overline{\mathfrak{p}}}^{-1},j}(\gamma)^{-1}\right)\tilde{\Omega}^{\epsilon}_{V,r,c}.$$

Then we consider the element

$$\boldsymbol{w} = (1\otimes\iota)\tilde{\Omega}^{\epsilon}_{V,1-r,1}(\boldsymbol{e}\otimes\omega) \in H^1_{\mathrm{Iw}}(\hat{H}_1, T)\otimes_{\hat{\Lambda}_1}\mathscr{K}_\infty(\mathcal{G}_{p^\infty}).$$

Then, by [21, Theorem 5.8] (a semi-local version of the explicit reciprocity law [32, Théorème 4.2.3]), for $g \in \mathscr{W}_1[[\mathcal{G}_{p^\infty}]]$, we have

$$\langle\tilde{\Omega}^{\epsilon}_{V,r,c}(g\boldsymbol{e}\otimes\eta), \boldsymbol{w}\rangle_{T,1} \in \mathscr{O}[[\mathcal{G}_{p^\infty}]]. \tag{4.33}$$

By [21, Theorem 5.7], there exists $u \in \hat{\mathscr{O}}^{\mathrm{ur}}[[\mathcal{G}_{p^\infty}]]^\times$ such that

$$u\langle\boldsymbol{z}[1], \boldsymbol{w}\rangle_{T,1} = \mathscr{L}^{\mathrm{BDP-B}}_{\mathfrak{p}}(f) \in \hat{\mathscr{O}}^{\mathrm{ur}}[[\mathcal{G}_{p^\infty}]]. \tag{4.34}$$

4.3. Twists of generalized Heegner cycles

In this subsection, we use Proposition 3.7 in order to twist the classes $\{z^{(i)}[c]_\alpha\}$ introduced in §4.1.

By [21, §7], there exists an integer $C \geq 0$ such that for $0 \leq i \leq 2r-2$, $n \geq 0$ and $c_0 \geq 1$ with $(c_0, pN) = 1$, the element $p^C z^{(i)}[c_0p^n]_\alpha$ lies in $H^1(K[c_0p^n], T(\psi_{\mathfrak{p}}^i\psi_{\overline{\mathfrak{p}}}^{k-i}))$ and we have

$$\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}\mathrm{Res}_\infty(p^C z^{(j+1)}[c_0p^n]_\alpha)\otimes e^{\otimes r-1-j} \equiv 0 \tag{4.35}$$

modulo $p^{in}H^1(K[c_0p^\infty], T)$ where e is the basis of $\mathscr{O}(\psi_{\mathfrak{p}}\psi_{\overline{\mathfrak{p}}}^{-1})$ and

$$\mathrm{Res}_\infty : H^1(K[c_0p^n], -) \to H^1([c_0p^\infty], -)$$

denotes the restriction map.

We use Proposition 3.7 as $\Gamma = \mathrm{Gal}(K[c_0p^\infty]/K[c_0p])$, $\rho = \psi_{\mathfrak{p}}\psi_{\overline{\mathfrak{p}}}^{-1}$, $L = K[c_0p]$, $L_\infty = K[c_0p^\infty]$ and $h = 2r-1$. (The conditions are verified by (4.31) and (4.35).) For a continuous character $\chi : \Gamma \to \overline{\mathbb{Q}}_p^\times$ and $n \geq 0$, we put

$$w_{i,n}(\chi) := \sum_{j=0}^{i}(-1)^{j-i}\binom{j}{i}\mathrm{Res}_\infty(z^{(j+1)}[c_0p^n]_\alpha)\otimes e_{\rho^{r-1-j}\chi}.$$

Then, by Proposition 3.11, the sequence

$$(4.36) \quad \frac{1}{\alpha^m} N_{n+m/n} \sum_{i=0}^{2r-2} \binom{\ell_{\rho,0}(\chi\rho^{r-1}(\gamma))}{i} w_{i,n+m}(\chi) \quad (m = 1, 2, \dots)$$

in $H^1(K[c_0p^\infty], V(\chi))$ converges to an element which lies in the image of $H^1(K[c_0p^n], V(\chi))$.

Definition 4.1. For $c_0 \geq 1$ such that $(c_0, pN) = 1$ and $n \geq 1$, we denote by

$$z^\chi[c_0p^n]_\alpha \in H^1(K[c_0p^n], V(\chi))$$

the element whose image in $H^1(K[c_0p^\infty], V(\chi))$ coincides with the limit of the sequence (4.36). (Note that since $H^0(L_\infty, V) = \{0\}$ (cf. [24, Lemma 3.10]), such an element is unique.)

Proposition 4.2. *Let n and c be positive integers such that $(c, N) = 1$. Then,*

(1) $\mathrm{Cor}_{K[cp^{n+1}]/K[cp^n]} z^\chi[cp^{n+1}]_\alpha = \alpha z^\chi[cp^n]_\alpha$.

(2) *For a prime $l \in \mathscr{P}$ such that $l \nmid c$, we have*

$$\mathrm{Cor}_{K[clp^n]/K[cp^n]} z^\chi[clp^n]_\alpha = a_l z^\chi[cp^n]_\alpha.$$

(3) *We have $\tau(z^\chi[cp^n]_\alpha) = w_f \chi(\sigma_{\mathfrak{N}})\sigma_{\overline{\mathfrak{N}}}(z^{\chi^{-1}}[cp^n]_\alpha)$, where $w_f \in \{\pm 1\}$ denotes the eigen value of the eigenform f with respect to the Fricke involution, τ denotes the complex conjugation, and $\sigma_{\mathfrak{a}}$ denotes the image of an ideal $\mathfrak{a}$ of K under the classical Artin map.*

Proof. The assertion (1) follows from Proposition 3.7 (2). The assertion (2) follows from the definition of $\{z^\chi[cp^n]_\alpha\}$ and the norm relation of generalized Heegner cycles $\{z^{(i)}[cp^n]\}$ (cf. [12, Proposition 4.4]). Similarly, the assertion (3) follows from [12, Lemma 4.6]. Q.E.D.

Lemma 4.3. *Let $\mathscr{O}_\chi$ be the ring of integers in $F(\mathrm{Im}(\chi))$ and $\mathfrak{m}_\chi$ the maximal ideal. (We give the trivial action of G_K on $\mathscr{O}_\chi$ (cf. $\mathscr{O}(\chi)$).) Put $\bar{T}(\chi) = T(\chi) \otimes_{\mathscr{O}_\chi} \mathscr{O}_\chi/\mathfrak{m}_\chi$. Under Condition 2.3, the following hold.*

(1) *The representation $\bar{T}(\chi)$ of G_K is absolutely irreducible.*

(2) *For a positive integer c, we have $H^0(K[c], \bar{T}(\chi)) = \{0\}$.*

(3) *The $\mathscr{O}_\chi$-module $H^1(K[c], T(\chi))$ is torsion-free, and for a non-zero ideal I of $\mathscr{O}_\chi$, the restriction map*

$$H^1(K, T(\chi)/I) \to H^0\left(K[c]/K, H^1(K[c], T(\chi)/I)\right)$$

is an isomorphism.

(4) *There exists a Galois extension $E/\mathbb{Q}$ such that $K \subseteq E$, G_E acts trivially on $T(\chi)$, and*

$$H^1(E(\mu_{p^\infty})/K, \bar{T}(\chi)) = \{0\}.$$

(5) *For $m \geq 1$, if L denotes the smallest finite extension $K(T/\mathfrak{m}^m)$ of K such that G_L acts on $T/\mathfrak{m}^m$ trivially, where $\mathfrak{m}$ denotes the maximal ideal of $\mathscr{O}$, then the restriction map $H^1(K, T/\mathfrak{m}^m) \to H^1(L, T/\mathfrak{m}^m)$ is injective.*

Proof. Since $\bar{T}(\chi)$ is isomorphic to $T/\mathfrak{m} \otimes \mathscr{O}_\chi/\mathfrak{m}_\chi$ as a G_K-module, in order to prove (1) and (2), we may assume that χ is the trivial character. Then, the assertion (1) follows from [6, Proposition 6.3 (1)]. To prove (2), it suffices to prove that $H^0(K[c], T/\mathfrak{m}) = \{0\}$. By [6, Lemma 6.2], Condition 2.3 (2) implies that the image of the representation $\varrho_f : G_\mathbb{Q} \to \mathrm{Aut}_\mathscr{O}(T) \cong \mathrm{GL}_2(\mathscr{O})$ contains a subgroup of $\mathrm{GL}_2(\mathscr{O})$ which is conjugate to $\mathrm{GL}_2(\mathbb{Z}_p)$, where we recall that $T = T_f(r)$. Hence, the image of the residual representation $G_\mathbb{Q} \to \mathrm{Aut}_{\mathscr{O}/\mathfrak{m}}(T/\mathfrak{m})$ contains a conjugation of $\mathrm{GL}_2(\mathbb{F}_p)$, which is not solvable by $p \geq 5$. Then, the assertion (2) follows from that $K[c]/\mathbb{Q}$ is a solvable extension.

(3) We first note that the torsion part $H^1(K[c], T(\chi))_{\mathrm{tors}}$ is canonically isomorphic to $H^0(K[c], W(\chi))$. Then, the assertion (2) implies that $H^0(K[c], W(\chi)) = \{0\}$, and hence $H^1(K[c], T(\chi))_{\mathrm{tors}} = \{0\}$. Since $T(\chi)/I$ may be regarded as a G_K-submodule of $W(\chi)$, we have that $H^0(K[c], T(\chi)/I) = \{0\}$. Hence, by the inflation-restriction exact sequence the restriction map

$$H^1(K, T(\chi)/I) \to H^0\left(K[c]/K, H^1(K[c], T(\chi)/I)\right)$$

is an isomorphism.

(4) We put $E = K_\infty(W)$, which contains μ_{p^∞} (we note that the determinant of T is the representation induced by the p-adic cyclotomic character). Since the restriction of χ to G_{K_∞} is trivial, G_E acts trivially on $T(\chi)$. As is explained in the proof of (2), the image $\varrho_f(G_\mathbb{Q})$ of $\varrho_f : G_\mathbb{Q} \to \mathrm{Aut}_\mathscr{O}(T) \cong \mathrm{GL}_2(\mathscr{O})$ contains a subgroup of $\mathrm{GL}_2(\mathscr{O})$ which is conjugate to $\mathrm{GL}_2(\mathbb{Z}_p)$. Then, via the diagonal embedding, we have $\mathbb{Z}_p^\times \subseteq \varrho_f(G_\mathbb{Q})$. Since $[\varrho_f(G_\mathbb{Q}) : \varrho_f(G_K)] \leq 2$, we have $\mu_{\frac{p-1}{2}} \subseteq \varrho_f(G_K)$. We let $\varrho_f(G_\mathbb{Q})$ act on $\bar{T}(\chi) = T/\mathfrak{m} \otimes \mathscr{O}_\chi/\mathfrak{m}_\chi$ where the action on the second component is trivial. Since $p \geq 5$, the image of $\mu_{\frac{p-1}{2}}$ in $\mathrm{Aut}_{\mathscr{O}_\chi/\mathfrak{m}_\chi}(\bar{T}(\chi))$ contains $\{\pm 1\}$, and then we have $H^i(\{\pm 1\}, \bar{T}(\chi)) = \{0\}$ for $i \geq 0$. Hence, by the inflation-restriction sequence relative to $\{\pm 1\} \subseteq \varrho_f(G_K)$, we have $H^1(\varrho_f(G_K), \bar{T}(\chi)) =$

$\{0\}$. Since $H^1(\varrho_f(G_K),\bar{T}(\chi)) = H^1(K(W)/K,\bar{T}(\chi))$, By the inflation-restriction sequence $0 \to H^1(E/K,\bar{T}(\chi)) \to H^1(K(W)/K,\bar{T}(\chi))$, we deduce (4).

(5) We denote by the representation $\varrho_{f,m} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathscr{O}/\mathfrak{m}^m)$ induced by $T/\mathfrak{m}^m$. Since $\varrho_f(G_{\mathbb{Q}})$ contains a subgroup of $\mathrm{GL}_2(\mathscr{O})$ which is conjugate to $\mathrm{GL}_2(\mathbb{Z}_p)$, $\varrho_{f,m}(G_{\mathbb{Q}})$ contains contains a subgroup of $\mathrm{GL}_2(\mathscr{O}/\mathfrak{m}^m)$ which is conjugate to $\mathrm{GL}_2(\mathbb{Z}_p/p^{m_1})$, where m_1 is the integer defined by $p^{m_1}\mathbb{Z}_p = \mathbb{Z}_p \cap \mathfrak{m}^m$. Hence, $\varrho_{f,m}(G_K)$ contains $\mu_{\frac{p-1}{2}}$, and then by the inflation-restriction sequence relative to $\mu_{\frac{p-1}{2}} \subseteq \varrho_{f,m}(G_K)$, we have $H^1(\varrho_{f,m}(G_K),T/\mathfrak{m}^m) = \{0\}$. Since $H^1(L/K,T/\mathfrak{m}^m)$ is isomorphic to $H^1(\varrho_{f,m}(G_K),T/\mathfrak{m}^m)$, we obtain the assertion (5). Q.E.D.

