MATHEMATICAL ENGINEERING TECHNICAL REPORTS

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METR 2008–41 October 2008

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Matching Structure of Symmetric Bipartite Graphs and a Generalization of Pólya's Problem

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Abstract

A bipartite graph is said to be *symmetric* if it has symmetry of reflecting two vertex sets. This paper investigates matching structure of symmetric bipartite graphs. We first apply the *Dulmage-Mendelsohn decomposition* to a symmetric bipartite graph. The resulting components, which are matching-covered, turn out to have symmetry. We then decompose a matching-covered bipartite graph via an *ear decomposition*, which is a sequence of subgraphs obtained by adding an odd-length path repeatedly. We show that, if a matching-covered bipartite graph is symmetric, an ear decomposition can retain symmetry by adding no more than two paths.

As an application of these decompositions to combinatorial matrix theory, we present a natural generalization of Pólya's problem. We introduce the problem of deciding whether a rectangular $\{0, 1\}$-matrix has a signing that is totally sign-nonsingular or not, where a rectangular matrix is *totally sign-nonsingular* if the sign of the determinant of each submatrix with the entire row set is uniquely determined by the signs of the nonzero entries. We show that this problem can be solved in polynomial time with the aid of the matching structure of symmetric bipartite graphs. In addition, we provide a characterization of this problem in terms of excluded minors.

1 Introduction

Let $G = (U, V; E)$ be a simple bipartite graph with two disjoint vertex sets $U = \{u_1, \ldots, u_m\}$, $V = \{v_1, \ldots, v_n\}$, and edge set $E \subseteq U \times V$. A bipartite graph $G = (U, V; E)$ with $|U| = |V|$ is said to be *symmetric* if $(u_j, v_i) \in E$ holds for any $(u_i, v_j) \in E$. A symmetric bipartite graph is associated with a combinatorially symmetric matrix [16], where a square matrix $A = (a_{ij})$ of order n is said to be *combinatorially symmetric* if $a_{ij} \neq 0$ implies $a_{ji} \neq 0$ for any two distinct indices i, j. Combinatorially symmetric matrices were studied in the contexts of matrix completion problems [7] and qualitative matrix theory [8, 10, 25, 27]. Another work related to symmetric bipartite graphs is given by Gabow [5], who discussed an upper degree-constrained

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partial orientation of graphs. This problem can be viewed as the problem of finding a degree-constrained maximum subgraph that has at most one edge of (u_i, v_j) and (u_j, v_i) for any indices i, j in a symmetric bipartite graph.

For a bipartite graph $G = (U, V; E)$, an edge subset $M \subseteq E$ is a *matching* if no two edges in M share a common vertex incident to them. A matching is *perfect* if $|M| = |U| = |V|$. For an edge subset $F \subseteq E$, we denote by $F^\top = \{(u_j, v_i) \mid (u_i, v_j) \in F\}$ the *transpose* of F. The matching structure of a symmetric bipartite graph has symmetry, since M is a matching if and only if so is $M^\top$. This paper aims at investigating decompositions related to the matching structure of symmetric bipartite graphs.

We first deal with the *Dulmage-Mendelsohn decomposition* (the *DM-decomposition* for short) [3, 4]. We say that a connected graph is *matching-covered* if every edge is contained in some perfect matching. The DM-decomposition is a unique decomposition of a bipartite graph with respect to the maximum matchings, which yields the matching-covered subgraphs and the remaining subgraphs. The subgraphs obtained by the DM-decomposition are called the *DM-components*. We show that, if a bipartite graph is symmetric, then each DM-component is the transpose of some DM-component, where the *transpose* of a subgraph $H = (U, V; F)$ is the subgraph $H^\top = (U, V; F^\top)$. A subgraph $H = (U, V; F)$ is called *symmetric* if $F = F^\top$. Our result means that a symmetric bipartite graph can be assembled from symmetric matching-covered subgraphs and pairs of subgraphs whose union is symmetric.

Each of DM-components, i.e., a matching-covered bipartite graph, is characterized by the *ear decomposition* [15]. An elementary path P of odd length is an *ear* of a subgraph G' if G' contains both of the end vertices of P, but no interior vertices and no edges. We denote by $G' + P$ the subgraph obtained from G' by adding an ear P. For a subgraph G' of a graph G, an *ear decomposition starting from* G' is a sequence $G_0, G_1, \ldots, G_k$ of subgraphs such that $G_0 = G'$, $G_k = G$, and $G_i = G_{i-1} + P_i$ for some ear P_i of G_{i-1} for $i = 1, \ldots, k$. It is known that a bipartite graph has an ear decomposition starting from an edge if and only if it is matching-covered.

Assume that a matching-covered bipartite graph G is symmetric. The symmetry of G motivates us to find an ear decomposition having symmetry. Unfortunately, G does not always have an ear decomposition in which every subgraph is itself symmetric. In fact, the complete bipartite graph with two vertex sets of size three has no such ear decomposition. Thus we may have to add more than one ears to maintain symmetry in an ear decomposition. We will see, however, that we can retain symmetry by adding no more than two ears. An ear decomposition $G_0, G_1, \ldots, G_k$ starting from G_0 is called *symmetric* if one of two consecutive subgraphs is symmetric, i.e., G_{l-1} or G_l is symmetric for $l = 1, \ldots, k$. We show that, if G is symmetric, G has a symmetric ear decomposition starting from an edge or a crossing pair, where a *crossing pair* is a pair of edges $(u_i, v_j) \in E$ and $(u_j, v_i) \in E$ for some distinct $i, j \in N = \{1, \ldots, n\}$. In addition, given a perfect matching, we describe a linear-time algorithm for finding a symmetric ear decomposition.

As an application of these decompositions to combinatorial matrix theory, we discuss a generalization of Pólya's problem. A square matrix is said to be *term-nonsingular* if the determinant has a nonzero expansion term. A term-nonsingular matrix is *sign-nonsingular* if all nonzero expansion terms of the determinant have the same sign. For a $\{0,1\}$-matrix A, a *signing* of A is a $\{0,\pm 1\}$-matrix obtained from A by replacing some ones with minus ones. *Pólya's problem* is the problem of deciding whether a given square $\{0,1\}$-matrix has a sign-nonsingular signing or not. Such a sign-nonsingular signing is called a *Pólya matrix.* Pólya's problem has a plenty of polynomial-time equivalent problems [1, 11, 15, 17, 22]. Robertson, Seymour, and Thomas [21] devised a polynomial-time algorithm for Pólya's problem. Excellent surveys on Pólya's problem can be found in [18, 26].

An $m \times n$ matrix with $m \leq n$ is said to be *totally sign-nonsingular* if each term-nonsingular submatrix of order m is sign-nonsingular. Totally sign-nonsingular matrices play an important role in the sign-solvability of linear systems of equations [2, 12, 13, 24], linear programming [6], and linear complementarity problems [9]. Total sign-nonsingularity can be recognized in polynomial time by testing sign-nonsingularity of a related symmetric matrix [6].

In this paper, we introduce the problem of deciding whether a rectangular $\{0,1\}$-matrix has a totally sign-nonsingular signing or not. If a square matrix is term-nonsingular, this problem is in fact Pólya's problem. It follows from [6] that this problem can be reduced to the problem of deciding whether a related symmetric matrix has a symmetric Pólya matrix with positive diagonals or not. We show that a symmetric Pólya matrix with a nonzero diagonal entry can be obtained in polynomial time with the aid of the DM-decomposition and ear decomposition for symmetric bipartite graphs. This implies that a totally sign-nonsingular signing can be found in polynomial time.

In addition, we characterize a matrix which has a totally sign-nonsingular signing in terms of excluded minors. Let $B_{m,n}$ denote the $m \times n$ matrix all of whose entries are equal to one. Little [14] proved that, for a square matrix, $B_{3,3}$ is the only obstruction to have a Pólya matrix (cf. [20]). By analogy with this result, we show that a rectangular matrix A has a totally sign-nonsingular signing if and only if A contains none of $B_{3,3}$, $B_{2,3}$, and the other specific matrix, as we will see in Section 6. Our result includes a forbidden configuration characterization for S-matrices by Brualdi and Shader [2] as a special case, where an *S-matrix* is an $m \times (m+1)$ matrix all of whose submatrices of order m are sign-nonsingular.

Before closing this section, we give some definitions and notations. For an $m \times n$ matrix $A = (a_{ij})$, we define the associated bipartite graph $G(A) = (U, V; E)$ with vertex sets $U = \{u_1, \ldots, u_m\}$, $V = \{v_1, \ldots, v_n\}$, and edge set $E = \{(u_i, v_j) \mid a_{ij} \neq 0, u_i \in U, v_j \in V\}$. Then A is combinatorially symmetric if and only if $G(A)$ is symmetric. A matrix A is term-nonsingular if and only if $G(A)$ has a perfect matching.

Let $G = (U, V; E)$ be a bipartite graph. For vertex subsets $I \subseteq U$ and $J \subseteq V$, we denote by $G[I, J]$ the subgraph induced by vertex subsets I and J. For a subgraph H, we denote by $U(H)$ and $V(H)$ the sets of vertices in H belonging to U and V, respectively, and by $E(H)$ the set of edges in H. Let $G \setminus H$ be the graph obtained from G by deleting $U(H)$ and $V(H)$ together with edges incident to them. For an edge subset $F \subseteq E$, we denote by $U(F)$ and $V(F)$ the set of the end vertices of F which belong to U and V, respectively. For a matching M, we say that a path

P of G is M-*alternating* if the elements of P alternate between elements of M and $E \setminus M$ along P. For two edge subsets F_1 and F_2, the *symmetric difference* $(F_1 \setminus F_2) \cup (F_2 \setminus F_1)$ is denoted by $F_1 \triangle F_2$. Notice that, for an M-alternating path P with a matching M, the symmetric difference $M \triangle E(P)$ is also a matching.

This paper is organized as follows. Section 2 discusses the DM-decomposition of symmetric bipartite graphs. In Section 3, we present the ear decomposition of matching-covered symmetric bipartite graphs. Sections 4 to 6 describe applications of results in Sections 2 and 3. Section 4 discusses Pólya matrices of combinatorially symmetric matrices. In Section 5, we introduce the problem of a totally sign-nonsingular signing of a rectangular matrix and discuss its computational complexity. In Section 6, we characterize matrices having a totally sign-nonsingular signing in terms of excluded minors.

2 DM-Decomposition of Symmetric Bipartite Graphs

In this section, we discuss symmetry of the DM-components of a symmetric bipartite graph.

We first review the Dulmage-Mendelsohn decomposition of a bipartite graph following the exposition in [19]. Let $G = (U, V; E)$ be a bipartite graph with $W = U \cup V$. A pair (I, J) of $I \subseteq U$ and $J \subseteq V$ is said to be a *cover* if no edges exist between $U \setminus I$ and $V \setminus J$. The *size* of a cover (I, J) is defined to be $|I| + |J|$. It is well-known that the maximum size of matchings is equal to the minimum size of covers. For convenience, we define the *cut function* $\kappa : 2^W \to \mathbb{Z} \cup \{+\infty\}$ as follows:

$$\kappa(X) = \begin{cases} |U \setminus X| + |V \cap X|, & \text{if } (U \setminus X, V \cap X) \text{ is a cover,} \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that $\kappa(X)$ is finite if and only if $(U \setminus X, V \cap X)$ is a cover. The function κ satisfies submodularity, i.e.,

$$\kappa(X) + \kappa(Y) \geq \kappa(X \cap Y) + \kappa(X \cup Y), \quad \forall X, Y \subseteq W.$$

The set of minimizers of a submodular function forms a distributive lattice. Hence there exist unique minimal and maximal minimizers.

