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UNCERTAINTY RELATIONS IN SIMULTANEOUS MEASUREMENTS FOR ARBITRARY OBSERVABLES

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In this paper we shall give mathematical foundations of the simultaneous measurements in quantum mechanics. Next we shall show the existence theorem of the simultaneous measurement for arbitrary observables. Furthermore, we deduce Heisenberg's uncertainty relation and approximate simultaneous uncertainty relation for a pair of arbitrary observables.

1. Introduction

Although the uncertainty relation (discovered by Heisenberg in 1927) has a long history, the various discussions about its interpretations are continued even now. Mainly, there are two interpretations of uncertainty relations. One is the statistical interpretation. By repeating the exact (i.e. the "error" $\Delta(q) = 0$) measurements of the position q of particles with same states, we can obtain its average value $\bar{q}$ and its variance $\mathrm{var}(q)$. Also, by repeating the exact (i.e. the "error" $\Delta(p) = 0$) measurements of the momentum p of the same particles, we can similarly get its average value $\bar{p}$ and its variance $\mathrm{var}(p)$. From the simple mathematical deduction, we can easily obtain the following uncertainty relation:

$$[\mathrm{var}(q)]^{1/2} \cdot [\mathrm{var}(p)]^{1/2} \geqslant \frac{\hbar}{2}, \tag{1}$$

where $\hbar =$ "Plank's constant"$/2\pi$. This is the statistical aspect of the uncertainty relation.

On the other hand, Heisenberg's uncertainty relation is rather individualistic. Most physicists will agree that the content of Heisenberg's uncertainty relation is roughly as stated in the following proposition (though it includes some ambiguous sentences as well as some ambiguous words, i.e. "simultaneous" and "error").

PROPOSITION 1 (Heisenberg's uncertainty relation). (i) *The particle position q and momentum p can be measured "simultaneously", if the "errors" $\Delta(q)$ and $\Delta(p)$ in determining the particle position and momentum are permitted to be non-zero.*

(ii) *Moreover, for any* $\varepsilon > 0$, *we can take the "simultaneous" measurement of the position* q *and momentum* p *such that* $\Delta(q) < \varepsilon$ (*or* $\Delta(p) < \varepsilon$).

(iii) *However, the following Heisenberg's uncertainty relation holds*:

$$\Delta(q)\cdot\Delta(p) \geqslant \frac{\hbar}{2}, \tag{2}$$

for all "simultaneous" measurements of the particle position and momentum.

Several authors have contributed to the problem of deducing Heisenberg's uncertainty relation. In [2] (Ali and Emach, 1974), [3] (Ali and Prugovečki, 1977) and [6] (Busch, 1984), these were done by means of the concept of modified observable which has been developed by Davies and Lewis (1970) [9]. Hence, a certain part of this problem has been already solved. In particular, the statements (i) and (ii) in the above Proposition 1 were deduced in a satisfactory way. However, with regard to the statement (iii), it seems there are still some questions. In order to deduce the statement (iii) it is necessary to clarify the class of all "simultaneous" measurements. However, this argument seems not to be sufficient. In this paper we shall make a proposal for the mathematical foundations for the "simultaneous" measurement and the "error" mentioned in Proposition 1. And we shall show the existence theorem of the simultaneous measurement for arbitrary observables $A_1, \ldots, A_n$, which corresponds to the statements (i) and (ii) in Proposition 1 in the case when (A_1, A_2) is a pair of conjugate observables (i.e. symbolically, $A_1A_2 - A_2A_1 = i\hbar$). Furthermore, we shall derive the Heisenberg's uncertainty relation (2) and the so-called approximate simultaneous uncertainty relation (which has been discovered in [5] and discussed in [21], [22]) for a pair of arbitrary observables as well as that of conjugate observables.

2. Definitions

Since the main purpose of this paper is to prove Proposition 1 (or the generalized theorem), we must clarify the ambiguous words in Proposition 1. For this, we prepare several definitions in this section.

DEFINITION 1 (Davies and Lewis [9]). Let Ω be a set with a σ-field $\mathscr{F}$ and let H be a Hilbert space. A positive operator valued measure E on Ω in H is defined to be a map $E: \mathscr{F} \to B(H) = \{L : L \text{ is a bounded linear operator on } H\}$ such that

(i) $0 = E(\phi) \leqslant E(G) \leqslant E(\Omega) = I$ for all $G \in \mathscr{F}$, where 0 and I are respectively a 0-operator and an identity operator on H,

(ii) for any countable decomposition $\{G_j\}_{j=1}^{\infty}$ of G, $(G_j,\ G \in \mathscr{F})$, $E(G) = \Sigma_{j=1}^{\infty} E(G_j)$ holds, where the series is weakly convergent.

Furthermore, a positive operator valued measure E on Ω is called a *projection valued measure*, if it satisfies the following additional requirement:

(iii)

$$E(G_1)E(G_2) = 0 \ (G_1 \cap G_2 = \phi).$$

A positive operator valued measure and a projection valued measure are also called an *observable* and a *standard observable*, respectively.

In this paper, a positive operator valued measure (or projection valued measure) E on an n-dimensional Euclidean space $\mathbf{R}^n$ with a Borel field $\mathscr{B}_n = \{G : G \text{ is a Borel set in } \mathbf{R}^n\}$ is usually called an observable (or standard observable) on $\mathbf{R}^n$.

According to the well-known spectral representation theorem, there is a bijective correspondence of a standard observable E on $\mathbf{R}^n$ in H to an n-tuple $(E_1, \ldots, E_n)$ of commutative self-adjoint operators in H such that $E_i = \int_{\mathbf{R}^n} \lambda_i E(d\lambda_1 \ldots d\lambda_n)$. So we sometimes identify E with $(E_1, \ldots, E_n)$ and we can write

$$E = (E_1, \ldots, E_n) = \Big(\int_{\mathbf{R}^n} \lambda_i E(d\lambda)\Big)_{i=1}^n = \int_{\mathbf{R}^n} \lambda E(d\lambda). \tag{3}$$

In particular, we frequently identify a standard observable on $\mathbf{R}$ in H with a self-adjoint operator in H.

DEFINITION 2. Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$.

(1) A quartet $\mathbf{M} = (K, s, (\Omega, \mathscr{F}, \tilde{A}), f = (f_1, \ldots, f_n))$ is called a *measurement in H*, if it satisfies the following conditions (i), (ii) and (iii):

(i) K is a Hilbert space and s is an element in K such that $\|s\| = 1$,

(ii) $\tilde{A}$ is a projection valued measure on Ω with a σ-field $\mathscr{F}$ in the tensor Hilbert space $H \otimes K$ with the inner product $\langle \cdot, \cdot \rangle_{H\otimes K}$ and

(iii) f: $\Omega \to \mathbf{R}^n$ is a measurable map from Ω into $\mathbf{R}^n$, i.e. f_i: $\Omega \to \mathbf{R}$, $(i = 1, \ldots, n)$, is a measurable function on Ω.

In particular, when $K = \mathbf{C}$ (so, $H \otimes \mathbf{C} = H$, where $\mathbf{C}$ is the complex field), $\mathbf{M}$ is called a *simple measurement.*

(2) A measurement $\mathbf{M} = (K, s, (\Omega, \mathscr{F}, \tilde{A}), f = (f_1, \ldots, f_n))$ is called the *measurement of an observable $\bar{A}$* on $\mathbf{R}^n$, if

$$\langle u \otimes s, \tilde{A}(f^{-1}(G))(u \otimes s)\rangle_{H\otimes K} = \langle u, \bar{A}(G)u\rangle_H \quad (u \in H, \ G \in \mathscr{B}_n). \tag{4}$$

The measurement $\mathbf{M}$ of $\bar{A}$ is also called the *realization of the measurement of $\bar{A}$* (cf. [13]).

The relation between measurements and observables is characterized by the following proposition.