Remark 4.4. Although in [6], the Hecke field is assumed to be unramified at p, the assumption is not needed for [6, Lemma 6.2, Proposition 6.3 (1)].

For a continuous character $\chi : \Gamma \to \overline{\mathbb{Q}}_p^\times$, we put

$$q_\chi = (2r-2)!p^{(2r-2)\delta(2r-1)+C}\alpha^{\max\{0,1-B(\chi\rho^{r-1})\}} \in F^\times,$$

where we use the notation as in Proposition 3.7 (with $h = 2r-1$), and C is the integer as in (4.35). Lemma 4.3 (3), Proposition 3.7 (3) and Remark 3.8 imply that if $p \mid c$ and $(c,N) = 1$, then we have

$$q_\chi z^\chi[c]_\alpha \in H^1(K[c],T(\chi)),$$

where we note that $\rho = \psi_{\mathfrak{p}}\psi_{\overline{\mathfrak{p}}}^{-1}$ on $G_{K[p]}$ satisfies $\mathrm{Im}(\rho) = 1 + p\mathbb{Z}_p$.

Lemma 4.5. *Let c be a positive integer relatively prime to N such that $p \mid c$. Then, the following assertions hold.*

(1) *For a prime $l \nmid p$, we have $\mathrm{loc}_\lambda(q_\chi z^\chi[c]_\alpha) \in H^1_{\mathrm{f}}(K[c]_\lambda,T(\chi))$, where λ is any prime of $K[c]$ above l.*

(2) *For $l \in \mathscr{P}$ such that $l \nmid c$, as elements of $H^1(K[cl]_\lambda,T(\chi))$,*

$$\mathrm{Res}_{K[cl]_\lambda,K[c]_\lambda}(\mathrm{Frob}_l \circ \mathrm{loc}_\lambda(q_\chi z^\chi[c]_\alpha)) = \mathrm{loc}_\lambda(q_\chi z^\chi[cl]_\alpha)$$

where we denote by the same symbol λ the prime of $K[cl]$ above a prime $\lambda \mid l$ of $K[c]$, and Frob_l is the Frobenius element at l over $\mathbb{Q}$. (Note that Frob_l acts on the cohomology since l totally splits in K_∞/K and χ is trivial on G_{K_l}.)

Proof. If we put

$$n = \max\{0,1-B(\chi\rho^{r-1})\}, \quad b = (2r-2)!p^{(2r-2)\delta(2r-1)+C},$$

then by Proposition 3.7 (2) we have

$$\mathrm{Cor}_{K[cp^n]/K[c]}(bz^{\chi}[cp^n]_{\alpha}) = q_{\chi} z^{\chi}[c]_{\alpha},$$
$$\mathrm{Cor}_{K[clp^n]/K[cl]}(bz^{\chi}[clp^n]_{\alpha}) = q_{\chi} z^{\chi}[cl]_{\alpha}.$$

Hence, it suffices to show that

$$\mathrm{loc}_{\lambda}(bz^{\chi}[cp^n]_{\alpha}) \in H^1_{\mathrm{f}}(K[cp^n]_{\lambda}, T(\chi)), \tag{4.37}$$

and

$$\mathrm{Res}_{K[clp^n]_{\lambda}, K[cp^n]_{\lambda}}(\mathrm{Frob}_l \mathrm{loc}_{\lambda}(bz^{\chi}[cp^n]_{\alpha})) = \mathrm{loc}_{\lambda}(bz^{\chi}[clp^n]_{\alpha}) \tag{4.38}$$

in $H^1(K[clp^n]_{\lambda}, T(\chi))$. Note that by Proposition 3.7 (3), the elements $bz^{\chi}[cp^n]_{\alpha}$ and $bz^{\chi}[clp^n]_{\alpha}$ are integral.

Let $P_m(X) \in \mathbb{Z}_p[X]$ be as in Lemma 3.4. We put

$$d'_m[c] := b \sum_{i=0}^{2r-2} \sum_{j=0}^{i} (-1)^{j-i} \binom{i}{j} \mathrm{Res}_{\infty}(z^{(j+1)}[cp^m]_{\alpha}) \otimes e_{\psi_{\mathfrak{p}}^{-j}\psi_{\bar{\mathfrak{p}}}^{j}} \otimes \binom{P_m(\gamma-1)}{i}$$

in $H^1(K[cp^{\infty}], T_f(\psi_{\mathfrak{p}}\psi_{\bar{\mathfrak{p}}}^{2r-1})) \otimes \mathscr{O}[\gamma - 1]$ where the integrality follows from Remark 3.9 and the definition of C. If we denote by

$$\mathrm{ev}_{\chi\rho^{r-1}} : H^1(K[cp^{\infty}], T_f(\psi_{\mathfrak{p}}\psi_{\bar{\mathfrak{p}}}^{2r-1})) \otimes \mathscr{O}[\gamma-1] \to H^1(K[cp^{\infty}], T)(\chi)$$

the homomorphism induced by the map $\mathscr{O}[\gamma - 1] \to \mathscr{O}$ sending γ to $\chi\rho^{r-1}(\gamma)$, where $\rho = \psi_{\mathfrak{p}}\psi_{\bar{\mathfrak{p}}}^{-1}$, then by definition, the sequence

$$\{\mathrm{ev}_{\chi\rho^{r-1}}(\alpha^{-m} N_{n+m/n} {d'}_{m+n}[c])\}_{m\geq 0}$$

converges to $\mathrm{Res}_{\infty}(bz^{\chi}[cp^n])$. We fix a prime ideal of $K[cp^{\infty}]$ above λ, which by abuse of notation, we denote by the same symbol λ, and its restriction to $K[cp^{n+m}]$ is also denote by λ. Since all $\mathrm{loc}_{\lambda}(z^{(j+1)}[cp^{n+m}]_{\alpha})$ lie in $H^1_{\mathrm{ur}}(K[cp^{n+m}]_{\lambda}, V_f(\psi_{\mathfrak{p}}^{j+1}\psi_{\bar{\mathfrak{p}}}^{2r-1-j}))$ (cf. [12, Remark 4.8]), we have

$$\mathrm{loc}_{\lambda}(\alpha^{-m} N_{n+m} d'_{n+m}[c]) \in H^1_{\mathrm{ur}}(K[cp^{\infty}]_{\lambda}, V_f(\psi_{\mathfrak{p}}\psi_{\bar{\mathfrak{p}}}^{2r-1})) \otimes \mathscr{O}[\gamma - 1].$$

Hence, the image of $z^{\chi}[cp^n]_{\alpha}$ in $H^1(K[cp^{\infty}]_{\lambda}, V(\chi))$ is unramified. Since $K[cp^{\infty}]_{\lambda} \subseteq K_{\lambda}^{\mathrm{ur}}$, we conclude the assertion (1).

We next consider (2). By (3.19) and (3.20), there exist

$$F_{n+m}[c](X),\ G_{n+m}[c](X) \in H^1(K[cp^{\infty}], T_f(\psi_{\mathfrak{p}}\psi_{\bar{\mathfrak{p}}}^{2r-1})) \otimes \mathscr{O}[X]$$

such that

$$(4.39)\quad \begin{aligned} &(N_{n+m/n} - N_{n+m-1/n}N_{n+m/n+m-1})d'_{n+m}[c](\gamma-1) \\ &\qquad = F_{n+m}[c](\gamma-1)\omega_{n+m,2r-1}(\gamma-1), \\ &N_{n+m+1/n+m}d'_{n+m+1}[c] - \alpha d'_{n+m}[c] \\ &\qquad = G_{n+m}[c](\gamma-1)\omega_{n+m,2r-1}(\gamma-1). \end{aligned}$$

To simplify the notation, we write

$$\begin{aligned} \omega_{n+m} &= \omega_{n+m,2r-1}(\gamma-1) \in \mathbb{Z}_p[\gamma-1], \\ F_{n+m}[c] &= F_{n+m}[c](\gamma-1) \in H^1(K[cp^\infty], T_f(\psi_{\mathfrak{p}}\psi_{\bar{\mathfrak{p}}}^{2r-1})) \otimes \mathscr{O}[\gamma-1], \\ G_{n+m}[c] &= G_{n+m}[c](\gamma-1) \in H^1(K[cp^\infty], T_f(\psi_{\mathfrak{p}}\psi_{\bar{\mathfrak{p}}}^{2r-1})) \otimes \mathscr{O}[\gamma-1]. \end{aligned}$$

By [12, Lemma 4.7], we have

$$(4.40)\quad \mathrm{Res}_{K[cp^\infty]_\lambda, K[clp^\infty]_\lambda}(\mathrm{Frob}_l(\mathrm{loc}_\lambda(d'_{n+m}[c]))) = \mathrm{loc}_\lambda(d'_{n+m}[cl]),$$

and by (4.39)

$$\begin{aligned} \mathrm{Frob}_l \circ \mathrm{loc}_\lambda(F_{n+m}[c])\omega_{n+m,2r-1} &= \mathrm{loc}_\lambda(F_{n+m}[cl])\omega_{n+m,2r-1}, \\ \mathrm{Frob}_l \circ \mathrm{loc}_\lambda(G_{n+m}[c])\omega_{n+m,2r-1} &= \mathrm{loc}_\lambda(G_{n+m}[cl])\omega_{n+m,2r-1}, \end{aligned}$$

where

$$\begin{aligned} \mathrm{loc}_\lambda : H^1(K[clp^\infty], T_f(\psi_{\mathfrak{p}}\psi_{\bar{\mathfrak{p}}}^{2r-1})) \otimes \mathscr{O}[\gamma-1] \\ \to H^1(K[clp^\infty]_\lambda, T_f(\psi_{\mathfrak{p}}\psi_{\bar{\mathfrak{p}}}^{2r-1})) \otimes \mathscr{O}[\gamma-1] \end{aligned}$$

denotes the scalar extension of the localization map at the prime λ of $K[clp^\infty]$ above λ of $K[cp^\infty]$, and Frob_l acts on $\mathscr{O}[\gamma-1]$ trivially. Since $\omega_{n+m,2r-1}(X) \in \mathscr{O}[X]$ is monic up to unit, we have

$$(4.41)\quad \begin{aligned} \mathrm{Frob}_l \circ \mathrm{loc}_\lambda(F_{n+m}[c]) &= \mathrm{loc}_\lambda(F_{n+m}[cl]), \\ \mathrm{Frob}_l \circ \mathrm{loc}_\lambda(G_{n+m}[c]) &= \mathrm{loc}_\lambda(G_{n+m}[cl]). \end{aligned}$$

We claim that for $m \geq 1$,

$$(4.42)\quad \begin{aligned} &N_{n+m/n}d'_{m+n}[c] - \alpha^m d'_n[c] \\ &= \sum_{0 \leq i \leq m-2} \alpha^i F_{n+m-i}[c]\omega_{n+m-i} \\ &\qquad + \sum_{0 \leq i \leq m-1} \alpha^i N_{n+m-1-i/n} G_{n+m-1-i}[c]\omega_{n+m-1-i}, \end{aligned}$$

where the first summation in the right hand side is regarded as zero when $m=1$.

We prove (4.42) by induction on m. We note that the case $m=1$ is (4.39). By (4.39), we have

$$\begin{aligned}&N_{n+m+1/n}d'_{n+m+1}[c]-\alpha^{m+1}d'_n[c]\\ =&N_{n+m/n}N_{n+m+1/n+m}d'_{n+m+1}[c]-\alpha^{m+1}d'_n[c]+F_{n+m+1}[c]\omega_{n+m+1}\\ =&N_{n+m/n}(\alpha d'_{n+m}[c]+G_{n+m}[c]\omega_{n+m})\\ &\qquad\qquad\qquad\qquad-\alpha^{m+1}d'_n[c]+F_{n+m+1}[c]\omega_{n+m+1}\\ =&\alpha(N_{n+m/n}d'_{n+m}[c]-\alpha^m d'_n[c])\\ &\qquad\qquad+F_{n+m+1}[c]\omega_{n+m+1}+N_{n+m/n}G_{n+m}[c]\omega_{n+m}.\end{aligned}$$

Then, the induction hypothesis implies (4.42).