Let $\mathcal{L}$ be the set of minimizers of κ. Take a maximal ascending chain $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_k$ in $\mathcal{L}$, where k is a nonnegative integer, and X_0 and X_k are the unique minimal and maximal minimizers, respectively. We put

$$\begin{aligned} W_0 &= X_0, \\ W_l &= X_l \setminus X_{l-1}, \quad l = 1, \ldots, k, \\ W_\infty &= W \setminus X_k. \end{aligned} \tag{1}$$

The family of the difference sets $\{W_l \mid l = 0, 1, \ldots, k, \infty\}$ is uniquely determined independently of the choice of the chain by a Jordan-Hölder type theorem. Define a partial order $\preceq$ on $\{W_l \mid l = 1, \ldots, k\}$ by

$$W_h \preceq W_l \iff [W_l \subseteq X \in \mathcal{L} \Rightarrow W_h \subseteq X].$$

Moreover, we extend this partial order to that on $\{W_l \mid l = 0, 1, \ldots, k, \infty\}$ by defining

$$W_0 \preceq W_l \preceq W_\infty, \quad l = 1, \ldots, k.$$

The pair of $\{W_l \mid l = 0, 1, \ldots, k, \infty\}$ and $\preceq$ defined above is called the *Dulmage-Mendelsohn decomposition* of G. Let $U_l = W_l \cap U$ and $V_l = W_l \cap V$ for $l = 0, 1, \ldots, k, \infty$. The subgraphs $G[U_l, V_l]\,(l = 0, 1, \ldots, k, \infty)$ are called the *DM-components*. Note that the subgraph $G[U_h, V_l]$ has no edges for $0 \le l < h \le \infty$.

We say that a bipartite graph with nonempty vertex set is *DM-irreducible* if it cannot be decomposed into more than one nonempty component via the DM-decomposition. Suppose that a bipartite graph with no vertices is DM-irreducible. Assume that $|U| \le |V|$. Since the DM-irreducibility means that $\mathcal{L}$ contains no proper subsets of W, the graph G is DM-irreducible if and only if $\kappa(X) \ge |U| + 1$ for any nonempty proper subset $X \subsetneq W$. Thus a bipartite graph $G = (U, V; E)$ with $|U| = |V|$ is DM-irreducible if and only if it is matching-covered.

We now obtain the following theorem for a symmetric bipartite graph. For a vertex subset $X \subseteq W$, we denote $X^\top = \{v_i \in V \mid u_i \in X \cap U\} \cup \{u_i \in U \mid v_i \in X \cap V\}$.

Theorem 2.1. *Let $G = (U, V; E)$ be a symmetric bipartite graph, and $(\{W_l\}, \preceq)$ be the DM-decomposition obtained by a maximal ascending chain $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_k$ in $\mathcal{L}$. Then the DM-decomposition satisfies the following.*

(1) *For each DM-component $G[U_l, V_l]\,(l = 0, 1, \ldots, k, \infty)$, there exists a DM-component $G[U_h, V_h]$ which is the transpose of $G[U_l, V_l]$.*

(2) *It holds that $W_l \preceq W_h$ if and only if $W_l^\top \succeq W_h^\top$.*

(3) *If $W_l = W_l^\top$ and $W_h = W_h^\top$ $(l \neq h)$, then there exists no partial order between W_l and W_h.*

Proof. Since G is symmetric, $(U \setminus X, V \cap X)$ is a cover if and only if so is $(U \cap X^\top, V \setminus X^\top)$. Hence $\kappa(X) = \kappa(W \setminus X^\top)$ holds for any $X \subseteq W$. This implies that $X \in \mathcal{L}$ if and only if $W \setminus X^\top \in \mathcal{L}$. Hence $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_k$ is a maximal ascending chain in $\mathcal{L}$ if and only if $W \setminus X_k^\top \subsetneq W^\top \setminus X_{k-1}^\top \subsetneq \cdots \subsetneq W^\top \setminus X_0^\top$ is that in $\mathcal{L}$. As (1), this ascending chain in $\mathcal{L}$ yields the partition $\{W_l' \mid l = 0, 1, \ldots, k, \infty\}$ of W:

$$\begin{aligned} W_0' &= W_\infty^\top, \\ W_l' &= (W \setminus X_{l-1}^\top) \setminus (W \setminus X_l^\top) = W_l^\top, \quad l = 1, \ldots, k, \\ W_\infty' &= W \setminus (W \setminus X_0^\top) = W_0^\top. \end{aligned}$$

By a Jordan-Hölder type theorem, this coincides with $\{W_l \mid l = 0, 1, \ldots, k, \infty\}$. Therefore, for each DM-component $G[U_l, V_l]\,(l = 0, 1, \ldots, k, \infty)$, the subgraph $G[V_l^\top, U_l^\top]$ is also a DM-component of G, where $V_l^\top = W_l^\top \cap U$ and $U_l^\top = W_l^\top \cap V$. Thus the statement (1) holds.

To prove (2), assume that $W_l \preceq W_h$. Let $X \in \mathcal{L}$ be a minimizer such that $W_l^\top \subseteq X$. Then $W_l \cap (W \setminus X^\top) = \emptyset$. Since $W \setminus X^\top \in \mathcal{L}$, it holds that $W_h \subseteq X^\top$ or $W_h \subseteq W \setminus X^\top$. By $W_l \preceq W_h$, we have $W_h \subseteq X^\top$. This implies that $W_l^\top \succeq W_h^\top$. The converse holds in a similar way.

The statement (3) immediately follows from (2). $\square$

The concept of the DM-decomposition is applied to matrices. Let A be a matrix and $G(A)$ be the associated bipartite graph. The *DM-decomposition of a matrix* A is the partition of rows and columns obtained by the DM-decomposition of $G(A)$. For $I \subseteq U$ and $J \subseteq V$, the submatrix corresponding to $G[I, J]$ is denoted by $A[I, J]$. Since $A[U_h, V_l] = O$ for $0 \leq l < h \leq \infty$, the matrix A can be rearranged into a block triangular matrix by row and column permutations. The DM-decomposition can be computed efficiently with the aid of bipartite matching algorithms. Note that, if $A[U_h, V_l] \neq O$ for $1 \leq h < l \leq k$ then $W_h \preceq W_l$ holds, and, conversely, if $W_h \prec W_l$ and there exists no $W_{l'}$ with $W_h \prec W_{l'} \prec W_l$ for $1 \leq h, l \leq k$, then $A[U_h, V_l] \neq O$. Thus the DM-decomposition of a matrix can be depicted as in Fig. 1.

Let A be a combinatorially symmetric matrix. It follows from Theorem 2.1 that the DM-decomposition of A can maintain symmetry. That is, for each DM-component $A[U_l, V_l]\,(l = 0, 1, \ldots, k, \infty)$, the block submatrix $A[U_l, V_l]$ is symmetric, or $A[U_l \cup U_h, V_l \cup V_h]$ is symmetric for some $h \in \{0, 1, \ldots, k, \infty\}$. Moreover, if both of $A[U_l, V_l]$ and $A[U_h, V_h]$ are symmetric, then $A[U_h, V_l] = A[U_h, V_l] = O$. Thus a combinatorially symmetric matrix A has a permutation matrix S such that $S^\top AS$ is a block triangular matrix depicted as in Fig. 2. Such a block triangular form of a combinatorially symmetric matrix can be obtained efficiently via the DM-decomposition.

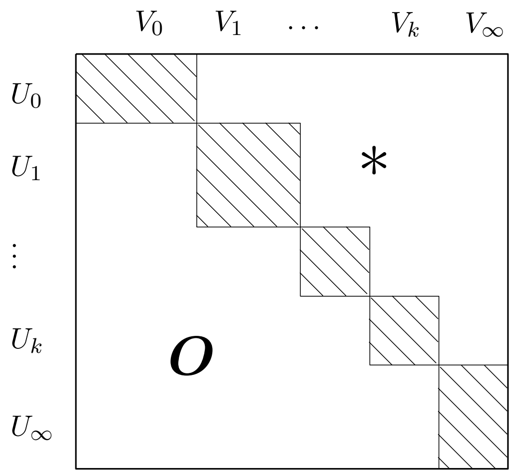

Figure 1: The DM-decomposition of a matrix

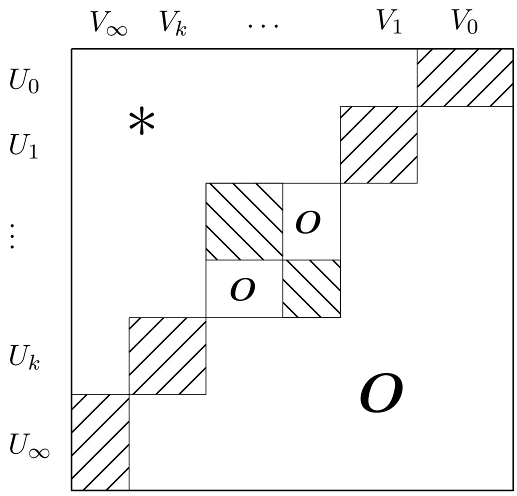

Figure 2: The DM-decomposition of a combinatorially symmetric matrix

There is another block-triangular decomposition for a square matrix, which employs a simultaneous permutation of rows and columns. For a square matrix A of order n, define the directed graph $D(A) = (W, E)$ with $W = \{w_1, \ldots, w_n\}$ and $E = \{(w_i, w_j) \mid a_{ij} \neq 0, i, j \in N\}$, where $N = \{1, \ldots, n\}$. Then the strongly-connected component decomposition of $D(A)$ leads to an upper-right block-triangularized form $S^\top AS$ for some permutation matrix S. A square matrix A is *indecomposable* if $D(A)$ is strongly connected. For a combinatorially symmetric matrix A, this decomposition is trivial, because A is indecomposable if and only if $D(A)$ is connected.

Theorem 2.1 suggests that, by the DM-decomposition of A, we can find a finer upper-left block-triangular form by simultaneous permutations of rows and columns. For example, consider the combinatorially symmetric matrix

$$A = \begin{pmatrix} +1 & -1 \\ -1 & 0 \end{pmatrix}.$$

Then A is indecomposable, while the DM-decomposition of A leads to two blocks of order one.

3 Ear Structure of Matching-Covered Symmetric Graphs

In this section, we discuss ear decomposition of a matching-covered symmetric bipartite graph. Let $G = (U, V; E)$ be a matching-covered symmetric bipartite graph with $|U| = |V| = n$. Recall that an ear decomposition $G_0, G_1, \dots, G_k$ is *symmetric* if G_l or G_{l+1} is symmetric for $l = 0, 1, \dots, k-1$. A *diagonal edge* is an edge $(u_i, v_i) \in E$ for some $i \in N = \{1, \dots, n\}$. The main purpose of this section is to prove the following theorem.

Theorem 3.1. *Let $G = (U, V; E)$ be a matching-covered symmetric bipartite graph. Then G has a symmetric ear decomposition starting from an edge or a crossing pair. In particular, if G has a diagonal edge, G has a symmetric one starting from the diagonal edge.*

We say that a subgraph G' is *central* if $G \setminus G'$ has a perfect matching. In order to prove Theorem 3.1, we first show that, for any central symmetric subgraph G', there exist an ear P of G' and an ear Q of $G' + P$ such that $G' + P + Q$ is symmetric and central, where Q may be empty.

Let $G' = (U', V'; E')$ be a central symmetric subgraph. If $U' = U$ and $V' = V$, then any diagonal edge and any crossing pair in $E \setminus E'$ are the desired ears. Hence we may assume that $U' \subsetneq U$ and $V' \subsetneq V$. Let $\bar{G}' = G[U \setminus U', V \setminus V']$ be the remaining symmetric subgraph. Since G' is central, $\bar{G}'$ has a perfect matching M.

