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# A shallow water approximation for water waves

*Dedicated to the late Professor Alexandre V. Kazhikhov*

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## 1 Introduction

In this paper we are concerned with the initial value problem for water waves in arbitrary space dimensions. The water wave is a model system for irrotational flow of an incompressible ideal fluid with a free surface under the gravitational field. The analysis of this problem is very hard because of the nonlinearity of the equations together with the presence of a unknown free surface. In order to understand various phenomena of water waves, one has approximated the equations by simple ones and analyzed the approximated equations. The simplest approximation is the linear one around the trivial flow by assuming that the amplitude of the free surface and the motion of the fluid are infinitesimal. However, this approximation could not explain the existence of solitary waves nor the breaking of the waves. In order to explain such phenomena we have to include nonlinear effects of the waves in the approximation. The shallow water equations are one of such approximations and derived from the water wave by assuming that the water depth is sufficiently small compared to the wave length. The aim of this paper is to give a mathematically rigorous justification of the shallow water approximation for water waves in Sobolev spaces.

By rewriting the equations in an appropriate non-dimensional form, we have two non-dimensional parameters $\delta$ and $\varepsilon$ the ratio of the water depth $h$ to the wave length $\lambda$ and the ratio of the amplitude of the free surface $a$ to the water depth $h$, respectively, in the equations. The shallow water equations are derived from the water wave in the limit $\delta \to +0$ by keeping $\varepsilon \simeq 1$. In the case of a flat bottom, they are of the same form as the compressible Euler equation for a barotropic gas and the solution generally has a singularity in finite time even if the initial data are sufficiently smooth. Therefore, this approximation is used to explain the breaking of the waves. The derivation of the shallow water equations goes back to G. B. Airy [1]. Then, K. O. Friedrichs [4] derived systematically the equations from the water wave by using an expansion of the solution with respect to $\delta^2$, which is called the Friedrichs expansion. See also H. Lamb [12] and J. J. Stoker [19]. A mathematically rigorous justification of the shallow water approximation for two-dimensional water waves was given by L. V. Ovsjannikov [15, 16] under the periodic boundary condition with respect to the horizontally spatial variable, and then by T. Kano and T. Nishida [8]. In order to guarantee the existence of solutions for water waves, they used an abstract Cauchy-Kowalevski theorem in a scaled Banach space so that analyticity of the initial data was required. A mathematical justification of the Friedrichs expansion was investigated by T. Kano and T. Nishida [9] and the justification in the three-dimensional case by T. Kano [7]. It is natural to ask if the approximation is valid in Sobolev spaces. However, this question was not resolved.

On the other hand, the Korteweg-de Vris (KdV) equation is also derived from the two-dimensional water wave in the limit $\varepsilon = \delta^2 \to +0$. It is well known that the solution of the KdV equation exists globally in time and the equation has solitary wave solutions. The derivation of the KdV equation goes back to D. J. Korteweg and G. de Vries [11]. Historically, the theory of long waves in shallow water gave rise to a paradox, because both the shallow water equations and the KdV equation are derived from water waves in the limit $\delta \to 0$ and the behavior of the solutions are completely different. We refer to F. Ursell [20] on this paradox. A mathematically rigorous justification of the KdV equation for the water wave was investigated by T. Kano and T. Nishida [10] in a class of analytic functions. Concerning this KdV approximation, a justification in Sobolev spaces was given by W. Craig [2] under a restriction that the wave is almost one-directional. Then, G. Schneider and C. E. Wayne [17] gave a justification without assuming one-directional motion of the wave. In the case with the surface tension on the free surface, G. Schneider and C. E. Wayne [18] and the author [5] gave justifications. An important part of the analysis in [2, 5] is to approximate a non-local operator, such as the Dirichlet-to-Neumann map for Laplace's equation and the Dirichlet-to-Dirichlet map for the Cauchy-Riemann equations, in terms of Fourier multipliers by expanding it with respect to a function which represents the surface elevation and to give a precise estimate for the remainder part. However, in the shallow water scaling we cannot obtain a good estimate for the remainder part so that we have to use another method in order to give a justification of the shallow water approximation.

In connection with the well-posedness of the initial value problem for water waves, the solvability in Sobolev spaces was given by several authors. In his pioneering work [14], V. I. Nalimov investigated the initial value problem in the case where the motion of the fluid is two-dimensional and the fluid has infinite depth. He showed that if the initial data are sufficiently small in a Sobolev space, that is, if the initial surface is almost flat and the initial movement of the fluid is sufficiently small, then there exists a unique solution of the problem locally in time in a Sobolev space. H. Yosihara [23] extended this result to the case of presence of an almost flat bottom. S. Wu [21] studied the problem in exactly the same situation as Nalimov's and gave the existence theorem locally in time without assuming the initial data to be small. It is known that the well-posedness of the problem may be broken unless a generalized Rayleight-Taylor sign condition $-\partial p/\partial N \geq c_0 > 0$ on the free surface is satisfied, where $N$ is the unit outward normal to the free surface. She showed surprising fact that this condition always holds for any smooth nonself-intersecting interface. In the above results, the proofs were based on the energy method. They first derived quasi-linear equations of the form

$$u_{tt} + a|D|u = f,$$

where the function $a$ is positively definite, and then defined a corresponding energy function by $E = \|u_t\|^2 + (a|D|^{1/2}u, |D|^{1/2}u) + \|u\|^2$. In the derivation of the above equation, an approximation of a non-local operator in terms of Fourier multipliers plays an important role. However, the extension of such approximation to the three-dimensional case was difficult. Instead, S. Wu [22] derived quasi-linear equations of the form

$$u_{tt} + a\Lambda u = f,$$

where $\Lambda$ is the Dirichlet-to-Neumann map for Laplace's equation and analyzed precisely the non-local operator $\Lambda$. She defined a corresponding energy function by $E = \|u_t\|^2 + (a\Lambda u, u) + \|u\|^2$. (Strictly speaking, she used a slightly different energy function.) As a result, she succeeded to give an existence theory in Sobolev spaces for three-dimensional water waves of infinite depth.

Note that in two-dimensional case a principal part of $\Lambda$ is equal to $|D|$ and that all of the three authors mentioned above used the Lagrangian coordinates. D. Lannes [13] studied the initial value problem for water waves of finite depth in arbitrary space dimensions. One of interesting features of his paper is that he did not use the Lagrangian coordinates but the Euler coordinates although the surface tension on the free surface was neglected. Another interesting feature is that he obtained a good expression of the Fréchet derivative of the operator $\Lambda$ with respect to a function which represents the surface elevation. As a result, he derived linearized equations of the form

$$\begin{cases} \eta_t + \nabla \cdot (v\eta) - \Lambda\phi = f_1, \\ \phi_t + v \cdot \nabla\phi + a\eta = f_2, \end{cases}$$

where the function $a$ is positively definite, defined a corresponding energy function by $E = (a\eta, \eta) + (\Lambda\phi, \phi) + \|\phi\|^2$, and gave an existence theory in Sobolev spaces. This energy function is very natural, because the water wave problem has a conserved energy defined by

$$H = \int_{\Omega(t)} \frac{1}{2}|\nabla_X \Phi(X)|^2 dX + \int_{\mathbf{R}^n} \frac{g}{2}|\eta(x)|^2 dx = \frac{1}{2}(\Lambda\phi, \phi) + \frac{g}{2}\|\eta\|^2.$$

See section 2 for the notation. We mention that the water wave problem has a Hamiltonian structure whose Hamiltonian is $H$ and the canonical variables are $\eta$ and $\phi$. The Hamiltonian formulation of water waves goes back to V. E. Zakharov [24] in the case of infinite depth. We refer to W. Craig, P. Guyenne, D. P. Nicholls, and C. Sulem [3] for an analysis of the Hamiltonian in long wave approximations. In calculation of the time evolution of the energy function $E$, we need to estimate commutators of the map $\Lambda$ and differential operators. S. Wu [22] obtained precise commutator estimates by using the theory of singular integral operators and Clifford analysis, whereas D. Lannes [13] used the theory of pseudo-differential operators and obtained commutator estimates by imposing much differentiability on the coefficients. This is one of the reasons why a Nash-Moser implicit function theorem was used to obtain the solution of the nonlinear equations in [13]. A relation between the generalized Rayleight-Taylor sign condition and the bottom topography was also analyzed in [13]. Under the shallow water scaling, such techniques in [22, 13] in estimating commutators do not give nice uniform estimates with respect to small $\delta$. In this paper, to obtain the uniform estimates, we only use the standard technique in estimating the solution of a boundary value problem for elliptic differential equations, so that the proof may become much simpler and elementary than the previous ones. We adopt the formulation of the problem used in [13]. However, thanks of a precise energy estimate for linearized equations it is not necessary to use the Nash-Moser implicit function theorem to obtain the solution of the nonlinear equations.

The contents of this paper are as follows. In section 2 we formulate the problem, rewrite it in a non-dimensional form, transform it into an equivalent problem on the free surface, and give one of our main results, which asserts the existence of the solution with uniform bounds in a Sobolev space. In section 3 we formally derive the shallow water equations from the water wave and give another main result, which justifies rigorously the shallow water approximation. In section 4 we analyze the Dirichlet-to-Neumann map for Laplace's equation. In the analysis, we transform a boundary value problem for Laplace's equation in the fluid domain $\Omega(t)$ to a problem on the simple fixed domain $\Omega_0 = \mathbf{R}^n \times (0, 1)$ by using a suitable diffeomorphism $\Theta : \Omega_0 \to \Omega(t)$. In sections 5 and 6 we derive estimates in a Sobolev space for the Dirichlet-to-Neumann map and its Fréchet derivatives with respect to the function which represents the surface elevation. In section 7, according to D. Lannes [13] we first linearize the full equations and derive an energy estimate for the linearized problem. In section 8 we reduce the full nonlinear equations to a quasi-linear equations. Finally, in section 9, by applying the energy estimates established in section 7 to the quasi-linear equations derived in section 8 we prove main theorems.

**Notation.** For a real number $s$, we denote by $H^s$ the Sobolev space of order $s$ on $\mathbf{R}^n$ equipped with the inner product $(u,v)_s = (2\pi)^{-n}\int_{\mathbf{R}^n}(1+|\xi|)^{2s}\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi$, where $\hat{u}$ is the Fourier transform of $u$, that is, $\hat{u}(\xi) = \int_{\mathbf{R}^n} u(x)\mathrm{e}^{-ix\cdot\xi}dx$. We put $\|u\|_s = \sqrt{(u,u)_s}$, $(u,v) = (u,v)_0$, and $\|u\| = \|u\|_0$. For $1 \le p \le \infty$, we denote by $|\cdot|_p$ the norm of the Lebesgue space $L^p = L^p(\mathbf{R}^n)$. The norm of a Banach space $X$ is denoted by $\|\cdot\|_X$. For $0 < T < \infty$, a non-negative integer $j$, and a Banach space $X$, we denote by $C^j([0,T];X)$ the Banach space of all functions of $C^j$-class on the interval $[0,T]$ with the value in $X$. We put $\partial_j = \partial/\partial x_j$, $\partial_{ij} = \partial_i\partial_j$, and $\partial_{ijk} = \partial_i\partial_j\partial_k$. A pseudo-differential operator $P(D)$, $D = (D_1,\dots,D_n)$ and $D_j = -i\partial_j$, with a symbol $P(\xi)$ is defined by $P(D)u(x) = (2\pi)^{-n}\int_{\mathbf{R}^n} P(\xi)\hat{u}(\xi)\mathrm{e}^{ix\cdot\xi}d\xi$. We put $J = 1+|D|$, so that $\|u\|_s = \|J^s u\|$. For operators $A$ and $B$, we denote by $[A,B] = AB - BA$ the commutator. Throughout this paper, we denote inessential constants by the same symbol $C$.

## 2 Formulation of the problem

Let $x = (x_1, x_2, \dots, x_n)$ be the horizontally spatial variables and $x_{n+1}$ the vertically spatial variable. We denote by $X = (x, x_{n+1}) = (x_1,\dots,x_n,x_{n+1})$ the whole spatial variables. We will consider a water wave in $(n+1)$-dimensional space and assume that the domain $\Omega(t)$ occupied by the fluid at time $t \ge 0$, the free surface $\Gamma(t)$, and the bottom $\Sigma$ are of the forms

$$\begin{aligned}
&\Omega(t) = \bigl\{X = (x,x_{n+1}) \in \mathbf{R}^{n+1}\,;\, b(x) < x_{n+1} < h + \eta(x,t)\bigr\},\\
&\Gamma(t) = \bigl\{X = (x,x_{n+1}) \in \mathbf{R}^{n+1}\,;\, x_{n+1} = h + \eta(x,t)\bigr\},\\
&\Sigma = \bigl\{X = (x,x_{n+1}) \in \mathbf{R}^{n+1}\,;\, x_{n+1} = b(x)\bigr\},
\end{aligned}$$

where $h$ is the mean depth of the fluid. The functions $b$ and $\eta$ represent the bottom topography and the surface elevation, respectively. In this paper $b$ is a given function, while $\eta$ is the unknown. In fact, our main interest is the behavior of the free surface.

We assume that the fluid is incompressible and inviscid, and that the flow is irrotational. Then, the fluid motion is described by the velocity potential $\Phi = \Phi(X,t)$ satisfying the equation

$$\triangle_X \Phi = 0 \qquad \text{in} \quad \Omega(t), \quad t > 0, \tag{2.1}$$

where $\triangle_X$ is the Laplacian with respect to $X$, that is, $\triangle_X = \triangle + \partial_{n+1}^2$ and $\triangle = \partial_1^2 + \cdots + \partial_n^2$. The boundary conditions on the free surface are given by

$$\begin{cases} \eta_t + \nabla\Phi\cdot\nabla\eta - \partial_{n+1}\Phi = 0, \\ \Phi_t + \frac{1}{2}|\nabla_X\Phi|^2 + g\eta = 0 \qquad \text{on} \quad \Gamma(t), \quad t > 0, \end{cases} \tag{2.2}$$

where $\nabla = (\partial_1,\dots,\partial_n)$ and $\nabla_X = (\partial_1,\dots,\partial_n,\partial_{n+1})$ are the gradients with respect to $x = (x_1,\dots,x_n)$ and to $X = (x,x_{n+1})$, respectively, and $g$ is the gravitational constant. The first equation is the kinematical condition and the second one is known as Bernoulli's law. The boundary condition on the bottom is given by

$$N \cdot \nabla_X \Phi = 0 \qquad \text{on} \quad \Sigma, \quad t > 0, \tag{2.3}$$

where $N$ is the normal vector to the bottom $\Sigma$. Finally, we impose the initial conditions

$$\eta(x,0) = \eta_0(x), \quad \Phi(X,0) = \Phi_0(X). \tag{2.4}$$

It should be assumed that the initial data satisfy the compatibility conditions, that is, $\triangle_X \Phi_0 = 0$ in $\Omega(0)$ and $N \cdot \nabla_X \Phi_0 = 0$ on $\Sigma$.

We proceed to rewrite the equations (2.1)–(2.4) in an appropriate non-dimensional form. Let $\lambda$ be the typical wave length and $h$ the mean depth. We introduce a non-dimensional parameter $\delta$ by $\delta = h/\lambda$ and rescale the independent and dependent variables by

$$x = \lambda \tilde{x}, \quad x_{n+1} = h \tilde{x}_{n+1}, \quad t = \frac{\lambda}{\sqrt{gh}} \tilde{t}, \quad \Phi = \lambda \sqrt{gh} \tilde{\Phi}, \quad \eta = h \tilde{\eta}, \quad b = h \tilde{b}. \tag{2.5}$$

Putting these into (2.1)–(2.4) and dropping the tilde sign in the notation we obtain

$$\delta^2 \triangle \Phi + \partial_{n+1}^2 \Phi = 0 \qquad \text{in} \quad \Omega(t), \quad t > 0, \tag{2.6}$$

$$\begin{cases} \delta^2 \big( \eta_t + \nabla \Phi \cdot \nabla \eta \big) - \partial_{n+1} \Phi = 0, \\ \delta^2 \big( \Phi_t + \frac{1}{2} |\nabla \Phi|^2 + \eta \big) + \frac{1}{2} (\partial_{n+1} \Phi)^2 = 0 \qquad \text{on} \quad \Gamma(t), \quad t > 0, \end{cases} \tag{2.7}$$

$$\partial_{n+1} \Phi - \delta^2 \nabla b \cdot \nabla \Phi = 0 \qquad \text{on} \quad \Sigma, \quad t > 0, \tag{2.8}$$

$$\eta(x, 0) = \eta_0^\delta(x), \quad \Phi(X, 0) = \Phi_0^\delta(X), \tag{2.9}$$

where

$$\begin{aligned} \Omega(t) &= \big\{ X = (x, x_{n+1}) \in \mathbf{R}^{n+1} \, ; \, b(x) < x_{n+1} < 1 + \eta(x,t) \big\}, \\ \Gamma(t) &= \big\{ X = (x, x_{n+1}) \in \mathbf{R}^{n+1} \, ; \, x_{n+1} = 1 + \eta(x,t) \big\}, \\ \Sigma &= \big\{ X = (x, x_{n+1}) \in \mathbf{R}^{n+1} \, ; \, x_{n+1} = b(x) \big\}. \end{aligned}$$

Since we are interested in asymptotic behavior of the solution when $\delta \to +0$, we always assume $0 < \delta \le 1$ in the following.

As in the usual way, we transform equivalently the initial value problem (2.6)–(2.9) to a problem on the free surface. To this end, we introduce new unknown function $\phi$ by

$$\phi(x, t) = \Phi(x, 1 + \eta(x,t), t), \tag{2.10}$$

which is the trace of the velocity potential on the free surface. Then, we see that

$$\begin{cases} \phi_t = \Phi_t|_{\Gamma(t)} + \partial_{n+1} \Phi|_{\Gamma(t)} \eta_t, \\ \nabla \phi = \nabla \Phi|_{\Gamma(t)} + \partial_{n+1} \Phi|_{\Gamma(t)} \nabla \eta. \end{cases} \tag{2.11}$$

We introduce the Dirichlet-to-Neumann map $\Lambda = \Lambda(\eta, b, \delta)$ for Laplace's equation in the following way. Under suitable assumptions on $\eta$ and $b$, for any function $\varphi$ on the free surface in some class there exits a unique solution $\Phi$ of the boundary value problem

$$\begin{cases} \delta^2 \triangle \Phi + \partial_{n+1}^2 \Phi = 0 & \text{in} \quad \Omega(t), \\ \Phi = \varphi & \text{on} \quad \Gamma(t), \\ \partial_{n+1} \Phi - \delta^2 \nabla b \cdot \nabla \Phi = 0 & \text{on} \quad \Sigma. \end{cases}$$

Using the solution $\Phi$ we define a linear operator $\Lambda = \Lambda(\eta, b, \delta)$ by

$$\Lambda(\eta, b, \delta) \varphi = (\delta^{-2} \partial_{n+1} \Phi - \nabla \eta \cdot \nabla \Phi) \big|_{\Gamma(t)}.$$

It is very important to study precisely the operator $\Lambda$ in the analysis of the initial value problem for water waves. By this definition, (2.6), (2.8), and (2.10), we have

$$\Lambda(\eta, b, \delta)\phi = (\delta^{-2}\partial_{n+1}\Phi - \nabla\eta\cdot\nabla\Phi)\big|_{\Gamma(t)}. \tag{2.12}$$

This and the second equation in (2.11) imply that

$$\begin{cases} \partial_{n+1}\Phi|_{\Gamma(t)} = \delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda\phi + \nabla\eta\cdot\nabla\phi), \\ \nabla\Phi|_{\Gamma(t)} = \nabla\phi - \delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda\phi + \nabla\eta\cdot\nabla\phi)\nabla\eta. \end{cases} \tag{2.13}$$

It follows from the first equation in (2.7) and (2.12) that $\eta_t - \Lambda\phi = 0$, so that by the first equation in (2.11) we get

$$\Phi_t|_{\Gamma(t)} = \phi_t - \delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda\phi + \nabla\eta\cdot\nabla\phi)\Lambda\phi.$$

Putting this and (2.13) into the second equation in (2.7) we obtain

$$\begin{cases} \eta_t - \Lambda(\eta, b, \delta)\phi = 0, \\ \phi_t + \eta + \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2}\delta^2(1+\delta^2|\nabla\eta|^2)^{-1}\big(\Lambda(\eta, b, \delta)\phi + \nabla\eta\cdot\nabla\phi\big)^2 = 0 \qquad \text{for} \quad t > 0, \end{cases} \tag{2.14}$$

$$\eta = \eta_0^\delta, \quad \phi = \phi_0^\delta \qquad \text{at} \quad t = 0, \tag{2.15}$$

where $\phi_0^\delta = \Phi_0^\delta(\cdot, 1 + \eta_0^\delta(\cdot))$. This is the initial value problem that we are going to investigate in this paper. The following theorem is one of the main results in this paper and asserts the existence of the solution with uniform bounds on a time interval independent of small $\delta > 0$ for the above initial value problem.