PROPOSITION 2. (i) *For any measurement* $\mathbf{M} = (K, s, (\Omega, \mathscr{F}, \tilde{A}), f = (f_1, \ldots, f_n))$, *there exists a unique observable $\bar{A}$ on $\mathbf{R}^n$ such that $\mathbf{M}$ is the measurement of $\bar{A}$. Also, $\bar{A}$ is determined by* (4).

(ii) *For any observable $\bar{A}$ on $\mathbf{R}^n$ in a Hilbert space H, there exists a measurement* $\mathbf{M} = (K, s, (\Omega, \mathscr{F}, \tilde{A}), f = (f_1, \ldots, f_n))$ *of* $\bar{A}$.

Proof: The statement (i) is trivial. Also the statement (ii) immediately follows from the following proposition.

PROPOSITION 3 (Holevo [12]). *Let $\bar{E}$ be a positive operator valued measure on Ω with a σ-field $\mathscr{F}$ in a Hilbert space H. Then, there exist a Hilbert space K, an element s ($\|s\|_K = 1$) in K and a projection valued measure $\tilde{E}$ on Ω in the tensor Hilbert space $H \otimes K$ satisfying*

$$\langle u \otimes s, \tilde{E}(G)(u \otimes s) \rangle_{H \otimes K} = \langle u, \bar{E}(G) u \rangle_H \quad (u \in H, G \in \mathscr{F}). \tag{5}$$

Conversely, any projection valued measure $\tilde{E}$ on Ω in $H \otimes K$ and $s \in K$ give rise to the unique positive operator valued measure $\bar{E}$ on Ω in H satisfying (5).

Now we postulate the following probabilistic interpretation of quantum mechanics: when we take the measurement $\mathbf{M} = (K, s, (\Omega, \mathscr{F}, \tilde{A}), f = (f_1, \ldots, f_n))$ of an observable $\bar{A}$ on $\mathbf{R}^n$ (also, we say briefly, "measurement $\mathbf{M}$" or "measurement of an observable $\bar{A}$") for a system with a state u ($u \in H$, $\|u\|_H = 1$), the probability that the value λ ($\in \mathbf{R}^n$) obtained in the measurement $\mathbf{M}$ belongs to a set G ($\in \mathscr{B}_n$) is given by

$$\langle u, \bar{A}(G) u \rangle_H \ \left(= \langle u \otimes s, \tilde{A}(f^{-1}(G))(u \otimes s) \rangle_{H \otimes K}\right).$$

Therefore, the expectation $\mathrm{Exp}[\mathbf{M}, u]$ $\left(= (\mathrm{Exp}[\mathbf{M}, u]_i)_{i=1}^n\right)$ of the measurement $\mathbf{M}$ of the observable $\bar{A}$ on $\mathbf{R}^n$ for the state u ($\|u\|_H = 1$) is given by

$$\mathrm{Exp}[\mathbf{M}, u]_i = \int_{\mathbf{R}^n} \lambda_i \langle u, \bar{A}(d\lambda) u \rangle_H$$

$$\left(= \int_{\Omega} f_i(\omega) \langle u \otimes s, \tilde{A}(d\omega)(u \otimes s) \rangle_{H \otimes K}\right), \quad i = 1, 2, \ldots, n \tag{6}$$

and its variance $\mathrm{var}[\mathbf{M}, u]$ $\left(= (\mathrm{var}[\mathbf{M}, u]_i)_{i=1}^n\right)$ is given by

$$\mathrm{var}[\mathbf{M}, u]_i = \int_{\mathbf{R}^n} |\lambda_i - \mathrm{Exp}[\mathbf{M}, u]_i|^2 \langle u, \bar{A}(d\lambda) u \rangle_H$$

$$\left(= \int_{\Omega} |f_i(\omega) - \mathrm{Exp}[\mathbf{M}, u]_i|^2 \langle u \otimes s, \tilde{A}(d\omega)(u \otimes s) \rangle_{H \otimes K}\right), \quad i = 1, 2, \ldots, n, \tag{7}$$

where $\mathrm{var}[\mathbf{M}, u]_i$ is defined by ∞ if $\int_{\mathbf{R}^n} |\lambda_i|^2 \langle u, \bar{A}(d\lambda) u \rangle_H = \infty$.

We shall use the following notation:

NOTATION 1. (1) Let $\mathbf{M} = (K, s, (\Omega, \mathscr{F}, \tilde{A}), f = (f_1, \ldots, f_n))$ be a measurement in H. Then

$$\hat{A}_i := \int_{\Omega} f_i(\omega) \tilde{A}(d\omega), \qquad \hat{A} := (\hat{A}_1, \ldots, \hat{A}_n) = \left(\int_{\mathbf{R}^n} \lambda_i \hat{A}(d\lambda)\right)_{i=1}^n.$$

(2) Let L and $\hat{L}$ be observables on $\mathbf{R}$ in H and $H \otimes K$, respectively. Let $s \in H$. Then

$$D(L) := \{u \in H: \int_{\mathbf{R}} |\lambda|^2 \langle u, L(\lambda) u\rangle < \infty\},$$

$$D(\hat{L}) := \{\omega \in H \otimes K: \int_{\mathbf{R}} |\lambda|^2 \langle \omega, \hat{L}(d\lambda)\omega\rangle_{H\otimes K} < \infty\},$$

$$D_s(\hat{L}) := \{u \in H: \int_{\mathbf{R}} |\lambda|^2 \langle u \otimes s, \hat{L}(d\lambda)(u \otimes s)\rangle_{H\otimes K} < \infty\},$$

where $D(L)$ (or $D(\hat{L})$) is called the *domain of* L (or $\hat{L}$).

(3) Let $\tilde{A}$ be an observable on $\mathbf{R}^n$ in a Hilbert space H. Then

$$\tilde{A}^{\mathrm{mar}(k)}(G) := \tilde{A}(\{x = (x_1, \ldots, x_k, \ldots, x_n) \in \mathbf{R}^n: x_k \in G\})$$
$$(G \in \mathscr{B}_1, k = 1, 2, \ldots, n).$$

$\tilde{A}^{\mathrm{mar}(k)}$ is called the *(k-th) marginal observable of* $\tilde{A}$.

Note that $\hat{A}(G) = \tilde{A}(f^{-1}(G))$ $(G \in \mathscr{B}_n)$ and $D_s(\hat{A}_i) = D(\tilde{A}^{\mathrm{mar}(i)})$ $(i = 1, \ldots, n)$ hold.

Presently, we shall give the definition of "simultaneous" measurements (Definitions 3 and 4).

DEFINITION 3. Let $A_1, \ldots, A_n$ be standard observables on $\mathbf{R}$ in a Hilbert space H. Then an observable $\tilde{A}$ on $\mathbf{R}^n$ such that $A_i = \tilde{A}^{\mathrm{mar}(i)}$ $(i = 1, \ldots, n)$ is called the *observable representing* $A_1, \ldots, A_n$ in the exact sense. Also the measurement $\mathbf{M} = (K, s, (\Omega, \mathscr{F}, \tilde{A}), f(\omega) = (f_1(\omega), f_2(\omega), \ldots, f_n(\omega))$ of the observable $\tilde{A}$ representing $A_1, \ldots, A_n$ in the exact sense is called the *exact simultaneous measurement of* $A_1, \ldots, A_n$, that is, $\mathbf{M}$ satisfies that $\langle u \otimes s, \tilde{A}(f_i^{-1}(G))(u \otimes s)\rangle_{H\otimes K} = \langle u, A_i(G) u\rangle_H$ $(u \in H, G \in \mathscr{B}_1, i = 1, \ldots, n)$.

We see by the following Proposition 4 that $A_1, \ldots, A_n$ commute if and only if there exists an exact simultaneous measurement of $A_1, \ldots, A_n$.

PROPOSITION 4. *Let $A_1, \ldots, A_n$ be standard observables on* $\mathbf{R}$ *in a Hilbert space H. Then, the following statements* (i) *and* (ii) *hold*:

(i) *If $A_1, \ldots, A_n$ commute, i.e.*

$$A_i(G_1) A_j(G_2) = A_j(G_2) A_i(G_1) \quad (i \neq j, G_1, G_2 \in \mathscr{B}_1),$$

then the standard observable $\tilde{A} = (A_1, \ldots, A_n)$ on $\mathbf{R}^n$ *defined by* (3) *satisfies $A_i = \tilde{A}^{\mathrm{mar}(i)}$* $(i = 1, \ldots, n)$.