We first assume that $M = M^\top$ holds. Note that, if a path P is M-alternating, then so is $P^\top$. The graph G has an edge (u_i, v_j) for some $u_i \in U'$ and $v_j \notin V'$. Since G is matching-covered, G has a perfect matching M' with $(u_i, v_j) \in M'$. The subgraph with edge set $M \cup M'$ consists of paths and circuits, in which the connected component having u_i forms an M-alternating ear $\hat{P}$ of G'. If the inner vertices in $\hat{P}$ and $\hat{P}^\top$ are disjoint, then $\hat{P}^\top$ is an ear of $G' + \hat{P}$ and $G' + \hat{P} + \hat{P}^\top$ is symmetric. Hence we may assume that $\hat{P}$ and $\hat{P}^\top$ have a common inner vertex. This implies that there exists an index $s \in N$ with $u_s \in U(\hat{P})$ and $v_s \in V(\hat{P})$ such that all vertices in P_{ss} have different indices, where P_{ss} is the path between u_s and v_s along $\hat{P}$. Among such s, we choose s such that the length of P_{is} is minimum, where P_{is} is the shorter one of the path from u_i to u_s along $\hat{P}$ and the path from u_i to v_s along $\hat{P}$. Define $P = P_{is} \cup P_{ss} \cup P_{is}^\top$, and Q to be empty if P_{ss} is a diagonal edge and $Q = P_{ss}^\top$ otherwise. Then P is an M-alternating ear of G', and, if Q is nonempty, Q is an M-alternating ear of $G' + P$. The subgraph $G' + P + Q$ has the edge set $E' \cup E(P_{is} \cup P_{ss}) \cup E((P_{is} \cup P_{ss})^\top)$, and hence $G' + P + Q$ is symmetric. Moreover, since P and Q are M-alternating paths of odd length, $G' + P + Q$ is central.

Therefore, the following lemma holds. Note that, if Q is empty, then P has exactly one diagonal edge, and, otherwise, P and Q have no diagonal edges.

Lemma 3.2. *Let $G = (U, V; E)$ be a matching-covered symmetric bipartite graph, and $G' = (U', V'; E')$ be a central symmetric subgraph. Assume that the remaining subgraph $\bar{G}' = G[U \setminus U', V \setminus V']$ has a perfect matching M with $M^\top = M$. Then there exist an ear P of G' and an ear Q of $G' + P$ such that $G' + P + Q$ is central and symmetric, where Q may be empty.*

We now discuss the case where M may not coincide with $M^\top$. For a bipartite graph $G = (U, V; E)$ with a matching M, we define *contracting an M-alternating circuit C to an edge (x, y)* as contracting $U(C)$ and $V(C)$ to vertices x and y, respectively, deleting resulting multiple edges, and replacing M with $M \setminus E(C) \cup \{(x, y)\}$. The reverse procedure is *expanding an edge to a circuit*. Note that, if G is matching-covered and M is a perfect matching of G, then the graph obtained by contracting an M-alternating circuit is also matching-covered.

Assume that $M \neq M^\top$. Then consider $M \cup M^\top$, which consists of diagonal edges, crossing pairs, pairs of asymmetric circuits, and symmetric circuits. By $M \neq M^\top$, the union $M \cup M^\top$ has pairs of asymmetric circuits, or symmetric circuits. For each pair of asymmetric circuits C and $C^\top$ in $M \cup M^\top$, replace M with $M \triangle E(C)$. Moreover, for each symmetric circuit C in $M \cup M^\top$, contract C to a diagonal edge e_C. Let F be the set of diagonal edges obtained by the contraction of all symmetric circuits in $M \cup M^\top$. The resulting graph G_* is symmetric and matching-covered, and G' is a central symmetric subgraph of G_*. Moreover, M is a perfect matching in $G_* \setminus G'$ with $M = M^\top$.

Therefore, it follows from Lemma 3.2 that G_* has an ear P_* of G' and an ear Q_* of $G' + P_*$ such that $G' + P_* + Q_*$ is symmetric and central, where Q_* may be empty. If P_* and Q_* have no edges in F, then $G' + P_* + Q_*$ is also a central symmetric subgraph of G. Assume that P_* has a diagonal edge e in F. Then Q_* is empty. We denote by C the contracted circuit corresponding to e. Since P_* has exactly one edge in F, the edge subset $E(P_*) \setminus \{e\} \cup E(C)$ forms an ear P of G' and an ear Q of $G' + P$ such that $G' + P + Q$ is symmetric and central.

By the above discussion, we obtain the following theorem.

Theorem 3.3. *Let G be a matching-covered symmetric bipartite graph, and G' be a central symmetric subgraph. Then there exist an ear P of G' and an ear Q of $G' + P$ such that $G' + P + Q$ is central and symmetric, where Q may be empty.*

For a symmetric bipartite graph with perfect matchings, the following proposition has been shown.

Proposition 3.4 (Kakimura and Iwata [10])**.** *Let G be a symmetric bipartite graph with perfect matchings. If G is not a disjoint union of symmetric circuits, then G satisfies the following* (a) *or* (b)*.*

(a) *The graph G has a perfect matching with a diagonal edge (u_i, v_i) for some $i \in N$.*

(b) *The graph G has a perfect matching with a crossing pair (u_i, v_j) and (u_j, v_i) for some distinct $i, j \in N$.*

Theorem 3.3, together with Proposition 3.4, implies Theorem 3.1.

Proof of Theorem 3.1. It is not difficult to see that a symmetric graph consisting of one circuit has a symmetric ear decomposition starting from an edge. Assume that G is not a circuit. If G has a diagonal edge, then the matching-coveredness of G implies that G has a perfect matching with this edge. Otherwise, G has a perfect matching with a crossing pair by Proposition 3.4. Hence G has a central subgraph G_0 consisting of a diagonal edge or a crossing pair. By applying Theorem 3.3 repeatedly, we obtain an ear decomposition $G_0, G_1, \ldots, G_k = G$ such that G_l or G_{l+1} is symmetric for $l = 0, 1, \ldots, k-1$. $\square$

This section concludes with a linear-time algorithm for finding a symmetric ear decomposition. The algorithm description is presented as follows.

Algorithm for symmetric ear decomposition.

Input: A matching-covered symmetric bipartite graph $G = (U, V; E)$ and a perfect matching M' of G.

Step 0: If G consists of a circuit, then halt (G_0 is the subgraph consisting of one edge and $G_1 = G$).

Step 1: Find a perfect matching M with a diagonal edge or a crossing pair using M'. Let G_0 be the subgraph consisting of a diagonal edge or a crossing pair in M.

Step 2: Do the following, so that $M = M^\top$.

- **2-1:** For each pair of asymmetric circuits C and $C^\top$ in $M \cup M^\top$, replace M with $M \triangle E(C)$.
- **2-2:** For each symmetric circuit C in $M \cup M^\top$, contract C to a diagonal edge e_C. Let $\mathcal{C}$ be the set of the contracted circuits.

Step 3: Set $i = 0$ and $M = M \setminus E(G_0)$. Repeat the following until $G_i = G$.

- **3-1:** Find an M-alternating ear $\hat{P}$ of G_i.
- **3-2:** Using $\hat{P}$, find at most two M-alternating paths P and Q such that $G_i + P + Q$ is symmetric, where Q may be empty.
- **3-3:** If P has an edge e_C obtained by contracting some $C \in \mathcal{C}$, then expand e_C to C and replace P and Q with two paths consisting of $E(P) \setminus \{e_C\} \cup E(C)$.
- **3-4:** If Q is empty, set $G_{i+1} = G_i + P$, $M = M \setminus E(G_{i+1})$, and $i = i + 1$. If Q is not empty, set $G_{i+1} = G_i + P$, $G_{i+2} = G_{i+1} + Q$, $M = M \setminus E(G_{i+2})$, and $i = i + 2$.

Note that, in Step 1, we can find a perfect matching with a diagonal edge or a crossing pair by using M' in $\mathrm{O}(|E|)$ time [10]. Therefore, the running time bound of this algorithm is presented as follows.

Theorem 3.5. *Let $G = (U, V; E)$ be a matching-covered symmetric bipartite graph, and M' be a perfect matching in G. Then we can find a symmetric ear decomposition starting from an edge or a crossing pair in $\mathrm{O}(|E|)$ time.*

Proof. Steps 1 and 2 require $\mathrm{O}(|E|)$ time. Before repeating Step 3, we find M-alternating paths from a vertex in G_0 to all vertices in G by the depth first search in advance. By using the depth first search tree, Step 3-1 requires $\mathrm{O}(|\hat{P}|)$ time to find an M-alternating ear $\hat{P}$. In Step 3, we can find all of ears that use $E(\hat{P} \cup \hat{P}^\top)$ in a symmetric ear decomposition in $\mathrm{O}(|\hat{P} \cup \hat{P}^\top|)$ time. Therefore, the total time complexity is $\mathrm{O}(|E|)$ time. □

4 Symmetric Pólya Matrices with a Nonzero Diagonal Entry

In this section, we discuss Pólya matrices of combinatorially symmetric matrices as an application of the two decompositions described in Sections 2 and 3.

Pólya's problem is equivalent to the problem of deciding whether a given bipartite graph has an orientation called *Pfaffian*. Let $G = (W, E)$ be a graph. An *orientation* $\overrightarrow{G}$ of G is a directed graph obtained from G by orienting its edges. For an orientation $\overrightarrow{G}$ of G, a circuit C of even length in G is said to be *oddly* (*evenly*) *oriented* in $\overrightarrow{G}$ if an odd (even) number of its edges are directed in the same direction along C. For a graph $G = (W, E)$, we say that an orientation of G is *Pfaffian* if every central circuit of even length is oddly oriented. For a square matrix A, it is known that A has a Pólya matrix if and only if $G(A)$ has a Pfaffian orientation. Robertson, Seymour, and Thomas [21] devised a polynomial-time algorithm to decide whether a given bipartite graph has a Pfaffian orientation (cf. McCuaig [18]).

Suppose that a bipartite graph $G = (U, V; E)$ with perfect matchings has Pfaffian orientations. We discuss constructing a Pfaffian orientation of G. We may assume that a bipartite graph $G = (U, V; E)$ is matching-covered, because G has a Pfaffian orientation if and only if so does each DM-component. Since G is matching-covered, G has an ear decomposition starting from an edge [15]. It is known that the following theorem holds.

Theorem 4.1 (Little [14], Seymour and Thomassen [23])**.** *Let G be a matching-covered bipartite graph which has Pfaffian orientations, and $G_0, G_1, \ldots, G_k = G$ be an ear decomposition starting from an edge with $G_l = G_{l-1} + P_l$ for $l = 1, \ldots, k$. Then an orientation is Pfaffian if and only if $C_1, \ldots, C_k$ are oddly oriented, where C_l is a central circuit of G_l which uses P_l for $l = 1, \ldots, k$.*

Theorem 4.1 suggests a polynomial-time algorithm for finding a Pfaffian orientation as follows. Let G be a matching-covered bipartite graph which has Pfaffian orientations. Obtain an ear decomposition $G_0, G_1, \ldots, G_k = G$ with $G_l = G_{l-1} + P_l$ starting from an edge for $l = 1, \ldots, k$. Orient the edge of G_0 arbitrary. For $l = 1, \ldots, k$, find a central circuit C_l of G_l which uses P_l, and orient all edges in P_l such that C_l is oddly oriented. Then the obtained orientation of $G_k = G$ is a Pfaffian orientation.

Let $G = (U, V; E)$ be a symmetric bipartite graph with perfect matchings. Suppose that G has a Pfaffian orientation. We discuss to find a symmetric Pfaffian orientation in G, where an orientation of a bipartite graph is *symmetric* if the two edges of any crossing pair are oriented in the same direction. Again, we may assume that G is matching-covered, because it follows from Theorem 2.1 that G has a symmetric Pfaffian orientation if and only if so does each symmetric DM-component and each non-symmetric DM-component has a Pfaffian orientation. Then we have the following theorem.