**Theorem 2.1.** *Let $M_0, c_0 > 0$ and $s > n/2 + 1$. There exist a time $T > 0$ and constants $C_0, \delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$, $\eta_0^\delta \in H^{s+3+1/2}$, $\phi_0^\delta \in H^{s+4}$, and $b \in H^{s+4+1/2}$ satisfying*

$$\begin{cases} \|\eta_0^\delta\|_{s+3+1/2} + \|\phi_0^\delta\|_{s+4} + \|b\|_{s+4+1/2} \le M_0, \\ 1 + \eta_0^\delta(x) - b(x) \ge c_0 \quad \text{for} \quad x \in \mathbf{R}^n, \end{cases}$$

*the initial value problem* (2.14) *and* (2.15) *has a unique solution $(\eta, \phi) = (\eta^\delta, \phi^\delta)$ on the time interval $[0, T]$ satisfying*

$$\begin{cases} \|(\eta^\delta(t), \phi^\delta(t))\|_{s+3} + \|(\eta_t^\delta(t), \phi_t^\delta(t))\|_{s+2} \le C_0, \\ 1 + \eta^\delta(x, t) - b(x) \ge c_0/2 \quad \text{for} \quad x \in \mathbf{R}^n, \ 0 \le t \le T. \end{cases}$$

# 3 The shallow water approximation

In this section we study formally asymptotic behavior of the solution $(\eta^\delta, \phi^\delta)$ to the initial value problem (2.14) and (2.15) when $\delta \to +0$ and derive the shallow water equation, whose solution approximates $(\eta^\delta, \phi^\delta)$ in a suitable sense. Then, we will give a theorem which ensures a rigorous approximation of the water wave by the shallow water equations.

It follows from the second equation in (2.14) that

$$\phi_t + \eta + \frac{1}{2}|\nabla\phi|^2 = O(\delta^2).$$

By (2.6) and (2.8),

$$(3.1)\qquad \begin{aligned}(\partial_{n+1}\Phi)(x,x_{n+1},t) &= (\partial_{n+1}\Phi)(x,b(x),t)+\int_{b(x)}^{x_{n+1}}(\partial_{n+1}^2\Phi)(x,y,t)dy\\ &= \delta^2\nabla b(x)\cdot\nabla\Phi(x,b(x),t)-\delta^2\int_{b(x)}^{x_{n+1}}(\triangle\Phi)(x,y,t)dy,\end{aligned}$$

which implies that $(\partial_{n+1}\Phi)(X,t)=O(\delta^2)$. Therefore,

$$\begin{aligned}\nabla\Phi(x,x_{n+1},t) &= \nabla\Phi(x,1+\eta(x,t),t)+\int_{1+\eta(x,t)}^{x_{n+1}}(\nabla\partial_{n+1}\Phi)(x,y,t)dy\\ &= \nabla\Phi(x,1+\eta(x,t),t)+O(\delta^2).\end{aligned}$$

Moreover, by the definition (2.10) it holds that

$$\begin{aligned}\nabla\phi(x,t) &= \nabla\Phi(x,1+\eta(x,t),t)+\nabla\eta(x)(\partial_{n+1}\Phi)(x,1+\eta(x),t)\\ &= \nabla\Phi(x,1+\eta(x,t),t)+O(\delta^2)\\ &= \nabla\Phi(X,t)+O(\delta^2).\end{aligned}$$

Similarly, we have

$$\triangle\phi(x,t)=\triangle\Phi(X,t)+O(\delta^2).$$

These relation and (3.1) imply that

$$\begin{aligned}(\partial_{n+1}\Phi)(x,1+\eta(x,t),t) &= \delta^2\nabla b(x)\cdot\nabla\phi(x,t)-\delta^2\int_{b(x)}^{1+\eta(x,t)}\triangle\phi(x,t)dy+O(\delta^4)\\ &= -\delta^2(1+\eta(x,t))\triangle\phi(x,t)+\delta^2\nabla\cdot\big(b(x)\nabla\phi(x,t)\big)+O(\delta^4).\end{aligned}$$

Hence, by the definition of the Dirichlet-to-Neumann map $\Lambda$ we have

$$(3.2)\qquad (\Lambda\phi)(x,t)=-\nabla\cdot\big((1+\eta(x,t)-b(x))\nabla\phi(x,t)\big)+O(\delta^2).$$

This and the first equation in (2.14) imply that

$$\eta_t+\nabla\cdot\big((1+\eta-b)\nabla\phi\big)=O(\delta^2).$$

To summarize, we have derived the partial differential equations

$$\begin{cases}\eta_t+\nabla\cdot\big((1+\eta-b)\nabla\phi\big)=O(\delta^2),\\ \phi_t+\eta+\frac{1}{2}|\nabla\phi|^2=O(\delta^2),\end{cases}$$

which approximate the equations in (2.14) up to order $\delta^2$. Letting $\delta\to 0$ in the above equations we finally obtain the shallow water equations

$$(3.3)\qquad \begin{cases}\eta^0_t+\nabla\cdot\big((1+\eta^0-b)\nabla\phi^0\big)=0,\\ \phi^0_t+\eta^0+\frac{1}{2}|\nabla\phi^0|^2=0.\end{cases}$$

Here, we remark that if we put $u^0:=\nabla\phi^0$ and take the gradient of the second equation in (3.3), then we obtain

$$\begin{cases}\eta^0_t+\nabla\cdot\big((1+\eta^0-b)u^0\big)=0,\\ u^0_t+(u^0\cdot\nabla)u^0+\nabla\eta^0=0.\end{cases}$$

The following theorem is another main result in this paper and gives a mathematically rigorous justification of the shallow water equations for water waves.

**Theorem 3.1.** *In addition to hypothesis of Theorem* 2.1 *we assume that as* $\delta \to +0$ *the initial data* $(\eta_0^\delta, \phi_0^\delta)$ *converge to* $(\eta_0^0, \phi_0^0)$ *in* $H^{s+3}$. *Then, as* $\delta \to +0$ *the solution obtained in Theorem* 2.1 *satisfies*

$$(\eta^\delta, \phi^\delta) \to (\eta^0, \phi^0) \quad \begin{array}{ll} \textit{weakly* in} & L^\infty(0,T; H^{s+3}), \\ \textit{strongly in} & C([0,T]; H^{s+3-\varepsilon}) \end{array}$$

*for each* $\varepsilon > 0$, *where* $(\eta^0, \phi^0)$ *is a unique solution of the shallow water equations* (3.3) *with initial conditions* $(\eta^0, \phi^0)|_{t=0} = (\eta_0^0, \phi_0^0)$.

*Moreover, if we also assume that* $\|\eta_0^\delta - \eta_0^0\|_s + \|\phi_0^\delta - \phi_0^0\|_{s+1} = O(\delta^2)$, *then for any* $\delta \in (0, \delta_0]$ *and* $t \in [0,T]$ *we have*

$$\|\eta^\delta(t) - \eta^0(t)\|_s + \|\phi^\delta(t) - \phi^0(t)\|_{s+1} \le C\delta^2$$

*with a constant* $C$ *independent of* $\delta$ *and* $t$.

# 4 The Dirichlet-to-Neumann map $\Lambda$

Throughout this and following two sections the time $t$ is arbitrarily fixed, so that $\Omega(t)$, $\Gamma(t)$, and $\eta(x,t)$ are simply denoted by $\Omega$, $\Gamma$, and $\eta(x)$, respectively. We consider the boundary value problem

$$\begin{cases} \delta^2 \triangle \Phi + \partial_{n+1}^2 \Phi = 0 & \text{in} \quad \Omega, \\ \Phi = \phi & \text{on} \quad \Gamma, \\ \partial_{n+1}\Phi - \delta^2 \nabla b \cdot \nabla \Phi = 0 & \text{on} \quad \Sigma. \end{cases}$$

Introducing a $(n+1)\times(n+1)$ matrix $I_\delta$ by

$$I_\delta = \begin{pmatrix} E_n & 0 \\ 0 & \delta^{-1} \end{pmatrix},$$

where $E_n$ is the $n \times n$ unit matrix, we can rewrite the above boundary value problem and the Dirichlet-to-Neumann map as

$$\begin{cases} \nabla_X \cdot I_\delta^2 \nabla_X \Phi = 0 & \text{in} \quad \Omega, \\ \Phi = \phi & \text{on} \quad \Gamma, \\ N \cdot I_\delta^2 \nabla_X \Phi = 0 & \text{on} \quad \Sigma. \end{cases} \tag{4.1}$$

and

$$\Lambda(\eta, b, \delta)\phi = (-\nabla\eta, 1) \cdot I_\delta^2 \nabla_X \Phi(x, 1+\eta(x)),$$

respectively.

**Definition 4.1.** The unique solution $\Phi$ of the boundary value problem (4.1) is denoted by $\phi^\hbar$.

**Lemma 4.1.** *The Dirichlet-to-Neumann map* $\Lambda = \Lambda(\eta, b, \delta)$ *is self-adjoint in* $L^2$, *that is, for any* $\phi, \psi \in H^1$ *it holds that*

$$(\Lambda\phi, \psi) = (\phi, \Lambda\psi).$$

**Proof**. Set $\Phi := \phi^\hbar$ and $\Psi := \psi^\hbar$. By Green's formula we have

$$\begin{aligned} 0 &= \int_\Omega \big( (\nabla_X \cdot I_\delta^2 \nabla_X \Phi)\Psi - \Phi(\nabla_X \cdot I_\delta^2 \nabla_X \Psi) \big) dX \\ &= \int_\Gamma \big( (N \cdot I_\delta^2 \nabla_X \Phi)\Psi - \Phi(N \cdot I_\delta^2 \nabla_X \Psi) \big) dS, \end{aligned}$$

where $N$ is the unit outward normal to the boundary $\partial\Omega$. In the above calculation we used the boundary condition on the bottom $\Sigma$. Since $\Phi = \phi$, $\Psi = \psi$, $\sqrt{1+|\nabla\eta|^2}N \cdot I_\delta^2\nabla_X\Phi = \Lambda\phi$, $\sqrt{1+|\nabla\eta|^2}N \cdot I_\delta^2\nabla_X\Psi = \Lambda\psi$, and $dS = \sqrt{1+|\nabla\eta|^2}dx$ on $\Gamma$, we obtain the desired identity. □

**Lemma 4.2.** *For any $\phi \in H^1$, it holds that $(\Lambda\phi, \phi) = \|I_\delta\nabla_X\Phi\|_{L^2(\Omega)}^2$, where $\Phi = \phi^\hbar$.*

**Proof**. By Green's formula we see that

$$0 = \int_\Omega (\nabla_X \cdot I_\delta^2\nabla_X\Phi)\Phi dX = \int_{\partial\Omega}(N \cdot I_\delta^2\nabla_X\Phi)\Phi dS - \int_\Omega |I_\delta\nabla_X\Phi|^2 dX.$$

This together with the boundary conditions yields the desired identity. □

By using an appropriate diffeomorphism $\Theta = (\Theta_1, \dots, \Theta_n, \Theta_{n+1}) : \overline{\Omega}_0 = \mathbf{R}^n \times [0,1] \to \overline{\Omega}$, we transform the boundary value problem (4.1) to a problem on the simple domain $\Omega_0$. We take functions $\theta = (\theta_1, \dots, \theta_n, \theta_{n+1})$ satisfying the conditions

$$\begin{cases} \theta_j(x,0) = \theta_j(x,1) = 0, \\ \partial_{n+1}\theta_j(x,0) = -\partial_j b(x), \quad \partial_{n+1}\theta_j(x,1) = -\partial_j\eta(x) \quad \text{for} \quad 1 \le j \le n, \\ \theta_{n+1}(x,0) = b(x), \quad \theta_{n+1}(x,1) = \eta(x), \\ \partial_{n+1}\theta_{n+1}(x,0) = \partial_{n+1}\theta_{n+1}(x,1) = 0, \end{cases} \tag{4.2}$$

and define the diffeomorphism $\Theta$ by

$$\begin{cases} \Theta_j(X) = x_j + \delta^2\theta_j(X) \quad \text{for} \quad 1 \le j \le n, \\ \Theta_{n+1}(X) = x_{n+1} + \theta_{n+1}(X). \end{cases} \tag{4.3}$$

It is easy to see that

$$\frac{\partial\Theta}{\partial X} = \begin{pmatrix} E_n + \delta^2\dfrac{\partial(\theta_1,\dots,\theta_n)}{\partial(x_1,\dots,x_n)} & (\nabla\theta_{n+1})^T \\ \delta^2\partial_{n+1}(\theta_1,\dots,\theta_n) & 1 + \partial_{n+1}\theta_{n+1} \end{pmatrix} \tag{4.4}$$

and that

$$\begin{cases} \Theta(x,0) = \big(x, b(x)\big), \quad \Theta(x,1) = \big(x, 1+\eta(x)\big), \\ \dfrac{\partial\Theta}{\partial X}(x,0) = \begin{pmatrix} E_n & (\nabla b(x))^T \\ -\delta^2\nabla b(x) & 1 \end{pmatrix}, \\ \dfrac{\partial\Theta}{\partial X}(x,1) = \begin{pmatrix} E_n & (\nabla\eta(x))^T \\ -\delta^2\nabla\eta(x) & 1 \end{pmatrix}. \end{cases}$$

We put $\tilde{\Phi} := \Phi \circ \Theta$ and

$$\begin{aligned} P \; &:= \; \det\Big(\frac{\partial\Theta}{\partial X}\Big) I_\delta^{-1}\bigg(\Big(\frac{\partial\Theta}{\partial X}\Big)^{-1}\bigg)^T I_\delta^2 \Big(\frac{\partial\Theta}{\partial X}\Big)^{-1} I_\delta^{-1} \\ &= \; \det\Big(\frac{\partial\Theta}{\partial X}\Big)\bigg(\Big(I_\delta\frac{\partial\Theta}{\partial X}I_\delta^{-1}\Big)\Big(I_\delta\frac{\partial\Theta}{\partial X}I_\delta^{-1}\Big)^T\bigg)^{-1}. \end{aligned} \tag{4.5}$$

Then, the boundary value problem (4.1) is transformed into

$$\begin{cases} \nabla_X \cdot I_\delta P I_\delta \nabla_X\tilde{\Phi} = 0 & \text{in} \quad 0 < x_{n+1} < 1, \\ \tilde{\Phi} = \phi & \text{on} \quad x_{n+1} = 1, \\ \partial_{n+1}\tilde{\Phi} = 0 & \text{on} \quad x_{n+1} = 0. \end{cases} \tag{4.6}$$

Here, we have

$$I_\delta \frac{\partial \Theta}{\partial X} I_\delta^{-1} = \begin{pmatrix} E_n + \delta^2 \dfrac{\partial(\theta_1,\dots,\theta_n)}{\partial(x_1,\dots,x_n)} & \delta(\nabla\theta_{n+1})^T \\ \delta\partial_{n+1}(\theta_1,\dots,\theta_n) & 1+\partial_{n+1}\theta_{n+1} \end{pmatrix}. \tag{4.7}$$

On the upper boundary $x_{n+1}=1$, we have $\det\big(\frac{\partial\Theta}{\partial X}\big) = 1+\delta^2|\nabla\eta|^2$ so that

$$\begin{aligned} P &= (1+\delta^2|\nabla\eta|^2)\left(\begin{pmatrix} E_n & \delta(\nabla\eta)^T \\ -\delta\nabla\eta & 1 \end{pmatrix}\begin{pmatrix} E_n & -\delta(\nabla\eta)^T \\ \delta\nabla\eta & 1 \end{pmatrix}\right)^{-1} \\ &= (1+\delta^2|\nabla\eta|^2)\begin{pmatrix} E_n+\delta^2(\nabla\eta)^T\nabla\eta & 0 \\ 0 & 1+\delta^2|\nabla\eta|^2 \end{pmatrix}^{-1}. \end{aligned}$$

Similar identity holds for the lower boundary $x_{n+1}=0$. Therefore, we see that

$$P(x,0) = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}, \quad P(x,1) = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.8}$$

Particularly, it holds that

$$\mathbf{e}_{n+1}\cdot I_\delta P I_\delta \nabla_X\tilde\Phi = \mathbf{e}_{n+1}\cdot I_\delta^2\nabla_X\tilde\Phi = \delta^{-2}\partial_{n+1}\tilde\Phi \quad \text{on} \quad x_{n+1}=0,1. \tag{4.9}$$

We also have the relation

$$I_\delta\nabla\tilde\Phi = \begin{pmatrix} E_n + \delta^2 \dfrac{\partial(\theta_1,\dots,\theta_n)}{\partial(x_1,\dots,x_n)} & \delta(\nabla\theta_{n+1})^T \\ \delta\partial_{n+1}(\theta_1,\dots,\theta_n) & 1+\partial_{n+1}\theta_{n+1} \end{pmatrix} I_\delta(\nabla\Phi)\circ\Theta. \tag{4.10}$$

**Assumption 4.1.** Let $r > n/2$.

(A1) There exists a $C^1$-diffeomorphism $\Theta : \overline{\Omega}_0 \to \overline{\Omega}$ satisfying (4.2), (4.3), and the conditions $\det\big(\frac{\partial\Theta}{\partial X}(X)\big) \ge c > 0$ and $|\nabla_X\theta(X)| \le M$ for $X\in\Omega_0$.

(A2) $\|\nabla_X\theta(\cdot,x_{n+1})\|_{r+1} \le M$ for $0\le x_{n+1}\le 1$.

(A3) $\|\nabla_X\theta(\cdot,x_{n+1})\|_{r+2} \le M$ for $0\le x_{n+1}\le 1$.

The construction of a diffeomorphism $\Theta$ satisfying the above conditions will be given later. By (4.10) we can easily obtain the following lemma.

**Lemma 4.3.** *Under Assumption* 4.1 (A1), *there exists a constant* $C=C(M,c)\ge 1$ *such that*

$$C^{-1}\|I_\delta\nabla_X\Phi\|_{L^2(\Omega)} \le \|I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)} \le C\|I_\delta\nabla_X\Phi\|_{L^2(\Omega)}.$$

**Lemma 4.4.** *Under Assumption* 4.1 (A1), *there exists a constant* $C=C(M,c)\ge 1$ *such that for any* $\phi\in H^1$ *we have*

$$C^{-1}\|\Lambda_0^{1/2}\phi\|^2 \le (\Lambda\phi,\phi) \le C\|\Lambda_0^{1/2}\phi\|^2,$$

*where* $\Lambda_0 = \Lambda(0,0,\delta) = \frac{1}{\delta}|D|\tanh(\delta|D|)$.