Since $|p^{2r-1}/\alpha|_p<1$, Lemma 3.2 implies that for sufficiently large m, we have that $\alpha^{i-m}\omega_{n+m-i}(\chi(\gamma)-1), \alpha^{i-m}\omega_{n+m-1-i}(\chi(\gamma)-1)\in\mathscr{O}_{\mathbb{C}_p}$. Hence, by (4.41), for such m we have

$$\begin{aligned}&\mathrm{Frob}_l\circ\mathrm{loc}_\lambda(\alpha^{i-m}\omega_{n+m-i}(\chi(\gamma)-1)F_{n+m-i}[c](\chi(\gamma)-1))\\ &\qquad=\mathrm{loc}_\lambda(\alpha^{i-m}\omega_{n+m-i}(\chi(\gamma)-1)F_{n+m-i}[cl](\chi(\gamma)-1)),\\ &\mathrm{Frob}_l\circ\mathrm{loc}_\lambda(\alpha^{i-m}\omega_{n+m-1-i}(\chi(\gamma)-1)G_{n+m-1-i}[c](\chi(\gamma)-1))\\ &\qquad=\mathrm{loc}_\lambda(\alpha^{i-m}\omega_{n+m-1-i}(\chi(\gamma)-1)G_{n+m-1-i}[cl](\chi(\gamma)-1))\end{aligned}$$

in $H^1(K[clp^\infty]_\lambda,T(\chi))$. Hence, by (4.40), (4.42) and Lemma 4.3 (3), for $m\geq 1$ we have

$$\begin{aligned}&\mathrm{Res}_{K[cp^\infty]_\lambda,K[clp^\infty]_\lambda}\left(\mathrm{Frob}_l\left(\mathrm{loc}_\lambda\circ\mathrm{ev}_{\chi\rho^{r-1}}\left(\alpha^{-m}N_{n+m/n}d'_{m+n}[c]\right)\right)\right)\\ &\qquad\qquad=\mathrm{loc}_\lambda\circ\mathrm{ev}_{\chi\rho^{r-1}}\left(\alpha^{-m}N_{n+m/n}d'_{n+m}[cl]\right)\end{aligned}$$

in $H^1(K[clp^\infty]_\lambda,T(\chi))$. Note that since our global Galois cohomology groups are torsion-free (cf. Lemma 4.3 (3)), it is harmless to invert α in the global cohomology groups. Therefore, we inverted α before the localization. Since the sequence

$$\{\mathrm{ev}_{\chi\rho^{r-1}}(\alpha^{-m}N_{n+m/n}d'_{m+n}[c])\}_{m\geq 0}$$

converges to $\mathrm{Res}_\infty bz^\chi[cp^n]$, we obtain (4.38). Q.E.D.

4.4. Local conditions of twisted Selmer groups

Following ideas in [12, §7.4], we introduce certain Selmer groups that are suite for Euler system arguments via twisted Heegner classes.

For a height-one prime ideal $\mathfrak{P}$ of Λ which is not equal to $p\Lambda$, we let $\Phi_{\mathfrak{P}}$ be the fractional field of $\Lambda/\mathfrak{P}$, which is a finite extension of $F = \mathscr{O}[1/p]$. We denote by $S_{\mathfrak{P}}$ the integral closure of $\Lambda/\mathfrak{P}$ in $\Phi_{\mathfrak{P}}$ and by $\mathfrak{m}_{\mathfrak{P}}$ the maximal ideal. We let G_K act on $S_{\mathfrak{P}}$ via the natural map $G_K \to \Lambda^\times \to S_{\mathfrak{P}}^\times$. Fixing an embedding $\Phi_{\mathfrak{P}} \hookrightarrow \mathbb{C}_p$ which is $\mathscr{O}$-linear, we denote by $\chi_{\mathfrak{P}} : G_K \to \mathscr{O}_{\mathbb{C}_p}^\times$ the character induced by the map $G_K \to S_{\mathfrak{P}}^\times \hookrightarrow \mathscr{O}_{\mathbb{C}_p}^\times$. For an $\mathscr{O}$-module M, we put $M_{\mathfrak{P}} = M \otimes_{\mathscr{O}} S_{\mathfrak{P}}$. In the case where M has a G_K-action, we let G_K act on $M_{\mathfrak{P}}$ by acting on both factors in the tensor product. Then, we have $M_{\mathfrak{P}} \cong M(\chi_{\mathfrak{P}})$ as G_K-modules.

In order to define Selmer structures, for a prime v of K, we define a perfect pairing

$$(-,-)_{\mathfrak{P},v} : H^1(K_v, V_{\mathfrak{P}}) \times H^1(K_{\bar{v}}, V_{\mathfrak{P}}) \to \Phi_{\mathfrak{P}} \tag{4.43}$$

as follows, where $\bar{v} := v^\tau$. By [28, Proposition 3.1 (ii)], there exists a $G_{\mathbb{Q}}$-equivariant skew-symmetric paring

$$[-,-]_f : T \times T \to \mathbb{Z}_p(1)$$

such that for $\lambda \in \mathscr{O}$ and $x, y \in T$, we have $[\lambda x, y]_f = [x, \lambda y]_f$, and the induced pairings $T/p^m \times T/p^m \to \mathbb{Z}/p^m(1)$ are perfect. If $\mathfrak{d} \in F^\times$ is a generator of the absolute inverse different of $\mathscr{O}$, then the homomorphism

$$\mathrm{Hom}_{\mathscr{O}}(T, \mathscr{O}(1)) \to \mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p(1)); \quad \phi \mapsto \mathrm{Tr}_{F/\mathbb{Q}_p} \circ (\mathfrak{d} \cdot \phi)$$

is an isomorphism. We note that the paring $[-,-]_f$ gives rise to a homomorphism $T \to \mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p(1))$, and by the isomorphism above, we have a perfect, skew-symmetric, $\mathscr{O}$-linear, $G_{\mathbb{Q}}$-equivariant paring

$$[-,-]_{\mathscr{O}} : T \times T \to \mathscr{O}(1).$$

Following [18, Lemma 2.1.1], we have a perfect $S_{\mathfrak{P}}$-bilinear pairing

$$[-,-]_{\mathfrak{P}} : T_{\mathfrak{P}} \times T_{\mathfrak{P}} \to S_{\mathfrak{P}}(1)$$

defined by $(t_1 \otimes \alpha_1, t_2 \otimes \alpha_2) \mapsto \alpha_1 \alpha_2 [t_1, t_2^\tau]_{\mathscr{O}}$. Then, for $\sigma \in G_K$ and $s, t \in T_{\mathfrak{P}}$, we have $[s^\sigma, t^{\tau\sigma\tau}]_{\mathfrak{P}} = [s,t]_{\mathfrak{P}}^\sigma$. Therefore, by the local duality, for a prime v of K it induces a perfect pairing

$$(-,-)_{\mathfrak{P},v} : H^1(K_v, V_{\mathfrak{P}}) \times H^1(K_{\bar{v}}, V_{\mathfrak{P}}) \to \Phi_{\mathfrak{P}}.$$

We note that for a prime v, $H^1(K_{\bar{v}}, V_{\mathfrak{P}}) \cong H^1(K_v, V_{\mathfrak{P}^\iota})$ (as $\Phi_{\mathfrak{P}}$-modules) via the complex conjugation. Here, we identify $\Phi_{\mathfrak{P}}$ with $\Phi_{\mathfrak{P}^\iota}$ via τ. By abuse of notation, we denote by the same symbol $(-,-)_{\mathfrak{P},v}$ the induced pairing $H^1(K_v, V_{\mathfrak{P}}) \times H^1(K_v, V_{\mathfrak{P}^\iota}) \to \Phi_{\mathfrak{P}}$.

We define $\mathscr{Z}_{\mathfrak{P}}^{\mathfrak{p}}$ as the $\Phi_{\mathfrak{P}}$-subspace of $H^1(K_v, V_{\mathfrak{P}})$ generated by the image of the map $\Omega_{\mathfrak{P}}$ defined as the composite

$$\begin{aligned}\Omega_{\mathfrak{P}} : \tilde{\mathfrak{R}}_1^{\psi=0} \otimes_{\mathbb{Z}_p} D_{\mathrm{crys}}(V_f|_{G_{K_{\mathfrak{p}}}})^{\varphi=\alpha}(r) &\xrightarrow{\tilde{\Omega}^{\epsilon}_{V,r,1}} H^1_{\mathrm{Iw}}(\hat{H}_1, T) \otimes_{\hat{\Lambda}_1} \mathscr{H}_\infty(\mathcal{G}_{cp^\infty}) \\ &\to H^1(\hat{H}_p, V_{\mathfrak{P}}) \\ &\to H^1(K_{\mathfrak{p}}, V_{\mathfrak{P}}),\end{aligned}$$

where the second map is induced by (3.8) and by regarding $\chi_{\mathfrak{P}}$ is a character on the Galois group $\mathrm{Gal}(\hat{H}_{p^\infty}/\hat{H}_p)$, and the last map is the corestriction map. If $\chi_{\mathfrak{P}}|_{G_{K_{\mathfrak{p}}}}$ is of the form $\rho^j\psi$, where $-r < j < r$ and ψ is a finite character of Γ, then the image $\mathscr{Z}_{\mathfrak{P}}^{\mathfrak{p}}$ coincides with the one-dimensional subspace $H^1_{\mathrm{f}}(K_{\mathfrak{p}}, V_{\mathfrak{P}})$. Under some assumption, we may also define the space $\mathscr{Z}_{\mathfrak{P}}^{\mathfrak{p}}$ in terms of twisted Heegner cycles. To explain it, we fix some notation. By following the notation as in the previous subsection, we put $q_{\mathfrak{P}} = q_{\chi_{\mathfrak{P}}}$, and for $c \in \mathscr{N}$, we put

$$\mathfrak{z}_c^{\mathfrak{P}} = \mathrm{Cor}_{K[cp]/K[c]}\,(q_{\mathfrak{P}} z^{\chi_{\mathfrak{P}}}[cp]_\alpha) \in H^1(K[c], T_{\mathfrak{P}}).$$

Then, by Proposition 4.2, for $l \in \mathscr{P}$ with $l \nmid c$, we have

$$\mathrm{Cor}_{K[cl]/K[c]}\left(\mathfrak{z}_{cl}^{\mathfrak{P}}\right) = a_l \mathfrak{z}_c^{\mathfrak{P}}. \tag{4.44}$$

We also put

$$\kappa_1^{\mathfrak{P}} = \mathrm{Cor}_{K[1]/K}(\mathfrak{z}_1^{\mathfrak{P}}) \in H^1(K, T_{\mathfrak{P}}).$$

Lemma 4.6. *For a height-one prime ideal $\mathfrak{P} \neq p\Lambda$ of Λ such that the image of $\kappa_1^{\mathfrak{P}}$ in $H^1(K_{\mathfrak{p}}, V_{\mathfrak{P}})$ is non-zero, the $\Phi_{\mathfrak{P}}$-vector space $\mathscr{Z}_{\mathfrak{P}}^{\mathfrak{p}}$ is generated by $\kappa_1^{\mathfrak{P}}$.*

Proof. The space $\mathscr{Z}_{\mathfrak{P}}^{\mathfrak{p}}$ is generated by

$$\Big(\Omega_{\mathfrak{P}}(\boldsymbol{e} \otimes \eta), \mathrm{Cor}_{\hat{H}_1/K_{\mathfrak{p}}}(\mathrm{pr}_{\chi_{\mathfrak{P}^\iota}}(\boldsymbol{w}))\Big)_{\mathfrak{P},\mathfrak{p}},$$

where $\mathrm{pr}_{\chi_{\mathfrak{P}^\iota}} : H^1_{\mathrm{Iw}}(\hat{H}_1, T) \otimes_{\hat{\Lambda}_1} \mathscr{H}_\infty(\mathcal{G}_{cp^\infty}) \to H^1(\hat{H}_p, V_{\mathfrak{P}^\iota})$ is induced by (3.8). Hence it is of dimension at most 1. (It is actually equal to 1 by the explicit reciprocity law.) Since $\boldsymbol{z}[1] = \tilde{\Omega}^{\epsilon}_{V,r,1}(\boldsymbol{g}_1)$, the element $\kappa_1^{\mathfrak{P}}$ lies on $\mathscr{Z}_{\mathfrak{P}}^{\mathfrak{p}}$, and hence generate this space if it is non-zero. Q.E.D.

We introduce a Selmer structure $\mathscr{F}_{\mathfrak{P}}$ as follows. For $v = \mathfrak{p}$, we define $H^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, V_{\mathfrak{P}}) \subseteq H^1(K_{\mathfrak{p}}, V_{\mathfrak{P}})$ as the orthogonal complement of $\mathscr{X}^{\mathfrak{p}}_{\mathfrak{P}^\iota}$ under the pairing $(-,-)_{\mathfrak{P},\mathfrak{p}}$. For $v = \bar{\mathfrak{p}}$, we define $H^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\bar{\mathfrak{p}}}, V_{\mathfrak{P}})$ as the complex conjugation of $H^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, V_{\mathfrak{P}})$ in $\oplus_{v|p} H^1(K_v, V_{\mathfrak{P}})$. For a prime v of K not dividing p, we put $H^1_{\mathscr{F}_{\mathfrak{P}}}(K_v, V_{\mathfrak{P}}) = H^1_{\mathrm{f}}(K_v, V_{\mathfrak{P}})$. One can show that if f is ordinary (*i.e.* $\alpha \in \mathscr{O}^\times$), then our $\mathscr{F}_{\mathfrak{P}}$ coincides with the Selmer structure introduced in [25, §3.4].

Lemma 4.7. *Let c be a positive integer relatively prime to p. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be elements in the image of $\tilde{\mathfrak{R}}_c^{\psi=0} \otimes D_{\mathrm{crys}}(V_f|_{G_{K_{\mathfrak{p}}}})^{\varphi=\alpha}(r)$ under $\tilde{\Omega}^\epsilon_{V,r,c}$. Then, the element $\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{T,c} \in \mathscr{H}_\infty(\mathcal{G}_{cp^\infty})$ is zero.*

Proof. This is proved in [21, §8]. Q.E.D.