Theorem 4.2. *Let $G = (U, V; E)$ be a matching-covered symmetric bipartite graph with a diagonal edge. If G has a Pfaffian orientation, then G has a symmetric one.*

Proof. By Theorem 3.1, G has a symmetric ear decomposition $G_0, G_1, \ldots, G_k = G$ starting from a diagonal edge. Let P_l be the path such that $G_l = G_{l-1} + P_l$ for $l = 1, \ldots, k$. The subgraph G_0, which consists of one diagonal edge, has a symmetric Pfaffian orientation. For an integer $l \in \{0, 1, \ldots, k-1\}$, assume that, if G_l is symmetric, it has a symmetric Pfaffian orientation $\overrightarrow{G}_l$.

Suppose that $G_{l+1} = G_l + P_{l+1}$ is symmetric. Since the length of P_{l+1} is odd and $P_{l+1} = P_{l+1}^\top$, the ear P_{l+1} has only one diagonal edge e. Let C_{l+1} be a central circuit of G_{l+1} which uses P_{l+1}. By orienting e properly, $\overrightarrow{G}_l$ can be extended to a symmetric orientation $\overrightarrow{G}_{l+1}$ of G_{l+1} such that C_{l+1} is oddly oriented. Since $\overrightarrow{G}_{l+1}$ is Pfaffian by Theorem 4.1, G_{l+1} has a symmetric Pfaffian orientation.

Next suppose that $G_{l+1} = G_l + P_{l+1}$ is not symmetric. Then $G_{l+2} = G_{l+1} + P_{l+2}$ is symmetric. Let C_{l+1} be a central circuit in G_{l+1} which uses P_{l+1}. Since G_{l+1} is not symmetric, there exists an edge $e = (u_i, v_j)$ with $(u_j, v_i) \notin E(P_{l+1})$. By orienting e properly, $\overrightarrow{G}_l$ can be extended to a symmetric orientation $\overrightarrow{G}_{l+1}$ of G_{l+1} such that C_{l+1} is oddly oriented, which implies that $\overrightarrow{G}_{l+1}$ is Pfaffian by Theorem 4.1. Consider the symmetric orientation $\overrightarrow{G}_{l+2}$ of G_{l+2} which includes $\overrightarrow{G}_{l+1}$. If P_{l+2} is also an ear of G_l, then $C_{l+1}^\top$ is an oddly oriented central circuit using P_{l+2} in $\overrightarrow{G}_{l+2}$. Otherwise, $P_{l+2} \cup P_{l+2}^\top$ forms a symmetric central circuit C with no diagonal edges by $P_{l+2}^\top \subseteq P_{l+1}$. Since C has a symmetric orientation in $\overrightarrow{G}_{l+2}$, the circuit C is oddly oriented. In both cases, $\overrightarrow{G}_{l+2}$ has an oddly oriented central circuit using P_{l+2}. Thus the symmetric orientation $\overrightarrow{G}_{l+2}$ is Pfaffian by Theorem 4.1.

Therefore, for any $l = 0, 1, \ldots, k$, if G_l is symmetric then G_l has a symmetric Pfaffian orientation by induction, and hence so does $G = G_k$. □

Since a symmetric ear decomposition can be obtained in linear time by Theorem 3.5, we have the following corollary.

Corollary 4.3. *Let $G = (U, V; E)$ be a matching-covered symmetric bipartite graph with a diagonal edge, and M be a perfect matching of G. Assume that G has a Pfaffian orientation. Then we can find a symmetric Pfaffian orientation in* $\mathrm{O}(|E|)$ *time.*

Theorem 4.2 can be written as the following corollary in terms of a Pólya matrix. Recall that a square matrix A is DM-irreducible if and only if $G(A)$ is matching-covered.

Corollary 4.4. *Let A be a DM-irreducible symmetric $\{0,1\}$-matrix with a nonzero diagonal entry. If A has a Pólya matrix, then A has a symmetric one.*

If A has no diagonal entries, then it is not necessarily true that A has a Pólya matrix which is symmetric. For example, consider the symmetric matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Then A has a Pólya matrix

$$\begin{pmatrix} 0 & +1 & +1 & +1 \\ +1 & 0 & -1 & +1 \\ -1 & -1 & 0 & +1 \\ +1 & -1 & +1 & 0 \end{pmatrix}.$$

However, A has no Pólya matrix which is symmetric. Indeed, if A has a Pólya matrix in the form of

$$\begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ a_1 & 0 & a_4 & a_5 \\ a_2 & a_4 & 0 & a_6 \\ a_3 & a_5 & a_6 & 0 \end{pmatrix},$$

where $a_1, \ldots, a_6 \in \{1, -1\}$, then the determinant has nonzero expansion terms $a_1^2 a_6^2$, $-a_1a_3a_4a_6$, $-a_2a_3a_4a_5$, and $-a_1a_2a_5a_6$. Since these nonzero expansion terms have the same sign, $a_1^2a_6^2 = -a_1a_3a_4a_6$ and $-a_1a_2a_5a_6 = -a_2a_3a_4a_5$ hold. The former implies $a_1a_6 = -a_3a_4$ while the latter $a_1a_6 = a_3a_4$, which is a contradiction.

5 Totally Sign-Nonsingular Signing

Recall that an $m \times n$ rectangular matrix is totally sign-nonsingular if each term-nonsingular submatrix of order m is sign-nonsingular. This section and Section 6 discuss the problem of deciding whether a given rectangular $\{0,1\}$-matrix has a totally sign-nonsingular signing or not. If a matrix is term-nonsingular, this problem is equivalent to Pólya's problem.

We first show the following theorem.

Theorem 5.1. *We can decide in polynomial time whether a given $m \times n$ $\{0,1\}$-matrix A with $m \leq n$ has a totally sign-nonsingular signing or not.*

For an $m \times n$ matrix A, we define the *augmented matrix of* A, denoted by A^*, as follows:

$$A^* = \begin{pmatrix} O & A \\ A^\top & I \end{pmatrix},$$

where I is the identity matrix of order n. The bipartite graph associated with A^* is denoted by G^*, called the *augmented graph of* G. That is, for a bipartite graph $G = (U, V; E)$ with $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$, the augmented graph G^* is defined to be $G^* = (U \cup \tilde{V}, \tilde{U} \cup V; E \cup \tilde{E} \cup E_{\rm d})$, where $\tilde{U} = \{\tilde{u}_1, \ldots, \tilde{u}_m\}$ and $\tilde{V} = \{\tilde{v}_1, \ldots, \tilde{v}_n\}$ are copies of U and V, respectively, and $\tilde{E}$ and $E_{\rm d}$ are the edge sets defined by $\tilde{E} = \{(\tilde{v}_j, \tilde{u}_i) \mid (u_i, v_j) \in E\}$ and $E_{\rm d} = \{(\tilde{v}_i, v_i) \mid i = 1, \ldots, n\}$.

The following proposition asserts the equivalence between the total sign-nonsingularity of a matrix A and the sign-nonsingularity of A^*. A matrix A is said to have *row-full term-rank* if A has a term-nonsingular submatrix with row size. Note that, if A does not have row-full term-rank, A is clearly totally sign-nonsingular.

Proposition 5.2 (Iwata and Kakimura [6])**.** *Let A be a matrix with row-full term-rank. Then A is totally sign-nonsingular if and only if the augmented matrix A^* is sign-nonsingular.*

We give the following lemma for signings of the augmented matrices.

Lemma 5.3. *Let A be an $m \times n$ rectangular matrix with $m < n$. If the augmented matrix A^* of A has a Pólya matrix, then A has a totally sign-nonsingular signing.*

Proof. It follows from Corollary 4.4 that A^* has a symmetric Pólya matrix, denoted by $\tilde{A}^*$. We denote $N = \{1, \ldots, n\}$. Let $\tilde{A}$ be the submatrix of $\tilde{A}^*$ corresponding to A, and d_i for $i \in N$ be the diagonal entry of column i in $\tilde{A}^*$. The determinant of $\tilde{A}^*$ is given by

$$\det \tilde{A}^* = \sum_{\substack{J \subseteq N, \\ |J| = m}} d_J (\det \tilde{A}[J])(\det \tilde{A}[J]),$$

where $d_J = \prod_{i \notin J} d_i$ and $\tilde{A}[J]$ is the square submatrix of $\tilde{A}$ with column subset J. Since $\tilde{A}^*$ is a Pólya matrix, all nonzero expansion terms of $\det \tilde{A}^*$ have the same sign. This implies that, for any $J \subseteq V$ such that $\tilde{A}[J]$ is term-nonsingular, $\tilde{A}[J]$ is sign-nonsingular. Thus $\tilde{A}$ is totally sign-nonsingular. □

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. If A is square, then we can find a totally sign-nonsingular signing, i.e., a Pólya matrix, in polynomial time. Assume that $m < n$. Note that A has a totally sign-nonsingular sining if and only if so does each DM-component. Hence we may assume without loss of generality that A is DM-irreducible, which implies that A has row-full term-rank. By Proposition 5.2, if A^* has no Pólya matrices, then A has no totally sign-nonsingular signings. It follows from Lemma 5.3 that, if A^* has a Pólya matrix, then A has a totally sign-nonsingular signing. Thus we can obtain a totally sign-nonsingular signing by testing whether A^* has a Pólya matrix or not. □

Testing sign-nonsingularity is polynomially equivalent to Pólya's problem [14, 23] (see also [28]). Theorem 5.1, together with Proposition 5.2, is summarized as the following corollary.

Corollary 5.4. *The following problems are polynomially equivalent.*

(1) *Deciding whether a given square matrix has a Pólya matrix or not (Pólya's problem).*

(2) *Deciding whether a given square matrix is sign-nonsingular or not.*

(3) *Deciding whether a given rectangular matrix has a totally sign-nonsingular signing or not.*

(4) *Deciding whether a given rectangular matrix is totally sign-nonsingular or not.*

We say that two matrices A and A' with same size are *equivalent* if A' can be obtained from A by multiplying -1 to some rows and columns, that is, if there exist two $\{1,-1\}$-diagonal matrices D_{r} and D_{c} with $A' = D_{\mathrm{r}} A D_{\mathrm{c}}$. It is known in [14] that, if a DM-irreducible square $\{0,1\}$-matrix has a Pólya matrix, then all of the Pólya matrices are equivalent. For totally sign-nonsingular signings, a similar statement holds.

Theorem 5.5. *If a DM-irreducible $\{0,1\}$-matrix A has a totally sign-nonsingular signing, then all of totally sign-nonsingular signings are equivalent.*

Let $G = (U, V; E)$ be a bipartite graph with two disjoint vertex sets $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$ $(m \leq n)$. We say that a matching M is *left-perfect* if $|M| = |U|$. For a bipartite graph $G = (U, V; E)$, the *neighbor* of $X \subseteq U$, denoted by $\Gamma_G(X)$, is the set of vertices in V that connect some vertex in X, that is, $\Gamma_G(X) = \{v_j \in V \mid \exists u_i \in X, (u_i, v_j) \in E\}$. We need the following well-known proposition (e.g., see [15, 19]).

Proposition 5.6. *Let $G = (U, V; E)$ be a bipartite graph with $|U| \leq |V|$.*

- *The graph G has a left-perfect matching if and only if $|\Gamma_G(X)| \geq |X|$ for any subset $X \subseteq U$.*
- *The graph G is DM-irreducible if and only if $|\Gamma_G(X)| \geq |X| + 1$ for any nonempty proper subset $X \subsetneq U$.*

Proposition 5.6 implies the following lemma.

Lemma 5.7. *Let $G = (U, V; E)$ be a connected bipartite graph with $|U| < |V|$. Then G is DM-irreducible if and only if G^* is DM-irreducible.*

Proof. By Proposition 5.6, if G is not DM-irreducible, then G has a proper subset $X \subsetneq U$ with $|\Gamma_G(X)| < |X| + 1$, which implies that $|\Gamma_{G^*}(X)| < |X| + 1$ holds. Thus the sufficiency holds.