(ii) *If there exists an observable $\tilde{A}$ on* $\mathbf{R}^n$ *such that $A_i = \tilde{A}^{\mathrm{mar}(i)}$ $(i = 1, \ldots, n)$, then $A_1, \ldots, A_n$ commute (so we can put $A = (A_1, \ldots, A_n)$ as* (3)*) and $\tilde{A} = A$ $(= (A_1, \ldots, A_n))$.*

Proof: The statement (i) is trivial. Also about (ii), see [8].

Now we have the following main definition.

DEFINITION 4. Let $A_1, \ldots, A_n$ be standard observables on $\mathbf{R}$ in a Hilbert space H.

(1) An observable $\bar{A}$ on $\mathbf{R}^n$ in H is called the *observable representing* $A_1, \ldots, A_n$ in the average sense, if it satisfies the following conditions:

(i) (domain condition) for each i $D(\bar{A}^{\mathrm{mar}(i)})$, the domain of $\bar{A}^{\mathrm{mar}(i)}$, is the core of the self-adjoint operator A_i $(= \int_{\mathbf{R}} \lambda A_i(d\lambda))$ $(i = 1, 2, \ldots, n)$, i.e. A_i is an essentially self-adjoint operator on $D(\bar{A}^{\mathrm{mar}(i)})$ $(\subseteq D(A_i))$,

(ii) (average value condition)

$$\langle u, A_i u\rangle = \int_{\mathbf{R}} \lambda \langle u, \bar{A}^{\mathrm{mar}(i)}(d\lambda)\, u\rangle \qquad \big(u \in D(\bar{A}^{\mathrm{mar}(i)}),\ i = 1, 2, \ldots, n\big). \tag{8}$$

(2) A measurement $\mathbf{M} = (K, s, (\Omega, \mathscr{F}, \tilde{A}), f(\omega) = (f_1(\omega), f_2(\omega), \ldots, f_n(\omega))$ of the observable representing $A_1, \ldots, A_n$ in the average sense is called the approximate simultaneous measurement of $A_1, \ldots, A_n$. Namely, $\mathbf{M}$ *satisfies the following conditions*:

(i) (domain condition) for each i, the set $D_s(\hat{A}_i)$ is the core of the self-adjoint operator A_i $(= \int_{\mathbf{R}} \lambda A_i(d\lambda))$, i.e. A_i is an essentially self-adjoint operator on $D_s(\hat{A}_i)$ $(\subseteq D(A_i))$,

(ii) (average value condition)

$$\langle u, A_i u\rangle = \int_{\Omega} f_i(\omega) \langle u \otimes s, \tilde{A}(d\omega)(u \otimes s)\rangle \qquad \big(u \in D_s(\hat{A}_i),\ i = 1, 2, \ldots, n\big). \tag{9}$$

Note that we have seen that $D(\bar{A}^{\mathrm{mar}(i)}) = D_s(\hat{A}_i)$. Also assume that $\mathbf{M}$ is the approximate simultaneous measurement of $A'_1, \ldots, A'_n$ as well as $A_1, \ldots, A_n$. Then we see, by (9), that $\langle u, A'_i u\rangle = \int_{\Omega} f_i(\omega) \langle u \otimes s, \tilde{A}(d\omega)(u \otimes s)\rangle = \langle u, A_i u\rangle$ $\big(u \in D_s(\hat{A}_i),$ $i = 1, 2, \ldots, n\big)$. So we see that $A'_i = A_i$ on $D_s(\hat{A}_i)$ $(i = 1, 2, \ldots, n)$. Therefore, by the domain condition we get $A'_i = A_i$ $(i = 1, 2, \ldots, n)$.

Now we shall define the notion of the "error" in an approximate simultaneous measurement. If we regard the approximate simultaneous measurement $\mathbf{M}$ of $A_1, \ldots, A_n$ as the substitute of the exact simultaneous measurement of $A_1, \ldots, A_n$, how can we represent the "fitness" (or "goodness") of this substitute? Though the approximate simultaneous measurement $\mathbf{M}$ is fit for $A_1, \ldots, A_n$ in the average sense (i.e. (i) and (ii) in Definition 4, (2) hold), it is not always fit for $A_1, \ldots, A_n$ in "every" sense (i.e. $\mathbf{M}$ is not always an exact simultaneous measurement of $A_1, \ldots, A_n$). However, we can say that it is fit for $A_1, \ldots, A_n$ in "every" sense if $\|(\hat{A}_i - A_i \otimes I)(u \otimes s)\| = 0$ $\big(u \in D(A_i),\ i = 1, 2, \ldots, n\big)$. Because this implies that for any $u \in \bigcap_{k=1}^{\infty} D(A_i^k)$ and any positive integer k

$$(\hat{A}_i)^k (u \otimes s) = (\hat{A}_i)^{k-1} \hat{A}_i (u \otimes s) = (\hat{A}_i)^{k-1} (A_i u \otimes s) = \ldots = (A_i^k u) \otimes s,$$

so we see that

$$\int_{\mathbf{R}} \lambda^k \langle u\otimes s,\ \hat{A}_i(d\lambda)(u\otimes s)\rangle = \int_{\mathbf{R}} \lambda^k \langle u,\ A_i(d\lambda)u\rangle,$$

hence

$$\langle u\otimes s,\ \hat{A}_i(G)(u\otimes s)\rangle = \langle u,\ A_i(G)u\rangle \quad (u\in H,\ G\in\mathscr{B}_1,\ i=1, 2, \ldots, n),$$

which implies that the measurement $\mathbf{M}$ is an exact simultaneous measurement of $A_1, \ldots, A_n$. Also if $\mathbf{M}$ is the exact simultaneous measurement of $A_1, \ldots, A_n$, clearly $\|(\hat{A}_i - A_i\otimes I)(u\otimes s)\| = 0$ holds for all $u\in D(A_i)$ and $i = 1, 2, \ldots, n$. Therefore, $(\|(\hat{A}_i - A_i\otimes I)(u\otimes s)\|)_{i=1}^n$ seems to represent the "unfitness" of $\mathbf{M}$ for $A_1, \ldots, A_n$ on a state u.

Also for a particular approximate simultaneous measurement $\mathbf{M} = (K, s, (\Omega, \mathscr{F}, \tilde{A}), f = (f_1, \ldots, f_n))$ of $A_1, \ldots, A_n$ satisfying that there exist functions $g_1, \ldots, g_n$ on Ω such that $A_i\otimes I = \int_{\Omega} g_i(\omega)\tilde{A}(d\omega)$, $(i = 1, \ldots, n)$, we can explain that the "unfitness" $(\|(\hat{A}_i - A_i\otimes I)(u\otimes s)\|)_{i=1}^n$ has the properties of the error in the measurement $\mathbf{M}$. Assume that someone is under the impression that he takes the exact simultaneous measurement of $A_1, \ldots, A_n$ for u (it should be noted that this measurement is identified with the exact simultaneous measurement of $A_1\otimes I, \ldots, A_n\otimes I$ for $u\otimes s$), although he actually takes an approximate simultaneous measurement $\mathbf{M}$ for u (i.e. an exact simultaneous measurement of $\hat{A}_1, \ldots, \hat{A}_n$ for $u\otimes s$). In this situation, he will think that the distance (or average distance in some sense) between the value a $(= (a_1, \ldots, a_n))$ obtained by the measurement $\mathbf{M}$ and the "true" value $\hat{a}$ $(= (\hat{a}_1, \ldots, \hat{a}_n))$ (i.e. the value obtained by the exact simultaneous measurement of $A_1\otimes I, \ldots, A_n\otimes I$ for $u\otimes s$) is the error $(\Delta_i)_{i=1}^n$ in the measurement $\mathbf{M}$, that is, $\Delta_i = \{\mathrm{Exp}[|\hat{a}_i - a_i|^2]\}^{1/2}$. Taking a measurement $\mathbf{M}' = (K, s, (\Omega, \mathscr{F}, \tilde{A}), h = (f_1, \ldots, f_n, g_1, \ldots, g_n))$, we can easily see that $\Delta_i = [\int_{\Omega} |f_i(\omega) - g_i(\omega)|^2 \langle u\otimes s,\ \tilde{A}(d\omega)(u\otimes s)\rangle]^{1/2} = \|(\hat{A}_i - A_i\otimes I)(u\otimes s)\|$. Therefore, in this particular case we can regard the "unfitness" $(\|(\hat{A}_i - A_i\otimes I)(u\otimes s)\|)_{i=1}^n$ as the error $(\Delta_i(u))_{i=1}^n$ in the measurement $\mathbf{M}$. However, in general, we can not define the error in the measurement $\mathbf{M}$ since we have no method to know the "true" value.