Remark 4.8. For a height-one prime ideal $\mathfrak{P} \neq p\Lambda$, since the specialization of $\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{T,1} \in \mathscr{H}_\infty(\mathcal{G}_{p^\infty})$ at $\chi_{\mathfrak{P}}$ is equal to the pairing $(\Omega_{\mathfrak{P}}(\boldsymbol{h}_{\boldsymbol{x}}), \Omega_{\mathfrak{P}^\iota}(\boldsymbol{h}_{\boldsymbol{y}}))_{\mathfrak{P},\mathfrak{p}}$, the lemma above (with $c = 1$) implies that $H^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, V_{\mathfrak{P}})$ is the orthogonal complement of $H^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\bar{\mathfrak{p}}}, V_{\mathfrak{P}})$.

For a prime $v \mid p$ of K, we put

$$\tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_v, T_{\mathfrak{P}}) = H^1_{\mathscr{F}_{\mathfrak{P}}}(K_v, T_{\mathfrak{P}})/H^1(K_v, T_{\mathfrak{P}})_{\mathrm{tors}},$$

which is a free $S_{\mathfrak{P}}$-module of rank one.

Lemma 4.9. *Let M be an $\mathscr{O}$-lattice of $D_{\mathrm{crys}}(V_f|_{G_{K_{\mathfrak{p}}}})^{\varphi=\alpha}(r)$. For a height-one prime ideal $\mathfrak{P} \neq p\Lambda$, there exists a non-zero element $\beta_{\mathfrak{P}} \in \mathscr{O}$ such that the following assertions hold.*

(1) *We have $\beta_{\mathfrak{P}}\Omega_{\mathfrak{P}}(\tilde{\mathfrak{R}}_1^{\psi=0} \otimes M) \subseteq \tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}})$.*

(2) *There exists $m_0 > 0$ which depends on $\mathfrak{P}$ such that for $m \geq m_0$, if $\mathfrak{Q} \neq p\Lambda$ is a height-one prime ideal of Λ with $\mathrm{Im}(\chi_{\mathfrak{P}}\chi_{\mathfrak{Q}}^{-1}) \in 1 + p^m S_{\mathfrak{P}}$, then $\beta_{\mathfrak{P}} = \beta_{\mathfrak{Q}}$ and*

$$\begin{aligned}\mathrm{length}_{S_{\mathfrak{P}}}&\left(\tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}})/\beta_{\mathfrak{P}}\Omega_{\mathfrak{P}}(\tilde{\mathfrak{R}}_1^{\psi=0} \otimes M)\right)\\ &= \mathrm{length}_{S_{\mathfrak{Q}}}\left(\tilde{H}^1_{\mathscr{F}_{\mathfrak{Q}}}(K_{\mathfrak{p}}, T_{\mathfrak{Q}})/\beta_{\mathfrak{Q}}\Omega_{\mathfrak{Q}}(\tilde{\mathfrak{R}}_1^{\psi=0} \otimes M)\right).\end{aligned}$$

Proof. (1) Let $\boldsymbol{h}$ be a $\mathscr{W}_1[[\mathcal{G}_{p^\infty}]] \otimes \mathscr{O}$-basis of $\tilde{\mathfrak{R}}_1^{\psi=0} \otimes M$. We write $\sum_{i=1}^k x_i \otimes g_i(\gamma - 1) \in H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}}, T) \otimes \mathscr{H}_{2r-1,F}(\Gamma)$ for the image of $\tilde{\Omega}^\epsilon_{V,r,1}(\boldsymbol{h})$ under the map induced by the trace map. Then, for $\mathfrak{P}$, we have

$$\Omega_{\mathfrak{P}}(\boldsymbol{h}) = \sum_i g_i(\chi_{\mathfrak{P}}(\gamma) - 1)\mathrm{pr}_{0,\chi_{\mathfrak{P}}}(x_i). \tag{4.45}$$

Hence, if we define

$$\beta_{\mathfrak{P}} = p^{\min\{\, n\in\mathbb{Z} \mid n\geq 0 \text{ and } n\geq \mathrm{ord}_p(g_i(\chi_{\mathfrak{P}}(\gamma-1))) \text{ for } 1\leq i\leq k\,\}},$$

then we obtain (1).

(2) We enlarge $\mathscr{O}$ so that $\mathscr{O} = S_{\mathfrak{P}} = S_{\mathfrak{Q}}$ (we note that the assumption on $\mathfrak{Q}$ implies $S_{\mathfrak{Q}} \subseteq S_{\mathfrak{P}}$). We first note that there exists a positive integer m_1 such that for $m \geq m_1$ and for $\mathfrak{Q}$ such that $\mathrm{Im}(\chi_{\mathfrak{P}}\chi_{\mathfrak{Q}}^{-1}) \in 1 + \mathfrak{m}^m\mathscr{O}$, where $\mathfrak{m}$ denotes the maximal ideal of $\mathscr{O}$, we have $\beta_{\mathfrak{P}} = \beta_{\mathfrak{Q}}$. We note that $\beta_{\mathfrak{P}}\Omega_{\mathfrak{P}}(\boldsymbol{h})$ is not torsion and that $\tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}})$ is free of rank one. Hence, if we put

$$m_2 = \max\left\{\, m\in\mathbb{Z} \;\middle|\; \beta_{\mathfrak{P}}\Omega_{\mathfrak{P}}(\boldsymbol{h}) \in \mathfrak{m}^m H^1(K_{\mathfrak{p}}, T_{\mathfrak{P}})\,\right\},$$

then we have

$$m_2 = \mathrm{length}_{\mathscr{O}}\left(\tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}})/\beta_{\mathfrak{P}}\Omega_{\mathfrak{P}}(\tilde{\mathfrak{R}}_1^{\psi=0}\otimes M)\right).$$

Under the identification $H^1(K_{\mathfrak{p}}, T_{\mathfrak{P}}/\mathfrak{m}^m) \cong H^1(K_{\mathfrak{p}}, T_{\mathfrak{Q}}/\mathfrak{m}^m)$, for $x \in H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}}/K_{\mathfrak{p}}, T)$ the images of $\mathrm{pr}_{0,\chi_{\mathfrak{P}}}(x)$ and $\mathrm{pr}_{0,\chi_{\mathfrak{P}}}(x)$ are the same. We put $m_0 = \max\{\, m_1, m_2\,\}$. Then, by (4.45), if $m \geq m_0$, the images of $\beta_{\mathfrak{P}}\Omega_{\mathfrak{P}}(h)$ and $\beta_{\mathfrak{Q}}\Omega_{\mathfrak{Q}}(h)$ in $H^1(K_{\mathfrak{p}}, T_{\mathfrak{P}}/\mathfrak{m}^m) \cong H^1(K_{\mathfrak{p}}, T_{\mathfrak{Q}}/\mathfrak{m}^m)$ are the same and non-trivial. Hence we have

$$\begin{aligned}\max&\left\{\, m\in\mathbb{Z} \;\middle|\; \beta_{\mathfrak{P}}\Omega_{\mathfrak{P}}(\boldsymbol{h}) \in \mathfrak{m}^m H^1(K_{\mathfrak{p}}, T_{\mathfrak{P}})\,\right\}\\ &= \max\left\{\, m\in\mathbb{Z} \;\middle|\; \beta_{\mathfrak{Q}}\Omega_{\mathfrak{Q}}(\boldsymbol{h}) \in \mathfrak{m}^m H^1(K_{\mathfrak{p}}, T_{\mathfrak{Q}})\,\right\},\end{aligned}$$

which implies the assertion (2). Q.E.D.

4.5. Bounding Selmer groups

In this subsection, we give a bound on the Selmer group $H^1_{\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}})$ in terms of twisted Heegner classes. We assume Condition 2.3. Let $\mathscr{P}$ be the set of rational primes that are inert in K and relatively prime to N. Let $\mathscr{N}$ be the set of square free natural numbers that are only divisible by primes in $\mathscr{P}$.

Let $\mathfrak{P} \neq p\Lambda$ be a height-one prime ideal of Λ. Since we use arguments in [18], we first show that our $T_{\mathfrak{P}}$ and $\mathscr{F}_{\mathfrak{P}}$ satisfy the hypotheses H.1) to H.5) in [18, §1.3]. Lemma 4.3 (1) implies H.1), and Lemma 4.3 (4) implies H.2). If we denote by $\Sigma(\mathscr{F}_{\mathfrak{P}})$ the set of primes of K dividing pN, then the hypothesis H.3) follows from the definition and from [27, Lemma 3.7.1]. The hypothesis H.4) is satisfied by the pairing $[-,-]_{\mathfrak{P}}$, the definition of $\mathscr{F}_{\mathfrak{P}}$ and Remark 4.8. Since $\bar{T}_{\mathfrak{P}} := T_{\mathfrak{P}}/\mathfrak{m}_{\mathfrak{P}}$ is naturally isomorphic to the $G_{\mathbb{Q}}$-module $T/\mathfrak{m}\otimes S_{\mathfrak{P}}/\mathfrak{m}_{\mathfrak{P}}$, the hypothesis H.5) a) follows from the fact that T is odd. Since the local condition at $\bar{\mathfrak{p}}$ is defined as the complex conjugation of that at $\mathfrak{p}$, the hypothesis H.5 b) holds. The hypothesis H.5) c) is satisfied by $\bar{T}_{\mathfrak{P}} \cong T/\mathfrak{m} \otimes S_{\mathfrak{P}}/\mathfrak{m}_{\mathfrak{P}}$.

For $c \in \mathscr{N}$, we put

$$\mathcal{G}_c = \mathrm{Gal}(K[c]/K), \quad G_c = \mathrm{Gal}(K[c]/K[1]).$$

For every $l \in \mathscr{P}$, we fix a generator σ_l of G_l and put $D_l = \sum_{i=1}^{l} i\sigma_l^i \in \mathbb{Z}[G_l]$. Then, by an straightforward computation, we have

$$(\sigma_l - 1)D_l = l + 1 - N_l, \tag{4.46}$$

where $N_l = \sum_{g \in G_l} g$. We note that $G_c = \prod_{l|c} G_l$ and put $D_c = \prod_{l|c} D_l \in \mathbb{Z}[G_c]$. For $l \in \mathscr{P}$, we denote by I_l the ideal of $S_{\mathfrak{P}}$ generated by $l+1$ and a_l, and we put $I_c = \sum_{l|c} I_l$. By convention, we put $I_1 = \{0\}$ and $G_1 = \mathbb{Z}$. We fix a coset representatives $S(c) \subseteq \mathcal{G}_c$ for $\mathcal{G}_c/G_c$, and put

$$\tilde{\kappa}_c^{\mathfrak{P}} = \sum_{s \in S(c)} sD_c(\mathfrak{z}_c^{\mathfrak{P}}) \in H^1(K[c], T_{\mathfrak{P}}).$$

By (4.44) and (4.46), the image of $\tilde{\kappa}_c^{\mathfrak{P}}$ in $H^1(K[c], T_{\mathfrak{P}}/I_c)$ is fixed by $\mathcal{G}_c$ and independent of the choice of $S(c)$. By Lemma 4.3 (3), the restriction map

$$H^1(K, T_{\mathfrak{P}}/I_c) \to H^0(\mathcal{G}_c, H^1(K[c], T_{\mathfrak{P}}/I_c))$$

is an isomorphism. Let $\kappa_c^{\mathfrak{P}} \in H^1(K, T_{\mathfrak{P}}/I_c)$ be the element whose image in $H^1(K[c], T_{\mathfrak{P}}/I_c)$ coincides with the image of $\tilde{\kappa}_c^{\mathfrak{P}}$.

Lemma 4.10. *For $c \in \mathscr{N}$, we have $\kappa_c^{\mathfrak{P}} \in H^1_{\mathscr{F}_{\mathfrak{P}}(c)}(K, T_{\mathfrak{P}}/I_c)$, where $\mathscr{F}_{\mathfrak{P}}(c)$ denotes the Selmer structure as in* [18, Definition 1.2.2] (*we recall that for $v \nmid c$, $H^1_{\mathscr{F}_{\mathfrak{P}}}(K_v, T_{\mathfrak{P}}) = H^1_{\mathscr{F}_{\mathfrak{P}}(c)}(K_v, T_{\mathfrak{P}})$*).

Proof. We need to prove that for a prime v of K we have

$$\mathrm{loc}_v(\kappa_c^{\mathfrak{P}}) \in H^1_{\mathscr{F}_{\mathfrak{P}}}(K_v, T_{\mathfrak{P}}/I_c). \tag{4.47}$$

If $v \nmid p$, then (4.47) follows from Lemma 4.5 (1), Proposition 4.2 and the same argument as in the proof of [18, Lemma 1.7.3]. Suppose that $v \mid p$. We first consider the case where $v = \mathfrak{p}$. We denote by

$$(-,-) : H^1(K_{\mathfrak{p}}, T_{\mathfrak{P}}/I_c) \times H^1(K_{\mathfrak{p}}, W_{\mathfrak{P}^\iota}[I_c]) \to S_{\mathfrak{P}}/I_c$$

the pairing induced by the local duality.