To show the necessity, assume that G is DM-irreducible, and that G^* is not DM-irreducible. By Proposition 5.6, G^* has a proper subset $X \subsetneq U \cup \tilde{V}$ with $|\Gamma_{G^*}(X)| < |X| + 1$. Since G has a left-perfect matching, so does G^*. Hence the subset X satisfies $|\Gamma_{G^*}(X)| = |X|$ by Proposition 5.6. We denote $X_U = X \cap U$ and $X_{\tilde{V}} = X \cap \tilde{V}$. Let $Y = \Gamma_{G^*}(X)$, $Y_{\tilde{U}} = Y \cap \tilde{U}$, and $Y_V = Y \cap V$ (See Fig. 3). Then $X_{\tilde{V}} \neq \emptyset$ holds by the DM-irreducibility of G. This implies that $Y_{\tilde{U}} \neq \emptyset$ because G has no isolated vertex. Since $|\Gamma_{G^*}(X_{\tilde{V}})| > |X_{\tilde{V}}|$ by the existence of diagonal edges in E_{d}, we have $X_U \neq \emptyset$. The definition of Γ_{G^*} implies that $G^*[X_{\tilde{V}}, \tilde{U} \setminus Y_{\tilde{U}}]$ and $G^*[X_U, V \setminus Y_V]$ have no edges.

We will show that $Y_V = X_V$ and $X_U = Y_U$, where $X_V = \{v_j \in V \mid \tilde{v}_j \in X_{\tilde{V}}\}$ and $Y_U = \{u_i \in U \mid \tilde{u}_i \in Y_{\tilde{U}}\}$. First, assume to the contrary that $Y_V \neq X_V$. Since $Y_V \supseteq X_V$, there exists a nonempty set $Z_V = Y_V \setminus X_V$. By $|Y| = |X|$ and $|Y_V| > |X_V|$, it holds that $Z_U = X_U \setminus Y_U$ is nonempty. The DM-irreducibility of G implies that $|\Gamma_{G^*}(Z_U)| \geq |Z_U| + 1$. Since $G^*[Z_U, X_V]$ is a subgraph of the transpose of $G^*[X_{\tilde{V}}, \tilde{U} \setminus Y_{\tilde{U}}]$, the subgraph $G^*[Z_U, X_V]$ has no edges. This implies that $|\Gamma_{G^*}(X)| \geq |\Gamma_{G^*}(Z_U)| + |X_V| + |Y_{\tilde{U}}| \geq |Z_U| + 1 + |X_{\tilde{V}}| + |Y_U| \geq |X| + 1$, which contradicts that $|\Gamma_{G^*}(X)| = |X|$. Thus $Y_V = X_V$ holds. By $|Y| = |X|$ and $|Y_V| = |X_V|$, it holds that $|X_U| = |Y_U|$. Since X_U has no isolated vertex, $X_U \subseteq Y_U$ holds, and hence we have $X_U = Y_U$.

By $Y_V = X_V$ and $X_U = Y_U$, the subgraph $G^*[U \setminus X_U, X_V]$ is the transpose of $G^*[X_{\tilde{V}}, \tilde{U} \setminus X_{\tilde{U}}]$. This implies that $G^*[U \setminus X_U, X_V]$ and $G^*[X_U, V \setminus X_V]$ have no edges, which contradicts that G is connected. Thus G^* is DM-irreducible. □

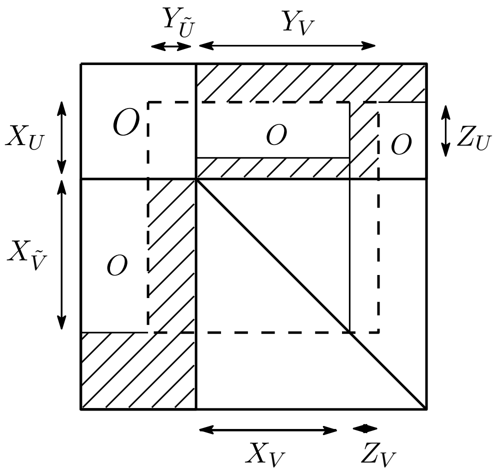

Figure 3: The matrix associated with an augmented bipartite graph

Theorem 5.5 immediately follows from Proposition 5.2 and Lemma 5.7.

Proof of Theorem 5.5. Since A is DM-irreducible, so is A^* by Lemma 5.7. Proposition 5.2 implies that each totally sign-nonsingular signing of A corresponds to a symmetric Pólya matrix of A^*. Since all of symmetric Pólya matrices of A^* are equivalent, so are all of totally sign-nonsingular signings of A. □

6 Excluded Minor Characterization for Totally Sign-Nonsingular Singing

We say that a graph G is a *subdivision* of a graph H if G is obtained from H by replacing some edges of H by an internally disjoint paths with at least two edge. A graph G is an *even subdivision* of a graph H if G is obtained from H by replacing some edges of H by internally disjoint paths of odd length. A bipartite graph $G = (U, V; E)$ with $|U| = |V|$ *contains* a graph H if G has a central subgraph which is isomorphic to some even subdivision of H.

Let $K_{m,n}$ denote the complete bipartite graph with two disjoint vertex sets of size m and n, respectively. Little [14] gave the following necessary and sufficient condition of a bipartite graph having Pfaffian orientations. Another proof is given in [20].

Proposition 6.1 (Little [14]). *A bipartite graph has a Pfaffian orientation if and only if the graph does not contain $K_{3,3}$.*

Let $G = (U, V; E)$ be a bipartite graph with $|U| \le |V|$. A subgraph G' is said to be *left-central* if $G \setminus G'$ has a left-perfect matching. We say that an orientation of G is *totally Pfaffian* if every left-central circuit is oddly oriented. If $|U| = |V|$ this is equivalent to Pfaffian orientations. By the definition, a matrix A has a totally sign-nonsingular signing if and only if the associated bipartite graph $G(A)$ has a totally Pfaffian orientation.

The main purpose of this section is to characterize a bipartite graph having totally Pfaffian orientations. We say that a bipartite graph $G = (U, V; E)$ with $|U| < |V|$ *contains* a graph H if G has a left-central subgraph which is isomorphic to some even subdivision of H. Let $L_{3,5}$ denote the bipartite graph associated with the following matrix:

$$\begin{pmatrix} +1 & 0 & 0 & +1 & +1 \\ 0 & +1 & 0 & +1 & +1 \\ 0 & 0 & +1 & +1 & +1 \end{pmatrix}.$$

Then we have the following theorem, which we will prove later.

Theorem 6.2. *Let $G = (U, V; E)$ be a DM-irreducible bipartite graph with $|U| < |V|$. Then G has a totally Pfaffian orientation if and only if G does not contain either $K_{2,3}$ or $L_{3,5}$.*

Figures 4 and 5 depict $K_{2,3}$ and $L_{3,5}$, respectively.

For a bipartite graph G, the graph G has a totally Pfaffian orientation if and only if so does each of the DM-components. Therefore, Theorem 6.2, together with Proposition 6.1, leads to the following corollary.

Corollary 6.3. *Let $G = (U, V; E)$ be a bipartite graph. Then G has a totally Pfaffian orientation if and only if G contains none of $K_{3,3}$, $K_{2,3}$, and $L_{3,5}$.*

Let A be an $m \times (m + 1)$ DM-irreducible matrix. Then A is totally sign-nonsingular if and only if all square submatrices of order m are sign-nonsingular. Such matrix is called an *S-matrix.*

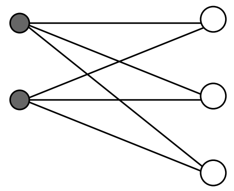

Figure 4: The bipartite graph $K_{2,3}$

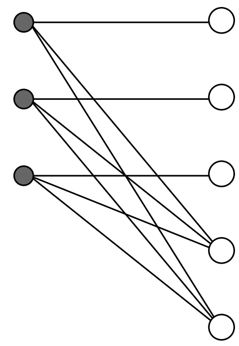

Figure 5: The bipartite graph $L_{3,5}$

The following characterization for S-matrices, given by Brualdi and Shader [2], is derived as a special case of Theorem 6.2.

Corollary 6.4 (Brualdi and Shader [2])**.** *A $m \times (m+1)$ $\{0,1\}$-matrix A has a signing which is an S-matrix if and only if $G(A)$ does not contain $K_{2,3}$.*

Proof of Theorem 6.2

The rest of this section is devoted to the proof of Theorem 6.2. It is obvious that $K_{2,3}$ and $L_{3,5}$ have no totally Pfaffian orientations. Hence the necessity of Theorem 6.2 follows from the following lemma.

Lemma 6.5. *Let G be a bipartite graph which contains a graph H. If G has a totally Pfaffian orientations, then so does H.*

Proof. The graph G has a left-central subgraph K isomorphic to an even subdivision of H. Let $\overrightarrow{G}$ be a totally Pfaffian orientation of G. We define an orientation of H as follows. Let $e = (u, v)$ be an edge of H. The subgraph K of G has the two vertices u' and v' corresponding to u and v, respectively, and the path between u' and v' corresponding to e. Consider traversing this path from u' to v'. If the number of edges in the forward direction is odd, then we orient the edge e from u to v, otherwise orient it from v to u. Since a left-central circuit in K is oddly oriented if and only if so is the corresponding left-central circuit in H, this orientation is a totally Pfaffian orientation of H. $\square$

Therefore, it suffices to prove the sufficiency. To do this, we provide the following proposition, which follows from Proposition 5.6.

Proposition 6.6. *Let $G = (U, V; E)$ be a bipartite graph with $|U| < |V|$, and M be a left-perfect matching of G. The graph G is DM-irreducible if and only if, for any $v \in V(M)$, there exists an M-alternating path from v to some vertex in $V \setminus V(M)$.*

For a path P and two vertices x, y in P, let $P[x, y]$ be the subpath of P between x and y. We denote $W(H) = U(H) \cup V(H)$ for a subgraph H. For two circuits C and C', we simply denote by $C \triangle C'$ the subgraph consisting of $E(C) \triangle E(C')$. The following claim is observed in [20].

Claim 1. *For a directed graph, let C and C' be two circuits of even length such that $P = C \cap C'$ is a path. Then $D = C \triangle C'$ is also a circuit of even length. Moreover, the followings hold.*

- *If P is an odd-length path, the number of evenly oriented circuits in $\{C, C', D\}$ is even.*
- *If P is an even-length path, the number of evenly oriented circuits in $\{C, C', D\}$ is odd.*

Let $G = (U, V; E)$ with $|U| < |V|$ be a DM-irreducible bipartite graph which does not have a totally Pfaffian orientation. We may assume that G is a minimal such graph with respect to the operations of edge and vertex deletion and replacing an odd path all of whose inner vertices have degree two with one edge. Then G is connected.

The following claim says that we can delete one edge preserving DM-irreducibility. Here we denote by G_e the bipartite graph obtained from G by deleting an edge e.