The above arguments lead us to the following definition.

DEFINITION 5. Let $A_1, \ldots, A_n$ be standard observables on $\mathbf{R}$ in a Hilbert space H. And let $\mathbf{M} = (K, s, (\Omega, \mathscr{F}, \tilde{A}), f(\omega) = (f_1(\omega), f_2(\omega), \ldots, f_n(\omega)))$ be an approximate simultaneous measurement of $A_1, \ldots, A_n$. Then, the unfitness $(\Delta_{\mathbf{M}}(A_i, u))_{i=1}^n$ of $\mathbf{M}$ for $A_1, \ldots, A_n$ on a state u $(\|u\|_H = 1)$ is defined by

$$\Delta_{\mathbf{M}}(A_i, u) = \|(\hat{A}_i - A_i\otimes I)(u\otimes s)\| \quad (u\in D(A_i)), \tag{10}$$

where (10) should be interpreted as $\Delta_{\mathbf{M}}(A_i, u) = \infty$ for $u\in D(A_i)\backslash D_s(\hat{A}_i)$ since $D_s(\hat{A}_i) \subseteq D(A_i)$.

Remark 1: (1) In Lemma 2 (iii) (cf. Section 4), we will see that $\|(\hat{A}_i - A_i \otimes I)(u \otimes s)\|^2 = \|\hat{A}_i(u \otimes s)\|^2 - \|A_i u\|^2 = \int_{\mathbf{R}} \lambda^2 \langle u, \bar{A}^{\mathrm{mar}(i)}(d\lambda) u\rangle - \int_{\mathbf{R}} \lambda^2 \langle u, A_i(d\lambda) u\rangle$ for all $u \in D_s(\hat{A}_i)$ $(= D(\bar{A}^{\mathrm{mar}(i)}))$ $(i = 1, 2)$.

(2) Since $\Delta_{\mathbf{M}}(A_i, u)$ is not defined for $u \in H \backslash D(A_i)$, we think that Definition 5 is not final but temporary. When $(\hat{A}_i - A_i \otimes I)$ on $D(\hat{A}_i) \cap D(A_i \otimes I)$ has a unique self-adjoint extension $[\hat{A}_i - A_i \otimes I]$ (for example, when $\hat{A}_i$ and $A_i \otimes I$ commute), it seems to be natural to define the unfitness $\Delta_{\mathbf{M}}(A_i, u)$ such that $\Delta_{\mathbf{M}}(A_i, u) = \|[\hat{A}_i - A_i \otimes I](u \otimes s)\|$ $(u \in D_s([\hat{A}_i - A_i \otimes I]))$, $= \infty$ (otherwise). Furthermore, we can define the "unfitness observable" $F_i(d\lambda)$ on $\mathbf{R}$ in H such that $\langle u, F_i(d\lambda) u\rangle_H = \langle u \otimes s, [\hat{A}_i - A_i \otimes I](d\lambda)(u \otimes s)\rangle_{H \otimes K}$ $(u \in H)$. So it seems to be reasonable to assume the essential self-adjointness of $(\hat{A}_i - A_i \otimes I)$ on $D(\hat{A}_i) \cap D(A_i \otimes I)$ in Definition 4 for approximate simultaneous measurements. However, without this assumption we can prove Theorem 3 (generalized approximate simultaneous uncertainty relation), which is one of our main results. So we proceed with our arguments without this assumption. We shall return to this problem again in Corollary 1.

Let $A_1, \ldots, A_n$ be standard observables on $\mathbf{R}$ in a Hilbert space H and let $g_1, \ldots, g_n$ be functions in $L^1(\mathbf{R})$ such that $g_i(x) \geqslant 0$, $\|g_i\|_{L^1} = 1$, $\int_{\mathbf{R}} x g_i(x)\, dx = 0$ and $\int_{\mathbf{R}} x^2 g_i(x)\, dx < \infty$ $(i = 1, \ldots, n)$. Now we shall consider the observable $\bar{A}$ on $\mathbf{R}^n$ such that $\bar{A}^{\mathrm{mar}(i)} = g_i * A_i$, where $(g_i * A_i)(d\lambda) := \int_{\mathbf{R}} g_i(x) A_i(d\lambda - x)\, dx$. We can easily see that

$$\int_{\mathbf{R}} \lambda^2 \langle u, (g_i * A_i)(d\lambda) u\rangle = \int_{\mathbf{R}^2} (\lambda + x)^2 \langle u, g_i(x) A_i(d\lambda) u\rangle\, dx$$

$$= \int_{\mathbf{R}} x^2 g_i(x)\, dx + \int_{\mathbf{R}} \lambda^2 \langle u, A_i(d\lambda) u\rangle.$$

So $D(A_i) = D(\bar{A}^{\mathrm{mar}(i)})$. Also, we see similarly that $\langle u, A_i u\rangle = \int_{\mathbf{R}} \lambda \langle u, (g_i * A_i)(d\lambda) u\rangle$ $(u \in D(\bar{A}^{\mathrm{mar}(i)}), i = 1, 2, \ldots, n)$. Therefore, $\bar{A}$ is the observable representing $A_1, \ldots, A_n$ in the average sense. Moreover, we see by Remark 2,(1) that $|\Delta_{\mathbf{M}}(A_i, u)|^2 = \int_{\mathbf{R}} x^2 g_i(x)\, dx$ $(u \in D(A_i), i = 1, \ldots, n)$. This particular kind of unfitness is also called "unsharpness", "indeterminacy" or "fuzziness" (see [2], [3], [6]).

The following result is obtained in [3].

PROPOSITION 5. *Let A_1 and A_2 be a pair of conjugate observables in a Hilbert space H. Let ε_1 and ε_2 be any positive numbers. Then, the following statements* (i) *and* (ii) *are equivalent*:

(i)

$$\varepsilon_1 \cdot \varepsilon_2 \geqslant \hbar/2,$$

(ii) *there exist* g_1 *and* g_2 *in* $L^1(\mathbf{R})$ *that satisfy the following conditions*:
(a) $g_i(x) \geqslant 0$, $\|g_i\|_{L^1} = 1$, $\int_{\mathbf{R}} x g_i(x)\,dx = 0$ *and* $\int_{\mathbf{R}} x^2 g_i(x)\,dx = \varepsilon_i$, $(i = 1, 2)$,
(b) *there exists an observable* $\bar{A}$ *on* $\mathbf{R}^2$ *such that* $\bar{A}^{\mathrm{mar}(i)} = g_i * A_i$ $(i = 1, 2)$.