It suffices to show that for $x \in H^1(K_{\mathfrak{p}}, W_{\mathfrak{P}^\iota}[I_c])$ whose image in $H^1(K_{\mathfrak{p}}, W_{\mathfrak{P}^\iota})$ coincides with the image of $\Omega_{\mathfrak{P}^\iota}(\boldsymbol{h})$ for some $\boldsymbol{h} \in \tilde{\mathfrak{R}}_1^{\psi=0} \otimes D_{\mathrm{crys}}(V_f|_{G_{K_{\mathfrak{p}}}})^{\varphi=\alpha}(r)$, we have

$$(\mathrm{loc}_{\mathfrak{p}}(\kappa_c^{\mathfrak{P}}), x) = 0. \tag{4.48}$$

Let M be an $\mathscr{O}$-lattice of $D_{\mathrm{crys}}(V_f|_{G_{K_{\mathfrak{p}}}})$ such that $\boldsymbol{h}$ lies in $\tilde{\mathfrak{R}}_1^{\psi=0} \otimes M$. Since the trace map $\mathrm{tr}_{c/1} : \mathscr{W}_c \to \mathscr{W}_1$ is surjective and $\tilde{\mathfrak{R}}_c$ is isomorphic to $\tilde{\mathfrak{R}}_1 \otimes_{\mathscr{W}_1} \mathscr{W}_c$, there exists $\boldsymbol{h}' \in \tilde{\mathfrak{R}}_c^{\psi=0} \otimes M$ whose image in $\tilde{\mathfrak{R}}_1^{\psi=0} \otimes M$ under the map induced by $\mathrm{tr}_{c/1}$ is equal to $\boldsymbol{h}$. Hence, if we put $\boldsymbol{y} = \tilde{\Omega}_{V,r,c}^{\epsilon}(\boldsymbol{h}') \in \mathscr{H}_\infty(\mathcal{G}_{cp^\infty}) \otimes_{\hat{\Lambda}_c} H^1_{\mathrm{Iw}}(\hat{H}_c, T)$ and $\boldsymbol{x} = \tilde{\Omega}_{V,r,1}^{\epsilon}(\boldsymbol{h})$, then

$$\pi_{K[cp^\infty]/K[p^\infty]}(\boldsymbol{y}) = \boldsymbol{x},$$

where $\pi_{K[cp^\infty]/K[p^\infty]} : H^1_{\mathrm{Iw}}(\hat{H}_c, T) \otimes_{\hat{\Lambda}_c} \mathscr{H}_\infty(\mathcal{G}_{cp^\infty}) \to H^1_{\mathrm{Iw}}(\hat{H}_1, T) \otimes_{\hat{\Lambda}_1} \mathscr{H}_\infty(\mathcal{G}_{p^\infty})$ denotes the natural map induced by the corestriction map. We denote by $y^{\mathfrak{P}^\iota} \in H^1(\hat{H}_c, W_{\mathfrak{P}^\iota})$ the image of $\boldsymbol{y}$. Then the element $\mathrm{Cor}_{\hat{H}_c/\hat{H}_1}(y^{\mathfrak{P}^\iota})$ of $H^1(\hat{H}_1, W_{\mathfrak{P}^\iota})$ coincides with the image of x. Hence, by a simple computation, we have

$$(\mathrm{loc}_{\mathfrak{p}}(\kappa_c^{\mathfrak{P}}), x) = \left(\mathrm{loc}_{\mathfrak{p},c}\left(\sum_{s \in S(c)} sD_c q_{\mathfrak{P}} z^{\chi_{\mathfrak{P}}}[c]_\alpha\right), y^{\mathfrak{P}^\iota}\right)_{\mathfrak{P},c}, \tag{4.49}$$

where $z^{\chi_{\mathfrak{P}}}[c]_\alpha$ is as in Definition 4.1, the pairing

$$(-,-)_{\mathfrak{P},c} : H^1(\hat{H}_c, T_{\mathfrak{P}}) \times H^1(\hat{H}_c, W_{\mathfrak{P}^\iota}) \to \Phi_{\mathfrak{P}}/S_{\mathfrak{P}}$$

is induced by the local duality, and $\mathrm{loc}_{\mathfrak{p},c} : H^1(K[c], T_{\mathfrak{P}}) \to H^1(\hat{H}_c, T_{\mathfrak{P}})$ denotes the localization map. Hence, we are reduced to showing that

$$\left(\mathrm{loc}_{\mathfrak{p},c}\left(q_{\mathfrak{P}} z^{\chi_{\mathfrak{P}}}[c]_\alpha\right), y^{\mathfrak{P}^\iota}\right)_{\mathfrak{P},c} = 0 \in \Phi_{\mathfrak{P}}/S_{\mathfrak{P}}. \tag{4.50}$$

We note that the left hand side coincides with the image of the specialization of the element $\langle q_{\mathfrak{P}} \boldsymbol{z}[c], \boldsymbol{y}\rangle \in \mathscr{H}_\infty(\mathcal{G}_{cp^\infty})$ at $\chi_{\mathfrak{P}}$, where $\boldsymbol{z}[c]$ is as in Subsection 4.2. Since $\boldsymbol{z}[c]$ and $\boldsymbol{y}$ are elements in $\tilde{\Omega}_{V,r,c}^{\epsilon}(\tilde{\mathfrak{R}}_c^{\psi=0} \otimes_{\mathbb{Z}_p} D_{\mathrm{crys}}(V_f|_{G_{K_{\mathfrak{p}}}})^{\varphi=\alpha}(r))$, the equation (4.50) follows from Lemma 4.7.

The case where $v = \bar{\mathfrak{p}}$ is completed by using the complex conjugation and Proposition 4.2. Q.E.D.

For $c \in \mathscr{N}$, we put $\underline{\kappa}_c^{\mathfrak{P}} = \kappa_c^{\mathfrak{P}} \otimes \prod_{l|c} \sigma_l \in H^1_{\mathscr{F}_{\mathfrak{P}}(c)}(K, T_{\mathfrak{P}}/I_c) \otimes_{\mathbb{Z}} G_c$. For $i \geq 1$, we denote by $\mathscr{P}^{(i)}$ the subset of $\mathscr{P}$ consisting of l such that

$I_l \subseteq \mathfrak{m}_{\mathfrak{P}}^i$. We similarly define $\mathscr{N}^{(i)}$. For $c \in \mathscr{N}^{(i)}$, we put $T_{\mathfrak{P}}^{(i)} = T_{\mathfrak{P}}/\mathfrak{m}_{\mathfrak{P}}^i$ and denote by $\underline{\kappa}_c^{\mathfrak{P},(i)}$ the image of $\underline{\kappa}_c^{\mathfrak{P}}$ in $H^1_{\mathscr{F}_{\mathfrak{P}}(c)}(K, T_{\mathfrak{P}}^{(i)})$. For a prime $v \nmid p$ of K, we put $H^1_{\mathrm{s}}(K_v, T_{\mathfrak{P}}^{(i)}) = H^1(K_v, T_{\mathfrak{P}}^{(i)})/H^1_{\mathrm{f}}(K_v, T_{\mathfrak{P}}^{(i)})$.

Lemma 4.11. *Let d^+ be an element of $H^1(K, \bar{T}_{\mathfrak{P}})^+$ and d^- an element of $H^1(K, \bar{T}_{\mathfrak{P}})^-$, where $H^1(K, \bar{T}_{\mathfrak{P}})^{\pm}$ denotes the subspace of $H^1(K, \bar{T}_{\mathfrak{P}})$ on which the complex conjugation acts by ± 1 (recall that $\bar{T}_{\mathfrak{P}} := T_{\mathfrak{P}}/\mathfrak{m}_{\mathfrak{P}}$). Then, for $c \in \mathscr{N}^{(2i-1)}$ there exist infinitely many primes $l \in \mathscr{P}^{(2i-1)}$ such that*

(1) *if $d^{\pm} \neq 0$, then $\mathrm{loc}_l(d^{\pm}) \neq 0 \in H^1(K_l, \bar{T}_{\mathfrak{P}})$,*
(2) *the image of $\mathrm{loc}_l(\underline{\kappa}_{cl}^{\mathfrak{P},(i)})$ in $H^1_{\mathrm{s}}(K_l, T_{\mathfrak{P}}^{(i)}) \otimes G_{cl}$ coincides with $(\phi_l^{\mathrm{fs}} \otimes 1) \circ \mathrm{loc}_l(\underline{\kappa}_c^{\mathfrak{P},(i)})$ up to a unit in $S_{\mathfrak{P}}^{\times}$, where $\phi_l^{\mathrm{fs}} \otimes 1 : H^1_{\mathrm{f}}(K_l, T_{\mathfrak{P}}^{(i)}) \otimes G_c \to H^1_{\mathrm{s}}(K_l, T_{\mathfrak{P}}^{(i)}) \otimes G_{cl}$ is as in* [18, §1.2].

Proof. We assume that d^+ and d^- are both nonzero (the other cases are proven by the same argument). We denote by $T_{\mathfrak{P}}^{(2i-1),\circ}$ the G_K-module whose underlying space is $T_{\mathfrak{P}}^{(2i-1)} = T \otimes_{\mathscr{O}} S_{\mathfrak{P}}/\mathfrak{m}_{\mathfrak{P}}^{2i-1}$ and whose G_K-action on the second factor is trivial. If we denote by ϖ a uniformizer of $S_{\mathfrak{P}}$, then we have a homomorphism of $S_{\mathfrak{P}}$-modules

$$H^1(K, \bar{T}_{\mathfrak{P}}) \to H^1(K, T_{\mathfrak{P}}^{(2i-1),\circ})[\mathfrak{m}_{\mathfrak{P}}] \tag{4.51}$$

induced by the map $\bar{T}_{\mathfrak{P}} \to T_{\mathfrak{P}}^{(2i-1),\circ}$ sending each element x to $\varpi^{2i-2}x$. Here, we used the isomorphism $\bar{T}_{\mathfrak{P}} \cong T/\mathfrak{m}_{\mathscr{O}} \otimes S_{\mathfrak{P}}/\mathfrak{m}_{\mathfrak{P}}$. We note that by the hypothesis H.1) in [18] the map (4.51) is injective, and by abuse of notation, we denote by the same symbols $d^{\pm}$ the corresponding elements in $H^1(K, T_{\mathfrak{P}}^{(2i-1),\circ})[\mathfrak{m}_{\mathfrak{P}}]$. Let $L = K(T_{\mathfrak{P}}^{(2i-1),\circ})$ be the smallest finite extension of K such that G_L acts on $T_{\mathfrak{P}}^{(2i-1),\circ}$ trivially. Then, L is a Galois extension of $\mathbb{Q}$, and by Lemma 4.3 (5) we have an injective map

$$\begin{aligned} H^1(K, T_{\mathfrak{P}}^{(2i-1),\circ}) &\to H^0(\mathrm{Gal}(L/K), H^1(L, T_{\mathfrak{P}}^{(2i-1),\circ})) \\ &\cong \mathrm{Hom}_{\mathrm{Gal}(L/K)}(G_L, T_{\mathfrak{P}}^{(2i-1),\circ}), \end{aligned}$$

where the first map is the restriction map. Regarding $d^{\pm}$ as homomorphisms $G_L \to T_{\mathfrak{P}}^{(2i-1),\circ}$, we denote by E the finite abelian extension of L such that $\ker(d^+) \cap \ker(d^-) = G_E$. Then, $\mathrm{Gal}(E/L)$ is an $\mathbb{F}_p$-vector space with $G_{\mathbb{Q}}$-action, and we denote by G^+ the subspace of $\mathrm{Gal}(E/L)$ on which the complex conjugation τ acts trivially. If we regard $d^{\pm}$ as elements of $\mathrm{Hom}_{\mathrm{Gal}(L/K)}(\mathrm{Gal}(E/L), T_{\mathfrak{P}}^{(2i-1),\circ})$, then by H.1 and H.5 in

[18], $d^+|_{G^+}$ and $d^-|_{G^+}$ are non-trivial. Hence, there exists $\eta \in G^+$ such that $d^+(\eta)$ and $d^-(\eta)$ are both nonzero. By [28, Proposition 12.2 (3)] or [12, Proposition 7.14], there exist infinitely many primes l inert in K such that

(a) The element $\mathrm{Frob}_l^2 \in \mathrm{Gal}(E/K)$ is conjugate to $\eta \in G^+$.
(b) $\varpi^{2i-1} \mid l+1+a_l$ and $\varpi^{2i-1} \mid l+1-a_l$,
(c) $\varpi^{2i} \nmid l+1+a_l$ and $\varpi^{2i} \nmid l+1-a_l$
(d) the elements $d^{\pm} \in H^1(K, \bar{T}_{\mathfrak{P}})$ are both unramified at l

(under Condition 2.3 (2), [6, Lemma 6.2] implies that the constant a in [28, Proposition 12.2] or B_1 in [12, Proposition 7.14] is zero). We note that (b) implies that $l \in \mathscr{P}^{(2i-1)}$. Since $H^1(K_l^{\mathrm{ur}}/K_l, \bar{T}_{\mathfrak{P}}) \cong \bar{T}_{\mathfrak{P}}$ by the map sending each cocycle c to $c(\mathrm{Frob}_l^2)$, the assertions (a) and (d) imply (1). We note that by Lemma 4.5 (2), the proof of [28, Proposition 10.2] also works in our case. Then, by [28, Proposition 10.2 (4)] or [12, (K2)], the assertions (b) and (c) imply that the image of $\mathrm{loc}_l(\underline{\kappa}_{cl}^{\mathfrak{P},(2i-1)})$ in $H^1_{\mathrm{s}}(K_l, T_{\mathfrak{P}}^{(2i-1)}) \otimes G_{cl}$ coincides with $(\phi_l^{\mathrm{fs}} \otimes 1) \circ \mathrm{loc}_l(\underline{\kappa}_c^{\mathfrak{P},(2i-1)})$ up to a unit in $S_{\mathfrak{P}}^{\times}$. By reduction modulo $\mathfrak{m}_{\mathfrak{P}}^i$, we deduce the assertion (2). Q.E.D.