**Proof**. We set $\Phi := \phi^\hbar$ and $\tilde\Phi := \Phi\circ\Theta$, and decompose $\tilde\Phi = \tilde\Phi_1 + \tilde\Phi_2$, where $\tilde\Phi_1$ and $\tilde\Phi_2$ are solutions of the boundary value problems

$$\begin{cases} \nabla_X\cdot I_\delta^2\nabla_X\tilde\Phi_1 = 0 & \text{in} \quad 0 < x_{n+1} < 1, \\ \tilde\Phi_1 = \phi & \text{on} \quad x_{n+1} = 1, \\ \partial_{n+1}\tilde\Phi_1 = 0 & \text{on} \quad x_{n+1} = 0 \end{cases}$$

and

$$\begin{cases} \nabla_X\cdot I_\delta^2\nabla_X\tilde\Phi_2 = \nabla_X\cdot I_\delta(I_1-P)I_\delta\nabla_X\tilde\Phi & \text{in} \quad 0 < x_{n+1} < 1, \\ \tilde\Phi_2 = 0 & \text{on} \quad x_{n+1} = 1, \\ \partial_{n+1}\tilde\Phi_2 = 0 & \text{on} \quad x_{n+1} = 0, \end{cases}$$

respectively. Then, it holds that

$$\Lambda\phi = \delta^{-2}\partial_{n+1}\tilde\Phi(\cdot,1) = \delta^{-2}\partial_{n+1}\tilde\Phi_1(\cdot,1) + \delta^{-2}\partial_{n+1}\tilde\Phi_2(\cdot,1) = \Lambda_0\phi + \delta^{-2}\partial_{n+1}\tilde\Phi_2(\cdot,1)$$

and, by Lemma 4.2, that

$$(\Lambda\phi,\phi) = \|I_\delta\nabla_X\Phi\|_{L^2(\Omega)}^2, \qquad \|\Lambda_0^{1/2}\phi\|^2 = (\Lambda_0\phi,\phi) = \|I_\delta\nabla_X\tilde\Phi_1\|_{L^2(\Omega_0)}^2.$$

By Green's formula we see that

$$\begin{aligned} &\big(\delta^{-2}\partial_{n+1}\tilde\Phi_2(\cdot,1),\phi\big) = \big(\delta^{-2}\partial_{n+1}\tilde\Phi_2(\cdot,1),\tilde\Phi_1(\cdot,1)\big) \\ &= \int_{\Omega_0} I_\delta\nabla_X\tilde\Phi_2\cdot I_\delta\nabla_X\tilde\Phi_1 dX + \int_{\Omega_0}(\nabla_X\cdot I_\delta^2\nabla_X\tilde\Phi_2)\tilde\Phi_1 dX \\ &= \int_{\Omega_0} I_\delta\nabla_X\tilde\Phi_2\cdot I_\delta\nabla_X\tilde\Phi_1 dX + \int_{\Omega_0}\big(\nabla_X\cdot I_\delta(I_1-P)I_\delta\nabla_X\tilde\Phi\big)\tilde\Phi_1 dX \\ &= \int_{\Omega_0} I_\delta\nabla_X\tilde\Phi_2\cdot I_\delta\nabla_X\tilde\Phi_1 dX - \int_{\Omega_0}(I_1-P)I_\delta\nabla_X\tilde\Phi\cdot I_\delta\nabla_X\tilde\Phi_1 dX, \end{aligned}$$

where we used (4.8). Therefore,

$$\big|\big(\delta^{-2}\partial_{n+1}\tilde\Phi_2(\cdot,1),\phi\big)\big| \le C(\|I_\delta\nabla_X\tilde\Phi_2\|_{L^2(\Omega_0)} + \|I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)})\|I_\delta\nabla_X\tilde\Phi_1\|_{L^2(\Omega_0)}.$$

Similarly, by the equations for $\tilde\Phi_2$ we see that

$$\begin{aligned} \|I_\delta\nabla_X\tilde\Phi_2\|_{L^2(\Omega_0)}^2 &= -\int_{\Omega_0}(\nabla_X\cdot I_\delta^2\nabla_X\tilde\Phi_2)\tilde\Phi_2 dX = -\int_{\Omega_0}(\nabla_X\cdot I_\delta(I_1-P)I_\delta\nabla_X\tilde\Phi)\tilde\Phi_2 dX \\ &= \int_{\Omega_0}(I_1-P)I_\delta\nabla_X\tilde\Phi\cdot I_\delta\nabla_X\tilde\Phi_2 dX \le C\|I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)}\|I_\delta\nabla_X\tilde\Phi_2\|_{L^2(\Omega_0)}, \end{aligned}$$

so that

$$\|I_\delta\nabla_X\tilde\Phi_2\|_{L^2(\Omega_0)} \le C\|I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)} \le C\|I_\delta\nabla_X\Phi\|_{L^2(\Omega)},$$

where we used Lemma 4.3. Summarizing the above estimates we obtain

$$\big|(\Lambda\phi,\phi) - (\Lambda_0\phi,\phi)\big| \le C\|I_\delta\nabla_X\Phi\|_{L^2(\Omega)}\|I_\delta\nabla_X\tilde\Phi_1\|_{L^2(\Omega_0)} \le C\sqrt{(\Lambda\phi,\phi)}\sqrt{(\Lambda_0\phi,\phi)},$$

which easily yields the desired inequalities. □

Now, let us construct the diffeomorphism $\Theta$ satisfying the conditions in Assumption 4.1.

**Lemma 4.5.** *Let $r > n/2$, $c_1, M_1 > 0$ and suppose that $\eta, b \in H^{1+r}$ satisfy the conditions*

$$\begin{cases} \|\eta\|_{1+r} + \|b\|_{1+r} \le M_1, \\ 1 + \eta(x) - b(x) \ge c_1 \quad \textit{for} \quad x \in \mathbf{R}^n. \end{cases}$$

*Then, there exists a constant $\delta_1 = \delta_1(M_1, c_1, r) > 0$ such that for any $\delta \in (0, \delta_1]$ we can construct a diffeomorphism $\Theta$ satisfying the conditions in Assumption* 4.1 (A1)*. Moreover, for any $s > 0$ and $k \in \mathbf{N}$ we have*

$$\begin{cases} \|J^s \nabla_X \theta\|_{L^2(\Omega_0)} \le C_1(\|\eta\|_{s+1/2} + \|b\|_{s+1/2}), \\ \sup\limits_{0<x_{n+1}<1} \|\partial_{n+1}^k \theta(\cdot, x_{n+1})\|_s \le C_2(\|\eta\|_{s+k} + \|b\|_{s+k}), \end{cases} \tag{4.11}$$

*where $C_1 = C_1(c_1) > 0$ and $C_2 = C_2(c_1, k) > 0$. In the case where $\eta$ depends also on the time $t$, for any $l \in \mathbf{N}$ we have*

$$\begin{cases} \|J^s \nabla_X \partial_t^l \theta(t)\|_{L^2(\Omega_0)} \le C_1 \|\partial_t^l \eta(t)\|_{s+1/2}, \\ \sup\limits_{0<x_{n+1}<1} \|\partial_{n+1}^k \partial_t^l \theta(\cdot, x_{n+1}, t)\|_s \le C_2 \|\partial_t^l \eta(t)\|_{s+k}. \end{cases} \tag{4.12}$$

**Proof**. Without loss of generality we can assume that $0 < c_1 < 1$. Since $\frac{1}{2}\big(1 + \frac{1}{1-c_1}\big) > 1$, we can take $\varphi \in C^\infty(\mathbf{R})$ satisfying the conditions

$$\varphi(x_{n+1}) = \begin{cases} 0 & \text{for} \quad x_{n+1} \le 0, \\ 1 & \text{for} \quad x_{n+1} \ge 1 \end{cases} \qquad \text{and} \qquad 0 \le \varphi'(x_{n+1}) \le \frac{1}{2}\left(1 + \frac{1}{1 - c_1}\right).$$

Then, it is easy to check that $1 + \big(\eta(x) - b(x)\big)\varphi'(x_{n+1}) \ge c_1/2$ holds for any $X = (x, x_{n+1}) \in \Omega_0$. Define the functions $\theta_j$, $1 \le j \le n+1$, by the relations

$$\begin{cases} \hat\theta_j(\xi, x_{n+1}) = \varphi(x_{n+1}) \mathrm{e}^{-|\xi|(1-x_{n+1})} i\xi_j (1 - x_{n+1}) \hat\eta(\xi) \\ \qquad\qquad - \big(1 - \varphi(x_{n+1})\big) \mathrm{e}^{-|\xi| x_{n+1}} i\xi_j x_{n+1} \hat b(\xi) \quad \text{for} \quad 1 \le j \le n, \\ \hat\theta_{n+1}(\xi, x_{n+1}) = \varphi(x_{n+1}) \mathrm{e}^{-\varepsilon|\xi|(1-x_{n+1})} \big(1 + \varepsilon|\xi|(1 - x_{n+1})\big) \hat\eta(\xi) \\ \qquad\qquad + \big(1 - \varphi(x_{n+1})\big) \mathrm{e}^{-\varepsilon|\xi| x_{n+1}} \big(1 + \varepsilon|\xi| x_{n+1}\big) \hat b(\xi), \end{cases}$$

where $\varepsilon > 0$ will be determined later. Obviously, (4.2) is satisfied. It is easy to see that

$$\begin{aligned} &|\partial_{n+1}^k \hat\theta(\xi, x_{n+1})|^2 \le C|\xi|^2 (1 + |\xi|)^{2(k-1)} \big(|\hat\eta(\xi)|^2 + |\hat b(\xi)|^2\big), \\ &\int_0^1 |\widehat{\nabla_X \theta}(\xi, x_{n+1})|^2 dx_{n+1} \le C|\xi| \big(|\hat\eta(\xi)|^2 + |\hat b(\xi)|^2\big), \end{aligned}$$

which yield (4.11). In the same way as above, we can show (4.12). It remains to show the estimates in Assumption 4.1 (A1). The latter estimate in (A1) comes from (4.11) and the Sobolev inequality. In view of the relation

$$\begin{aligned} &\hat\theta_{n+1}(\xi, x_{n+1}) - \varphi(x_{n+1})\hat\eta(\xi) - \big(1 - \varphi(x_{n+1})\big)\hat b(\xi) \\ &\quad= \varepsilon\big\{\varphi(x_{n+1}) \mathrm{e}^{-\varepsilon|\xi|(1-x_{n+1})} |\xi|(1 - x_{n+1})\hat\eta(\xi) + \big(1 - \varphi(x_{n+1})\big) \mathrm{e}^{-\varepsilon|\xi| x_{n+1}} |\xi| x_{n+1} \hat b(\xi)\big\} \\ &\qquad + \varphi(x_{n+1})(\mathrm{e}^{-\varepsilon|\xi|(1-x_{n+1})} - 1)\hat\eta(\xi) + \big(1 - \varphi(x_{n+1})\big)(\mathrm{e}^{-\varepsilon|\xi| x_{n+1}} - 1)\hat b(\xi), \end{aligned}$$

we obtain

$$\begin{aligned} |\partial_{n+1}\theta_{n+1}(x, x_{n+1}) - \big(\eta(x) - b(x)\big)\varphi'(x_{n+1})| &\le \varepsilon C \int_{\mathbf{R}^n} |\xi| \big(|\hat\eta(\xi)| + |\hat b(\xi)|\big) d\xi \\ &\le \varepsilon C\big(\|\eta\|_{1+r} + \|b\|_{1+r}\big) \le \varepsilon C M_1. \end{aligned}$$

Therefore, if we take $\varepsilon > 0$ so small that $\varepsilon C M_1 \le c_1/4$, then

$$\begin{aligned} 1+\partial_{n+1}\theta_{n+1}(x,x_{n+1}) &\ge 1+\big(\eta(x)-b(x)\big)\varphi'(x_{n+1}) \\ &\quad -|\partial_{n+1}\theta_{n+1}(x,x_{n+1})-\big(\eta(x)-b(x)\big)\varphi'(x_{n+1})| \\ &\ge \frac{c_1}{2}-\frac{c_1}{4}=\frac{c_1}{4}. \end{aligned}$$

On the other hand, it follows from (4.4) that

$$\det\left(\frac{\partial\Theta}{\partial X}\right)=1+\partial_{n+1}\theta_{n+1}+\delta^2 J_1, \tag{4.13}$$

where $J_1$ is a polynomial of $\nabla_X\theta$ with coefficients which are polynomials of $\delta^2$. Hence, we have

$$\det\left(\frac{\partial\Theta}{\partial X}(X)\right)\ge\frac{c_1}{4}-\delta^2 C.$$

Therefore, if we take $\delta_1 > 0$ so small that $\delta^2 C \le c_1/8$, then we obtain the former estimate in (A1). Particularly, we see that $\Theta : \overline{\Omega}_0 \to \overline{\Omega}$ is a $C^1$-diffeomorphism. □

# 5 Estimates of the Dirichlet-to-Neumann map

**Lemma 5.1.** *Let $r > n/2$. There exists a constant $C = C(r) > 0$ such that we have*

$$\|[\Lambda_0^{1/2}, a]u\| \le C\|\nabla a\|_r\|u\|.$$

**Proof**. Put $v := [\Lambda_0^{1/2}, a]u$. Then, we have

$$\hat v(\xi)=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\big(\sqrt{\delta^{-1}|\xi|\tanh(\delta|\xi|)}-\sqrt{\delta^{-1}|\eta|\tanh(\delta|\eta|)}\big)\hat a(\xi-\eta)\hat u(\eta)d\eta.$$

It is easy to see that $|\sqrt{\alpha\tanh\alpha}-\sqrt{\beta\tanh\beta}|\le C|\alpha-\beta|$ for $\alpha,\beta\ge 0$, so that

$$\big|\sqrt{\delta^{-1}|\xi|\tanh(\delta|\xi|)}-\sqrt{\delta^{-1}|\eta|\tanh(\delta|\eta|)}\big|\le C|\xi-\eta|, \tag{5.1}$$

and that

$$|\hat v(\xi)|\le C\int_{\mathbf{R}^n}|\xi-\eta||\hat a(\xi-\eta)||\hat u(\eta)|d\eta.$$

This and Hausdorff-Young's inequality give the desired estimate. □

**Lemma 5.2.** *For any real $s$, we have*

$$\begin{cases} \|\nabla\phi\|_s \le \sqrt{2(1+\delta)}\|\Lambda_0^{1/2}\phi\|_{s+1/2}, \\ \|\Lambda_0^{1/2}\phi\|_s \le \min\{\|\nabla\phi\|_s, \delta^{-1/2}\|\phi\|_{s+1/2}\}. \end{cases}$$

**Proof**. By the inequalities $(1+\sqrt{\alpha})^{-1}\alpha \le \sqrt{\alpha\tanh\alpha}\le\min\{\alpha,\sqrt{\alpha}\}$ for $\alpha\ge 0$, it holds that

$$\big(1+\sqrt{\delta|\xi|}\big)^{-1}|\xi|\le\sqrt{\delta^{-1}|\xi|\tanh(\delta|\xi|)}\le\min\{|\xi|,\delta^{-1/2}|\xi|^{1/2}\}\quad\text{for}\quad \xi\in\mathbf{R}^n,\ \delta>0,$$

which yields the desired estimates. □

**Lemma 5.3.** *Under Assumption* 4.1 (A1) *and* (A2), *there exists a constant* $C = C(M, c, r) > 0$ *such that we have*

$$\|\Lambda\phi\| \le C\big(\|\Lambda_0\phi\| + \|\Lambda_0^{1/2}\phi\|\big).$$

**Proof**. Set $\Phi := \phi^\hbar$ and $\tilde{\Phi} := \Phi \circ \Theta$. We take $\psi \in H^0$ arbitrarily and define $\tilde{\Psi}$ by $\tilde{\Psi}(\cdot, x_{n+1}) = \mathrm{e}^{-\delta|D|(1-x_{n+1})}\psi$. By Green's formula, we see that

$$(\Lambda\phi, \psi) = \int_{\Omega_0} PI_\delta\nabla_X\tilde{\Phi} \cdot I_\delta\nabla_X\tilde{\Psi}dX = \int_{\Omega_0} \Lambda_0^{1/2}PI_\delta\nabla_X\tilde{\Phi} \cdot I_\delta\Lambda_0^{-1/2}\nabla_X\tilde{\Psi}dX. \tag{5.2}$$

In view of the relations

$$|D|\Lambda_0^{-1/2}\tilde{\Psi}(\cdot, x_{n+1}) = \delta^{-1}\partial_{n+1}\Lambda_0^{-1/2}\tilde{\Psi}(\cdot, x_{n+1}) = \sqrt{\frac{\delta|D|}{\tanh(\delta|D|)}}\mathrm{e}^{-\delta|D|(1-x_{n+1})}\psi,$$

we have

$$\int_{\Omega_0} |I_\delta\Lambda_0^{-1/2}\nabla_X\tilde{\Psi}|^2dX \le C\|\psi\|^2.$$

This and (5.2) imply that

$$\begin{aligned}
\|\Lambda\phi\|^2 &\le C\int_{\Omega_0} |\Lambda_0^{1/2}PI_\delta\nabla_X\tilde{\Phi}|^2dX \\
&\le C\bigg(\int_{\Omega_0} |PI_\delta\nabla_X\Lambda_0^{1/2}\tilde{\Phi}|^2dX + \int_{\Omega_0} |[\Lambda_0^{1/2}, P]I_\delta\nabla_X\tilde{\Phi}|^2dX\bigg).
\end{aligned} \tag{5.3}$$

Set $\Phi_1 := (\Lambda_0^{1/2}\phi)^\hbar$ and $\tilde{\Phi}_1 := \Phi_1 \circ \Theta$. Then, it holds that

$$\begin{cases}
\nabla_X \cdot I_\delta PI_\delta\nabla_X(\Lambda_0^{1/2}\tilde{\Phi} - \tilde{\Phi}_1) = -\nabla_X \cdot I_\delta[\Lambda_0^{1/2}, P]I_\delta\nabla_X\tilde{\Phi}, \\
(\Lambda_0^{1/2}\tilde{\Phi} - \tilde{\Phi}_1)(\cdot, 1) = 0, \\
\mathbf{e}_{n+1} \cdot I_\delta^2\nabla(\Lambda_0^{1/2}\tilde{\Phi} - \tilde{\Phi}_1)(\cdot, 0) = 0.
\end{cases}$$

Therefore, by Green's formula we see that

$$\begin{aligned}
&\int_{\Omega_0} PI_\delta\nabla_X(\Lambda_0^{1/2}\tilde{\Phi} - \tilde{\Phi}_1) \cdot I_\delta\nabla_X(\Lambda_0^{1/2}\tilde{\Phi} - \tilde{\Phi}_1)dX \\
&\quad = -\int_{\Omega_0} \big(\nabla_X \cdot I_\delta PI_\delta\nabla_X(\Lambda_0^{1/2}\tilde{\Phi} - \tilde{\Phi}_1)\big)(\Lambda_0^{1/2}\tilde{\Phi} - \tilde{\Phi}_1)dX \\
&\quad = \int_{\Omega_0} \big(\nabla_X \cdot I_\delta[\Lambda_0^{1/2}, P]I_\delta\nabla_X\tilde{\Phi}\big)(\Lambda_0^{1/2}\tilde{\Phi} - \tilde{\Phi}_1)dX \\
&\quad = -\int_{\Omega_0} [\Lambda_0^{1/2}, P]I_\delta\nabla_X\tilde{\Phi} \cdot I_\delta\nabla_X(\Lambda_0^{1/2}\tilde{\Phi} - \tilde{\Phi}_1)dX,
\end{aligned}$$

where we used (4.8). This implies that

$$\int_{\Omega_0} \big|I_\delta\nabla_X(\Lambda_0^{1/2}\tilde{\Phi} - \tilde{\Phi}_1)\big|^2dX \le C\int_{\Omega_0} \big|[\Lambda_0^{1/2}, P]I_\delta\nabla_X\tilde{\Phi}\big|^2dX.$$

Hence, by (5.3) we obtain

$$\|\Lambda\phi\|^2 \le C\bigg(\int_{\Omega_0} |I_\delta\nabla_X\tilde{\Phi}_1|^2dX + \int_{\Omega_0} |[\Lambda_0^{1/2}, P]I_\delta\nabla_X\tilde{\Phi}|^2dX\bigg).$$