Claim 2. *There exists an edge $e \in E$ such that G_e is DM-irreducible.*

Proof. Consider the augmented graph $G^* = (U^*, V^*; E \cup \tilde{E} \cup E_{\mathrm{d}})$. Since G is DM-irreducible, so is G^* by Lemma 5.7. Since G^* is symmetric, G^* has a symmetric ear decomposition $G_0^*, G_1^*, \ldots, G_k^* = G^*$ starting from a diagonal edge by Theorem 3.1. We denote $G_l^* = G_{l-1}^* + P_l$ for $l = 1, \ldots, k$. Let h be the last index such that P_h contains some edge in $E \cup \tilde{E}$. Then each of $P_{h+1}, \ldots, P_k$ is an ear consisting of one diagonal edge in E_{d}. Since G is minimal, P_h is either a path of three length using a diagonal edge or a path consisting of one edge in $E \cup \tilde{E}$. If P_h is a path of three length using a diagonal edge e, then G_{h-1}^* is symmetric, and we define G'^* to be the subgraph consisting of $G_{h-1}^* + P_{h+1} + \cdots + P_k$. Otherwise, if P_h consists of one edge in $E \cup \tilde{E}$, then G_{h-2}^* is symmetric, and we define $G'^* = G_{h-2}^* + P_{h+1} + \cdots + P_k$. Let G' be the bipartite graph whose augmented graph is G'^*. Since G'^* is DM-irreducible, so is G' by Lemma 5.7. The graph G' is obtained from G by deleting an edge. $\square$

Let $e = (u, v)$ be an edge such that G_e is DM-irreducible. The minimality of G implies that G_e has a totally Pfaffian orientation $\overrightarrow{G_e}$. Consider an orientation $\overrightarrow{G}$ of G such that the edge e is directed arbitrarily and the other edges are directed in the same directions as those in $\overrightarrow{G_e}$. Since $\overrightarrow{G}$ is not totally Pfaffian, there exists an evenly oriented left-central circuit C.

We divide the proof into the following two cases: (1) the case where G has an evenly oriented left-central circuit C with $e \notin E(C)$ and (2) the other case, i.e., all evenly oriented left-central circuits have the edge e.

Case (1): G has an evenly oriented left-central circuit not having the edge e

Assume that G has an evenly oriented left-central circuit C with $e \notin E(C)$. Let M be a left-perfect matching such that C is M-alternating. Since G_e is totally Pfaffian, C is not left-central in G_e, which implies by Proposition 5.6 that there exists a vertex subset $X \subseteq U \setminus U(C)$ such that $|\Gamma_{G_e \setminus C}(X)| \leq |X| - 1$. We may suppose that we choose X such that $|X|$ is minimum. On the other hand, C is left-central in G, and hence $|\Gamma_{G \setminus C}(X)| \geq |X|$. These inequalities imply that

$u \in X$, $v \notin \Gamma_{G_e \setminus C}(X)$, and $|\Gamma_{G_e \setminus C}(X)| = |X| - 1$ hold. We denote $G'_e = G_e[U \setminus X, V \setminus \Gamma_{G_e \setminus C}(X)]$. The subgraph G'_e has a left-perfect matching $M \cap E(G'_e)$.

Let $Y = \Gamma_{G_e}(X) \setminus \Gamma_{G_e \setminus C}(X)$. Then $Y \subseteq V(C)$ holds. Since G_e is DM-irreducible, we have $|\Gamma_{G_e}(X)| \geq |X| + 1$, which implies that $|Y| \geq 2$ by $|\Gamma_{G_e \setminus C}(X)| = |X| - 1$. For a graph G and a vertex y, we denote by $G - y$ the subgraph obtained from G by deleting y together with edges incident to y. Then the following claim holds.

Claim 3. *There exist $y_1, y_2 \in Y$ such that $G'_e - y_1 - y_2$ has a left-perfect matching.*

Proof. Since G is DM-irreducible, G has an M-alternating path P from the vertex $v \in V(M)$ to some vertex $w \notin V(M)$ by Proposition 6.6. Then $w \in V(G'_e)$. Hence P has a vertex in Y. Let y_1 be the vertex in $Y \cap V(P)$ which is closest to w along P. Then $P[y_1, w]$ is an M-alternating path in G'_e. The graph $G'_e - y_1$ has a left-perfect matching $M \triangle E(P[y_1, w])$.

We denote $G''_e = G'_e - y_1$ and $Y' = Y \setminus \{y_1\}$. Let $J \subseteq U \setminus X$ be the maximum subset such that $|\Gamma_{G''_e}(J)| = |J|$. Note that $V(G''_e) \setminus \Gamma_{G''_e}(J)$ is nonempty because of $|U| + 1 \leq |V|$. Since G_e is DM-irreducible, it holds that $|\Gamma_{G_e}(X \cup J)| \geq |X| + |J| + 1$. Moreover, $|\Gamma_{G_e}(X \cup J)| = |\Gamma_{G_e \setminus C}(X)| + |Y \cup \Gamma_{G''_e}(J)| = |\Gamma_{G_e \setminus C}(X)| + |Y' \setminus \Gamma_{G''_e}(J)| + |\Gamma_{G''_e}(J)| + 1$ holds. Hence we have $|Y' \setminus \Gamma_{G''_e}(J)| \geq 1$ by $|\Gamma_{G_e \setminus C}(X)| = |X| - 1$ and $|\Gamma_{G''_e}(J)| = |J|$. Take $y_2 \in Y' \setminus \Gamma_{G''_e}(J)$. Then $G''_e - y_2$ has a left-perfect matching by the maximality of J. $\square$

It follows from Claim 3 that G'_e has a left-perfect matching M' with $y_1, y_2 \notin V(M')$. Taking $M \cup M'$, we obtain two disjoint M-alternating paths P_i in G'_e from y_i to some two vertices $w_i \notin V(M)$ for $i = 1, 2$, respectively. We may assume that C, P_1, and P_2 have been chosen to minimize $|E(C \cup P_1 \cup P_2)|$.

Since we have chosen X such that $|X|$ is minimum, $G_e[X, \Gamma_{G_e \setminus C}(X)]$ is DM-irreducible, which implies by Proposition 6.6 that $G_e[X, \Gamma_{G_e \setminus C}(X) \cup \{y_1, y_2\}]$ has an M-alternating path R_i from u to y_i for $i = 1, 2$. Define $T_i = P_i \cup R_i \cup \{e\}$ for $i = 1, 2$. The path T_i is an M-alternating path from v to w_i.

For $i = 1, 2$, the subgraph with edge set $E(P_i) \setminus E(C)$ is the set of paths, denoted by $Q_i^1, Q_i^2, \ldots, Q_i^{p_i}$, where p_i is a positive integer. We may assume that $Q_i^1, Q_i^2, \ldots, Q_i^{p_i}$ appear in this order along P_i from y_i to w_i. Then the path Q_i^j for $1 \leq j \leq p_i - 1$ is an M-alternating ear of C, and $Q_i^{p_i}$ is an M-alternating path from a vertex in $U(C)$ to w_i. We denote the end vertices of Q_i^j by $s_i^j \in U(C)$ and $t_i^j \in V(C)$ for $i = 1, 2$ and $j = 1, \ldots, p_i$.

Claim 4. *If $p_1 \geq 2$ or $p_2 \geq 2$, then G contains $K_{2,3}$.*

Proof. It suffices to show the case of $p_1 \geq 2$. Then $Q_1^{p_1-1}$ is an M-alternating ear of C. Let C^{p_1-1} be the path along C from $s_1^{p_1-1}$ to $t_1^{p_1-1}$ such that $D = Q_1^{p_1-1} \cup C^{p_1-1}$ is an M-alternating circuit. The other path from $s_1^{p_1-1}$ to $t_1^{p_1-1}$ along C is denoted by $\bar{C}^{p_1-1}$ (see Fig. 6).

First assume that there exist $s_i^j \in U(\bar{C}^{p_1-1})$ and $t_i^j \in V(C^{p_1-1})$ for some $i \in \{1, 2\}$ and $j \in \{1, \ldots, p_i - 1\}$. Let D' be the M-alternating circuit consisting of Q_i^j and C. Then $D' \cup Q_1^{p_1-1} \cup \bar{C}^{p_1-1}$ is an even subdivision of $K_{2,3}$, denoted by H. By taking $M \triangle E(P_1[t_1^{p_1-1}, w_1])$, we know that H is left-central.

Next assume that there exist no $i \in \{1,2\}$ and $j \in \{1,\ldots,p_i-1\}$ such that $s_i^j \in U(\bar{C}^{p_1-1})$ and $t_i^j \in V(C^{p_1-1})$. By $t_1^{p_1-2} \in V(C^{p_1-1})$, where $t_1^0 = y_1$, this assumption implies that $t_1^{j-1} \in V(C^{p_1-1})$ and $s_1^j \in V(C^{p_1-1})$ for any $j = 1,\ldots,p_1-1$. Hence $Q_1^1, Q_1^2, \ldots, Q_1^{p_i-1}$ are ears of C^{p_1-1}. Since $M \triangle E(T_1)$ is a left-perfect matching in G_e and $C \triangle D$ is $(M \triangle E(T_1))$-alternating, the circuit $C \triangle D$ is oddly oriented. Hence D is evenly oriented by Claim 1. Since $M \triangle E(T_2)$ is a left-perfect matching in G_e, the circuit D is not $(M \triangle E(T_2))$-alternating. This implies that P_2 has an edge in C^{p_1-1}. Moreover, by the choice of P_1 and P_2, the path P_2 also has an edge in $\bar{C}^{p_1-1}$. Hence there exists an ear Q_2^k with some $k \in \{1,\ldots,p_2-1\}$ from $s_2^k \in U(C^{p_1-1})$ to $t_2^k \in V(\bar{C}^{p_1-1})$. It follows from the assumption that $t_2^j \in V(\bar{C}^{p_1-1})$ for any $j = k,\ldots,p_2-1$ and $t_2^j \in V(C^{p_1-1})$ for any $j = 0,\ldots,k-1$, where $t_2^0 = y_1$. Then we obtain $|E(D \cup P_1 \cup P_2)| < |E(C \cup P_1 \cup P_2)|$ since the terminal edges in $\bar{C}^{p_1-1}$ are not contained in $D \cup P_1 \cup P_2$. This contradicts the choice of C, P_1, and P_2. □

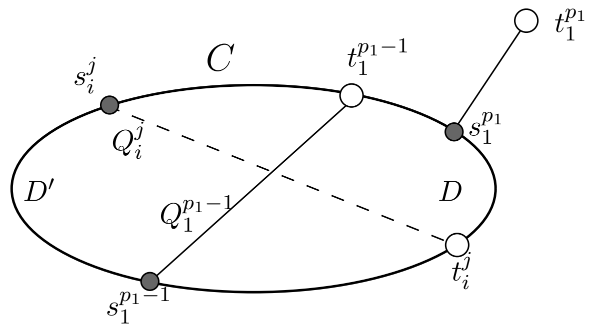

Figure 6: The case of $p_1 \geq 2$ (Claim 4)

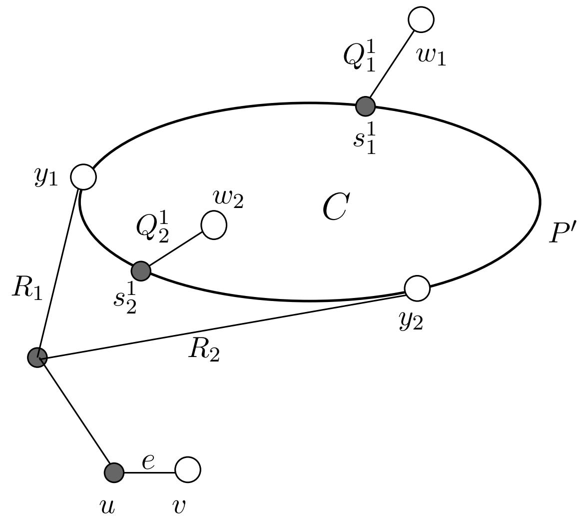

Figure 7: The case of $p_1 = p_2 = 1$ (Claim 5)

Claim 5. *If $p_1 = p_2 = 1$, then G contains $K_{2,3}$ or $L_{3,5}$.*

Proof. By $p_1 = p_2 = 1$, the vertices y_1, s_1^1, y_2, s_1^2 appear in this order along C. First assume that neither of w_1 and w_2 coincide with v (see Fig. 7). Then $H = R_1 \cup R_2 \cup \{e\} \cup C \cup Q_1^1 \cup Q_1^2$ is an even subdivision of $L_{3,5}$. Moreover, H is left-central, because $M \setminus E(H)$ is a left-perfect matching in $G \setminus H$. Thus G contains $L_{3,5}$. Next assume that either of w_1 and w_2 coincides with v. We may assume that $w_1 = v$. Then $H' = R_1 \cup R_2 \cup \{e\} \cup P' \cup Q_1^1$ an even subdivision of $K_{2,3}$, where P' is the path along C from y^1 to y^2 with $s_1^1 \in U(P')$ and $s_2^1 \notin U(P')$. We know that H' is left-central, because $(M \triangle E(P_2)) \setminus E(H')$ is a left-perfect matching in $G \setminus H'$. Thus G contains $K_{2,3}$. □

Case (2): all evenly oriented left-central circuits have the edge e

Assume that all evenly oriented left-central circuits have the edge $e = (u, v)$. Let C_0 be one of such evenly oriented left-central circuits. Since G has no totally Pfaffian orientation even if e is oriented oppositely, there exists an oddly oriented left-central circuit C_1 which uses e. We choose C_0 and C_1 such that $|E(C_{01})|$ is maximum, where C_{01} is the connected component of $C_0 \cap C_1$ having e. For $i = 0, 1$, let M_i be a left-perfect matching such that C_i is M_i-alternating. We may assume that $e \notin M_i$ for $i = 0, 1$ and that $|V(M_0) \cap V(M_1)|$ is maximum.