The measurement $\mathbf{M} = (L^2(\mathbf{R}), s(x), (\mathbf{R}^2, \mathscr{B}_2, \tilde{A}), f(\lambda_1, \lambda_2) = (\lambda_1, \lambda_2))$ of the $\bar{A}$ above can be easily constructed as follows: put $\hat{A}_1 = A_1 \otimes I + I \otimes x$ and $\hat{A}_2 = A_2 \otimes I + I \otimes \dfrac{\hbar d}{i dx}$ in $H \otimes L^2(\mathbf{R})$. From the commutativity of $\hat{A}_1$ and $\hat{A}_2$, we can define the unique standard observable $\tilde{A}$ on $\mathbf{R}^2$ in $H \otimes L^2(\mathbf{R})$ such that $\hat{A}_i = \int_{\mathbf{R}^2} \lambda_i \tilde{A}(d\lambda_1 d\lambda_2)$ $(i = 1, 2)$, i.e. $\tilde{A} = \hat{A}$. Also choose $s(x) \in L^2(\mathbf{R})$ such that $g_1(x) = |s(x)|^2$ and $g_2(x) = |\hat{s}(x)|^2$ a.e., where $\hat{s}(x)$ is a Fourier transform of $s(x)$. Then, by a simple calculation, we can see that $\mathbf{M}$ is the measurement of $\bar{A}$, that is, $\mathbf{M}$ is the approximate simultaneous measurement of A_1 and A_2. Note that $(\hat{A}_i - A_i \otimes I)$ on $D(\hat{A}_i) \cap D(A_i \otimes I)$ has the unique self-adjoint extension $[\hat{A}_i - A_i \otimes I]$ since $\hat{A}_i$ and $A_i \otimes I$ commute. Also note that $[\hat{A}_1 - A_1 \otimes I] = I \otimes x$ and $[\hat{A}_2 - A_2 \otimes I] = I \otimes \dfrac{\hbar d}{i dx}$, so $D_s([\hat{A}_i - A_i \otimes I]) = H$ $(i = 1, 2)$, $\|[\hat{A}_1 - A_1 \otimes I](u \otimes s)\| = \|u\|_H \cdot \|x s(x)\|_{L^2}$ and $\|[\hat{A}_2 - A_2 \otimes I](u \otimes s)\| = \|u\|_H \cdot \|\hbar x \hat{s}(x)\|_{L^2}$. Furthermore, it is clear that $D_s(\hat{A}_i) = D_s(A_i)$ $(i = 1, 2)$.

Compared with Proposition 1, the above Proposition 5 seems to give the satisfactory solution to the problem of deducing the statements (i) and (ii) in Proposition 1. However, the class of the approximate simultaneous measurements considered in Proposition 5 is smaller than the class of all "simultaneous" measurements. Hence, we think there are some questions concerning the statement (iii) in Proposition 1.

3. Existence theorem

Now we shall mention the following theorem, which assures the existence of approximate simultaneous measurements of arbitrary observables $A_1, \ldots, A_n$.

THEOREM 1. *Let* $A_1, \ldots, A_n$ *be standard observables on* $\mathbf{R}$ *in a Hilbert space* H. *Let* $a_1, \ldots, a_n$ *be any positive numbers such that* $\sum_{i=1}^{n} (1 + a_i^2)^{-1} = 1$. *Then there exists an approximate simultaneous measurement* $\mathbf{M}$ *of* $A_1, \ldots, A_n$ *such that*

$$\Delta_{\mathbf{M}}(A_i, u) = a_i \|A_i u\| \quad (u \in D(A_i),\ i = 1, 2, \ldots, n).$$

Proof: Let $\mathbf{C}^n = \{z = (z_1, \ldots, z_n) : z_i \in \mathbf{C}\ (i = 1, 2, \ldots, n)\}$ be the n-dimensional Hilbert space with the norm $\|z\|_n = [\sum_{i=1}^{n} |z_i|^2]^{1/2}$. Put $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, 0, \ldots, 1) \in \mathbf{C}^n$. And put $P_i : \mathbf{C}^n \to \mathbf{C}^n$, $(i = 1, 2, \ldots, n)$, the projection such that $P_i e_i = e_i$, $P_i e_k = 0$ $(k \neq i)$. Put $b_i = (1 + a_i^2)^{1/2}$ and $B_i = b_i^2 A_i$ $(i = 1, 2, \ldots, n)$.

We take the spectral representations $A_i = \int_{\mathbf{R}} \lambda A_i(d\lambda)$, $B_i = \int_{\mathbf{R}} \lambda B_i(d\lambda)$ in H and $0 = \int_{\mathbf{R}} \lambda 0(d\lambda)$ in H. Note that $A_i(d(\lambda/b_i^2)) = B_i(d\lambda)$. Put $\hat{U}$: $H \otimes \mathbf{C}^n \to H \otimes \mathbf{C}^n$ a unitary operator such that $\hat{U} = I \otimes U$, where the unitary operator U on $\mathbf{C}^n$ satisfies that $Ue_1 = \sum_{i=1}^{n} e_i/b_i$. And define a projection valued measure $\tilde{A}_i$ on $\mathbf{R}$ in $H \otimes \mathbf{C}^n$ by

$$\tilde{A}_i(d\xi) = \hat{U}^*[B_i(d\xi) \otimes P_i + 0(d\xi) \otimes (I - P_i)]\, \hat{U} \quad (i = 1, 2, \ldots, n).$$

Since $\tilde{A}_1, \ldots, \tilde{A}_n$ commute, we can define a projection valued measure $\tilde{A}$ on $\mathbf{R}^n$ in $H \otimes \mathbf{C}^n$ such that

$$\tilde{A}(d\xi_1 d\xi_2 \ldots d\xi_n) = \prod_{i=1}^{n} \tilde{A}_i(d\xi_i).$$

Now, we shall show that the measurement $\mathbf{M} = (\mathbf{C}^n, e_1, (\mathbf{R}^n, B_n, \tilde{A}), f(\xi_1, \xi_2, \ldots \ldots, \xi_n) = (\xi_1, \xi_2, \ldots, \xi_n))$ is an approximate simultaneous measurement of $A_1, \ldots, A_n$. Put $A_i = \int_{\mathbf{R}^n} \xi_i \tilde{A}(d\xi_1 d\xi_2 \ldots d\xi_n)$ $(i = 1, \ldots, n)$. Then we see that

$$\begin{aligned}
&\int_{\mathbf{R}^n} |\xi_i|^2 \langle u \otimes e_1, \tilde{A}(d\xi_1 d\xi_2 \ldots d\xi_n)(u \otimes e_1)\rangle \\
&= \int_{\mathbf{R}} |\xi_i|^2 \langle u \otimes e_1, \tilde{A}_i(d\xi_i)(u \otimes e_1)\rangle \\
&= \int_{\mathbf{R}} |\xi_i|^2 \langle u \otimes e_1, \left[\hat{U}^*\big(B_i(d\xi_i) \otimes P_i + 0(d\xi_i) \otimes (I - P_i)\big)\, \hat{U}\right](u \otimes e_1)\rangle \\
&= \int_{\mathbf{R}} |\xi|^2 \langle u, B_i(d\xi)u\rangle \cdot \langle Ue_1, P_i Ue_1\rangle \\
&= \int_{\mathbf{R}} |\xi|^2 \langle u, B_i(d\xi)u\rangle \cdot \left\langle \sum_{j=1}^{n} \frac{e_j}{b_j}, P_i \sum_{k=1}^{n} \frac{e_k}{b_k} \right\rangle \\
&= |b_i|^{-2} \int_{\mathbf{R}} |\lambda|^2 \langle u, B_i(d\lambda)u\rangle = |b_i|^2 \int_{\mathbf{R}} |\lambda|^2 \langle u, A_i(d\lambda)u\rangle.
\end{aligned}$$

Hence, $D_s(\hat{A}_i) = D(A_i)$ (where $s = e_1$), so $\mathbf{M}$ satisfies the condition (i) in Definition 4, (2). Also, we see that for each i $(i = 1, 2, \ldots, n)$ and $G_k \in \mathscr{B}_1$ $(k = 1, 2)$,

$$\begin{aligned}
&\tilde{A}_i(G_1) \cdot (A_i(G_2) \otimes I)) \\
&\quad = (I \otimes U^*)\big(B_i(G_1) \otimes P_i + 0(G_1) \otimes (I - P_i)\big)(I \otimes U)\big(A_i(G_2) \otimes I\big)
\end{aligned}$$

$$= (A_i(G_2)\otimes I)(I\otimes U^*)(B_i(G_1)\otimes P_i + 0(G_1)\otimes(I-P_i))(I\otimes U)$$
$$= (A_i(G_2)\otimes I)\cdot \tilde{A}_i(G_1).$$