Here, by Lemma 5.1 and the hypothesis on $P$ we have $\|[\Lambda_0^{1/2}, P]u\| \le C\|u\|$, so that

$$\begin{aligned}
\|\Lambda\phi\|^2 &\le C\Big(\int_{\Omega_0} |I_\delta \nabla_X \tilde{\Phi}_1|^2 dX + \int_{\Omega_0} |I_\delta \nabla_X \tilde{\Phi}|^2 dX\Big) \\
&\le C\big((\Lambda\Lambda_0^{1/2}\phi, \Lambda_0^{1/2}\phi) + (\Lambda\phi, \phi)\big) \\
&\le C\big(\|\Lambda_0\phi\|^2 + \|\Lambda_0^{1/2}\phi\|^2\big),
\end{aligned}$$

where we used Lemmas 4.2–4.4. This shows the desired estimate. □

**Lemma 5.4.** *Let $s > n/2+1$. Under Assumption* 4.1 (A1) *and*

$$\sup_{0\le x_{n+1}\le 1} \|\nabla_X \theta(\cdot, x_{n+1})\|_{s+1} \le M,$$

*there exists a constant $C = C(M, c, s) > 0$ such that we have*

$$\|[J^s, \Lambda]\phi\| \le C\|\Lambda_0^{1/2}\phi\|_s.$$

**Proof**. Set $\Phi := \phi^\hbar$, $\Phi_s := (J^s\phi)^\hbar$, $\tilde{\Phi} := \Phi \circ \Theta$, and $\tilde{\Phi}_s := \Phi_s \circ \Theta$. Then, we have

$$\begin{cases}
\nabla_X \cdot I_\delta P I_\delta \nabla_X (J^s\tilde{\Phi} - \tilde{\Phi}_s) = -\nabla_X \cdot I_\delta [J^s, P] I_\delta \nabla_X \tilde{\Phi}, \\
(J^s\tilde{\Phi} - \tilde{\Phi}_s)(\cdot, 1) = 0, \\
\mathbf{e}_{n+1} \cdot I_\delta^2 \nabla_X (J^s\tilde{\Phi} - \tilde{\Phi}_s)(\cdot, 0) = 0.
\end{cases} \tag{5.4}$$

and

$$[J^s, \Lambda]\phi = \mathbf{e}_{n+1} \cdot I_\delta^2 \nabla_X (J^s\tilde{\Phi} - \tilde{\Phi}_s)(\cdot, 1).$$

We take $\psi \in H^0$ arbitrarily and define $\tilde{\Psi}$ by $\tilde{\Psi}(\cdot, x_{n+1}) = \mathrm{e}^{-\delta|D|(1-x_{n+1})}\psi$. Taking the inner product of the equation in (5.4) and $\tilde{\Psi}$ in $L^2(\Omega_0)$ and using Green's formula, we see that

$$([J^s, \Lambda]\phi, \psi) = \int_{\Omega_0} [J^s, P] I_\delta \nabla_X \tilde{\Phi} \cdot I_\delta \nabla_X \tilde{\Psi} dX + \int_{\Omega_0} P I_\delta \nabla_X (J^s\tilde{\Phi} - \tilde{\Phi}_s) \cdot I_\delta \nabla_X \tilde{\Psi} dX.$$

In view of $\|J^{-1} I_\delta \nabla_X \tilde{\Psi}\|_{L^2(\Omega_0)}^2 \le C\|\psi\|^2$, we obtain

$$\begin{aligned}
\|[J^s, \Lambda]\phi\| &\le C\big(\|J[J^s, P] I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} + \|JPI_\delta \nabla_X (J^s\tilde{\Phi} - \tilde{\Phi}_s)\|_{L^2(\Omega_0)}\big) \\
&\le C\big(\|J^s I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} + \|JI_\delta \nabla_X (J^s\tilde{\Phi} - \tilde{\Phi}_s)\|_{L^2(\Omega_0)}\big) \\
&\le C\big(\|I_\delta \nabla_X \tilde{\Phi}_s\|_{L^2(\Omega_0)} + \|JI_\delta \nabla_X (J^s\tilde{\Phi} - \tilde{\Phi}_s)\|_{L^2(\Omega_0)}\big).
\end{aligned} \tag{5.5}$$

In the above calculation we used a commutator estimate $\|[J^s, a]u\|_1 \le C\|a\|_{s+1}\|u\|_s$.

On the other hand, taking the inner product of the equation in (5.4) and $J^2(J^s\tilde{\Phi} - \tilde{\Phi}_s)$ in $L^2(\Omega_0)$ and using Green's formula we obtain

$$\begin{aligned}
&\int_{\Omega_0} PJI_\delta \nabla_X (J^s\tilde{\Phi} - \tilde{\Phi}_s) \cdot JI_\delta \nabla_X (J^s\tilde{\Phi} - \tilde{\Phi}_s) dX \\
&= -\int_{\Omega_0} [J, P] I_\delta \nabla_X (J^s\tilde{\Phi} - \tilde{\Phi}_s) \cdot JI_\delta \nabla_X (J^s\tilde{\Phi} - \tilde{\Phi}_s) dX \\
&\quad -\int_{\Omega_0} J[J^s, P] I_\delta \nabla_X \tilde{\Phi} \cdot JI_\delta \nabla_X (J^s\tilde{\Phi} - \tilde{\Phi}_s) dX,
\end{aligned}$$

which implies that

$$\begin{aligned}\|JI_\delta\nabla_X(J^s\tilde\Phi-\tilde\Phi_s)\|_{L^2(\Omega_0)} &\leq C\big(\|I_\delta\nabla_X(J^s\tilde\Phi-\tilde\Phi_s)\|_{L^2(\Omega_0)}+\|J^sI_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)}\big)\\ &\leq C\big(\|I_\delta\nabla_X(J^s\tilde\Phi-\tilde\Phi_s)\|_{L^2(\Omega_0)}+\|I_\delta\nabla_X\tilde\Phi_s\|_{L^2(\Omega_0)}\big).\end{aligned}$$

Here, by the interpolation inequality for any $\varepsilon>0$ we have

$$\begin{aligned}&\|I_\delta\nabla_X(J^s\tilde\Phi-\tilde\Phi_s)\|_{L^2(\Omega_0)}\\ &\leq \varepsilon\|JI_\delta\nabla_X(J^s\tilde\Phi-\tilde\Phi_s)\|_{L^2(\Omega_0)}+C_\varepsilon\|J^{-s}I_\delta\nabla_X(J^s\tilde\Phi-\tilde\Phi_s)\|_{L^2(\Omega_0)}\\ &\leq \varepsilon\|JI_\delta\nabla_X(J^s\tilde\Phi-\tilde\Phi_s)\|_{L^2(\Omega_0)}+C_\varepsilon\big(\|I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)}+\|I_\delta\nabla_X\tilde\Phi_s\|_{L^2(\Omega_0)}\big).\end{aligned}$$

Therefore,

$$\|JI_\delta\nabla_X(J^s\tilde\Phi-\tilde\Phi_s)\|_{L^2(\Omega_0)}\leq C\big(\|I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)}+\|I_\delta\nabla_X\tilde\Phi_s\|_{L^2(\Omega_0)}\big),$$

which together with (5.5) implies that

$$\|[J^s,\Lambda]\phi\|\leq C\big(\|I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)}+\|I_\delta\nabla_X\tilde\Phi_s\|_{L^2(\Omega_0)}\big).$$

This and Lemmas 4.2–4.4 show the desired estimate. □

In view of $\|\Lambda\phi\|_s\leq\|\Lambda J^s\phi\|+\|[J^s,\Lambda]\phi\|$ and Lemmas 5.3–5.4, we can obtain the following lemma.

**Lemma 5.5.** *Under the hypothesis of Lemma* 5.4*, we have*

$$\|\Lambda\phi\|_s\leq C(\|\Lambda_0\phi\|_s+\|\Lambda_0^{1/2}\phi\|_s),$$

*where $C=C(M,c,s)>0$. Particularly, it holds that $\|\Lambda\phi\|_s\leq C\delta^{-1}\|\phi\|_{s+1}$.*

**Lemma 5.6.** *Let $s\geq 0$ and set $\Phi:=\phi^\hbar$ and $\tilde\Phi:=\Phi\circ\Theta$. Under Assumption* 4.1 (A1), (A2)*, and*

$$\|J^s\nabla_X\theta\|_{L^2(\Omega_0)}+\sup_{0\leq x_{n+1}\leq 1}\|\nabla_X\theta(\cdot,x_{n+1})\|_{s-1/2}\leq M,$$

*there exists a constant $C=C(M,c,r,s)>0$ such that we have*

$$\begin{cases}\|J^sI_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)}\leq C\big(\|\Lambda_0^{1/2}\phi\|_s+\|I_\delta\nabla_X\tilde\Phi\|_{L^\infty(\Omega_0)}\big),\\ \|J^{r+1}I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)}\leq C\|\Lambda_0^{1/2}\phi\|_{r+1}.\end{cases}$$

**Proof**. Set $\Phi_s:=(J^s\phi)^\hbar$ and $\tilde\Phi_s:=\Phi_s\circ\Theta$. Then, we have (5.4). Taking the inner product of (5.4) with $J^s\tilde\Phi-\tilde\Phi_s$ in $L^2(\Omega_0)$ and using Green's formula, we see that

$$\int_{\Omega_0}PI_\delta\nabla_X(J^s\tilde\Phi-\tilde\Phi_s)\cdot I_\delta\nabla_X(J^s\tilde\Phi-\tilde\Phi_s)dX=-\int_{\Omega_0}[J^s,P]I_\delta\nabla_X\tilde\Phi\cdot I_\delta\nabla_X(J^s\tilde\Phi-\tilde\Phi_s)dX,$$

which implies that

$$\begin{aligned}&\|I_\delta\nabla_X(J^s\tilde\Phi-\tilde\Phi_s)\|_{L^2(\Omega_0)}\\ &\leq C\|[J^s,P]I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)}\\ &\leq C\big(\|J^sP\|_{L^2(\Omega_0)}\|I_\delta\nabla_X\tilde\Phi\|_{L^\infty(\Omega_0)}+\|\nabla P\|_{L^\infty(\Omega_0)}\|J^{s-1}I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)}\big),\end{aligned}\tag{5.6}$$

where we used the well-known commutator estimate $\|[J^s,a]u\| \le C(|\nabla a|_\infty \|J^{s-1}u\| + \|J^s a\| |u|_\infty)$. Note that Assumption 4.1 (A2) and the Sobolev inequality imply that $\|\nabla\nabla_X\theta\|_{L^\infty(\Omega_0)} \le C$. Hence, it holds that $\|J^s P\|_{L^2(\Omega_0)} + \|\nabla P\|_{L^\infty(\Omega_0)} \le C$, so that

$$\begin{aligned}\|J^s I_\delta \nabla_X \tilde\Phi\|_{L^2(\Omega_0)} &\le \|I_\delta\nabla_X\tilde\Phi_s\|_{L^2(\Omega_0)} + \|I_\delta\nabla_X(J^s\tilde\Phi - \tilde\Phi_s)\|_{L^2(\Omega_0)} \\ &\le \|I_\delta\nabla_X\tilde\Phi_s\|_{L^2(\Omega_0)} + C\big(\|I_\delta\nabla_X\tilde\Phi\|_{L^\infty(\Omega_0)} + \|J^{s-1}I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)}\big).\end{aligned}$$

This and the interpolation inequality yields that

$$\|J^s I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)} \le C\big(\|I_\delta\nabla_X\tilde\Phi_s\|_{L^2(\Omega_0)} + \|I_\delta\nabla_X\tilde\Phi\|_{L^\infty(\Omega_0)} + \|I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)}\big). \tag{5.7}$$

It follows from Lemmas 4.2–4.4 that $\|I_\delta\nabla_X\tilde\Phi_s\|_{L^2(\Omega_0)}$ and $\|I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)}$ are equivalent to $\|\Lambda_0^{1/2}\phi\|_s$ and $\|\Lambda_0^{1/2}\phi\|$, respectively, so that we obtain the first estimate of the lemma.

Set $\Phi_{r+1} := (J^{r+1}\phi)^\hbar$ and $\tilde\Phi_{r+1} := \Phi_{r+1}\circ\Theta$. Since $\|J^{r+1}P(\cdot,x_{n+1})\| \le C$ for $0 \le x_{n+1} \le 1$, in place of (5.6) we have

$$\begin{aligned}\|I_\delta\nabla_X(J^{r+1}\tilde\Phi - \tilde\Phi_{r+1})\|_{L^2(\Omega_0)} &\le C\|[J^{r+1},P]I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)} \\ &\le C\|J^r I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)},\end{aligned}$$

where we used the commutator estimate $\|[J^{r+1},a]u\| \le C\|J^{r+1}a\|\|J^r u\|$. Therefore, in place of (5.7) we obtain

$$\|J^{r+1}I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)} \le C\big(\|I_\delta\nabla_X\tilde\Phi_{r+1}\|_{L^2(\Omega_0)} + \|I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)}\big),$$

which together with Lemmas 4.2–4.4 yields the second estimate of the lemma. □

We proceed to give a $L^\infty$-estimate of $I_\delta\nabla_X\tilde\Phi$. To this end, we use (4.6), where the matrix $P$ is defined by (4.5). It follows from (4.7) that

$$A := \left(I_\delta\frac{\partial\Theta}{\partial X}I_\delta^{-1}\right)\left(I_\delta\frac{\partial\Theta}{\partial X}I_\delta^{-1}\right)^T = A_1 + \delta^2 A_2,$$

where $A_2$ is a matrix whose elements are polynomials of $\nabla_X\theta$ with coefficients which are also polynomials of $\delta^2$, and

$$A_1 = \begin{pmatrix} E_n & \delta\mathbf{a}^T \\ \delta\mathbf{a} & b \end{pmatrix},$$

where

$$\begin{cases} \mathbf{a} = \partial_{n+1}\theta + (1+\partial_{n+1}\theta_{n+1})\nabla\theta_{n+1}, \\ b = (1+\partial_{n+1}\theta_{n+1})^2. \end{cases}$$

By the definition of $P$ we have $P = \big(\det(\frac{\partial\Theta}{\partial X})\big)^{-1}\widetilde{A}$, where $\widetilde{A}$ is the adjoint matrix of $A$ and has the form

$$\widetilde{A} = \widetilde{A}_1 + \delta^2 A_3,$$

where $A_3$ is a matrix whose elements are polynomials of $\nabla_X\theta$. Moreover, we see that

$$\widetilde{A}_1 = \begin{pmatrix} (b-\delta^2|\mathbf{a}|^2)E_n + \delta^2\mathbf{a}^T\mathbf{a} & -\delta\mathbf{a}^T \\ -\delta\mathbf{a} & 1 \end{pmatrix}.$$

By these relations and (4.13) we see that the matrix $P$ has the form

$$P = \begin{pmatrix} (1+\partial_{n+1}\theta_{n+1})E_n + \delta^2 P_{11} & \delta\mathbf{p}_{12}^T \\ \delta\mathbf{p}_{12} & (1+\partial_{n+1}\theta_{n+1})^{-1} + \delta^2 p_{22} \end{pmatrix}, \tag{5.8}$$

where $P_{11}$, $\mathbf{p}_{12}$, and $p_{22}$ are $n \times n$, $1 \times n$, and $1 \times 1$ matrixes whose elements are polynomials of $\nabla_X \theta$. Moreover, it follows from (4.8) that

$$\mathbf{p}_{12}(x,0) = \mathbf{p}_{12}(x,1) = \mathbf{0}, \quad p_{22}(x,0) = p_{22}(x,1) = 0. \tag{5.9}$$

Using these notations we can rewrite the first equation in (4.6) as

$$\begin{aligned} &\partial_{n+1}\big( (\delta^{-2}(1+\partial_{n+1}\theta_{n+1})^{-1} + p_{22})\partial_{n+1}\tilde{\Phi}\big) \\ &\quad = -\nabla \cdot \big( ((1+\partial_{n+1}\theta_{n+1})E_n + \delta^2 P_{11})\nabla\tilde{\Phi}\big) - \nabla \cdot (\mathbf{p}_{12}\partial_{n+1}\tilde{\Phi}) - \partial_{n+1}(\mathbf{p}_{12}\cdot\nabla\tilde{\Phi}). \end{aligned} \tag{5.10}$$

It follows from this, the boundary condition on $x_{n+1} = 0$, (4.2), and (5.9) that

$$\begin{aligned} \partial_{n+1}\tilde{\Phi} &= \int_0^{x_{n+1}} \partial_{n+1}\big( ((1+\partial_{n+1}\theta_{n+1})^{-1} + \delta^2 p_{22})\partial_{n+1}\tilde{\Phi}\big) dx_{n+1} \\ &= -\delta^2 \int_0^{x_{n+1}} \nabla \cdot \big( ((1+\partial_{n+1}\theta_{n+1})E_n + \delta^2 P_{11})\nabla\tilde{\Phi}\big) dx_{n+1} \\ &\quad -\delta^2 \int_0^{x_{n+1}} \nabla \cdot (\mathbf{p}_{12}\partial_{n+1}\tilde{\Phi}) dx_{n+1} - \delta^2 \mathbf{p}_{12}\cdot\nabla\tilde{\Phi}. \end{aligned} \tag{5.11}$$

We also have

$$\nabla\tilde{\Phi} = \nabla\phi - \int_{x_{n+1}}^1 \nabla\partial_{n+1}\tilde{\Phi} dx_{n+1}. \tag{5.12}$$

**Lemma 5.7.** *Let $\Phi = \phi^\hbar$ and $\tilde{\Phi} = \Phi \circ \Theta$. Under Assumption* 4.1 (A1) *and* (A2)*, there exists a constant $C = C(M, c, r) > 0$ such that we have*

$$\|I_\delta \nabla_X \tilde{\Phi}\|_{L^\infty(\Omega_0)} \le C\big( \|\nabla\phi\|_r + \delta\|\Lambda_0^{1/2}\phi\|_{r+1}\big).$$

**Proof**. Note that the assumptions imply the uniform boundedness of $P_{11}$, $p_{22}$, $\mathbf{p}_{12}$, and their first derivatives with respect to $x$. It follows from (5.12), the Sobolev inequality, and Lemma 5.6 that

$$\|\nabla\tilde{\Phi}\|_{L^\infty(\Omega_0)} \le \|\nabla\phi\|_r + \delta\|J^{r+1}I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)} \le \|\nabla\phi\|_r + C\delta\|\Lambda_0^{1/2}\phi\|_{r+1}.$$

Similarly, it follows from (5.11) that

$$\delta^{-1}\|\partial_{n+1}\tilde{\Phi}\|_{L^\infty(\Omega_0)} \le C\delta\big( \|J^{r+1}I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)} + \|\nabla\tilde{\Phi}\|_{L^\infty(\Omega_0)}\big) \le C\delta\|\Lambda_0^{1/2}\phi\|_{r+1},$$

where we used Lemma 5.2. These yield the desired estimate. □

**Remark 5.1.** In the case of a flat bottom, by appying the maximal principle to the subharmonic function $|I_\delta\nabla_X\Phi|^2$ we see that $\|I_\delta^2\nabla_X\Phi\|_{L^\infty(\Omega)} = \|I_\delta^2\nabla_X\Phi\|_{L^\infty(\Gamma)} \le \sqrt{|\nabla\phi|_\infty^2 + (\delta|\Lambda\phi|_\infty)^2}$.

# 6 Fréchet derivatives of the Dirichlet-to-Neumann map

The following lemma was obtained by D. Lannes [13].

**Lemma 6.1.** *The Fréchet derivative of $\Lambda(\eta, b, \delta)$ with respect to $\eta$ has the form*

$$D_\eta\Lambda(\eta,b,\delta)[\zeta]\phi = -\delta^2\Lambda(\eta,b,\delta)(Z\zeta) - \nabla\cdot(v\zeta),$$

*where*

$$\begin{cases} Z = (1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda(\eta,b,\delta)\phi + \nabla\eta\cdot\nabla\phi), \\ v = \nabla\phi - \delta^2 Z\nabla\eta. \end{cases} \tag{6.1}$$

We proceed to give estimates of the Fréchet derivatives of $\Lambda$ in the Sobolev spaces.