Remark 4.12. Although the Kolyvagin system of Heegner points in [18] satisfies the assertion (2) for *all* $c \in \mathscr{N}^{(2i-1)}$ and $l \in \mathscr{P}^{(2i-1)}$, our proof implies only (2) as in Lemma 4.11. However, the lemma is sufficient for our application.

Lemma 4.13. *If* $c \in \mathscr{N}^{(2i-1)}$, *then* $\underline{\kappa}_c^{\mathfrak{P},(i)} \in \mathcal{S}^{(i)}(c)$, *where* $\mathcal{S}^{(i)}(c)$ *is the stub Selmer group contained in* $H^1_{\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}}^{(i)})$ (*see* [18, §1.6] *for the details*).

Proof. Although this is essentially [18, Lemma 1.4], we slightly need to modify the proof as follows. When we choose a prime $l \in \mathscr{P}^{(2i-1)}$ in Cases i and ii in the proof of [18], we use Lemma 4.11 instead of [18, Lemma 1.6.2]. Q.E.D.

By Lemma 4.13 the proof of [18, Theorem 1.6.1] also works in our case, and we obtain the following theorem.

Theorem 4.14. *Under Condition 2.3, if* $\mathfrak{P} \neq p\Lambda$ *is a height-one prime ideal of* Λ *such that* $\kappa_1^{\mathfrak{P}}$ *is not torsion, then the* $S_{\mathfrak{P}}$*-module* $H^1_{\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}})$ *is free of rank one,* $\mathrm{rank}_{S_{\mathfrak{P}}}(X_{\mathscr{F}_{\mathfrak{P}}}(K, W_{\mathfrak{P}})) = 1$ *and*

$$2\ \mathrm{length}_{S_{\mathfrak{P}}} \left(\frac{H^1_{\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}})}{S_{\mathfrak{P}} \kappa_1^{\mathfrak{P}}} \right) \geq \mathrm{length}_{S_{\mathfrak{P}}} \left(X_{\mathscr{F}_{\mathfrak{P}}}(K, W_{\mathfrak{P}})_{\mathrm{tors}} \right)$$

(see Definition 2.1 for the notation on Selmer groups).

4.6. Twisted Heegner classes and the p-adic L-function

By using the explicit reciprocity law for generalized Heegner classes explained in §4.2, we construct integral maps relating twisted Heegner classes with specializations of the p-adic L-function.

Proposition 4.15. *Let $\mathscr{L}$ be a generator of the ideal $(\mathscr{L}_{\mathfrak{p}}(f)\Lambda^{\mathrm{ur}})\cap\Lambda$ of Λ. For a height-one prime ideal $\mathfrak{P}\neq p\Lambda$ there exist a homomorphism of $S_{\mathfrak{P}}$-modules $\varphi_{\mathfrak{P}} : H^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}}) \to S_{\mathfrak{P}}$ and an element $b_{\mathfrak{P}} \in F^{\times}$ such that the following assertions hold.*

(1) *We have*

$$\varphi_{\mathfrak{P}}(\mathrm{loc}_{\mathfrak{p}}(\kappa_1^{\mathfrak{P}})) = b_{\mathfrak{P}}\chi_{\mathfrak{P}}(\mathscr{L}) \tag{4.52}$$

up to a unit in $S_{\mathfrak{P}}$.

(2) *There exists $m_0 \geq 1$ depending on $\mathfrak{P}$ such that for $m \geq m_0$ and for every height-one prime ideal $\mathfrak{Q} \neq p\Lambda$ of Λ satisfying $\mathrm{Im}(\chi_{\mathfrak{P}}\chi_{\mathfrak{Q}}^{-1}) \subseteq 1 + p^m S_{\mathfrak{P}}$, we have $b_{\mathfrak{P}} = b_{\mathfrak{Q}}$.*

Remark 4.16. We recall that for every height-one prime ideal $\mathfrak{P} \neq p\Lambda$, we have fixed an embedding $S_{\mathfrak{P}} \hookrightarrow \mathscr{O}_{\mathbb{C}_p}$, by which the inclusion $\mathrm{Im}(\chi_{\mathfrak{P}}\chi_{\mathfrak{Q}}^{-1}) \subseteq 1 + p^m S_{\mathfrak{P}}$ in (2) above is taken in $\mathscr{O}_{\mathbb{C}_p}^{\times}$.

Proof. Let M be the $\mathscr{O}$-submodule of $D_{\mathrm{crys}}(V_f|_{G_{K_{\mathfrak{p}}}})^{\varphi=\alpha}(r)$ generated by η (cf. §4.2). By (4.33), there exists an element $\boldsymbol{w}$ of $H^1_{\mathrm{Iw}}(\hat{H}_1, T)\otimes \mathscr{K}_\infty(\mathcal{G}_{p^\infty})$ such that the composite

$$\Psi_{\boldsymbol{w}} : \tilde{\mathfrak{R}}_1^{\psi=0} \otimes M \to \mathscr{O}[[\mathcal{G}_{p^\infty}]] \to \mathscr{O}[[\Gamma]]$$

sends the element $\boldsymbol{g}_1$ to $\mathscr{L} \in \mathscr{O}[[\Gamma]]$ up to a unit in $\mathscr{O}[[\Gamma]]$ (see §4.2 for the notation), where the first map is given by $h \mapsto \langle \tilde{\Omega}^{\epsilon}_{V,r,1}(h), \boldsymbol{w}\rangle$, and the second one is the natural projection.

By using Lemma 4.9 with the M fixed above, we put

$$b'_{\mathfrak{P}} = \mathrm{length}_{S_{\mathfrak{P}}}\left(\tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}})/\beta_{\mathfrak{P}}\Omega_{\mathfrak{P}}(\tilde{\mathfrak{R}}_1^{\psi=0} \otimes M)\right).$$

We define $\varphi_{\mathfrak{P}}$ as follows. Let z be an element of $H^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}})$. Then, there exists an element $\boldsymbol{h}_z \in \tilde{\mathfrak{R}}_1^{\psi=0} \otimes M$ such that $\beta_{\mathfrak{P}}\Omega_{\mathfrak{P}}(\boldsymbol{h}_z)$ coincides with the image of $p^{b'_{\mathfrak{P}}}z$. By denoting by $\varphi_{\mathfrak{P}}(z)$ the specialization of $\Psi_{\boldsymbol{w}}(\boldsymbol{h}_z) \in \mathscr{O}[[\mathcal{G}_{p^\infty}]]$ at $\chi_{\mathfrak{P}}$, we obtain $\varphi_{\mathfrak{P}} : H^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}}) \to S_{\mathfrak{P}}$ under which the image of $\kappa_1^{\mathfrak{P}}$ coincides with $p^{b'_{\mathfrak{P}}}q_{\mathfrak{P}}\beta_{\mathfrak{P}}^{-1}\chi_{\mathfrak{P}}(\mathscr{L})$ up to a unit in $S_{\mathfrak{P}}$. Hence, if we put $b_{\mathfrak{P}} = p^{b'_{\mathfrak{P}}}q_{\mathfrak{P}}\beta_{\mathfrak{P}}^{-1}$, then we have the assertion (1). The assertion (2) follows from Lemma 4.9 (2) and from that if m is sufficiently large, then for $\mathfrak{Q}$ such that $\mathrm{Im}(\chi_{\mathfrak{Q}}\chi_{\mathfrak{P}}^{-1}) \subseteq 1 + p^m S_{\mathfrak{P}}$, we have $q_{\mathfrak{P}} = q_{\mathfrak{Q}}$. Q.E.D.

Remark 4.17. If $\mathrm{loc}_{\mathfrak{p}}(\kappa_1^{\mathfrak{P}})$ is not torsion, then the kernel of $\varphi_{\mathfrak{P}}$ is $H^1(K_{\mathfrak{p}}, T_{\mathfrak{P}})_{\mathrm{tors}}$.

Lemma 4.18. *For a height-one prime ideal $\mathfrak{P} \neq p\Lambda$ of Λ, if the localization $\mathrm{loc}_{\mathfrak{p}}(\kappa_1^{\mathfrak{P}}) \in H^1(K_{\mathfrak{p}}, T_{\mathfrak{P}})$ is not torsion, then the $S_{\mathfrak{P}}$-module*

$$\mathrm{Coker}\left(H^1_{\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}}) \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} \tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}})\right)$$

is torsion.

Proof. Since $\mathrm{loc}_{\mathfrak{p}}(\kappa_1^{\mathfrak{P}})$ is not torsion, the lemma follows from the fact that the rank of $\tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}})$ is one. Q.E.D.

Lemma 4.19. *Suppose that $\chi_{\mathfrak{P}}(\mathscr{L}) \neq 0$. Then for a height-one prime ideal $\mathfrak{P} \neq p\Lambda$ of Λ, we have*

$$\mathrm{length}_{S_{\mathfrak{P}}}\left(\frac{S_{\mathfrak{P}}}{b_{\mathfrak{P}}\chi_{\mathfrak{P}}(\mathscr{L})S_{\mathfrak{P}}}\right) \geq \mathfrak{l}_{\mathfrak{P}} + \mathrm{length}_{S_{\mathfrak{P}}}\left(\frac{H^1_{\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}})}{S_{\mathfrak{P}}\kappa_1^{\mathfrak{P}}}\right),$$

where $\mathfrak{l}_{\mathfrak{P}} := \mathrm{length}_{S_{\mathfrak{P}}}\left(\mathrm{Coker}\left(H^1_{\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}}) \to \tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}})\right)\right)$, *and* $b_{\mathfrak{P}} \in F^{\times}$ *is as in Proposition 4.15.*

Proof. By our assumption and Proposition 4.15, $\mathrm{loc}_{\mathfrak{p}}(\kappa_1^{\mathfrak{P}})$ is not a torsion element. Hence the $S_{\mathfrak{P}}$-module $H^1_{\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}})$ is free of rank one by Theorem 4.14, and the restriction map $H^1_{\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}}) \to \tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}})$ is a non-zero map between free modules of rank 1. In particular, its kernel $H^1_{0,\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}})$ is trivial. Then, we have an exact sequence

$$\begin{aligned} 0 \to \frac{H^1_{\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}})}{S_{\mathfrak{P}}\kappa_1^{\mathfrak{P}}} &\to \frac{\tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}})}{S_{\mathfrak{P}}\mathrm{loc}_{\mathfrak{p}}(\kappa_1^{\mathfrak{P}})} \\ &\to \mathrm{Coker}(H^1_{\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}}) \to \tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}})) \to 0. \end{aligned} \tag{4.53}$$

By Proposition 4.15, the map

$$\frac{\tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}})}{S_{\mathfrak{P}}\mathrm{loc}_{\mathfrak{p}}(\kappa_1^{\mathfrak{P}})} \to \frac{S_{\mathfrak{P}}}{b_{\mathfrak{P}}\chi_{\mathfrak{P}}(\mathscr{L})S_{\mathfrak{P}}}$$

is injective. Combining this with (4.53) implies the lemma. Q.E.D.

§5. Proof of the main result

The aim of this section is to prove Theorem 2.5. We keep the same notation as in the previous section. In particular, we assume Condition 2.3.

5.1. Comparison of Selmer groups

The aim of this subsection is to prove Proposition 5.4, which is combined with the control theorem in the next subsection to deduce a comparison between specialization of both sides of Conjecture 2.2 (2).