For $i = 0, 1$, we denote by $\hat{C}_i$ the path obtained from C_i by deleting e. Let D_0 be the subgraph with edge set $E(C_0) \setminus E(C_1)$, and D_1 the subgraph with $E(C_1) \setminus E(C_0)$, and $D = D_0 \cup D_1$. We denote the connected components of D_0 by $D_0^1, \ldots, D_0^p$, where p is a positive integer. Each component D_0^i is a path. We may assume that $D_0^1, \ldots, D_0^p$ appear in this order along $\hat{C}_0$ from u to v. For $i = 1, \ldots, p$, the end vertices of D_0^i are denoted by s^i and t^i, where s^i is closer to u along $\hat{C}_0$ than t^i. Let D_1^i be the path of D_1 whose end vertices are s^i and t^i, and D^i be the circuit consisting of D_0^i and D_1^i for $i = 1, \ldots, p$.

We first show the following claims.

Claim 6. *Let P be a path with end vertices $w \in W(D_0) \setminus \{s^1\}$ and $z \in W(D_1) \setminus \{t^p\}$. Assume that $W(P \cap \hat{C}_0[u, w]) = \{w\}$ and $W(P \cap \hat{C}_1[z, v]) = \{z\}$. Then the circuit $C' = \hat{C}_0[u, w] \cup P \cup \hat{C}_1[z, v] \cup \{e\}$ is not left-central.*

Proof. Assume that C' is left-central. If C' is oddly oriented, then C' and C_0 contradict the maximality of $|E(C_{01})|$. Otherwise, C' and C_1 contradict the maximality of $|E(C_{01})|$. □

Claim 7. *The path D_i^1 is not M_j-alternating ear of C_j, where $(i, j) = (0, 1)$ and $(1, 0)$.*

Proof. Assume to the contrary that D_0^1 is an M_1-alternating ear of C_1. Then the two circuits D^1 and $C_1 \triangle D^1$ are left-central by taking $M_1 \triangle E(D^1)$ if necessary. If $p \geq 2$, then either C_0 and $C_1 \triangle D^1$ or C_1 and $C_1 \triangle D^1$ contradict the choice of C_0 and C_1. Hence we have $p = 1$ and $C_0 = C_1 \triangle D^1$. By Claim 1, D^1 is evenly oriented, which contradicts the assumption of Case (2). Thus D_0^1 is not an M_1-alternating ear of C_1. In a similar way, D_1^1 is not an M_0-alternating ear of C_0. □

A path P is said to be (M_0, M_1)-*path* if the elements of P alternate between elements of M_0 and M_1 along P. An (M_0, M_1)-path is both M_0-alternating and M_1-alternating. An (M_0, M_1)-path is *maximal* if one of its end vertices is in $V(M_0) \setminus V(M_1)$ and the other is in $V(M_1) \setminus V(M_0)$.

We will next show in Claims 9 and 10 that G contains $K_{2,3}$ or $L'_{3,5}$, where $L'_{3,5}$ is the bipartite graph obtained from $L_{3,5}$ by deleting one vertex with degree one. For that purpose, we need the following claim.

Claim 8. *Assume that $E(C_i) \setminus E(C_{01})$ and $M_i \setminus E(C_{01})$ for $i = 0, 1$ have been chosen to minimize $|E(C_0 \cup C_1) \cup M_0 \cup M_1|$. Let R be an (M_0, M_1)-ear of C_i with end vertices $w \in U(D_i)$ and $z \in V(D_i)$ for some $i \in \{0, 1\}$. Then $R \cup \hat{C}_i[w, z]$ is M_i-alternating, and $w \in U(\hat{C}_i[s^1, z])$ holds.*

Proof. It suffices to show the case of $i = 0$. Assume that $C' = R \cup \hat{C}_0[w, z]$ is not M_0-alternating. Since C' is left-central by taking $M_0 \triangle E(C_0)$, the circuit C' is oddly oriented by the assumption of Case (2). Claim 1 implies that $C'_0 = C_0 \triangle C'$ is evenly oriented. The circuit C'_0 satisfies that $|E(C'_0 \cup C_1) \cup M_0 \cup M_1| < |E(C_0 \cup C_1) \cup M_0 \cup M_1|$, since the terminal edges of $\hat{C}_0[w, z]$ are not contained in $E(C'_0 \cup C_1) \cup M_0 \cup M_1$. This contradicts the minimality of $|E(C_0 \cup C_1) \cup M_0 \cup M_1|$. Thus C' is not M_0-alternating, which implies that $w \in U(\hat{C}_i[s^1, z])$. □

Using Claim 8, we obtain Claims 9 and 10 as follows. Figures 8 and 9 will be helpful to understand the proofs of these claims.

Claim 9. *If $s^1 \in U$, then G contains $K_{2,3}$.*

Proof. We may suppose that $E(C_i) \setminus E(C_{01})$ and $M_i \setminus E(C_{01})$ for $i = 0, 1$ have been chosen to minimize $|E(C_0 \cup C_1) \cup M_0 \cup M_1|$. We will show that such C_0 and C_1 form a left-central even subdivision of $K_{2,3}$.

Let P be the maximal (M_0, M_1)-path with $s^1 \in U(P)$. The path P is denoted by a sequence of edges $e_0^1, e_1^1, \ldots, e_0^r, e_1^r$, where $e_0^j = (x^j, y^{j-1}) \in M_0$ and $e_1^j = (x^j, y^j) \in M_1$ with $x^j \in U$ and $y^0, y^j \in V$ for $j = 1, \ldots, r$. We denote $s^1 = x^k$ for some $k \in \{1, \ldots, r\}$.

We will first show that $P[x^k, y^0] \subsetneq D_0^1$. Assume to the contrary that there exists an edge of $P[x^k, y^0]$ not in $E(D_0^1)$. Let l in $1, \ldots, k-1$ be the maximum index such that $e_1^l \notin E(D_0^1)$. Note that $P[x^k, y^l] \subseteq D_0^1$ holds. Since D^1 is not M_1-alternating by Claim 7, $y^l \neq t^1$ holds. If $P[y^l, y^0]$ does not have an edge in $E(C_1)$, then M_0 and $M_1 \triangle E(P[y^l, y^0])$ contradict the minimality of $|E(C_0 \cup C_1) \cup M_0 \cup M_1|$. Hence $P[y^l, y^0]$ has an edge in $E(C_1)$. Since $P[x^k, y^l] \subseteq D_0^1$ and $y^l \in V$, there exists no (M_0, M_1)-ear of C_0 from y^l to a vertex in $U(D_0)$ by Claim 8. Hence the subpath $P[y^l, y^0]$ includes an (M_0, M_1)-path Q from y^l to a vertex $y^{l'}$ in $V(D_1)$ such that $W(Q \cap C_0) = \{y^l\}$ and $W(Q \cap C_1) = \{y^{l'}\}$. The vertex $y^{l'}$ is not equal to t^p, and $P[x^k, y^{l'}]$ is an M_1-alternating ear of C_1. Let $D' = P[x^k, y^{l'}] \cup \hat{C}_1[x^k, y^{l'}]$. Since the circuit $C_1 \triangle D'$ is M'_1-alternating, where $M'_1 = M_1 \triangle E(D')$, this circuit contradicts Claim 6. Thus $P[x^k, y^0] \subsetneq D_0^1$ and $y^0 \in V(D_0^1) \setminus \{t^1\}$ hold.

We next show that, for any edge $f \in M_0$ in $E(D_0^1[y^0, t^1])$, we have $f \in M_1$. Indeed, if $w \in U(D_0^1[y^0, t^1])$ has two distinct edges $f = (w, z) \in M_0$ and $f' = (w, z') \in M_1$, then $P' = D_0^1[y^0, w] \cup \{f'\}$ is an M_1-alternating path by choosing w that is closest to y^0 along D_0^1, and hence M_0 and $M_1 \triangle E(P')$ contradict the minimality of $|E(C_0 \cup C_1) \cup M_0 \cup M_1|$. Thus $D_0^1[y^0, t^1]$ is also M_1-alternating, and hence $t^1 \in U$ holds. This implies that $C_1 \triangle D^1$ is an M''_1-alternating circuit, where $M''_1 = M_1 \triangle E(P[y^0, y^k])$. By the maximality of $|E(C_{01})|$, we have $p = 1$ and $C_0 = C_1 \triangle D^1$.

Therefore, by $p = 1$ and $s^1, t^1 \in U$, the subgraph $C_0 \cup C_1$ is an even subdivision of $K_{2,3}$ (see Fig. 8). Since $G \setminus (C_0 \cup C_1)$ has a left-perfect matching $M_1 \setminus E(C_0 \cup C_1)$, this is left-central. □

Claim 10. *If $s^1 \in V$, then there exist M_i and C_i for $i = 0, 1$ such that $M_1 = M_0 \triangle E(P)$ for some (M_0, M_1)-path P, and $C_0 \cup C_1 \cup P$ forms a left-central even subdivision of $L'_{3,5}$.*

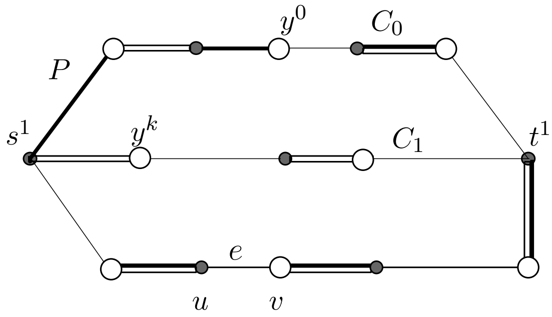

Figure 8: The case of $s^1 \in U$ (Claim 9)

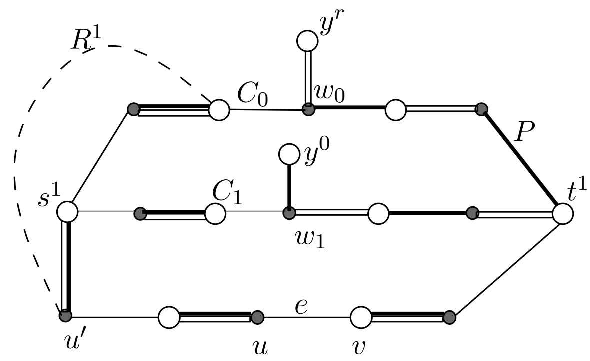

Figure 9: The case of $s^1 \in V$ (Claims 10 & 12)

Proof. In a similar way to Claim 9, suppose that $E(C_i) \setminus E(C_{01})$ and $M_i \setminus E(C_{01})$ for $i = 0, 1$ have been chosen to minimize $|E(C_0 \cup C_1) \cup M_0 \cup M_1|$. Let $w_0 \in U(C_0)$ be the vertex that has two distinct edges of M_0 and M_1. Choose w_0 that is closest to s^1 along $\hat{C}_0$. Note that $D_0[s^1, w_0]$ is M_1-alternating. Claim 7 implies that $w_0 \in U(D_0^1)$. Let P be the maximal (M_0, M_1)-path having w_0. The path P is denoted by a sequence of edges $e_0^1, e_1^1, \ldots, e_0^r, e_1^r$, where $e_0^j = (x^j, y^{j-1}) \in M_0$ and $e_1^j = (x^j, y^j) \in M_1$ with $x^j \in U$ and $y^0, y^j \in V$ for $j = 1, \ldots, r$. We denote $w_0 = x^k$ for some $k \in \{1, \ldots, r\}$.