**Lemma 6.2.** *Let $s > n/2$. Under Assumption* 4.1 (A1) *and*

$$\sup_{0 \le x_{n+1} \le 1} \|\nabla_X \theta(\cdot, x_{n+1})\|_{s+1} \le M,$$

*there exists a constant $C = C(M, c, s) > 0$ such that we have*

$$\|D_\eta^n \Lambda[\zeta_1, \ldots, \zeta_n]\phi\|_s \le C\|\zeta_1\|_{s+3/2} \cdots \|\zeta_n\|_{s+3/2} \|\Lambda_0^{1/2}\phi\|_{s+1}.$$

*Similar estimate holds for the Fréchet derivative of $\Lambda$ with respect to $b$.*

**Proof**. We only show the estimate in the case $n = 1$, and the general case can be proved in the same way. Set $\Phi := \phi^\hbar$ and $\tilde{\Phi} := \Phi \circ \Theta$. Then, it holds that

$$\begin{cases} \nabla_X \cdot I_\delta P I_\delta \nabla_X \tilde{\Phi} = 0, \\ \tilde{\Phi}(\cdot, 1) = \phi, \quad \mathbf{e}_{n+1} \cdot I_\delta^2 \nabla_X \tilde{\Phi}(\cdot, 1) = \Lambda\phi, \\ \mathbf{e}_{n+1} \cdot I_\delta^2 \nabla_X \tilde{\Phi}(\cdot, 0) = 0. \end{cases}$$

For simplicity, we write $\Lambda_\eta \phi = D_\eta \Lambda[\zeta]\phi$, $\tilde{\Phi}_\eta = D_\eta \tilde{\Phi}[\zeta]$, and $P_\eta = D_\eta P[\zeta]$. Taking the Fréchet derivative of the above equations, we obtain

$$\begin{cases} \nabla_X \cdot I_\delta P I_\delta \nabla_X \tilde{\Phi}_\eta = -\nabla_X \cdot I_\delta P_\eta I_\delta \nabla_X \tilde{\Phi}, \\ \tilde{\Phi}_\eta(\cdot, 1) = 0, \quad \mathbf{e}_{n+1} \cdot I_\delta^2 \nabla_X \tilde{\Phi}_\eta(\cdot, 1) = \Lambda_\eta \phi, \\ \mathbf{e}_{n+1} \cdot I_\delta^2 \nabla_X \tilde{\Phi}_\eta(\cdot, 0) = 0. \end{cases} \tag{6.2}$$

We take $\psi \in H^0$ arbitrarily and define $\tilde{\Psi}$ by $\tilde{\Psi}(\cdot, x_{n+1}) = \mathrm{e}^{-\delta|D|(1-x_{n+1})}\psi$. Taking the inner product of the above equation and $J^s \tilde{\Psi}$ in $L^2(\Omega_0)$ and using Green's formula, we see that

$$(J^s \Lambda_\eta \phi, \psi) = \int_{\Omega_0} J^s P I_\delta \nabla_X \tilde{\Phi}_\eta \cdot I_\delta \nabla_X \tilde{\Psi} dX + \int_{\Omega_0} J^s P_\eta I_\delta \nabla_X \tilde{\Phi} \cdot I_\delta \nabla_X \tilde{\Psi} dX.$$

In view of $\|\Lambda_0^{-1/2} I_\delta \nabla_X \tilde{\Psi}\|_{L^2(\Omega_0)} \le C\|\psi\|$, we obtain

$$\begin{aligned} \|\Lambda_\eta \phi\|_s &\le C\big(\|\Lambda_0^{1/2} J^s P I_\delta \nabla_X \tilde{\Phi}_\eta\|_{L^2(\Omega_0)} + \|\Lambda_0^{1/2} J^s P_\eta I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}\big) \\ &\le C\big(\|J^{s+1} P I_\delta \nabla_X \tilde{\Phi}_\eta\|_{L^2(\Omega_0)} + \|J^{s+1} P_\eta I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}\big). \end{aligned} \tag{6.3}$$

On the other hand, taking the inner product of the first equation in (6.2) and $J^{2(s+1)}\tilde{\Phi}_\eta$ in $L^2(\Omega_0)$ and using Green's formula, we see that

$$\begin{aligned} &\int_{\Omega_0} P J^{s+1} I_\delta \nabla_X \tilde{\Phi}_\eta \cdot J^{s+1} I_\delta \nabla_X \tilde{\Phi}_\eta dX \\ &= -\int_{\Omega_0} [J^{s+1}, P] I_\delta \nabla_X \tilde{\Phi}_\eta \cdot J^{s+1} I_\delta \nabla_X \tilde{\Phi}_\eta dX - \int_{\Omega_0} J^{s+1} P_\eta I_\delta \nabla_X \tilde{\Phi} \cdot J^{s+1} I_\delta \nabla_X \tilde{\Phi}_\eta dX, \end{aligned}$$

which implies that

$$\begin{aligned} \|J^{s+1} I_\delta \nabla_X \tilde{\Phi}_\eta\|_{L^2(\Omega_0)} &\le C\big(\|[J^{s+1}, P] I_\delta \nabla_X \tilde{\Phi}_\eta\|_{L^2(\Omega_0)} + \|J^{s+1} P_\eta I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}\big) \\ &\le C\big(\|J^s I_\delta \nabla_X \tilde{\Phi}_\eta\|_{L^2(\Omega_0)} + \|J^{s+1} P_\eta I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}\big). \end{aligned} \tag{6.4}$$

Similarly, taking the inner product of the first equation in (6.2) and $\tilde{\Phi}_\eta$ in $L^2(\Omega_0)$ and using Green's formula yield that $\|I_\delta \nabla_X \tilde{\Phi}_\eta\|_{L^2(\Omega_0)} \leq C\|P_\eta I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}$. Therefore, by the interpolation inequality we obtain

$$\begin{aligned} &\|J^{s+1} I_\delta \nabla_X \tilde{\Phi}_\eta\|_{L^2(\Omega_0)} \leq C\|J^{s+1} P_\eta I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \\ &\leq \ C\big(\|J^{s+1}P_\eta\|_{L^2(\Omega_0)}\|I_\delta \nabla_X \tilde{\Phi}\|_{L^\infty(\Omega_0)} + \|\nabla P_\eta\|_{L^\infty(\Omega)}\|J^{s+1} I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}\big) \\ &\leq \ C\|\zeta\|_{s+3/2}\|\Lambda_0^{1/2}\phi\|_{s+1}, \end{aligned}$$

where we used Lemmas 5.2, 5.6 and 5.7. Hence, we obtain the desired estimate. □

**Lemma 6.3.** *Let $s > (n+1)/2$. Under Assumption* 4.1 (A1) *and*

$$\sup_{0 \leq x_{n+1} \leq 1} \|\nabla_X \theta(\cdot, x_{n+1})\|_{s+1/2} \leq M,$$

*there exists a constant $C = C(M, c, s) > 0$ such that we have*

$$\|D_\eta^n \Lambda[\zeta_1, \ldots, \zeta_n]\phi\|_s \leq C\delta^{-1/2}\|\zeta_1\|_{s+1} \cdots \|\zeta_n\|_{s+1}\|\Lambda_0^{1/2}\phi\|_{s+1/2}.$$

*Similar estimate holds for the Fréchet derivative of $\Lambda$ with respect to $b$.*

**Proof**. By (6.3) we have

$$\|\Lambda_\eta \phi\|_s \leq C\delta^{-1/2}\big(\|J^{s+1/2} P I_\delta \nabla_X \tilde{\Phi}_\eta\|_{L^2(\Omega_0)} + \|J^{s+1/2} P_\eta I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}\big).$$

Therefore, by the same argument as in the proof of the previous lemma we obtain the desired estimate. □

**Lemma 6.4.** *Let $s > n/2 + 2$. Under Assumption* 4.1 (A1) *and*

$$\|(\eta, b)\|_{s+1} + \sup_{0 \leq x_{n+1} \leq 1} \|\nabla_X \theta(\cdot, x_{n+1})\|_s \leq M,$$

*there exists a constant $C = C(M, c, s) > 0$ such that we have*

$$\|\Lambda \phi\|_s \leq C\delta^{-1/2}\|\Lambda_0^{1/2}\phi\|_{s+1/2}.$$

**Proof**. It is sufficient to evaluate $\|\Lambda\phi\|_{s-1}$ and $\|\nabla\Lambda\phi\|_{s-1}$. By Lemmas 5.5 and 5.2 we have $\|\Lambda\phi\|_{s-1} \leq C\delta^{-1/2}\|\Lambda_0^{1/2}\phi\|_{s-1/2}$. By the relation $\nabla\Lambda\phi = \Lambda\nabla\phi + D_\eta\Lambda[\nabla\eta]\phi + D_b\Lambda[\nabla b]\phi$ and Lemma 6.3, we see that

$$\|\nabla\Lambda\phi\|_{s-1} \leq C\delta^{-1/2}\big(\|\Lambda_0^{1/2}\nabla\phi\|_{s-1/2} + \|(\nabla\eta, \nabla b)\|_s\|\Lambda_0^{1/2}\phi\|_{s-1/2}\big) \leq C\delta^{-1/2}\|\Lambda_0^{1/2}\phi\|_{s+1/2}.$$

Therefore, we obtain the desired estimate. □

**Lemma 6.5.** *Let $s > n/2 + 1$. Under Assumption* 4.1 (A1) *and*

$$\|\eta\|_{s+3} + \sup_{0 \leq x_{n+1} \leq 1} \|\nabla_X \theta(\cdot, x_{n+1})\|_{s+2} \leq M,$$

*there exists a constant $C = C(M, c, s) > 0$ such that we have*

$$\|D_\eta^2 \Lambda[\zeta_1, \zeta_2]\phi\|_s \leq C\|\zeta_1\|_{s+2}\|\zeta_2\|_{s+1}(\|\nabla\phi\|_{s+1} + \delta^{1/2}\|\Lambda_0^{1/2}\phi\|_{s+3/2}).$$

**Proof**. By Lemma 6.1 we have

$$D_\eta\Lambda[\zeta_2]\phi = -\delta^2\Lambda(Z\zeta_2) - \nabla\cdot(v\zeta_2),$$

where $Z = (1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda\phi + \nabla\eta\cdot\nabla\phi)$ and $v = \nabla\phi - \delta^2 Z\nabla\eta$, so that

$$D_\eta^2\Lambda[\zeta_1,\zeta_2]\phi = -\delta^2 D_\eta\Lambda[\zeta_1](Z\zeta_2) - \delta^2\Lambda\big((D_\eta Z[\zeta_1])\zeta_2\big) - \nabla\cdot\big((D_\eta v[\zeta_1])\zeta_2\big).$$

By Lemmas 5.5, 6.3, and 5.2 we see that

$$\begin{aligned}\|D_\eta^2\Lambda[\zeta_1,\zeta_2]\phi\|_s &\le C\big(\delta^{3/2}\|\zeta_1\|_{s+1}\|\Lambda_0^{1/2}(Z\zeta_2)\|_{s+1/2} \\ &\quad +\delta\|(D_\eta Z[\zeta_1])\zeta_2\|_{s+1} + \|(D_\eta v[\zeta_1])\zeta_2\|_{s+1}\big) \\ &\le C\|\zeta_2\|_{s+1}\big(\delta(\|\zeta_1\|_{s+1}\|Z\|_{s+1} + \|D_\eta Z[\zeta_1]\|_{s+1}) + \|D_\eta v[\zeta_1]\|_{s+1}\big).\end{aligned}$$

Here, by Lemmas 5.5 and 5.2 it holds that

$$\|Z\|_{s+1} \le C\big(\|\Lambda\phi\|_{s+1} + \|\nabla\phi\|_{s+1}\big) \le C\delta^{-1}\|\nabla\phi\|_{s+1}.$$

The Fréchet derivative of $Z$ can be written as

$$\begin{aligned}D_\eta Z[\zeta_1] &= -2\delta^2(1+\delta^2|\nabla\eta|^2)^{-2}\nabla\eta\cdot\nabla\zeta_1(\Lambda\phi + \nabla\eta\cdot\nabla\phi) \\ &\quad +(1+\delta^2|\nabla\eta|^2)^{-1}(D_\eta\Lambda[\zeta_1]\phi + \nabla\zeta_1\cdot\nabla\phi),\end{aligned}$$

so that by Lemmas 5.5, 6.3, and 5.2 we get

$$\begin{aligned}\|D_\eta Z[\zeta_1]\|_{s+1} &\le C\big(\delta^2\|\zeta_1\|_{s+2}(\|\Lambda\phi\|_{s+1} + \|\nabla\phi\|_{s+1}) + \|D_\eta\Lambda[\zeta_1]\phi\|_{s+1} + \|\zeta_1\|_{s+2}\|\nabla\phi\|_{s+1}\big) \\ &\le C\|\zeta_1\|_{s+2}\big(\|\nabla\phi\|_{s+1} + \delta^{-1/2}\|\Lambda_0^{1/2}\phi\|_{s+3/2}\big).\end{aligned}$$

Similarly, we see that

$$\begin{aligned}\|D_\eta v[\zeta_1]\|_{s+1} &= \delta^2\|(D_\eta Z[\zeta_1])\nabla\eta + Z\nabla\zeta_1\|_{s+1} \\ &\le C\delta\|\zeta_1\|_{s+2}(\|\nabla\phi\|_{s+1} + \delta^{1/2}\|\Lambda_0^{1/2}\phi\|_{s+3/2}).\end{aligned}$$

Therefore, we obtain the desired estimate. □

## 7 Energy estimates of a linear system

Following D. Lannes [13], we linearize the equations in (2.14) around $(\eta,\phi)$. Taking the derivative $\partial$ of the second equation in (2.14), we see that

$$\begin{aligned}(7.1)\quad 0 &= \partial\phi_t + \partial\eta + \nabla\phi\cdot\nabla\partial\phi + \delta^4\nabla\eta\cdot\nabla\partial\eta(1+\delta^2|\nabla\eta|^2)^{-2}(\Lambda\phi+\nabla\eta\cdot\nabla\phi)^2 \\ &\quad -\delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda\phi+\nabla\eta\cdot\nabla\phi)\big(\partial\Lambda\phi + \partial(\nabla\eta\cdot\nabla\phi)\big) \\ &= \partial\phi_t + \partial\eta + \nabla\phi\cdot\nabla\partial\phi + \delta^4 Z^2\nabla\eta\cdot\nabla\partial\eta - \delta^2 Z(\partial\Lambda\phi + \nabla\partial\eta\cdot\nabla\phi + \nabla\eta\cdot\nabla\partial\phi) \\ &= \partial\phi_t + \partial\eta + (\nabla\phi - \delta^2 Z\nabla\eta)\cdot\nabla\partial\phi - \delta^2 Z(\nabla\phi - \delta^2 Z\nabla\eta)\cdot\nabla\partial\eta - \delta^2 Z\partial\Lambda\phi,\end{aligned}$$

so that

$$(\partial\phi - \delta^2 Z\partial\eta)_t + (\nabla\phi - \delta^2 Z\nabla\eta)\cdot\nabla(\partial\phi - \delta^2 Z\partial\eta) + (1 + \delta^2 Z_t + \delta^2 v\cdot\nabla Z)\partial\eta = 0,$$

where we used the equation $\partial\eta_t = \partial\Lambda\phi$, which comes from the first equation in (2.14). By Lemma 6.1 we also have

$$\partial\eta_t = \Lambda(\partial\phi - \delta^2 Z\partial\eta) - \nabla\cdot(v\partial\eta) + D_b\Lambda[\partial b]\phi.$$

Introducing new functions $\zeta$ and $\psi$ by

$$\zeta := \partial\eta, \qquad \psi := \partial\phi - Z\partial\eta,$$

we obtain

$$\left\{\begin{array}{l} \zeta_t + \nabla\cdot(v\zeta) - \Lambda\psi = D_b\Lambda[\partial b]\phi, \\ \psi_t + v\cdot\nabla\psi + (1+\delta^2 Z_t + \delta^2 v\cdot\nabla Z)\zeta = 0. \end{array}\right. \tag{7.2}$$

Taking these equations into account, we will consider the following system of linear equations

$$\left\{\begin{array}{l} \zeta_t + b_1\cdot\nabla\zeta - \Lambda\psi = f_1, \\ \psi_t + b_2\cdot\nabla\psi + a\zeta = f_2, \end{array}\right. \tag{7.3}$$

where $a$, $b_1 = (b_{11},\ldots,b_{1n})$, $b_2 = (b_{21},\ldots,b_{2n})$, $f_1$, $f_2$ are given function of $x$ and $t$ and may depend on $\delta$, and $\Lambda = \Lambda(\eta, b, \delta)$ is the Dirichlet-to-Neumann map. We assume the function $a$ to be positively definite and define an energy function $E(t)$ by

$$E(t) = (a\zeta(t), \zeta(t)) + (\Lambda\psi(t), \psi(t)) + \|\psi(t)\|^2. \tag{7.4}$$

Let $(\zeta, \psi)$ be a solution of (7.3). Then, it holds that

$$\begin{aligned} \frac{d}{dt}E(t) &= (a_t\zeta,\zeta) + 2(a\zeta_t,\zeta) + ([\partial_t,\Lambda]\psi,\psi) + 2(\psi_t,\Lambda\psi) + 2(\psi_t,\psi) \\ &= (a_t\zeta,\zeta) + ((\nabla\cdot(ab_1))\zeta,\zeta) + 2(af_1,\zeta) + ([\partial_t,\Lambda]\psi,\psi) \\ &\quad -2(b_2\cdot\nabla\psi,\Lambda\psi) + 2(f_2,\Lambda\phi_2) + ((\nabla\cdot b_2)\psi,\psi) - 2(a\zeta,\psi) + 2(f_2,\psi). \end{aligned} \tag{7.5}$$

Here, we have

$$-2(b_2\cdot\nabla\psi,\Lambda\psi) = (\psi,(\nabla\cdot b_2)\Lambda\psi) + (\psi, b_2\cdot[\nabla,\Lambda]\psi) + (\psi,[b_2,\Lambda]\cdot\nabla\psi). \tag{7.6}$$

**Lemma 7.1.** *Under Assumption* 4.1 (A1) *and*

$$\|\nabla_X\theta_t(\cdot,t)\|_{L^\infty(\Omega_0)} \le M, \tag{7.7}$$

*there exists a constant $C = C(M,c) > 0$ such that we have*

$$|([\partial_t,\Lambda]\phi,\phi)| \le C(\Lambda\phi,\phi).$$

**Proof**. Set $\Phi := \phi^\hbar$ and $\tilde\Phi := \Phi\circ\Theta$. Then, by Lemma 4.2 we have

$$(\Lambda\phi,\phi) = \int_\Omega |I_\delta\nabla_X\Phi|^2 dX = \int_{\Omega_0} PI_\delta\nabla_X\tilde\Phi\cdot I_\delta\nabla_X\tilde\Phi dX,$$

so that

$$([\partial_t,\Lambda]\phi,\phi) = \frac{d}{dt}(\Lambda\phi,\phi) = 2\int_{\Omega_0} PI_\delta\nabla_X\tilde\Phi\cdot I_\delta\nabla_X\tilde\Phi_t dX + \int_{\Omega_0} P_t I_\delta\nabla_X\tilde\Phi\cdot I_\delta\nabla_X\tilde\Phi dX.$$

Since $\tilde{\Phi}(\cdot,1)=\phi$, we have $\tilde{\Phi}_t(\cdot,1)=0$. Therefore, by Green's formula we see that

$$\begin{aligned}\int_{\Omega_0} PI_\delta\nabla_X\tilde{\Phi}\cdot I_\delta\nabla_X\tilde{\Phi}_t dX &= -\int_{\Omega_0}(\nabla_X\cdot I_\delta PI_\delta\nabla_X\tilde{\Phi})\tilde{\Phi}_t dX \\ &\quad +(\mathbf{e}_2\cdot I_\delta^2\nabla_X\tilde{\Phi}(\cdot,1),\tilde{\Phi}_t(\cdot,1))-(\mathbf{e}_2\cdot I_\delta^2\nabla_X\tilde{\Phi}(\cdot,0),\tilde{\Phi}_t(\cdot,0)) \\ &= 0.\end{aligned}$$

Hence, we obtain

$$|([\partial_t,\Lambda]\phi,\phi)|\le |P_t|_\infty\|I_\delta\nabla_X\tilde{\Phi}\|^2_{L^2(\Omega_0)}.$$

This together with Lemmas 4.2 and 4.3 implies the desired estimate. □

The following lemma was given by S. Wu [22]. For the completeness, we will give the proof.