Lemma 5.1. *Let $\mathfrak{P} \neq p\Lambda$ be a height-one prime ideal of Λ. Then, we have*

$$\mathrm{length}_{S_{\mathfrak{P}}}(X_{\mathscr{F}_{\mathfrak{P}},\emptyset}(K, W_{\mathfrak{P}})_{\mathrm{tors}}) = \mathrm{length}_{S_{\mathfrak{P}}}(X_{0,\mathscr{F}_{\mathfrak{P}}}(K, W_{\mathfrak{P}})_{\mathrm{tors}}).$$

Proof. By [27, Lemma 3.5.3 and Theorem 4.1.13], there exists an integer $r \geq 0$ such that for $i \geq 1$

$$H^1_{\mathscr{F}_{\mathfrak{P}},\emptyset}(K, W_{\mathfrak{P}})[p^i] \cong (\Phi_{\mathfrak{P}}/S_{\mathfrak{P}})^{\oplus r}[p^i] \oplus H^1_{\mathscr{F}_{\mathfrak{P}^\iota},0}(K, W_{\mathfrak{P}^\iota})[p^i]. \tag{5.54}$$

By a formula of Wiles (cf. [16, Theorem 2.19]), we have

$$r = \mathrm{corank}_{S_{\mathfrak{P}}}(H^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, W_{\mathfrak{P}})) + \mathrm{corank}_{S_{\mathfrak{P}}}(H^1(K_{\bar{\mathfrak{p}}}, W_{\mathfrak{P}})) - \mathrm{corank}_{S_{\mathfrak{P}}}(H^0(K \otimes \mathbb{R}, W_{\mathfrak{P}})).$$

Hence, $r = 1 + 2 - 2 = 1$. Since (5.54) holds with $r = 1$ for all $i \geq 0$, by the isomorphism $H^1_{0,\mathscr{F}_{\mathfrak{P}}}(K, W_{\mathfrak{P}}) \cong H^1_{\mathscr{F}_{\mathfrak{P}^\iota},0}(K, W_{\mathfrak{P}^\iota})$ induced by the complex conjugation, we obtain the lemma. Q.E.D.

Lemma 5.2. *If $\mathfrak{P} \neq p\Lambda$ is a height-one prime ideal of Λ such that the image of $\kappa_1^{\mathfrak{P}}$ in $H^1(K_{\mathfrak{p}}, T_{\mathfrak{P}})$ is not torsion, then*

(1) $H^1_{\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}}) = H^1_{\mathscr{F}_{\mathfrak{P}},\emptyset}(K, T_{\mathfrak{P}})$.

(2) *The $S_{\mathfrak{P}}$-module $X_{0,\mathscr{F}_{\mathfrak{P}}}(K, W_{\mathfrak{P}})$ is torsion.*

Proof. (1) We consider the exact sequence

$$0 \to H^1_{\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}}) \to H^1_{\mathscr{F}_{\mathfrak{P}},\emptyset}(K, T_{\mathfrak{P}}) \xrightarrow{\mathrm{loc}_{/\mathscr{F}_{\mathfrak{P}},\bar{\mathfrak{p}}}} \frac{H^1(K_{\bar{\mathfrak{p}}}, T_{\mathfrak{P}})}{H^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\bar{\mathfrak{p}}}, T_{\mathfrak{P}})} \to \mathrm{Coker}(\mathrm{loc}_{/\mathscr{F}_{\mathfrak{P}},\bar{\mathfrak{p}}}) \to 0, \tag{5.55}$$

where $\mathrm{loc}_{/\mathscr{F}_{\mathfrak{P}},\bar{\mathfrak{p}}}$ is the natural map induced by the localization map at $\bar{\mathfrak{p}}$. Since the $S_{\mathfrak{P}}$-modules $H^1(K_{\bar{\mathfrak{p}}}, T_{\mathfrak{P}})/H^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\bar{\mathfrak{p}}}, T_{\mathfrak{P}})$ and $H^1_{\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}})$ are free of rank one (cf. Theorem 4.14), it suffices to show that the rank of $H^1_{\mathscr{F}_{\mathfrak{P}},\emptyset}(K, T_{\mathfrak{P}})$ is one. First note that the rank of $H^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}})$ is one. Then, by the exact sequence

$$0 \to H^1_{0,\emptyset}(K, T_{\mathfrak{P}}) \to H^1_{\mathscr{F}_{\mathfrak{P}},\emptyset}(K, T_{\mathfrak{P}}) \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} \tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}}),$$

it is sufficient to show that $H^1_{0,\emptyset}(K, V_{\mathfrak{P}}) = \{0\}$.

By Proposition 4.2 (3) and the assumption that $\mathrm{loc}_{\mathfrak{p}}(\kappa_1^{\mathfrak{P}})$ is not torsion, the localization $\mathrm{loc}_{\bar{\mathfrak{p}}}(\kappa_1^{\mathfrak{P}^{\iota}})$ is also not torsion. Then, by the exact sequence

$$0 \to H^1_{\mathscr{F}_{\mathfrak{P}^{\iota}},0}(K, T_{\mathfrak{P}^{\iota}}) \to H^1_{\mathscr{F}_{\mathfrak{P}^{\iota}}}(K, T_{\mathfrak{P}^{\iota}}) \to \tilde{H}^1_{\mathscr{F}_{\mathfrak{P}^{\iota}}}(K_{\bar{\mathfrak{p}}}, T_{\mathfrak{P}})$$

and by Theorem 4.14, we have $H^1_{\mathscr{F}_{\mathfrak{P}^{\iota}},0}(K, T_{\mathfrak{P}^{\iota}}) = \{0\}$. Hence, by the global duality, we have an exact sequence

$$H^1_{\mathscr{L}_{\mathfrak{P}},\emptyset}(K, V_{\mathfrak{P}}) \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} \mathscr{L}_{\mathfrak{P}}^{\mathfrak{p}} \to H^1_{\emptyset,0}(K, V_{\mathfrak{P}^{\iota}})^* \to 0.$$

Here $*$ denotes the $\Phi_{\mathfrak{P}^{\iota}}$-linear dual, and $\mathscr{L}_{\mathfrak{P}}$ denotes the Selmer structure whose local conditions at $v \mid p$ are given by the subspace $\mathscr{L}_{\mathfrak{P}}^{v}$ generated by $\mathrm{loc}_v(\kappa_1^{\mathfrak{P}})$. Since $\kappa_1^{\mathfrak{P}} \in H^1_{\mathscr{L}_{\mathfrak{P}},\emptyset}(K, V_{\mathfrak{P}})$ and $\mathrm{loc}_{\mathfrak{p}}(\kappa_1^{\mathfrak{P}})$ is assumed to be non-zero, by the exact sequence above we have $H^1_{\emptyset,0}(K, V_{\mathfrak{P}^{\iota}}) = \{0\}$. Hence by the complex conjugation, the assertion is proved.

(2) By (1) and (5.55), the cokernel of $\mathrm{loc}_{/\mathscr{F}_{\mathfrak{P}},\bar{\mathfrak{p}}}$ is free of rank one. Hence, by the exact sequence (induced by the global duality and (4.43))

$$0 \to \mathrm{Coker}(\mathrm{loc}_{/\mathscr{F}_{\mathfrak{P}},\bar{\mathfrak{p}}}) \to X_{\mathscr{F}_{\mathfrak{P}}}(K, W_{\mathfrak{P}}) \to X_{0,\mathscr{F}_{\mathfrak{P}}}(K, W_{\mathfrak{P}}) \to 0,$$

Theorem 4.14 implies the assertion (2). Q.E.D.

Lemma 5.3. *If $\mathfrak{P} \neq p\Lambda$ is a height one prime ideal of Λ such that the image of $\kappa_1^{\mathfrak{P}}$ in $H^1(K_{\mathfrak{p}}, T_{\mathfrak{P}})$ is not torsion, then*

(1) *the $S_{\mathfrak{P}}$-module $X_{0,\emptyset}(K, W_{\mathfrak{P}})$ is torsion,*

(2) $\mathrm{length}_{S_{\mathfrak{P}}}(X_{0,\emptyset}(K, W_{\mathfrak{P}})) = \mathrm{length}_{S_{\mathfrak{P}}}(X_{\mathscr{F}_{\mathfrak{P}}}(K, W_{\mathfrak{P}})_{\mathrm{tors}}) + 2\mathfrak{l}_{\mathfrak{P}}$, *where $\mathfrak{l}_{\mathfrak{P}}$ is as in Lemma 4.19.*

Proof. By the global duality, we have an exact sequence

$$0 \to \mathrm{Coker}\left(H^1_{\mathscr{F}_{\mathfrak{P}},\emptyset}(K, T_{\mathfrak{P}}) \to \tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_v, T_{\mathfrak{P}})\right) \to X_{0,\emptyset}(K, W_{\mathfrak{P}}) \to X_{0,\mathscr{F}_{\mathfrak{P}}}(K, W_{\mathfrak{P}}) \to 0.$$

Then, by Lemma 5.2 (1) we have an exact sequence

$$0 \to \mathrm{Coker}\left(H^1_{\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}}) \to \tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_v, T_{\mathfrak{P}})\right) \to X_{0,\emptyset}(K, W_{\mathfrak{P}}) \to X_{0,\mathscr{F}_{\mathfrak{P}}}(K, W_{\mathfrak{P}}) \to 0. \tag{5.56}$$

Hence, since the image of $\kappa_1^{\mathfrak{P}}$ in $H^1_{\mathscr{F}_{\mathfrak{P}}}(K_{\mathfrak{p}}, T_{\mathfrak{P}})$ is not torsion, Lemma 5.2 (2) implies the assertion (1).

(2) By the global duality, we have an exact sequence

$$0 \to \mathrm{Coker}\left(H^1_{\mathscr{F}_{\mathfrak{P}}}(K, T_{\mathfrak{P}}) \to \tilde{H}^1_{\mathscr{F}_{\mathfrak{P}}}(K_v, T_{\mathfrak{P}})\right) \to X_{\mathscr{F}_{\mathfrak{P}},\emptyset}(K, W_{\mathfrak{P}}) \to X_{\mathscr{F}_{\mathfrak{P}}}(K, W_{\mathfrak{P}}) \to 0.$$

By Lemma 4.18 and Theorem 4.14, taking $S_{\mathfrak{P}}$-torsion in the sequence above implies that

(5.57)
$$\mathrm{length}_{S_{\mathfrak{P}}}(X_{\mathscr{F}_{\mathfrak{P}},\emptyset}(K, W_{\mathfrak{P}})_{\mathrm{tors}}) = \mathrm{length}_{S_{\mathfrak{P}}}(X_{\mathscr{F}_{\mathfrak{P}}}(K, W_{\mathfrak{P}})_{\mathrm{tors}}) + \mathfrak{l}_{\mathfrak{P}}.$$

By (5.56), Lemma 5.1 implies that

$$\begin{aligned}\mathrm{length}_{S_{\mathfrak{P}}}(X_{0,\emptyset}(K, W_{\mathfrak{P}})) &= \mathrm{length}_{S_{\mathfrak{P}}}(X_{0,\mathscr{F}_{\mathfrak{P}}}(K, W_{\mathfrak{P}})) + \mathfrak{l}_{\mathfrak{P}} \\ &= \mathrm{length}_{S_{\mathfrak{P}}}(X_{\mathscr{F}_{\mathfrak{P}},\emptyset}(K, W_{\mathfrak{P}})_{\mathrm{tors}}) + \mathfrak{l}_{\mathfrak{P}}.\end{aligned}$$

Hence, by (5.57) we obtain the assertion (2). Q.E.D.

By combining Lemmas 4.19 and 5.3, we obtain the following.

Proposition 5.4. *Let $\mathscr{L}$ be a generator of the ideal $(\mathscr{L}_{\mathfrak{p}}(f)\Lambda^{\mathrm{ur}}) \cap \Lambda$ of Λ. If $\mathfrak{P} \neq p\Lambda$ is a height-one prime ideal of Λ such that $\chi_{\mathfrak{P}}(\mathscr{L}_{\mathfrak{p}}(f)) \in \mathbb{C}_p$ is non-zero, then the $S_{\mathfrak{P}}$-module $X_{0,\emptyset}(K, W_{\mathfrak{P}})$ is torsion, and*

$$2\ \mathrm{length}_{S_{\mathfrak{P}}}\left(\frac{S_{\mathfrak{P}}}{b_{\mathfrak{P}}\chi_{\mathfrak{P}}(\mathscr{L})S_{\mathfrak{P}}}\right) \geq \mathrm{length}_{S_{\mathfrak{P}}}(X_{0,\emptyset}(K, W_{\mathfrak{P}})).$$

5.2. Control theorem

In this subsection, we prove a certain control theorem (Proposition 5.7), which plays an important role in the gluing up Proposition 5.4 to deduce the main theorem.