We first note that there exists an edge in $E(P[x^k, y^0]) \setminus E(D_0^1)$. Indeed, if $P[x^k, y^0] \subseteq D_0^1$, then M_0 and $M_1 \triangle E(P[y^k, y^0])$ contradict the minimality of $|E(C_0 \cup C_1) \cup M_0 \cup M_1|$. Let l in $1, \ldots, k-1$ be the maximum index such that $e_1^l = (x^l, y^l) \notin E(D_0)$. Then $y^l \in V(D_0^1)$ holds. If $P[y^l, y^0]$ does not have an edge in $E(C_1)$, then M_0 and $M_1 \triangle E(P[y^l, y^0])$ contradict the minimality of $|E(C_0 \cup C_1) \cup M_0 \cup M_1|$. Hence $P[y^0, y^l]$ has an edge in $E(C_1)$. Moreover, since $D_0^1[s^1, x^k]$ is M_1-alternating and $y^l \in V$, there exists no (M_0, M_1)-ear of C_0 from y^l to a vertex in $U(D_0)$ by Claim 8, which implies that $P[y^0, y^l]$ has no edges in $E(C_0)$.

Let l' in $1, \ldots, k-1$ be the index such that $y^{l'} \in V(P)$ is the vertex of $V(C_1)$ that is closest to v along $\hat{C}_1$. Then $C' = \hat{C}_0[u, y^l] \cup P[y^l, y^{l'}] \cup \hat{C}_1[y^{l'}, v] \cup \{e\}$ is an M_1'-alternating circuit, where $M_1' = M_1 \triangle E(P[y^k, y^0])$. Hence C' is left-central. Since $y^l \in V(D_0^1) \setminus \{s^1\}$, it follows from Claim 6 that $y^{l'} = t^p$. This implies that $e_0^{l'+1} \in E(D_0)$ and $e_1^{l'} \in E(D_1)$. Since $P[y^0, y^l]$ has no edges in $E(C_0)$, we have $l = l'$. Therefore, by $y^l \in V(D_0^1)$ and $y^{l'} = t^p$, we obtain $p = 1$ and $y^l = y^{l'} = t^1$. Thus P includes $\hat{C}_0[w_0, t^1]$.

In a similar way, let $w_1 \in U(C_1)$ be the vertex that has two distinct edges of M_0 and M_1. Choose w_1 that is closest to s^1 along $\hat{C}_1$. Note that $D_1^1[s^1, w_1]$ is M_0-alternating. Let P' be the maximal (M_0, M_1)-path having w_1. Then P' includes $\hat{C}_1[w_1, t^1]$. Since P' has the vertex t^1, the path P' coincides with P.

By $p = 1$, the subgraph with edge set $E(P) \setminus E(C_0 \cup C_1)$ consists of two paths from $V \setminus V(C_0 \cup C_1)$ to $U(D_0)$ and $U(D_1)$, respectively (see Fig. 9). Therefore, by $s^1, t^1 \in V$, the subgraph $C_0 \cup C_1 \cup P$ is an even subdivision of $L_{3,5}'$, denoted by L. The subgraph L is left-central, since $G \setminus L$ has a left-perfect matching $M_0 \setminus E(L)$. The minimality of $|E(C_0 \cup C_1) \cup M_0 \cup M_1|$ implies that $M_0 = M_1 \triangle E(P)$. $\square$

Claim 9 implies that, if $s^1 \in U$, then G contains $K_{2,3}$. Thus we henceforth assume $s^1 \in V$. By Claim 10, take C_i and M_i for $i = 0, 1$ such that $M_1 = M_0 \triangle E(P)$ for some (M_0, M_1)-path P, and $C_0 \cup C_1 \cup P$, denoted by L, forms a left-central even subdivision of $L'_{3,5}$. Since D is one circuit, we simply denote $s^1 = s$ and $t^1 = t$. The path P is denoted by a sequence of edges $e_0^1, e_1^1, \ldots, e_0^r, e_1^r$, where $e_0^j = (x^j, y^{j-1}) \in M_0$ and $e_1^j = (x^j, y^j) \in M_1$ with $x^j \in U$ and $y^0, y^j \in V$ for $j = 1, \ldots, r$. Let x^k be the vertex in $U(D_0)$ that is closest to y^r along P, and $x^{k'}$ be the vertex in $U(D_1)$ that is closest to y^0. We may assume that $k = r$ and $k' = 1$ by taking $M_0 \triangle E(P[y^k, y^r])$ and $M_1 \triangle E(P[y^0, y^{k'-1}])$.

Claim 11. *There is no M_0-alternating ear of L between $(W(D_0) \setminus \{s, t\}) \cup \{y^r\}$ and $(W(D_1) \setminus \{s, t\}) \cup \{y^0\}$.*

Proof. If suffices to show that there is no M_0-alternating ear of L from $U(D_0)$ to $V(D_1) \cup \{y^0\}$. Assume that L has such an M_0-alternating ear Q from $w \in U(D_0)$ to $z \in V(D_1) \cup \{y^0\}$.

First assume that $z \in V(D_1[s, x^1])$. Then $C' = D_1[u, z] \cup Q \cup D_0[w, v] \cup \{e\}$ is an M_0-alternating circuit, which contradicts Claim 6. Next assume that $z \in V(\hat{C}_1[x^1, t])$. Then $C'' = \hat{C}_1[t, z] \cup Q \cup \hat{C}_0[w, t]$ is M_0-alternating. Hence $C_0 \triangle C''$ is left-central, which contradicts Claim 6. Finally, assume that $z = y^0$. Let $D' = \hat{C}_1[s, z] \cup \{e_0^1\} \cup Q \cup \hat{C}_0[w, s]$ and $D'' = D_1 \triangle D'$. Then one of D' and D'' is evenly oriented by Claim 1, because D_1 is oddly oriented. Since D' is $(M_0 \triangle E(C_0))$-alternating and D'' is M_0-alternating, both of D' and D'' are left-central. This contradicts the assumption of Case (2). □

Since G_e is DM-irreducible, Proposition 6.6 implies that there exists an M_0-alternating path R from some vertex u' in $U(\hat{C}_0[u, s])$ to a vertex $v' \notin V(M_0)$ such that $e \notin E(R)$ and $W(Q \cap \hat{C}_0[u, s]) = \{u'\}$. We may suppose that D_0, D_1, and R have been chosen to minimize $|E(R \cup C_0 \cup C_1)|$.

The subgraph with edge set $E(R) \setminus E(L)$ is the set of M_0-alternating paths, denoted by $R^1, R^2, \ldots, R^q$, where q is a positive integer. We may assume that $R^1, R^2, \ldots, R^q$ appear in this order along R from u' to v'. Then the path R^j for $1 \leq j \leq q-1$ is an M_0-alternating ear of L, and R^q is an M_0-alternating path from a vertex in $U(L)$ to v'. Note that R^j is also M_1-alternating. We denote the end vertices of R^j by $w^j \in U$ and $z^j \in V$ for $j = 1, \ldots, q$. In the same way as Claim 8, if R^j is an M_0-alternating ear of L with $w^j, z^j \in W(D_0)$, then $C_0[w^j, z^j] \cup R^j$ is M_0-alternating and $w_j \in U(\hat{C}_0[s, z_j])$ by the minimality of $|E(R \cup C_0 \cup C_1)|$.

We next show the following claim, which completes the proof of Case (2) in Theorem 6.2. The proof of this claim uses the same technique as that of Proposition 6.1 by Norine, Little, and Teo [20].

Claim 12. *The graph G contains $K_{2,3}$ or $L_{3,5}$.*

Proof. Suppose that $z^1 \in V(\hat{C}_0[t, v])$. Then $\hat{C}_i[w^1, z^1] \cup R^1$ is an M_i-alternating circuit for $i = 0, 1$, and hence these two circuits are oddly oriented by the assumption of Case (2). This implies that D is evenly oriented by Claim 1, which is a contradiction. Thus $z^1 \notin V(\hat{C}_0[t, v])$ holds.

We first consider the case of $q \geq 2$. By $z^1 \notin V(\hat{C}_0[t,v])$, we have $z^1 \in V(D)$. Since R^1 is also M_1-alternating, we may assume that $z^1 \in V(D_0)$. Then $C' = \hat{C}_0[w^1, z^1] \cup R^1$ is an M_0-alternating circuit not having the edge e. Hence C' is oddly oriented by the assumption of Case (2). Since D is oddly oriented, $D \triangle C'$ is evenly oriented by Claim 1. If there exists an M_0-alternating path P' from a vertex $w' \in U(D_0[s, z^1])$ to a vertex $z' \notin V(M_0)$ with $W(P' \cap (D \cup C')) = \{w'\}$, then the circuit $D \triangle C'$ is left-central by taking $M_0 \triangle E(P' \cup D_0[w', t] \cup P[t, y^0])$, which contradicts the assumption of Case (2). Thus there are no such M_0-alternating paths from $U(D_0[s, z^1])$ to $V \setminus V(M)$. This implies that $x^r \in U(D_0[z^1, t])$ and $q \neq 2$.

Assume that $z^2 \in V(\hat{C}_0[t,v])$. Then the subgraph with edge set $E(C_0 \cup R^1 \cup R^2) \setminus E(\hat{C}_0[z^1, z^2])$ forms an even subdivision of $K_{2,3}$. By $x^r \in U(D_0[z^1, t])$, this subdivision is left-central by taking $M_0 \triangle E(D_0[z^2, x^r] \cup \{e_1^r\})$. Hence we may assume that R^2 is an M_0-alternating ear of D_0. Note that $R^j \cup D_0[w^j, z^j]$ is an M_0-alternating circuit for $1 < j < q$. Hence we have $z^{j+1} \in V(D_0[z^j, t])$ if $j < q-1$ and $w^{j+1} \in V(D_0[z^{j-1}, z^j])$ if $1 < j < q$. Letting $C'' = R^2 \cup D_0[w^2, z^2]$, we define $M_0' = M_0 \triangle E(C'')$, $C_0' = C_0 \triangle C''$, and R' to be the subgraph with $E(R) \triangle E(C'')$. Then R' is an M_0-alternating path from u' to v' and C_0' is an evenly oriented circuit. They satisfy $|E(R' \cup C_0' \cup C_1)| < |E(R \cup C_0 \cup C_1)|$, since the terminal edges of $D_0[w^2, z^2]$ are not in $R' \cup C_0' \cup C_1$. This contradicts the choice of C_0, C_1, and R. Thus, if $q \geq 2$, then G contains $K_{2,3}$.

It remains to discuss the case of $q = 1$. If $z^1 = y^r$, then $H = C_0 \cup \{e_1^r\} \cup R^1$ is an even subdivision of $K_{2,3}$. Otherwise, $H = L \cup R$ is an even subdivision of $L_{3,5}$. In both cases, H is left-central because $M_0 \setminus E(H)$ is a left-perfect matching of $G \setminus H$. $\square$

Acknowledgements

The author is very obliged to Satoru Iwata for helpful suggestions and valuable comments on the manuscript. This work is supported in part by Global COE Program "The research and training center for new development in mathematics" from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

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