So $\hat{A}_i$ and $A_i\otimes I$ commute since $\hat{A}_i = \int_{\mathbf{R}} \xi \tilde{A}_i(d\xi)$ and $A_i\otimes I = \int_{\mathbf{R}} \xi\, (A_i(d\xi)\otimes I)$. Hence, $\hat{A}_i - A_i\otimes I$ on $D(\hat{A}_i)\cap D(A_i\otimes I)$ has the unique self-adjoint extension $[\hat{A}_i - A_i\otimes I]$ which has the spectral representation

$$[\hat{A}_i - A_i\otimes I] = \int_{\mathbf{R}^2} (\xi_1-\xi_2)\tilde{A}_i(d\xi_1)(A_i(d\xi_2)\otimes I).$$

Then we see that

$$\|[\hat{A}_i - A_i\otimes I](u\otimes e_1)\|^2$$
$$= \int_{\mathbf{R}^2} |\xi_1-\xi_2|^2 \langle u\otimes e_1, \tilde{A}_i(d\xi_1)(A_i(d\xi_2)\otimes I)(u\otimes e_1)\rangle$$
$$= \int_{\mathbf{R}} |\xi_1|^2 \langle u\otimes e_1, \tilde{A}_i(d\xi_1)(u\otimes e_1)\rangle$$
$$-2\int_{\mathbf{R}^2} \xi_1\xi_2 \langle u\otimes e_1, \tilde{A}_i(d\xi_1)(A_i(d\xi_2)\otimes I)(u\otimes e_1)\rangle$$
$$+\int_{\mathbf{R}} |\xi_2|^2 \langle u\otimes e_1, (A_i(d\xi_2)\otimes I)(u\otimes e_1)\rangle$$
$$= (|b_i|^2 - 2 + 1)\int_{\mathbf{R}} |\xi|^2 \langle u, A_i(d\xi)u\rangle$$
$$= |a_i|^2\, \|A_i u\|^2,$$

which implies that $D_s([\hat{A}_i - A_i\otimes I]) = D(A_i)$ (where $s = e_1$) and $\Delta_{\mathbf{M}}(A_i, u) = a_i\|A_i u\|$. Therefore, the proof of theorem is complete.

Remark 2: (1). Even when A_1 and A_2 are a pair of conjugate observables, the above measurement $\mathbf{M}$ is clearly another one considered in Proposition 5. Moreover, since we can take a_i arbitrary positive for any fixed i, this theorem seems to give the solution to the problem of deducing the statements (i) and (ii) in Proposition 1 in the general case when $A_1, \ldots, A_n$ are arbitrary standard observables. When $a_i = (n-1)^{1/2}$ $(i = 1, 2, \ldots, n)$, this theorem was essentially proved in [1].

(2). In the proof above the following statements were also proved:

(i) $\hat{A}_i$ and $A_i\otimes I$ commute, so $\hat{A}_i - A_i\otimes I$ on $D(\hat{A}_i)\cap D(A_i\otimes I)$ has the unique self-adjoint extension $[\hat{A}_i - A_i\otimes I]$ $(i = 1, 2)$,

(ii) $D_s(\hat{A}_i) = D_s([\hat{A}_i - A_i\otimes I]) = D(A_i)$ $(i = 1, 2)$.

4. Uncertainty relations

In this section we shall discuss uncertainty relations in a quantum measurement. The following well-known Lemma gives a foundation for the statistical uncertainty relations (1).

LEMMA 1. *Let A_1 and A_2 be any symmetric operators on a Hilbert space H. Then*

$$[\|A_1u\|^2-|\langle u, A_1u\rangle|^2]\cdot[\|A_2u\|^2-|\langle u, A_2u\rangle|^2] \geqslant \tfrac{1}{4}|\langle A_1u, A_2u\rangle-\langle A_2u, A_1u\rangle|^2$$

for all $u\in D(A_1)\cap D(A_2)$.

Proof: See, for example, [19].

LEMMA 2. *Let A_1 and A_2 be any self-adjoint operators in a Hilbert space H. Let $(K, s, (\Omega, \mathscr{F}, \tilde{A}), f=(f_1, f_2)$ be the approximate simultaneous measurement for A_1 and A_2. Put $\hat{A}_i=\int_\Omega f_i(\omega)\tilde{A}(d\omega)$ $(i=1, 2)$. Then, the following equalities* (i)–(iii) *hold*

(i)

$$\langle v, A_iu\rangle=\langle v\otimes s, \hat{A}_i(u\otimes s)\rangle=\int_\Omega f_i(\omega)\langle v\otimes s, \tilde{A}(d\omega)(u\otimes s)\rangle$$

for all $u\in D_s(\hat{A}_i)$ and all $v\in H$ $(i=1, 2)$,

(ii)

$$\begin{aligned}&\int_\Omega f_1(\omega)f_2(\omega)\langle u\otimes s, \tilde{A}(d\omega)(u\otimes s)\rangle\\ &\quad=\langle\hat{A}_1(u\otimes s), \hat{A}_2(u\otimes s)\rangle\\ &\quad=\langle A_1u, A_2u\rangle+\langle(\hat{A}_1-A_1\otimes I)(u\otimes s), (\hat{A}_2-A_2\otimes I)(u\otimes s)\rangle\end{aligned}$$

for all $u\in D_s(\hat{A}_1)\cap D_s(\hat{A}_2)$,

(iii)

$$\begin{aligned}&\int_\Omega |f_i(\omega)|^2\langle u\otimes s, \tilde{A}(d\omega)(u\otimes s)\rangle\\ &\quad=\|\hat{A}_i(u\otimes s)\|^2=\|A_iu\|^2+\|(\hat{A}_i-A_i\otimes I)(u\otimes s)\|^2\end{aligned}$$

for all $u\in D_s(\hat{A}_i)$ $(i=1, 2)$.

Proof: Fix $k\in\{1, 2\}$. We can see that, for any $v, u\in D_s(\hat{A}_k)$,

$$\begin{aligned}&\langle v, A_ku\rangle\\ &\quad=\tfrac{1}{4}\{\langle(v+u), A_k(v+u)\rangle-\langle(v-u), A_k(v-u)\rangle\\ &\qquad-i\langle(v+iu), A_k(v+iu)\rangle+i\langle(v-iu), A_k(v-iu)\rangle\}\\ &\quad=\tfrac{1}{4}\{\langle(v+u)\otimes s, \hat{A}_k((v+u)\otimes s)\rangle-\langle(v-u)\otimes s, \hat{A}_k((v-u)\otimes s)\rangle\end{aligned}$$

$$-i\langle(v+iu)\otimes s,\ \hat{A}_k((v+iu)\otimes s)\rangle+i\langle(v-iu)\otimes s,\ \hat{A}_k((v-iu)\otimes s)\rangle\}$$
$$=\langle v\otimes s,\ \hat{A}_k(u\otimes s)\rangle$$
$$=\langle v\otimes s,\ \int_\Omega f_k(\omega)\tilde{A}_k(d\omega)(u\otimes s)\rangle=\int_\Omega f_k(\omega)\langle v\otimes s,\ \tilde{A}_k(d\omega)(u\otimes s)\rangle.$$

Since $D_s(\hat{A}_k)$ is dense in H, we see that

$$\langle v,\ A_k u\rangle=\langle v\otimes s,\ \hat{A}_k(u\otimes s)\rangle=\int_\Omega f_k(\omega)\langle v\otimes s,\ \hat{A}_k(d\omega)(u\otimes s)\rangle$$

for all $u\in D_s(\hat{A}_k)$ and all $v\in H$. This completes the proof of (i).