**Lemma 7.2.** *For the Dirichlet-to-Neumann map $\Lambda=\Lambda(\eta,b,\delta)$ it holds that*

$$\begin{cases} |(\phi,a\Lambda\psi)+(a\Lambda\phi,\psi)|\le \big(|a|_\infty+2|\Lambda a|_\infty\big)\sqrt{\|\phi\|^2+(\phi,\Lambda\phi)}\sqrt{\|\psi\|^2+(\psi,\Lambda\psi)}, \\ |(\phi,\Lambda\psi)|\le\sqrt{(\phi,\Lambda\phi)}\sqrt{(\psi,\Lambda\psi)}. \end{cases}$$

**Proof**. Set $\Phi:=\phi^\hbar$, $\Psi:=\psi^\hbar$, and $A:=a^\hbar$. Putting $u=A$ and $v=\Phi\Psi$ in Green's formula

$$\int_\Omega\big\{u(\nabla_X\cdot I_\delta^2\nabla_X v)-v(\nabla_X\cdot I_\delta^2\nabla_X u)\big\}dX=\int_{\partial\Omega}\big\{u(N\cdot I_\delta^2\nabla_X v)-v(N\cdot I_\delta^2\nabla_X u)\big\}dS, \tag{7.8}$$

we obtain

$$\begin{aligned}&2\int_\Omega AI_\delta\nabla_X\Phi\cdot I_\delta\nabla_X\Psi dX \\ &= \int_{\partial\Omega}\big\{A\big(\Phi(N\cdot I_\delta^2\nabla_X\Psi)+\Psi(N\cdot I_\delta^2\nabla_X\Phi)\big)-\Phi\Psi(N\cdot I_\delta^2\nabla_X A)\big\}dS \\ &= (a,\phi\Lambda\psi+\psi\Lambda\phi)-(\phi\psi,\Lambda a).\end{aligned}$$

This implies that

$$|(\phi,a\Lambda\psi)+(a\Lambda\phi,\psi)|\le|\Lambda a|_\infty\|\phi\|\|\psi\|+2\|A\|_{L^\infty(\Omega)}\|I_\delta\nabla_X\Phi\|_{L^2(\Omega)}\|I_\delta\nabla_X\Psi\|_{L^2(\Omega)}.$$

Here, by the weak maximal principle we have $\|A\|_{L^\infty(\Omega)}=\max\{|a|_\infty,|A(\cdot,b(\cdot))|_\infty\}$. If $a$ is not constant and $A$ attains its maximal value at the bottom $(x_0,b(x_0))$, then the strong maximal principle implies that $N\cdot I_\delta^2\nabla_X A(x_0,b(x_0))>0$. However, this contradicts with the definition of $A$. Therefore, it holds that $\|A\|_{L^\infty(\Omega)}=|a|_\infty$. This together with Lemma 4.2 shows the first estimate. By taking $a\equiv1$ in the above argument, we obtain the second one. □

**Lemma 7.3.** *Let $r>n/2$. Under Assumption* 4.1 (A1) *and* (A2)*, there exists a constant $C=C(M,c)>0$ such that for any $j=1,\dots,n$ we have*

$$|(\phi,[\psi,\Lambda]\partial_j\phi)|\le\frac12|\partial_j(\Lambda\psi)|_\infty\|\phi\|^2+C(\|\nabla\psi\|_r+\delta\|\Lambda_0^{1/2}\psi\|_{r+1})(\Lambda\phi,\phi).$$

**Proof**. Set $\Phi:=\phi^\hbar$, $\Phi_j:=(\partial_j\phi)^\hbar$, and $\Psi:=\psi^\hbar$. Putting $u=\Phi_j$ and $v=\Phi\Psi$ in Green's formula (7.8), we see that

$$\begin{aligned}2\int_\Omega\Phi_j I_\delta\nabla_X\Phi\cdot I_\delta\nabla_X\Psi dX &= (\partial_j\phi,\phi\Lambda\psi+\psi\Lambda\phi)-(\phi\psi,\Lambda\partial_j\phi) \\ &= -\frac12(\phi^2,\partial_j(\Lambda\psi))-(\phi,[\psi,\Lambda]\partial_j\phi).\end{aligned}$$

This implies that

$$|(\phi,[\psi,\Lambda]\partial_j\phi)| \le \frac{1}{2}|\partial_j(\Lambda\psi)|_\infty\|\phi\|^2 + 2\|I_\delta\nabla_X\Psi\|_{L^\infty(\Omega)}\|\Phi_j\|_{L^2(\Omega)}\|I_\delta\nabla_X\Phi\|_{L^2(\Omega)}.$$

Set $\tilde\Phi := \Phi\circ\Theta$ and $\tilde\Phi_j := \Phi_j\circ\Theta$. Then,

$$\begin{aligned}\|\Phi_j\|_{L^2(\Omega)} &\le C\|\tilde\Phi_j\|_{L^2(\Omega_0)} \le C(\|\tilde\Phi_j-\partial_j\tilde\Phi\|_{L^2(\Omega_0)} + \|\partial_j\tilde\Phi\|_{L^2(\Omega_0)})\\ &\le C(\delta\|I_\delta\nabla_X(\tilde\Phi_j-\partial_j\tilde\Phi)\|_{L^2(\Omega_0)} + \|I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)}),\end{aligned}$$

where we used Poincaré's inequality. Here, as in the proof of Lemma 5.6 we can show that $\|I_\delta\nabla_X(\tilde\Phi_j-\partial_j\tilde\Phi)\|_{L^2(\Omega_0)} \le C\|I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)}$. Therefore,

$$\|\Phi_j\|_{L^2(\Omega)} \le C\|I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)} \le C\|I_\delta\nabla_X\Phi\|_{L^2(\Omega)}.$$

These estimates together with Lemmas 4.2 and 5.7 yield the desired estimate. □

**Lemma 7.4.** *Under Assumption* 4.1 (A1) *and* (A2)*, there exists a constant* $C = C(M,c) > 0$ *such that for any* $j = 1,\dots,n$ *we have*

$$|(\psi,[\partial_j,\Lambda]\phi)| \le C\sqrt{(\Lambda\psi,\psi)}\sqrt{(\Lambda\phi,\phi)}.$$

**Proof**. Set $\Phi := \phi^\hbar$, $\Phi_j := (\partial_j\phi)^\hbar$, $\Psi := \psi^\hbar$, $\tilde\Phi := \Phi\circ\Theta$, $\tilde\Phi_j := \Phi_j\circ\Theta$, and $\tilde\Psi := \Psi\circ\Theta$. By Green's formula we see that

$$\begin{aligned}(\psi,[\partial_j,\Lambda]\phi) &= \int_{\Omega_0}\nabla_X\cdot\big(\tilde\Psi I_\delta P I_\delta\nabla_X(\partial_j\tilde\Phi-\tilde\Phi_j)\big)dX\\ &= \int_{\Omega_0}\big\{PI_\delta\nabla_X\tilde\Psi\cdot I_\delta\nabla_X(\partial_j\tilde\Phi-\tilde\Phi_j) + \tilde\Psi\nabla_X\cdot I_\delta PI_\delta\nabla_X\partial_j\tilde\Phi\big\}dX\\ &= -\int_{\Omega_0}\tilde\Psi\nabla_X\cdot I_\delta(\partial_jP)I_\delta\nabla_X\tilde\Phi dX = \int_{\Omega_0}(\partial_jP)I_\delta\nabla_X\tilde\Psi\cdot I_\delta\nabla_X\tilde\Phi dX,\end{aligned}$$

where we used the equation $\nabla_X\cdot I_\delta PI_\delta\nabla_X\partial_j\tilde\Phi = -\nabla_X\cdot I_\delta(\partial_jP)I_\delta\nabla_X\tilde\Phi$ and (4.8). This implies that

$$|(\psi,[\partial_j,\Lambda]\phi)| \le C\|I_\delta\nabla_X\tilde\Psi\|_{L^2(\Omega_0)}\|I_\delta\nabla_X\tilde\Phi\|_{L^2(\Omega_0)},$$

which together with Lemmas 4.2 and 4.3 yields the desired estimate. □

**Lemma 7.5.** *It holds that*

$$\sqrt{(\Lambda(a\phi),a\phi)} \le \sqrt{|a|_\infty|\Lambda a|_\infty}\|\phi\| + 3|a|_\infty\sqrt{(\Lambda\phi,\phi)}.$$

**Proof**. Set $\Phi := \phi^\hbar$, $A := a^\hbar$, and $\Psi := (a\phi)^\hbar$. Then, we see that

$$\begin{aligned}(\Lambda(a\phi),a\phi) &= \int_\Gamma(N\cdot I_\delta^2\nabla_X\Psi)A\Phi dS = \int_\Omega\nabla_X\cdot(A\Phi I_\delta^2\nabla_X\Psi)dX\\ &= \int_\Omega I_\delta\nabla_X(A\Phi)\cdot I_\delta\nabla_X\Psi dS \le \|I_\delta\nabla_X(A\Phi)\|_{L^2(\Omega)}\|I_\delta\nabla_X\Psi\|_{L^2(\Omega)}.\end{aligned}$$

By Lemma 4.2 we have $(\Lambda(a\phi),a\phi) = \|I_\delta\nabla_X\Psi\|^2_{L^2(\Omega)}$, so that the above inequality implies that

$$\sqrt{(\Lambda(a\phi),a\phi)} \le \|I_\delta\nabla_X(A\Phi)\|_{L^2(\Omega)} \le \|AI_\delta\nabla_X\Phi\|_{L^2(\Omega)} + \|\Phi I_\delta\nabla_X A\|_{L^2(\Omega)}.$$

Here, by Green's formula it holds that

$$\begin{aligned}\|\Phi I_\delta \nabla_X A\|_{L^2(\Omega)}^2 &= (a\phi^2, \Lambda a) - 2\int_\Omega A\Phi I_\delta \nabla_X A \cdot I_\delta \nabla_X \Phi dX \\ &\le |a|_\infty |\Lambda a|_\infty \|\phi\|^2 + 2\|AI_\delta \nabla_X \Phi\|_{L^2(\Omega)} \|\Phi I_\delta \nabla_X A\|_{L^2(\Omega)},\end{aligned}$$

which implies that

$$\|\Phi I_\delta \nabla_X A\|_{L^2(\Omega)} \le \sqrt{|a|_\infty |\Lambda a|_\infty} \|\phi\| + 2\|AI_\delta \nabla_X \Phi\|_{L^2(\Omega)}.$$

These estimates and $\|AI_\delta \nabla_X \Phi\|_{L^2(\Omega)} \le \|A\|_{L^\infty(\Omega)} \|I_\delta \nabla_X \Phi\|_{L^2(\Omega)} \le |a|_\infty \sqrt{(\Lambda\phi, \phi)}$, which comes from the maximal principle and Lemma 4.2, yield the desired estimate. □

**Lemma 7.6.** *Let $r > n/2$. In addition to Assumption* 4.1 (A1) *and* (A3)*, and* (7.7)*, we assume that*

$$M^{-1} \le a(x,t) \le M, \quad \|(a_t, \nabla a)\|_r + \|(b_1, b_2, \Lambda_0 b_2)\|_{r+1} \le M. \tag{7.9}$$

*Then, there exists a constant $C = C(M, c, r) > 0$ such that for any smooth solution $(\zeta, \psi)$ of* (7.3) *we have*

$$E(t) \le \mathrm{e}^{Ct} E(0) + \int_0^t \mathrm{e}^{C(t-\tau)} \big( \|f_1(\tau)\|^2 + \|f_2(\tau)\|^2 + \|\Lambda_0^{1/2} f_2(\tau)\|^2 \big) d\tau.$$

**Proof**. By the Sobolev inequality and Lemmas 5.5 and 5.2 we see that

$$\begin{cases} |\Lambda(\nabla\cdot b_2)|_\infty \le C(\|\Lambda_0(\nabla\cdot b_2)\|_r + \|\Lambda_0^{1/2}(\nabla\cdot b_2)\|_r) \le C(\|\Lambda_0 b_2\|_{r+1} + \|b_2\|_{r+1}) \le C, \\ |\nabla \Lambda b_2|_\infty \le C(\|\Lambda_0 b_2\|_{r+1} + \|\Lambda_0^{1/2} b_2\|_{r+1}) \le C. \end{cases}$$

Therefore, by (7.6) and Lemmas 7.2–7.5 we get

$$|(b_2 \cdot \nabla\psi, \Lambda\psi)| \le C\big(\|\psi\|^2 + (\psi, \Lambda\psi)\big). \tag{7.10}$$

Hence, by (7.5) and Lemmas 7.1–7.2 and 4.4 we obtain

$$\frac{d}{dt} E(t) \le CE(t) + \|f_1(t)\|^2 + \|f_2(t)\|^2 + \|\Lambda_0^{1/2} f_2(t)\|^2,$$

so that the desired energy estimate comes from Gronwall's inequality. □

We proceed to estimate a high order energy function $E_s(t)$ defined by

$$E_s(t) := (aJ^s\zeta(t), J^s\zeta(t)) + (\Lambda J^s\psi(t), J^s\psi(t)) + \|\psi(t)\|_s^2. \tag{7.11}$$

Let $(\zeta, \psi)$ be a solution of (7.3). Then, it holds that

$$\begin{aligned}\frac{d}{dt} E_s(t) &= (a_t J^s\zeta, J^s\zeta) + 2(aJ^s\zeta_t, J^s\zeta) + ([\partial_t, \Lambda]J^s\psi, J^s\psi) \\ &\quad +2(\Lambda J^s\psi_t, J^s\psi) + 2(J^s\psi_t, J^s\psi) \\ &\le |a^{-1}a_t|_\infty (aJ^s\zeta, J^s\zeta) + C(\Lambda J^s\psi, J^s\psi) \\ &\quad -2(aJ^s b_1\cdot\nabla\zeta, J^s\zeta) + 2(aJ^s\Lambda\psi, J^s\zeta) + 2(aJ^s f_1, J^s\zeta) \\ &\quad -2(\Lambda J^s b_2\cdot\nabla\psi, J^s\psi) - 2(\Lambda J^s a\zeta, J^s\psi) + 2(\Lambda J^s f_2, J^s\psi) \\ &\quad -2(J^s b_2\cdot\nabla\psi, J^s\psi) - 2(J^s a\zeta, J^s\psi) + 2(J^s f_2, J^s\psi).\end{aligned} \tag{7.12}$$

**Lemma 7.7.** *Let $s > n/2+1$. Then, there exists a constant $C = C(s) > 0$ such that for any $j = 1, \dots, n$ we have*

$$\|\Lambda_0^{1/2}[J^s,\psi]\partial_j\phi\| \le C\|\psi\|_{s+1}\bigl(\|\Lambda_0^{1/2}\phi\|_s + \|\phi\|_s\bigr).$$

**Proof**. Put $u := \Lambda_0^{1/2}[J^s,\psi]\partial_j\phi$. Then, we have

$$\begin{aligned}\hat{u}(\xi) &= \frac{\sqrt{\delta^{-1}|\xi|\tanh(\delta|\xi|)}}{(2\pi)^n}\int_{\mathbf{R}^n}\hat{\psi}(\xi-\eta)\bigl((1+|\xi|)^s-(1+|\eta|)^s\bigr)i\eta_j\hat{\phi}(\eta)d\eta \\ &= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\hat{\psi}(\xi-\eta)\bigl((1+|\xi|)^s-(1+|\eta|)^s\bigr)i\eta_j\sqrt{\delta^{-1}|\eta|\tanh(\delta|\eta|)}\hat{\phi}(\eta)d\eta \\ &\quad +\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\bigl(\sqrt{\delta^{-1}|\xi|\tanh(\delta|\xi|)}-\sqrt{\delta^{-1}|\eta|\tanh(\delta|\eta|)}\bigr) \\ &\qquad\qquad \times\hat{\psi}(\xi-\eta)\bigl((1+|\xi|)^s-(1+|\eta|)^s\bigr)i\eta_j\hat{\phi}(\eta)d\eta.\end{aligned}$$

Therefore, by (5.1) we obtain

$$\begin{aligned}|\hat{u}(\xi)| &\le \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}|\hat{\psi}(\xi-\eta)|\bigl|(1+|\xi|)^s-(1+|\eta|)^s\bigr||\eta||\widehat{\Lambda_0^{1/2}\phi}(\eta)|d\eta \\ &\quad +C\int_{\mathbf{R}}|\xi-\eta||\hat{\psi}(\xi-\eta)|\bigl|(1+|\xi|)^s-(1+|\eta|)^s\bigr||\eta||\hat{\phi}(\eta)|d\eta.\end{aligned}$$

In view of the inequality $\bigl|(1+|\xi|)^s-(1+|\eta|)^s\bigr| \le C|\xi-\eta|\bigl((1+|\xi-\eta|)^{s-1}+(1+|\eta|)^{s-1}\bigr)$, the above estimate and Hausdorff-Young's inequality give the desired estimate. □

**Lemma 7.8.** *Let $s > n/2+1$. In addition to Assumption* 4.1 (A1), *(7.7), and (7.9), we assume that*

$$\|(\nabla a, b_1)\|_s + \|b_2\|_{s+1} + \sup_{0\le x_{n+1}\le 1}\|\nabla_X\theta(\cdot,x_{n+1})\|_{s+1} \le M. \tag{7.13}$$

*Then, there exists a constant $C = C(M,c,s) > 0$ such that for any smooth solution $(\zeta,\psi)$ of* (7.3) *we have*

$$E_s(t) \le \mathrm{e}^{Ct}E_s(0) + \int_0^t \mathrm{e}^{C(t-\tau)}\bigl(\|f_1(\tau)\|_s^2 + \|f_2(\tau)\|_s^2 + \|\Lambda_0^{1/2}f_2(\tau)\|_s^2\bigr)d\tau.$$

**Proof**. We will evaluate each term in the right hand side of (7.12). It is easy to see that

$$\begin{aligned}|(aJ^sb_1\cdot\nabla\zeta, J^s\zeta)| &= \bigl|-\tfrac{1}{2}((\nabla\cdot ab_1)J^s\zeta, J^s\zeta) + (a[J^s,b_1]\cdot\nabla\zeta, J^s\zeta)\bigr| \\ &\le \tfrac{1}{2}|\nabla\cdot(ab_1)|_\infty\|\zeta\|_s^2 + C|a|_\infty\|b_1\|_s\|\zeta\|_s^2,\end{aligned}$$

$$|(J^sb_2\cdot\nabla\psi, J^s\psi)| = \bigl|-\tfrac{1}{2}((\nabla\cdot b_2)J^s\psi, J^s\psi) + ([J^s,b_2]\cdot\nabla\psi, J^s\psi)\bigr| \le C\|b_2\|_s\|\psi\|_s,$$

$$|(J^sa\zeta, J^s\psi)| \le \|a\phi_1\|_s\|\phi_2\|_s \le C(|a|_\infty + \|\nabla a\|_{s-1})\|\phi_1\|_s\|\phi_2\|_s.$$

By Lemma 7.2, we have

$$\begin{aligned}&|(aJ^s\Lambda\psi, J^s\zeta) - (\Lambda J^sa\zeta, J^s\psi)| \\ &= |([J^s,\Lambda]\psi, aJ^s\zeta) - ([J^s,a]\zeta, \Lambda J^s\psi)| \\ &\le |a|_\infty\|\zeta\|_s\|[J^s,\Lambda]\psi\| + \sqrt{(\Lambda J^s\psi, J^s\psi)}\sqrt{(\Lambda[J^s,a]\zeta, [J^s,a]\zeta)}.\end{aligned}$$

Here, by Lemmas 5.4 and 4.4

$$\|[J^s,\Lambda]\psi\|^2 \le C\|\Lambda_0^{1/2}\psi\|_s^2 = C(\Lambda_0 J^s\psi, J^s\psi) \le C(\Lambda J^s\psi, J^s\psi),$$

by Lemmas 4.4 and 5.2

$$\sqrt{(\Lambda[J^s,a]\zeta,[J^s,a]\zeta)|} \le C\|\Lambda_0^{1/2}[J^s,a]\zeta\| \le C\|[J^s,a]\zeta\|_1 \le C\|\nabla a\|_s\|\zeta\|_s,$$

so that we obtain

$$|(aJ^s\Lambda\psi, J^s\zeta) - (\Lambda J^s a\zeta, J^s\psi)| \le C(|a|_\infty + \|\nabla a\|_s)\|\zeta\|_s\sqrt{(\Lambda J^s\psi, J^s\psi)}.$$