We put $\mathbb{T} = T \otimes_{\mathscr{O}} \Lambda$ and $\mathbb{W} = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{T}, \mu_{p^\infty})$. Then, we may canonically identify

$$H^1(K, \mathbb{W}) = \varinjlim_m H^1(K_m, W)$$

(cf. [18, Definition 2.2.3 and Proposition 2.2.4]). We introduce a Selmer structure $\mathscr{F}_\Lambda$ on $\mathbb{W}$ so that the resulting Selmer group $H^1_{\mathscr{F}_\Lambda}(K, \mathbb{W})$ is isomorphic to the Selmer group $H^1_{\emptyset,0}(K_\infty, W)$ in (1.1). In other words, $H^1_{\mathscr{F}_\Lambda}(K_{\mathfrak{p}}, \mathbb{W}) = H^1_{\mathscr{F}_\Lambda}(K_{\mathfrak{p}}, \mathbb{W})$, $H^1_{\mathscr{F}_\Lambda}(K_{\overline{\mathfrak{p}}}, \mathbb{W}) = \{0\}$, and for $v \neq \mathfrak{p}$, $H^1_{\mathscr{F}_\Lambda}(K_v, \mathbb{W})$ is the kernel of the restriction map from $H^1(K_v, \mathbb{W})$ to $H^1(K_v^{\mathrm{ur}}, \mathbb{W})$. For a height-one prime ideal $\mathfrak{P} \neq p\Lambda$ of Λ, we denote by the same symbol $\mathscr{F}_\Lambda$ the Selmer structure on $\mathbb{W}[\mathfrak{P}]$ such that for every prime v of K,

$$H^1_{\mathscr{F}_\Lambda}(K_v, \mathbb{W}[\mathfrak{P}]) = \operatorname{Ker}\left(H^1(K_v, \mathbb{W}[\mathfrak{P}]) \to \frac{H^1(K_v, \mathbb{W})}{H^1_{\mathscr{F}_\Lambda}(K_v, \mathbb{W})} \right).$$

We note that unlike in the previous section, the Selmer structure $\mathscr{F}_\Lambda$ on $\mathbb{W}[\mathfrak{P}]$ is not given by the image of a Selmer structure on a vector space. We also note that by the duality, the natural map $\mathbb{T}/\mathfrak{P}^\iota \to T_{\mathfrak{P}^\iota}$ induces a homomorphism $W_\mathfrak{P} \to \mathbb{W}[\mathfrak{P}]$, and we have homomorphisms

$$s_{\mathfrak{P},v} : H^1_{\emptyset,0}(K_v, W_\mathfrak{P}) \to H^1_{\mathscr{F}_\Lambda}(K_v, \mathbb{W}[\mathfrak{P}]),$$
$$s_\mathfrak{P} : H^1_{\emptyset,0}(K, W_\mathfrak{P}) \to H^1_{\mathscr{F}_\Lambda}(K, \mathbb{W})[\mathfrak{P}].$$

Lemma 5.5. *Let $\mathfrak{P} \neq p\Lambda$ be a height-one prime ideal of Λ. Then, for every prime $v \neq \mathfrak{p}$ of K, the kernel and cokernel of $s_{\mathfrak{P},v}$ are finite, and their orders are bounded by a constant depending only on T and the index $[S_\mathfrak{P} : \Lambda/\mathfrak{P}]$ and independent of v.*

Proof. This is [27, Lemma 5.3.13]. Although representations of $G_\mathbb{Q}$ are considered in *loc. cit.*, the proof also works in our setting. Q.E.D.

Lemma 5.6. *Let $\mathfrak{P} \neq p\Lambda$ be a height-one prime ideal of Λ. Then, the modules* $\ker(s_{\mathfrak{P},\mathfrak{p}})$ *and* $\operatorname{coker}(s_{\mathfrak{P},\mathfrak{p}})$ *are finite, and their orders are less than a constant which depends only on T, the index $[S_\mathfrak{P} : \Lambda/\mathfrak{P}]$ and* $\operatorname{rank}_\mathscr{O}(\Lambda/\mathfrak{P})$.

Proof. By the local duality, it suffices to bound the kernel and cokernel of the homomorphism

$$H^1(K_\mathfrak{p}, \mathbb{T}/\mathfrak{P}^\iota) \to \frac{H^1(K_\mathfrak{p}, T_{\mathfrak{P}^\iota})}{H^1(K_\mathfrak{p}, T_{\mathfrak{P}^\iota})_{\mathrm{tors}}}.$$

We note that

$$H^1(K_\mathfrak{p}, T_{\mathfrak{P}^\iota})_{\mathrm{tors}} \cong H^0(K_\mathfrak{p}, W_{\mathfrak{P}^\iota}) \subseteq H^0(K_{\infty,\mathfrak{p}}, W_{\mathfrak{P}^\iota}) = H^0(K_{\infty,\mathfrak{p}}, W) \otimes_\mathscr{O} S_{\mathfrak{P}^\iota},$$

where by abuse of notation, we denote by $\mathfrak{p}$ the prime of K_∞ above $\mathfrak{p}$. Since $H^0(K_{\infty,\mathfrak{p}}, W)$ is finite (cf. Lemma 2.7), the order of the group $H^1(K_\mathfrak{p}, T_{\mathfrak{P}^\iota})_{\mathrm{tors}}$ is less than a constant depending only on T, $[S_\mathfrak{P} : \Lambda/\mathfrak{P}]$ and $\operatorname{rank}_\mathscr{O}(S_\mathfrak{P}) = \operatorname{rank}_\mathscr{O}(\Lambda/\mathfrak{P})$. Hence, to complete the proof, it suffices to bound the kernel and cokernel of the homomorphism $H^1(K_\mathfrak{p}, \mathbb{T}/\mathfrak{P}^\iota) \to H^1(K_\mathfrak{p}, T_{\mathfrak{P}^\iota})$. We note that the kernel (resp. cokernel) is bounded by $H^0(K_\mathfrak{p}, T \otimes (S_\mathfrak{P}/(\Lambda/\mathfrak{P})))$ (resp. $H^1(K_\mathfrak{p}, T \otimes (S_\mathfrak{P}/(\Lambda/\mathfrak{P}))))$, and then we complete the proof. Q.E.D.

Proposition 5.7. *There is a finite set Σ_Λ of height-one prime ideals of Λ such that for a height-one prime ideal $\mathfrak{P} \notin \Sigma_\Lambda \cup \{p\Lambda\}$ of Λ, the kernel and cokernel of $s_{\mathfrak{P}}$ are finite, and their orders are less than a constant which depends only on T, $[S_{\mathfrak{P}} : \Lambda/\mathfrak{P}]$ and $\mathrm{rank}_{\mathscr{O}}(\Lambda/\mathfrak{P})$.*

Proof. This is an analogue of [27, Proposition 5.3.14]. The difference is our local condition at $\mathfrak{p}$, which is not covered in [27]. However, thanks to Lemma 5.6, the proposition is deduced from Lemmas 5.5 and as in the proof of [27, Proposition 5.3.14]. Q.E.D.

5.3. Proof of the main theorem

Proof of Theorem 2.5 (1). Let $\mathscr{L} \in \Lambda$ be a generator of the ideal $(\mathscr{L}_{\mathfrak{p}}(f)\Lambda^{\mathrm{ur}}) \cap \Lambda$. Since $\mathscr{L}_{\mathfrak{p}}(f)$ is not zero as an element of Λ^{ur} (cf. [12, Theorem 3.7]), for almost all height-one prime ideals $\mathfrak{P} \neq p\Lambda$, we have $\chi_{\mathfrak{P}}(\mathscr{L}) \neq 0 \in \mathscr{O}_{\mathbb{C}_p}$. Let Σ be a finite set of height-one prime ideals of Λ such that Σ contains all prime ideals $\mathfrak{Q}$ which satisfy at least one of the following conditions:

(1) $\mathfrak{Q}^\iota \in \Sigma_\Lambda$, where Σ_Λ is as in Proposition 5.7,
(2) $\chi_{\mathfrak{Q}}(\mathscr{L}_{\mathfrak{p}}(f)) = 0$,
(3) $\mathfrak{Q} = p\Lambda$.

Let $\mathfrak{P}$ be a prime ideal of Λ which does not lie in Σ. Then, Proposition 5.4 implies that $H^1_{\emptyset,0}(K, W_{\mathfrak{P}^\iota})$ is finite. Hence, by Proposition 5.7, $H^1_{\mathscr{F}_\Lambda}(K, \mathbb{W})[\mathfrak{P}^\iota]$ is finite. Therefore, since $H^1_{\mathscr{F}_\Lambda}(K, \mathbb{W}) = H^1_{\emptyset,0}(K_\infty, W)$, we have that $X_{\emptyset,0}(K_\infty, W)/\mathfrak{P}$ is finite. Then, $X_{\emptyset,0}(K_\infty, W)$ is Λ-torsion. Q.E.D.

Proof of Theorem 2.5 (2). As in the proof of [18, Theorem 2.2.10], we can prove Theorem 2.5 (2) by combining Propositions 5.4 and 5.7.

Let γ be a generator of Γ, and we identify Λ with $\mathscr{O}[[S]]$ via $\gamma - 1 \mapsto S$. We may enlarge $\mathscr{O} \subseteq \overline{\mathbb{Q}}_p$ so that we have

$$\mathrm{Char}(X_{\emptyset,0}(K_\infty, W)) = \left(\prod_{i=1}^{c}(S - \xi_i)\Lambda\right), \quad \mathscr{L}\Lambda = \left(\prod_{j=1}^{d}(S - \eta_j)\Lambda\right),$$

as ideals of Λ where ξ_i and η_j are elements of the maximal ideal of $\mathscr{O}$.

Let $\mathfrak{Q} \neq p\Lambda$ be a height-one prime ideal of Λ such that $\mathfrak{Q}$ divides $\mathrm{Char}(X_{\emptyset,0}(K_\infty, W))$ or $\mathscr{L}$. Then, we may write $\mathfrak{Q} = (S - \theta)\Lambda$, where θ is an element of $\{\xi_i, \eta_j\}_{i,j}$, and we have $\chi_{\mathfrak{Q}}(\gamma) = 1 + \theta$ (note that $\theta \in \mathscr{O}$ and $S_{\mathfrak{Q}} = \Lambda/\mathfrak{Q} = \mathscr{O}$). For $m \geq 1$, we put $\mathfrak{P}^{(m)} = (S - \theta - p^m)\Lambda$, which satisfies $\mathrm{Im}(\chi_{\mathfrak{Q}}\chi^{-1}_{\mathfrak{P}^{(m)}}) \subseteq 1 + p^m\mathscr{O}$. For a sufficiently large $m > 0$, the following assertions hold:

(i) $\mathfrak{P}^{(m)} \notin \Sigma$, where Σ is the same as in the proof of Theorem 2.5 (1),

(ii) $b_{\mathfrak{P}^{(m)}} = b_{\mathfrak{Q}}$ (cf. Proposition 4.15),

(iii) $\mathfrak{P}^{(m)}$ does not divide $\mathrm{Char}(X_{\emptyset,0}(K_\infty, W))$ or $\mathscr{L}$.

We note that as ideals of Λ we have $(\mathfrak{P}^{(m)}, \mathfrak{Q}^n) = (\mathfrak{P}^{(m)}, p^{mn})$ for $n \geq 0$. Then,

$$\begin{aligned} & 2\ \mathrm{length}_{\mathscr{O}}(S_{\mathfrak{P}^{(m)}}/b_{\mathfrak{P}^{(m)}}\chi_{\mathfrak{P}^{(m)}}(\mathscr{L})S_{\mathfrak{P}^{(m)}}) \\ =& 2\ \mathrm{length}_{\mathscr{O}}\left(\Lambda/(\mathscr{L}, \mathfrak{P}^{(m)})\right) + O(1) \\ =& 2\ \mathrm{length}_{\mathscr{O}}(\Lambda/(\mathfrak{Q}^{\mathrm{ord}_{\mathfrak{Q}}(\mathscr{L})}, \mathfrak{P}^{(m)})) + O(1) \\ =& 2m\ \mathrm{ord}_{\mathfrak{Q}}(\mathscr{L}) + O(1), \end{aligned} \tag{5.58}$$

where each $O(1)$ is described in terms of $b_{\mathfrak{P}^{(m)}} = b_{\mathfrak{Q}}$ and is independent of m. To simplify the notation, we write $X_{\emptyset,0} = X_{\emptyset,0}(K_\infty, W)$. By Proposition 5.7, we have

$$\begin{aligned} \mathrm{length}_{\mathscr{O}}(H^1_{0,\emptyset}(K, W_{\mathfrak{P}^{(m)}})^\vee) &= \mathrm{length}_{\mathscr{O}}(H^1_{\emptyset,0}(K, W_{\mathfrak{P}^{(m)},\iota})^\vee) \\ &= \mathrm{length}_{\mathscr{O}}((H^1_{\mathscr{F}_\Lambda}(K, \mathbb{W})[\mathfrak{P}^{(m),\iota}])^\vee) + O(1) \\ &= \mathrm{length}_{\mathscr{O}}(X_{\emptyset,0}/\mathfrak{P}^{(m)}) + O(1) \\ &= m\ \mathrm{ord}_{\mathfrak{Q}}(\mathrm{Char}(X_{\emptyset,0})) + O(1), \end{aligned} \tag{5.59}$$

where each $O(1)$ is independent of m and described in terms of T. Then by (5.58), (5.59) and Proposition 5.4, we have

$$2\ \mathrm{ord}_{\mathfrak{Q}}(\mathscr{L}) \geq \mathrm{ord}_{\mathfrak{Q}}(\mathrm{Char}(X_{\emptyset,0})) + m^{-1}O(1).$$

Letting $m \to \infty$, we obtain

$$2\ \mathrm{ord}_{\mathfrak{Q}}(\mathscr{L}) \geq \mathrm{ord}_{\mathfrak{Q}}(\mathrm{Char}(X_{\emptyset,0}).$$

Since this inequality holds for every height-one prime ideal $\mathfrak{Q} \neq p\Lambda$ of Λ, we conclude Theorem 2.5 (2). Q.E.D.

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