Next we shall prove (ii). Let u be any element in $D_s(\hat{A}_1)\cap D_s(\hat{A}_2)$. Then we see by (i) in this lemma that

$$\int_\Omega f_1(\omega)f_2(\omega)\langle u\otimes s,\ \tilde{A}(d\omega)(u\otimes s)\rangle$$
$$=\langle\int_\Omega f_1(\omega)\tilde{A}(d\omega)(u\otimes s),\ \int_\Omega f_2(\omega)\tilde{A}(d\omega)(u\otimes s)\rangle$$
$$=\langle\hat{A}_1(u\otimes s),\ \hat{A}_2(u\otimes s)\rangle$$
$$=\langle(\hat{A}_1-A_1\otimes I)(u\otimes s)+(A_1u\otimes s),\ (\hat{A}_2-A_2\otimes I)(u\otimes s)+(A_2u\otimes s)\rangle$$
$$=\langle(\hat{A}_1-A_1\otimes I)(u\otimes s),\ (\hat{A}_2-A_2\otimes I)(u\otimes s)\rangle$$
$$+\langle(\hat{A}_1-A_1\otimes I)(u\otimes s),\ A_2u\otimes s\rangle$$
$$\langle A_1u\otimes s,\ (\hat{A}_2-A_2\otimes I)(u\otimes s)\rangle+\langle A_1u\otimes s,\ A_2u\otimes s\rangle$$
$$=\langle(\hat{A}_1-A_1\otimes I)(u\otimes s),\ (\hat{A}_2-A_2\otimes I)(u\otimes s)\rangle$$
$$+\langle\hat{A}_1(u\otimes s),\ A_2u\otimes s\rangle-\langle A_1u,\ A_2u\rangle$$
$$+\langle A_1u\otimes s,\ \hat{A}_2(u\otimes s)\rangle-\langle A_1u,\ A_2u\rangle+\langle A_1u,\ A_u\rangle$$
$$=\langle(\hat{A}_1-A_1\otimes I)(u\otimes s),\ (\hat{A}_2-A_2\otimes I)(u\otimes s)\rangle-\langle A_1u,\ A_2u\rangle$$
$$+\int_\Omega f_2(\omega)\langle A_1u\otimes s,\ \tilde{A}(d\omega)(u\otimes s)\rangle+\int_\Omega f_1(\omega)\langle\tilde{A}(d\omega)(u\otimes s),\ A_2u\otimes s\rangle$$
$$=\langle A_1u,\ A_2u\rangle+\langle(\hat{A}_1-A_1\otimes I)(u\otimes s),\ (\hat{A}_2-A_2\otimes I)(u\otimes s)\rangle.$$

Hence, the proof of (ii) is completed. Also the proof of (iii) is carried out just in a similar way.

Now we have the following theorem, which is one of our main results.

THEOREM 2 (generalized Heisenberg's uncertainty relation). *Let A_1 and A_2 be any self-adjoint operators on a Hilbert space H. Then for any simultaneous measurement $\mathbf{M}=(K,\ s,\ (\Omega,\ \mathscr{F},\ \tilde{A}),f=(f_1,f_2))$ of A_1 and A_2 the following inequality holds:*

$$\Delta_{\mathbf{M}}(A_1, u)\cdot\Delta_{\mathbf{M}}(A_2, u) \geqslant \tfrac{1}{2}|\langle A_1 u, A_2 u\rangle - \langle A_2 u, A_1 u\rangle| \tag{11}$$

for all $u \in D(A_1)\cap D(A_2)$, *where the left-hand-side of* (11) *is defined to be* ∞ *if* $\Delta_{\mathbf{M}}(A_i, u) = \infty$ *for some* i.

Proof: Put $\hat{A}_i = \int_\Omega f_i(\omega)\tilde{A}(d\omega)$ $(i = 1, 2)$ as Notation 1,(1). Let $u \in D(A_1)\cap D(A_2)$. If $u \notin D_s(\hat{A}_i)$ for some i, we see by the definition of the unfitness that $\Delta_{\mathbf{M}}(A_i, u) = \infty$, so (11) clearly holds. Hence, it is sufficient to prove (11) for $u \in D_s(\hat{A}_1)\cap D_s(\hat{A}_2)$. Let u be any element in $u \in D_s(\hat{A}_1)\cap D_s(\hat{A}_2)$. We see, by the part (ii) of Lemma 2, that

$$\begin{aligned}
&\langle A_1 u, A_2 u\rangle + \langle(\hat{A}_1 - A_1\otimes I)(u\otimes s), (\hat{A}_2 - A_2\otimes I)(u\otimes s)\rangle \\
&= \int_\Omega f_1(\omega) f_2(\omega)\langle u\otimes s, \tilde{A}(d\omega)(u\otimes s)\rangle \\
&= \langle A_2 u, A_1 u\rangle + \langle(\hat{A}_2 - A_2\otimes I)(u\otimes s), (\hat{A}_1 - A_1\otimes I)(u\otimes s)\rangle,
\end{aligned}$$

from which we get by Schwarz inequality

$$\begin{aligned}
&\tfrac{1}{2}|\langle A_1 u, A_2 u\rangle - \langle A_2 u, A_1 u\rangle| \\
&= \tfrac{1}{2}|\langle(\hat{A}_1 - A_1\otimes I)(u\otimes s), (\hat{A}_2 - A_2\otimes I)(u\otimes s)\rangle \\
&\quad - \langle(\hat{A}_2 - A_2\otimes I)(u\otimes s), (\hat{A}_1 - A_1\otimes I)(u\otimes s)\rangle| \\
&\leqslant \|(\hat{A}_1 - A_1\otimes I)(u\otimes s)\|\cdot\|(\hat{A}_2 - A_2\otimes I)(u\otimes s)\|.
\end{aligned}$$

Hence, the proof is completed.

The analogue of the following theorem was first discovered by Arthurs and Kelly [5] and discussed by She and Heffner in [21], when (A_1, A_2) is a pair of conjugate observables. Also Yuen [22] discussed the general case. However, their observations are rather physical and the class of all "simultaneous" measurements considered in these papers is too narrow.

THEOREM 3 (generalized approximate simultaneous uncertainty relation). *Let* A_1 *and* A_2 *be any self-adjoint operators in a Hilbert space* H. *Then for any approximate simultaneous measurement* $\mathbf{M} = (K, s, (\Omega, \mathcal{F}, \tilde{A}), f = (f_1, f_2))$ *of* (A_1, A_2) *the following inequality holds*:

$$(\mathrm{var}[\mathbf{M}, u]_1)^{1/2}\cdot(\mathrm{var}[\mathbf{M}, u]_2)^{1/2} \geqslant |\langle A_1 u, A_2 u\rangle - \langle A_2 u, A_1 u\rangle| \tag{12}$$

for all $u \in H$, *where the left-hand-side of* (12) *is defined to be* ∞ *if* $\mathrm{var}[\mathbf{M}, u]_i = \infty$ *for some* i; *also the right-hand-side of* (12) *is defined to be* ∞ *if* $u \notin D(A_1)\cap D(A_2)$.

Proof: Put $\hat{A}_i = \int_\Omega f_i(\omega)\tilde{A}(d\omega)$ $(i = 1, 2)$. If $u \notin D_s(\hat{A}_i)$ for some i, we see by the definition of the variance that $\mathrm{var}[\mathbf{M}, u]_i = \infty$, so (12) clearly holds. Hence, it is sufficient to prove (12) in the case that $u \in D_s(\hat{A}_1) \cap D_s(\hat{A}_2)$. Let u be any element in $D_s(\hat{A}_1) \cap D_s(\hat{A}_2)$. Then we see by (iii) in Lemma 2 that

$$\begin{aligned} \mathrm{var}[\mathbf{M}, u]_i &= \|\hat{A}_i(u \otimes s)\|^2 - |\langle u \otimes s, \hat{A}_i(u \otimes s)\rangle|^2 \\ &= \|A_i u\|^2 + \|(\hat{A}_i - A_i \otimes I)(u \otimes s)\|^2 - |\langle u, A_i u\rangle|^2, \\ &\qquad i = 1, 2, \end{aligned}$$

so by the arithmetic-geometric mean inequality, Lemma 1 and Theorem 2 we get

$$\begin{aligned} &\mathrm{var}[\mathbf{M}, u]_1 \cdot \mathrm{var}[\mathbf{M}, u]_2 \\ &\geqslant 4(\|A_1 u\|^2 - |\langle u, A_1 u\rangle|^2)^{1/2} \cdot (\|A_2 u\|^2 - |\langle u, A_2 u\rangle|^2)^{1/2} \cdot \|(\hat{A}_1 - A_1 \otimes I)(u \otimes s)\| \\ &\qquad \cdot \|(\hat{A}_2 - A_2 \otimes I)(u \otimes s)\| \geqslant |\langle A_1 u, A_2 u\rangle - \langle A_2 u, A_1 u\rangle|^2. \end{aligned}$$

Hence, the proof is complete.