By using the same procedure of the derivation of (7.10) and Lemma 7.2 we have

$$\begin{aligned}
&|(\Lambda J^s b_2\cdot\nabla\psi, J^s\psi)| \\
&\quad = \ |(\Lambda[J^s,b_2]\cdot\nabla\psi, J^s\psi) + (\Lambda b_2\cdot\nabla J^s\psi, J^s\psi)| \\
&\quad \le \ \sqrt{(\Lambda J^s\psi, J^s\psi)}\sqrt{(\Lambda[J^s,b_2]\cdot\nabla\psi,[J^s,b_2]\cdot\nabla\psi)} + C\big(\|\psi\|_s^2 + (\Lambda J^s\psi, J^s\psi)\big).
\end{aligned}$$

Here, by Lemmas 4.4 and 7.7 we get

$$\begin{aligned}
(\Lambda[J^s,b_2]\cdot\nabla\psi,[J^s,b_2]\cdot\nabla\psi) &\le C\|\Lambda_0^{1/2}[J^s,b_2]\cdot\nabla\psi\|^2 \\
&\le C\|b_2\|_{s+1}^2\big(\|\psi\|_s^2 + (\Lambda J^s\psi, J^s\psi)\big).
\end{aligned}$$

By the above estimates and Lemma 7.2, it follows from (7.12) that

$$\frac{d}{dt}E_s(t) \le E_s(t) + \|f_1(t)\|_s^2 + \|f_2(t)\|_s^2 + \|\Lambda_0^{1/2}f_2(t)\|_s^2,$$

so that the desired energy estimate comes from Gronwall's inequality. □

# 8 Reduction to a quasi-linear system

In this section we reduce the equations

$$(8.1)\qquad \begin{cases} \eta_t - \Lambda\phi = 0, \\ \phi_t + \eta + \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2}\delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda\phi + \nabla\eta\cdot\nabla\phi)^2 = 0 \end{cases}$$

to a quasi-linear system of equations. By the same way as in (7.1), differentiating the second equation in (8.1) with respect to $x_i$ we obtain

$$\partial_i\phi_t + \partial_i\eta + (\nabla\phi - \delta^2 Z\nabla\eta)\cdot(\nabla\partial_i\phi - \delta^2 Z\nabla\partial_i\eta) - \delta^2 Z\partial_i\Lambda\phi = 0.$$

Differentiating this with respect to $x_j$ and $x_k$, we see that

$$\begin{aligned}
&\partial_{ijk}\phi_t + \partial_{ijk}\eta + v\cdot\big\{\nabla\partial_{ijk}\phi - \delta^2\big(Z\nabla\partial_{ijk}\eta + (\partial_{jk}Z)\nabla\partial_i\eta + (\partial_j Z)\nabla\partial_{ki}\eta + (\partial_k Z)\nabla\partial_{ij}\eta\big)\big\} \\
&\quad +(\partial_j v)\cdot\big\{\nabla\partial_{ik}\phi - \delta^2\big(Z\partial_{ik}\eta + (\partial_k Z)\nabla\partial_i\eta\big)\big\} + (\partial_k v)\cdot\big\{\nabla\partial_{ij}\phi - \delta^2\big(Z\partial_{ij}\eta + (\partial_j Z)\nabla\partial_i\eta\big)\big\} \\
&\quad +\big\{\nabla\partial_{jk}\phi - \delta^2\big(Z\nabla\partial_{jk}\phi + (\partial_j Z)\nabla\partial_k\eta + (\partial_k Z)\nabla\partial_j\eta + (\partial_{jk}Z)\nabla\eta\big)\big\}\cdot(\nabla\partial_i\phi - \delta^2 Z\nabla\partial_i\eta) \\
&\quad -\delta^2\big\{(\partial_j Z)\partial_{ik}\Lambda\phi + (\partial_k Z)\partial_{ij}\Lambda\phi + (\partial_{jk}Z)\partial_i\Lambda\phi + Z\partial_{ijk}\Lambda\phi\big\} = 0.
\end{aligned}$$

Here, by Lemma 6.1 we have

$$\partial_{ik}\Lambda = \partial_k\big(\Lambda(\partial_i\phi - \delta^2 Z\partial_i\eta) - (\nabla\cdot v)\partial_i\eta + D_b\Lambda[\partial_i b]\phi\big) - (\partial_k v)\cdot\nabla\partial_i\eta - v\cdot\nabla\partial_{ik}\eta,$$

so that

$$\begin{aligned}
&(\partial_{ijk}\phi - \delta^2 Z\partial_{ijk}\eta)_t + v\cdot\nabla(\partial_{ijk}\phi - \delta^2 Z\partial_{ijk}\eta) + (1+\delta^2 Z_t + \delta^2 v\cdot\nabla Z)\partial_{ijk}\eta \\
&\quad = \delta^2(\partial_{jk}Z)v\cdot\nabla\partial_i\eta - (\partial_j v)\cdot\big\{\big(\nabla\partial_{ik}\phi - \delta^2 Z\nabla\partial_{ik}\eta\big) - \delta^2(\partial_k Z)\nabla\partial_i\eta\big\} \\
&\qquad -(\partial_k v)\cdot\big\{\big(\nabla\partial_{ij}\phi - \delta^2 Z\partial_{ij}\eta\big) - \delta^2(\partial_j Z)\nabla\partial_i\eta\big\} \\
&\qquad -(\nabla\partial_i\phi - \delta^2 Z\nabla\partial_i\eta)\cdot\big\{\big(\nabla\partial_{jk}\phi - \delta^2 Z\nabla\partial_{jk}\eta\big) \\
&\qquad\qquad -\delta^2\big((\partial_j Z)\nabla\partial_k\eta + (\partial_k Z)\nabla\partial_j\eta + (\partial_{jk}Z)\nabla\eta\big)\big\} + \delta^2(\partial_{jk}Z)\partial_i\Lambda\phi \\
&\qquad +\delta^2(\partial_j Z)\big\{\partial_k\big(\Lambda(\partial_i\phi - \delta^2 Z\partial_i\eta) - (\nabla\cdot v)\partial_i\eta + D_b\Lambda[\partial_i b]\phi\big) - (\partial_k v)\cdot\nabla\partial_i\eta\big\} \\
&\qquad +\delta^2(\partial_k Z)\big\{\partial_j\big(\Lambda(\partial_i\phi - \delta^2 Z\partial_i\eta) - (\nabla\cdot v)\partial_i\eta + D_b\Lambda[\partial_i b]\phi\big) - (\partial_j v)\cdot\nabla\partial_i\eta\big\}.
\end{aligned}$$

Now, we write $u = (\eta, b)$ and denote by $\Lambda_n$ the $n$-th Fréchet derivative of the Dirichlet-to-Neumann map $\Lambda$ with respect to $u$. Differentiating the first equation in (8.1) yields that

$$\begin{aligned}
\partial_{ijk}\Lambda\phi \;=\;& \Lambda\partial_{ijk}\phi + \Lambda_1[\partial_{ijk}u]\phi + \Lambda_1[\partial_i u]\partial_{jk}\phi + \Lambda_1[\partial_j u]\partial_{ki}\phi + \Lambda_1[\partial_k u]\partial_{ij}\phi \\
&+\Lambda_1[\partial_{ij}u]\partial_k\phi + \Lambda_1[\partial_{jk}u]\partial_i\phi + \Lambda_1[\partial_{ki}u]\partial_j\phi \\
&+\Lambda_2[\partial_i u, \partial_j u]\partial_k\phi + \Lambda_2[\partial_j u, \partial_k u]\partial_i\phi + \Lambda_2[\partial_k u, \partial_i u]\partial_j\phi \\
&+\Lambda_2[\partial_{ij}u, \partial_k u]\phi + \Lambda_2[\partial_{jk}u, \partial_i u]\phi + \Lambda_2[\partial_{ki}u, \partial_j u]\phi + \Lambda_3[\partial_i u, \partial_j u, \partial_k u]\phi.
\end{aligned}$$

Here, by Lemma 6.1 we have

$$\Lambda_1[\partial_{ijk}u]\phi = -\delta^2\Lambda(Z\partial_{ijk}\eta) - \nabla\cdot(v\partial_{ijk}\eta) + D_b\Lambda[\partial_{ijk}b]\phi.$$

Therefore, introducing new functions $\zeta_{ijk}$ and $\psi_{ijk}$ by

$$\zeta_{ijk} := \partial_{ijk}\eta, \qquad \psi_{ijk} := \partial_{ijk}\phi - \delta^2 Z\partial_{ijk}\eta,$$

we obtain

$$\left\{\begin{array}{l} \partial_t\zeta_{ijk} + v\cdot\nabla\zeta_{ijk} - \Lambda\psi_{ijk} = f_1^{ijk}, \\ \partial_t\psi_{ijk} + v\cdot\nabla\psi_{ijk} + a\zeta_{ijk} = f_2^{ijk}, \end{array}\right. \tag{8.2}$$

where $a = 1 + \delta^2 Z_t + \delta^2 v\cdot\nabla Z$ and

$$\begin{aligned}
f_1^{ijk} =& -(\nabla\cdot v)\zeta_{ijk} + D_b\Lambda[\partial_{ijk}b]\phi + \Lambda_1[\partial_i u]\partial_{jk}\phi + \Lambda_1[\partial_j u]\partial_{ki}\phi + \Lambda_1[\partial_k u]\partial_{ij}\phi \\
&+\Lambda_1[\partial_{ij}u]\partial_k\phi + \Lambda_1[\partial_{jk}u]\partial_i\phi + \Lambda_1[\partial_{ki}u]\partial_j\phi \\
&+\Lambda_2[\partial_i u, \partial_j u]\partial_k\phi + \Lambda_2[\partial_j u, \partial_k u]\partial_i\phi + \Lambda_2[\partial_k u, \partial_i u]\partial_j\phi \\
&+\Lambda_2[\partial_{ij}u, \partial_k u]\phi + \Lambda_2[\partial_{jk}u, \partial_i u]\phi + \Lambda_2[\partial_{ki}u, \partial_j u]\phi + \Lambda_3[\partial_i u, \partial_j u, \partial_k u]\phi, \\
f_2^{ijk} =& \,\delta^2(\partial_{jk}Z)v\cdot\nabla\partial_i\eta - (\partial_j v)\cdot\big\{\big(\nabla\partial_{ik}\phi - \delta^2 Z\nabla\partial_{ik}\eta\big) - \delta^2(\partial_k Z)\nabla\partial_i\eta\big\} \\
&-(\partial_k v)\cdot\big\{\big(\nabla\partial_{ij}\phi - \delta^2 Z\partial_{ij}\eta\big) - \delta^2(\partial_j Z)\nabla\partial_i\eta\big\} \\
&-(\nabla\partial_i\phi - \delta^2 Z\nabla\partial_i\eta)\cdot\big\{\big(\nabla\partial_{jk}\phi - \delta^2 Z\nabla\partial_{jk}\eta\big) \\
&\qquad -\delta^2\big((\partial_j Z)\nabla\partial_k\eta + (\partial_k Z)\nabla\partial_j\eta + (\partial_{jk}Z)\nabla\eta\big)\big\} + \delta^2(\partial_{jk}Z)\partial_i\Lambda\phi \\
&+\delta^2(\partial_j Z)\big\{\partial_k\big(\Lambda(\partial_i\phi - \delta^2 Z\partial_i\eta) - (\nabla\cdot v)\partial_i\eta + D_b\Lambda[\partial_i b]\phi\big) - (\partial_k v)\cdot\nabla\partial_i\eta\big\} \\
&+\delta^2(\partial_k Z)\big\{\partial_j\big(\Lambda(\partial_i\phi - \delta^2 Z\partial_i\eta) - (\nabla\cdot v)\partial_i\eta + D_b\Lambda[\partial_i b]\phi\big) - (\partial_j v)\cdot\nabla\partial_i\eta\big\}.
\end{aligned}$$

Setting $\zeta := (\zeta_{ijk})$ and $\psi := (\psi_{ijk})$, we can rewrite (8.2) as

$$\left\{\begin{array}{l} \partial_t\zeta + v\cdot\nabla\zeta - \Lambda\psi = f_1, \\ \partial_t\psi + v\cdot\nabla\psi + a\zeta = f_2, \end{array}\right. \tag{8.3}$$

where $f_1$ and $f_2$ can be written symbolically as

$$
\begin{aligned}
f_1 &= -(\nabla\cdot v)\zeta + D_b\Lambda[\partial^3 b]\phi + 3\Lambda_1[\partial u]\partial^2\phi + 3\Lambda_1[\partial^2 u]\partial\phi + 3\Lambda_2[\partial u,\partial u]\partial\phi \\
&\quad +3\Lambda_2[\partial^2 u,\partial u]\phi + \Lambda_3[\partial u,\partial u,\partial u]\phi, \\
f_2 &= \delta^2(\partial^2 Z)v\cdot\nabla\partial\eta - 2(\partial v)\cdot\big(\psi - \delta^2(\partial Z)\nabla\partial\eta\big) \\
&\quad -(\nabla\partial\phi - \delta^2 Z\nabla\partial\eta)\cdot\big\{\psi - \delta^2\big(2(\partial Z)\nabla\partial\eta + (\partial^2 Z)\nabla\eta\big)\big\} + \delta^2(\partial^2 Z)\partial\Lambda\phi \\
&\quad +2\delta^2(\partial Z)\big\{\partial\big(\Lambda v - (\nabla\cdot v)\partial\eta + D_b\Lambda[\partial b]\phi\big) - (\partial v)\cdot\nabla\partial\eta\big\}.
\end{aligned}
$$

We proceed to give a uniform estimate of the coefficients $v$ and $a$, and the remainder terms $f_1$ and $f_2$.

**Lemma 8.1.** *Let $s > n/2$. There exists a constant $C = C(s) > 0$ such that we have*

$$\|\Lambda_0^{1/2}(\phi\psi)\|_s \le C(\|\phi\|_s\|\Lambda_0^{1/2}\psi\|_s + \|\Lambda_0^{1/2}\phi\|_s\|\psi\|_s).$$

**Proof**. In view of the fact that $\sqrt{\alpha\tanh\alpha}$ is equivalent to $\alpha/\sqrt{1+\alpha}$ for $\alpha \ge 0$, we easily obtain $\sqrt{(\alpha+\beta)\tanh(\alpha+\beta)} \le C\big(\sqrt{\alpha\tanh\alpha} + \sqrt{\beta\tanh\beta}\big)$ for $\alpha,\beta \ge 0$. Therefore, for any $\xi,\eta \in \mathbf{R}^n$ we have

$$\sqrt{\delta^{-1}|\xi|\tanh(\delta|\xi|)} \le C\big(\sqrt{\delta^{-1}|\xi-\eta|\tanh(\delta|\xi-\eta|)} + \sqrt{\delta^{-1}|\eta|\tanh(\delta|\eta|)}\big).$$

Put $u := \Lambda_0^{1/2}(\phi\psi)$. Then, we have

$$
\begin{aligned}
|\hat{u}(\xi)| &= \left|\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\sqrt{\delta^{-1}|\xi|\tanh(\delta|\xi|)}\hat{\phi}(\xi-\eta)\hat{\psi}(\eta)d\eta\right| \\
&\le C\int_{\mathbf{R}^n}\big(|\widehat{\Lambda_0^{1/2}\phi}(\xi-\eta)||\hat{\psi}(\eta)| + |\hat{\phi}(\xi-\eta)||\widehat{\Lambda_0^{1/2}\psi}(\eta)|\big)d\eta.
\end{aligned}
$$

Therefore, Hausdorff-Young's inequality gives the desired estimate. □

In the following we will use the notation $\partial\phi = (\partial_j\phi)$, $\partial^2\phi = (\partial_{ij}\phi)$, $\partial^3\phi = (\partial_{ijk}\phi)$, etc.

**Lemma 8.2.** *Let $s > (n+1)/2$, $M, c_1 > 0$ and suppose that*

$$
\left\{
\begin{array}{l}
\|b\|_{s+3} \le M, \quad E \equiv \|(\eta,\phi)\|_{s+3} + \|\big(\partial^3\phi - \delta^2 Z\partial^3\eta, \Lambda_0^{1/2}(\partial^3\phi - \delta^2 Z\partial^3\eta)\big)\|_s \le M, \\
1 + \eta(x) - b(x) \ge c_1 \quad \text{for} \quad x \in \mathbf{R}^n,
\end{array}
\right.
\tag{8.4}
$$

*where $\partial^3\phi - \delta^2 Z\partial^3\eta = \big(\partial_{ijk}\phi - \delta^2 Z\partial_{ijk}\eta\big)$. Then, there exist positive constants $\delta_2 = \delta_2(M, c_1, s)$ and $C = C(M, c_1, s)$ such that for any $\delta \in (0, \delta_2]$ we have*

$$
\left\{
\begin{array}{l}
\|Z\|_{s+1} + \delta^{1/2}\|Z\|_{s+2} + \delta\|\Lambda_0^{1/2}Z\|_{s+2} \le CE, \\
\|v\|_{s+2} + \|\Lambda_0^{1/2}v\|_{s+2} \le CE.
\end{array}
\right.
$$

**Proof**. By Lemma 4.5, for any $\delta \in (0, \delta_1]$ we can construct a diffeomorphism $\Theta$ satisfying the Assumption 4.1 (A1)–(A3) and

$$\|J^{s+5/2}\nabla_X\theta\|_{L^2(\Omega_0)} + \sup_{0\le x_{n+1}\le 1}\|\nabla_X\theta(\cdot, x_{n+1})\|_{s+2} \le C.$$

Therefore, by the definition (6.1) of $(v, Z)$ and Lemmas 5.5 and 5.2 we easily obtain $\|(v, Z)\|_{s+1} \le CE$. By Lemmas 6.4 and 5.2 we obtain $\delta^{1/2}\|Z\|_{s+2} \le CE$. Therefore,

$$\|v\|_{s+2} \le \|v\|_s + \|\partial^2 v\|_s \le \|v\|_s + \|\partial^3\phi - \delta^2 Z\partial^3\eta\|_s + C\delta^2\|Z\|_{s+2}\|\eta\|_{s+2} \le CE.$$

It remains to evaluate $\|\Lambda_0^{1/2}v\|_{s+2}$ and $\delta\|\Lambda_0^{1/2}\nabla Z\|_{s+1}$. By the definition of $Z$ and Lemma 6.1, we see that

$$\begin{aligned}
\partial_j Z &= -2\delta^2(\nabla\eta\cdot\nabla\partial_j\eta)(1+\delta^2|\nabla\eta|^2)^{-2}(\Lambda\phi + \nabla\eta\cdot\nabla\phi) \\
&\quad +(1+\delta^2|\nabla\eta|^2)^{-1}\big(\Lambda(\partial_j\phi - \delta^2 Z\partial_j\eta) - \nabla\cdot(v\partial_j\eta) + D_b\Lambda[\partial_j b]\phi + \partial_j(\nabla\eta\cdot\nabla\phi)\big) \\
&= (1+\delta^2|\nabla\eta|^2)^{-1}\big\{\Lambda(\partial_j\phi - \delta^2 Z\partial_j\eta) - \nabla\cdot(v\partial_j\eta) \\
&\qquad - 2\delta^2 Z\nabla\eta\cdot\nabla\partial_j\eta + D_b\Lambda[\partial_j b]\phi + \partial_j(\nabla\eta\cdot\nabla\phi)\big\} \\
&= (1+\delta^2|\nabla\eta|^2)^{-1}\big\{\delta^2|\nabla\eta|^2\partial_j Z + \Lambda(\partial_j\phi - \delta^2 Z\partial_j\eta) \\
&\qquad + D_b\Lambda[\partial_j b]\phi + \nabla\eta\cdot\partial_j v - (\nabla\cdot v)\partial_j\eta\big\},
\end{aligned}$$

which implies the expression

$$\partial_j Z = \Lambda(\partial_j\phi - \delta^2 Z\partial_j\eta) + D_b\Lambda[\partial_j b]\phi + \nabla\eta\cdot\partial_j v - (\nabla\cdot v)\partial_j\eta. \tag{8.5}$$