Compared with Theorem 3, Theorem 2 seems not to be sufficiently satisfactory, since the inequality (11) is not assured for all $u \in H$ but only for $u \in D(A_1) \cap D(A_2)$. Of course, if we assume that $\Delta_{\mathbf{M}}(A_i, u) = \infty$ if $u \notin D(A_i)$ in Definition 5, the inequality (11) holds for all $u \in H$. However, this assumption seems not to be natural. So we shall consider natural conditions when the inequality (11) holds for all $u \in H$ in the case when A_1 and A_2 form a pair of conjugate observables. Let $\mathbf{M} = (K, s, (\Omega, \mathscr{F}, \tilde{A}), f(\omega) = (f_1(\omega), f_2(\omega)))$ be the approximate simultaneous measurement of a pair of conjugate observables A_1 and A_2 in a Hilbert space H. Put $\hat{A}_i = \int_\Omega f_i(\omega) \tilde{A}(d\omega)$ $(i = 1, 2)$. Assume that $\mathbf{M}$ satisfies the following additional conditions:

(C1) for each i $(\hat{A}_i - A_i \otimes I)$ on $D(\hat{A}_i) \cap D(A_i \otimes I)$ has the unique self-adjoint extension $[\hat{A}_i - A_i \otimes I]$ (so $\Delta_{\mathbf{M}}(A_i, u)$ is defined in the sense of Remark 1,(2)),

(C2) put $K = D_s([\hat{A}_1 - A_1 \otimes I]) \cap D_s([\hat{A}_2 - A_2 \otimes I])$ and define $\|u\|_K = \|u\|_H + \|[\hat{A}_1 - A_1 \otimes I](u \otimes s)\|_H + \|[\hat{A}_2 - A_2 \otimes I](u \otimes s)\|_H$ for all $u \in K$ (we can easily see by usual arguments that K is a Banach space with the norm $\|\cdot\|_K$). Then assume that $D_s(\hat{A}_1) \cap D_s(\hat{A}_2)$ is dense in K.

Note that approximate simultaneous measurements considered in Proposition 5 and Theorem 1 satisfy the above conditions (C1) and (C2) (cf. the arguments below Proposition 5 and Remark 2,(2)).

Now we have the following corollary.

Corollary 1. *Let A_1 and A_2 be a pair of conjugate observables in a Hilbert space H. Let $\mathbf{M} = (K, s, (\Omega, \mathscr{F}, \tilde{A}), f(\omega) = (f_1(\omega), f_2(\omega)))$ be any approximate simultaneous measurement of A_1 and A_2 satisfying additional conditions* (C1) *and* (C2). *Then the following inequalities hold*:

(i) (*approximate simultaneous uncertainty relation*)

$$(\mathrm{var}[\mathbf{M}, u]_1)^{1/2} \cdot (\mathrm{var}[\mathbf{M}, u]_2)^{1/2} \geqslant \hbar \tag{13}$$

for all $u \in H$ ($\|u\|_H = 1$),

(ii) (*Heisenberg's uncertainty relation*)

$$\Delta_{\mathbf{M}}(A_1, u) \cdot \Delta_{\mathbf{M}}(A_2, u) \geqslant \hbar/2 \tag{14}$$

for all $u \in H$ ($\|u\|_H = 1$), *where the left-hand-sides of these inequalities are defined as in Theorem* 2 *and* 3.

Proof: Note that $\langle A_1 u, A_2 u\rangle - \langle A_2 u, A_1 u\rangle = i\hbar$ $\big(u \in D(A_1) \cap D(A_2),\ \|u\|_H = 1\big)$. Then (i) is a special case of Theorem 2. Also (ii) clearly holds under additional conditions (C1) and (C2).

Remark 3: Let A_1 and A_2 be a pair of conjugate observables in a Hilbert space H. As shown in [10], there exists a simple measurement $\mathbf{M} = \big(\mathbf{C}, 1, (\mathbf{R}^2, \mathscr{B}_2, \tilde{A}), f(\lambda_1, \lambda_2) = (\lambda_1, \lambda_2)\big)$ satisfying that for any $\varepsilon > 0$ there exists a state u_ε $\big(u \in D(A_1) \cap D(A_2),\ \|u_\varepsilon\|_H = 1\big)$ such that

$$\|(\hat{A}_1 - A_1 \otimes I)(u \otimes 1)\|_{H \otimes \mathbf{C}} = \varepsilon \quad \text{and} \quad \|(\hat{A}_2 - A_2 \otimes I)(u \otimes 1)\|_{H \otimes \mathbf{C}} = 0,$$

where $\hat{A}_i = \int_{\mathbf{R}^2} \lambda_i \tilde{A}(d\lambda_1 d\lambda_2)$ ($i = 1, 2$). The example mentioned in [10] is essentially as follows: Put $H = L^2(\mathbf{R}^2)$, $A_1 = x_1$ and $A_2 = \dfrac{\hbar\partial}{i\partial x_1}$. Therefore, (A_1, A_2) is a pair of conjugate observables in H $\big(= L^2(\mathbf{R}^2)\big)$. Put $B_1 = x_2$. Since $B_1 \otimes I$ and $A_2 \otimes I$ commute in $H \otimes \mathbf{C}$, we can define $\tilde{A}$ by $\tilde{A}(d\lambda_1 d\lambda_2) = \big(B_1(d\lambda_1) \otimes I\big)\big(A_2(d\lambda_2) \otimes I\big)$ and put $f(\lambda_1, \lambda_2) = (\lambda_1, \lambda_2)$. So $\hat{A}_1 = \int_{\mathbf{R}^2} \lambda_1 \tilde{A}(d\lambda_1 d\lambda_2) = B_1 \otimes I$, $\hat{A}_2 = \int_{\mathbf{R}^2} \lambda_2 \tilde{A}(d\lambda_1 d\lambda_2) = A_2 \otimes I$. Put $u_\varepsilon(x_1, x_2) = \left[\dfrac{1}{4\pi\varepsilon}\exp\left[-\dfrac{(x_1 - x_2)^2}{4\varepsilon^2} - \dfrac{(x_1 + x_2)^2}{4}\right]\right]^{1/2}$. Now we can easily see that $\|(\hat{A}_1 - A_1 \otimes I)(u_\varepsilon \otimes 1)\|^2_{H \otimes \mathbf{C}} = \|(B_1 - A_1) u_\varepsilon\|^2_H = \int_{\mathbf{R}^2} (x_2 - x_1)^2 |u_\varepsilon(x_1, x_2)|^2 dx_1 dx_2 = \varepsilon^2$ and also, clearly, $\|(\hat{A}_2 - A_2 \otimes I)(u_\varepsilon \otimes 1)\|_{H \otimes \mathbf{C}} = \|(A_2 - A_2) u_\varepsilon\|_H = 0$. Of course, this example does not contradict our result (Corollary 1 or Theorem 2) since $\mathbf{M}$ is not an approximate simultaneous measurement of A_1 $(= x_1)$ and A_2 $\left(= \dfrac{\hbar\partial}{i\partial x_1}\right)$ but the exact simultaneous measurement of B_1 $(= x_2)$ and A_2 $\left(= \dfrac{\hbar\partial}{i\partial x_1}\right)$. However, since Proposition 1 includes some ambiguous sentences, there seem to be a few confusions in some books.

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Note added in proof: For the further arguments of this paper, see the additional references [23] and [24].

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