Hence, by Lemma 8.1 we obtain

$$\begin{aligned}
&\delta\|\Lambda_0^{1/2}\nabla Z\|_{s+1} \\
&\le C\delta\big(\|\Lambda_0^{1/2}\Lambda v\|_{s+1} + \|D_b\Lambda[\nabla b]\phi\|_{s+1} + \|\Lambda_0^{1/2}\nabla v\|_{s+1}\|\nabla\eta\|_{s+1} + \|\nabla v\|_{s+1}\|\Lambda_0^{1/2}\nabla\eta\|_{s+1}).
\end{aligned}$$

Here, by Lemmas 5.2 and 6.4 $\delta\|\Lambda_0^{1/2}\Lambda v\|_{s+1} \le \delta^{1/2}\|\Lambda v\|_{s+3/2} \le C\|\Lambda_0^{1/2}v\|_{s+2}$, and by Lemmas 6.3 and 5.2 $\delta\|D_b\Lambda[\nabla b]\phi\|_{s+1} \le C\delta^{1/2}\|\nabla b\|_{s+2}\|\Lambda_0^{1/2}\phi\|_{s+3/2} \le C\|b\|_{s+3}\|\phi\|_{s+2}$. Therefore, we get

$$\delta\|\Lambda_0^{1/2}\nabla Z\|_{s+1} \le C(\|\Lambda_0^{1/2}v\|_{s+2} + E).$$

Similarly, we see that

$$\begin{aligned}
\|\Lambda_0^{1/2}v\|_{s+2} &\le \|\Lambda_0^{1/2}v\|_s + \|\Lambda_0^{1/2}\partial^2 v\|_s \\
&\le C\big(\|v\|_{s+1} + \|\Lambda_0^{1/2}(\partial^3\phi - \delta^2 Z\partial^3\phi)\|_s \\
&\quad +\delta^2(\|\Lambda_0^{1/2}\nabla Z\|_{s+1}\|\eta\|_{s+2} + \|Z\|_{s+2}\|\Lambda_0^{1/2}\eta\|_{s+2})\big) \\
&\le C(\delta^2\|\Lambda_0^{1/2}\nabla Z\|_{s+1} + E).
\end{aligned}$$

These two estimates imply that if we take $\delta_2 \in (0, \delta_1]$ sufficiently small, then for any $\delta \in (0, \delta_2]$ we have $\delta\|\Lambda_0^{1/2}\nabla Z\|_{s+1} + \|\Lambda_0^{1/2}v\|_{s+2} \le CE$. The proof is complete. □

**Lemma 8.3.** *In addition to hypothesis of Lemma 8.2 we assume that $\|b\|_{s+9/2} \le M$. Then, there exists a constant $C = C(M, c, s) > 0$ such that we have*

$$\|(f_1, f_2)\|_s + \|\Lambda_0^{1/2}f_2\|_s \le CE.$$

**Proof**. By Lemmas 6.2 and 5.2 we have $\|D_b\Lambda[\partial^3 b]\phi\|_s \le C\|b\|_{s+9/2}\|\phi\|_{s+2}$. By Lemmas 6.2 and 6.3 we see that

$$\begin{aligned}
\|\Lambda_1[\partial u]\partial^2\phi\|_s &\le \|\Lambda_1[\partial u](\partial^2\phi - \delta^2 Z\partial\eta\|_s + \delta^2\|\Lambda_1[\partial u](Z\partial^2\eta)\|_s \\
&\le C\big(\|\partial u\|_{s+3/2}\|\Lambda_0^{1/2}(\partial^2\phi - \delta^2 Z\partial^2\eta)\|_{s+1} + \delta^{3/2}\|\partial u\|_{s+1}\|\Lambda_0^{1/2}(Z\partial^2\eta)\|_{s+1/2}\big) \\
&\le C\|u\|_{s+5/2}\big(\|\Lambda_0^{1/2}\psi\|_s + \|\phi\|_{s+3} + \delta\|Z\|_{s+1}\|\eta\|_{s+3}\big).
\end{aligned}$$

By Lemma 6.1 we have

$$\Lambda_1[\partial^2 u]\partial\phi = -\delta^2\Lambda(Z_1\partial^2\eta) - \nabla\cdot(v_1\partial^2\eta) + D_b\Lambda[\partial^2 b]\partial\phi,$$

where

$$\begin{cases} Z_1 = (1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda\partial\phi + \nabla\eta\cdot\nabla\partial\phi), \\ v_1 = \nabla\partial\phi - \delta^2 Z_1\nabla\eta. \end{cases}$$

Hence, by Lemmas 5.5, 5.2, and 6.2 we obtain

$$\begin{aligned} \|\Lambda_1[\partial^2 u]\partial\phi\|_s &\le C(\delta\|Z_1\partial^2\eta\|_{s+1} + \|v_1\partial^2\eta\|_{s+1} + \|\partial^2 b\|_{s+3/2}\|\Lambda_0^{1/2}\partial\phi\|_{s+1}) \\ &\le C(\delta\|\Lambda\partial\phi\|_{s+1} + \|\phi\|_{s+3}) \le C\|\phi\|_{s+3}. \end{aligned}$$

We can directly evaluate $\Lambda_2[\partial u,\partial u]\partial\phi$ and $\Lambda_3[\partial u,\partial u,\partial u]\phi$ by Lemma 6.2, and $\Lambda_2[\partial^2 u,\partial u]\phi$ by Lemma 6.5. Combining the above estimates and those obtained in Lemma 8.2 yields the estimate for $f_1$. By the same way as in the proof of Lemma 8.2, we can obtain the estimate for $\|f_2\|_s$. By Lemmas 8.1, 6.2–6.4 and 5.2, and the estimates obtained in Lemma 8.2, we see that

$$\begin{aligned} \|\Lambda_0^{1/2} f_2\|_s &\le C\big\{\delta^2\|\Lambda_0^{1/2}Z\|_{s+2} + \|\Lambda_0^{1/2}v\|_{s+1} + \|\Lambda_0^{1/2}\psi\|_s + \|\Lambda_0^{1/2}(\partial^2\phi - \delta^2 Z\partial^2\eta)\|_s \\ &\quad +\delta^2\big(\|Z\|_{s+2}\|\Lambda_0^{1/2}\Lambda\phi\|_{s+1} + \|\Lambda_0^{1/2}\Lambda v\|_{s+1} + \|\Lambda_0^{1/2}D_b\Lambda[\partial b]\phi\|_{s+1}\big)\big\} \\ &\le C\big\{E + \|\partial^2\phi - \delta^2 Z\partial^2\eta\|_{s+1} \\ &\quad +\delta^{3/2}(\|\Lambda\phi\|_{s+3/2} + \|\Lambda v\|_{s+3/2} + \|D_b\Lambda[\partial b]\phi\|_{s+3/2}\big)\big\} \\ &\le C\big(E + \delta(\|\Lambda_0^{1/2}\phi\|_{s+2} + \|\Lambda_0^{1/2}v\|_{s+2} + \|\partial b\|_{s+5/2}\|\Lambda_0^{1/2}\phi\|_{s+2})\big) \le CE. \end{aligned}$$

The proof is complete. □

In the following lemma we consider the case where $\eta$ and $\phi$ depends also on the time $t$.

**Lemma 8.4.** *Under the hypothesis of Lemma* 8.2*, there exists a constant* $C = C(M,c,s) > 0$ *such that for any* $\delta\in(0,\delta_2]$ *we have*

$$\begin{cases} \delta\|Z_t\|_{s+1} + \|v_t\|_{s+1} \le C\|(\eta_t,\phi_t)\|_{s+2}, \\ \delta\|Z_{tt}\|_s + \|v_{tt}\|_s \le C(\|\eta_{tt}\|_s + \delta\|\eta_{tt}\|_{s+1} + \|\phi_{tt}\|_{s+1} + \|(\eta_t,\phi_t)\|_{s+2}^2). \end{cases}$$

**Proof**. By the definition (6.1) of $(Z,v)$ we have

$$\begin{aligned} Z_t &= (1+\delta^2|\nabla\eta|^2)^{-1}\big(\Lambda\phi_t + D_\eta\Lambda[\eta_t]\phi + \nabla\eta_t\cdot\nabla\phi + \nabla\eta\cdot\nabla\phi_t - 2\delta^2 Z\nabla\eta\cdot\nabla\eta_t\big), \\ v_t &= \nabla\phi_t - \delta^2(Z\nabla\eta_t + Z_t\nabla\eta). \end{aligned}$$

Therefore, by Lemmas 5.5, 6.3, and 8.2 we obtain the estimate for $(Z_t, v_t)$. Similarly, in view of

$$\begin{aligned} Z_{tt} &= (1+\delta^2|\nabla\eta|^2)^{-1}\big(\Lambda\phi_{tt} + D_\eta\Lambda[\eta_{tt}]\phi + 2D_\eta\Lambda[\eta_t]\phi_t + D_\eta^2\Lambda[\eta_t,\eta_t]\phi \\ &\quad +\nabla\eta_{tt}\cdot\nabla\phi + \nabla\eta\cdot\nabla\phi_{tt} + 2\nabla\eta_t\cdot\nabla\phi_t - 4\delta^2 Z_t\nabla\eta\cdot\nabla\eta_t - 2\delta^2 Z(\nabla\eta\cdot\nabla\eta_{tt} + |\nabla\eta_t|^2)\big), \\ v_{tt} &= \nabla\phi_{tt} - \delta^2(Z_{tt}\nabla\eta + Z\nabla\eta_{tt} + 2Z_t\nabla\eta_t), \end{aligned}$$

we obtain the estimate for $(Z_{tt}, v_{tt})$. □

# 9 Proof of the main theorems

In this section we will give a proof of Theorems 2.1 and 3.1. The existence of the solution for fixed $\delta > 0$ can be proved by using approximate equations and taking the limit. See [23, 21, 22, 13] for details. We can also show the dependence of the solution on the initial data, that is, the well-posedness of the initial value problem. Therefore, to show Theorem 2.1 it is sufficient to derive a priori estimates of the smooth solution $(\eta^\delta, \phi^\delta)$ for a time interval $[0, T]$ independent of $\delta$.

Suppose that $(\eta^\delta, \phi^\delta)$ is the solution of (2.14) and (2.15) and satisfies

$$\tag{9.1} \begin{cases} \mathscr{E}(t) \equiv \|(\eta^\delta(t), \phi^\delta(t))\|_{s+3}^2 + \|\Lambda_0^{1/2}(\partial^3\phi^\delta(t) - \delta^2 Z^\delta \partial^3 \eta^\delta(t))\|_s^2 \le N_1, \\ \|(\eta^\delta(t), \phi^\delta(t))\|_{s+2}^2 \le N_2, \\ 1 + \eta^\delta(x,t) - b(x) > c_0/2 \quad \text{for} \quad x \in \mathbf{R}^n,\ 0 \le t \le T,\ 0 < \delta \le \delta_0, \end{cases}$$

where $Z^\delta$ is determined by (6.1) from $(\eta^\delta, \phi^\delta)$ and positive constants $N_1$, $N_2$, $T$, and $\delta_0$ will be determined later. In the following we simply write the constants depending only on $(M_0, N_1, c_0, s)$ and $(M_0, N_2, c_0, s)$ by $C_1$ and $C_2$, respectively. By Lemmas 4.5 and 4.4, there exists a small $\delta_1 = \delta_1(M_0, N_2, c_0, s) > 0$ such that for any $\delta \in (0, \delta_1]$ and $\varphi \in H^1$ we have

$$\tag{9.2} C_2^{-1}(\Lambda\varphi, \varphi) \le (\Lambda_0\varphi, \varphi) \le C_2(\Lambda\varphi, \varphi).$$

By Lemmas 8.2 and 8.3, there exists a small $\delta_2 = \delta_2(M_0, N_1, c_0, s) \le \delta_1$ such that we have

$$\tag{9.3} \begin{cases} \|Z^\delta\|_{s+1} + \delta^{1/2}\|Z^\delta\|_{s+2} + \delta\|\Lambda_0^{1/2} Z^\delta\|_{s+2} + \|v^\delta\|_{s+2} + \|\Lambda_0^{1/2} v^\delta\|_{s+2} \le C_1, \\ \|(f_1, f_2)\|_s^2 + \|\Lambda_0^{1/2} f_2\|_s^2 \le C_1 \mathscr{E} \quad \text{for} \quad 0 \le t \le T,\ 0 < \delta \le \min\{\delta_0, \delta_2\}. \end{cases}$$

In view of

$$\begin{aligned} \eta_t^\delta &= \Lambda\phi^\delta, \quad \partial_j \eta_t^\delta = \Lambda(\partial_j\phi^\delta - \delta^2 Z^\delta \partial_j \eta^\delta) - \nabla\cdot(v^\delta \partial_j \eta^\delta) + D_b\Lambda[\partial_j b]\phi^\delta, \\ \eta_{tt}^\delta &= \Lambda(\phi_t - \delta^2 Z^\delta \eta_t) - \nabla\cdot(v^\delta \eta_t), \\ \phi_t^\delta &= -\eta^\delta - \tfrac{1}{2}|\nabla\phi^\delta|^2 + \tfrac{1}{2}\delta^2(1 + \delta^2|\nabla\eta^\delta|^2)(Z^\delta)^2, \\ \phi_{tt}^\delta &= -\eta_t^\delta - \nabla\phi^\delta\cdot\nabla\phi_t^\delta + \delta^4(\nabla\eta^\delta\cdot\nabla\eta_t^\delta)(Z^\delta)^2 + \delta^2(1 + \delta^2|\nabla\eta^\delta|^2) Z^\delta Z_t^\delta, \end{aligned}$$

in the same way as the proof of Lemma 8.2 we obtain

$$\tag{9.4} \|(\eta_t^\delta(t), \phi_t^\delta(t))\|_{s+2} + \|\eta_{tt}^\delta(t)\|_s + \delta\|\eta_{tt}^\delta(t)\|_{s+1} + \|\phi_{tt}^\delta(t)\|_{s+1} \le C_1$$

for $0 \le t \le T$ and $0 \le \delta \le \min\{\delta_2, \delta_0\}$. Therefore, by Lemma 8.4

$$\delta(\|Z_t^\delta\|_{s+1} + \|Z_{tt}^\delta\|_s) + \|v_t^\delta\|_{s+1} + \|v_{tt}^\delta\|_s \le C_1.$$

Particularly, for $a^\delta = 1 + \delta^2(Z_t^\delta + v^\delta\cdot\nabla Z^\delta)$ we have $\|\nabla a^\delta\|_s + \|a_t^\delta\|_s \le C_1$ for $0 \le t \le T$ and $0 \le \delta \le \min\{\delta_2, \delta_0\}$. Moreover, by the Sobolev inequality

$$\delta|Z_t^\delta + v^\delta\cdot\nabla Z^\delta|_\infty \le C\delta(\|Z_t^\delta\|_s + \|v^\delta\|_s\|Z^\delta\|_{s+1}) \le C_1.$$

Hence, setting $\delta_3 = \min\{\delta_2, (2C_1)^{-1}\}$ we obtain $1/2 \le a^\delta(x,t) \le 2$ for $x \in \mathbf{R}^n$, $0 \le t \le T$, and $0 < \delta \le \min\{\delta_3, \delta_0\}$. Now, we can apply the energy estimate obtained in Lemma 7.8 to the quasi-linear system (8.3) and obtain

$$\begin{aligned} &\|\zeta^\delta(t)\|_s^2 + \|\psi^\delta(t)\|_s^2 + \|\Lambda_0^{1/2}\psi^\delta(t)\|_s^2 \\ &\le C_2 e^{C_1 t}\big(\|\zeta^\delta(0)\|_s^2 + \|\psi^\delta(0)\|_s^2 + \|\Lambda_0^{1/2}\psi^\delta(0)\|_s^2\big) + C_1\int_0^t e^{C_1(t-\tau)}\mathscr{E}(\tau)d\tau, \end{aligned}$$

where we used (9.2). By (9.4) we easily obtain

$$\|(\eta^\delta(t),\phi^\delta(t))\|_{s+2}^2 \le \mathrm{e}^{C_1 t}\|(\eta_0^\delta,\phi_0^\delta)\|_{s+2}^2.$$

Therefore, we obtain

$$\mathscr{E}(t) \le (C_2 + C_1\delta)\mathrm{e}^{C_1 t}(\|\eta_0^\delta\|_{s+3+1/2} + \|\phi_0^\delta\|_{s+4}) + C_1\int_0^t \mathrm{e}^{C_1(t-\tau)}\mathscr{E}(\tau)d\tau,$$

which together with Gronwall's inequality yields that

$$\mathscr{E}(t) \le (C_2 + C_1\delta)\mathrm{e}^{C_1 T}(\|\eta_0^\delta\|_{s+3+1/2} + \|\phi_0^\delta\|_{s+4}).$$

Moreover, we have $|\eta^\delta(t) - \eta_0^\delta|_\infty \le C\int_0^t \|\eta_t^\delta(\tau)\|_s d\tau \le c_0 C_1 t$. By setting $N_2 = 2\|(\eta_0^\delta,\phi_0^\delta)\|_{s+2}^2$, $N_1 = 4C_2(\|\eta_0^\delta\|_{s+3+1/2} + \|\phi_0^\delta\|_{s+4})$, $\delta_0 = \min\{\delta_3, C_1^{-1}C_2\}$, and $T = (2C_1)^{-1}(< C_1^{-1}\log 2)$, we see that the estimates in (9.1) holds for $0 \le t \le T$ and $0 < \delta \le \delta_0$. By (9.4) we also obtain a uniform bound for $\|(\eta_t^\delta(t),\phi_t^\delta(t))\|_{s+2}$. The proof of Theorem 2.1 is complete.

We proceed to prove Theorem 3.1. To this end, we first expand the Dirichlet-to-Neumann map $\Lambda(\eta,\delta)$ with respect to $\delta^2$. The next lemma is a mathematically rigorous version of the formal expansion (3.2).

**Lemma 9.1.** *Let $s > n/2$. Under Assumption* 4.1 (A1) *and*

$$\|J^{s+2}\nabla_X\theta\|_{L^2(\Omega_0)} + \sup_{0\le x_{n+1}\le 1}\|\nabla_X\theta(\cdot,x_{n+1})\|_{s+3/2} \le M,$$

*there exists a constant $C = C(M,c,s) > 0$ such that we have*

$$\|\Lambda\phi + \nabla\cdot((1+\eta-b)\nabla\phi)\|_s \le C\delta^2(\|\Lambda_0^{1/2}\phi\|_{s+2} + \|\nabla\phi\|_{s+1}).$$

**Proof**. Set $\Phi := \phi^\hbar$ and $\tilde\Phi := \Phi\circ\Theta$. Then, we have (4.6). Since $\partial_{n+1}\tilde\Phi(\cdot,0) = 0$ and $\delta^{-2}\partial_{n+1}\tilde\Phi(\cdot,1) = \Lambda\phi$, we see that

$$\begin{aligned}\Lambda\phi &= \int_0^1 \partial_{n+1}\big((\delta^{-2}(1+\partial_{n+1}\theta_{n+1})^{-1} + p_{22})\partial_{n+1}\tilde\Phi\big)dx_{n+1}\\ &= -\int_0^1 \nabla\cdot\big(((1+\partial_{n+1}\theta_{n+1})E_n + \delta^2 P_{11})\nabla\tilde\Phi\big)dx_{n+1} - \int_0^1 \nabla\cdot(\mathbf{p}_{12}\partial_{n+1}\tilde\Phi)dx_{n+1},\end{aligned}$$

where we used (5.9) and (5.10). By (4.2) we see that $\tilde\Phi(\cdot,x_{n+1}) = \phi - \int_{x_{n+1}}^1 \partial_{n+1}\tilde\Phi(\cdot,y)dy$ and $\int_0^1(1+\partial_{n+1}\theta_{n+1})dx_{n+1} = 1+\eta-b$, so that

$$\begin{aligned}\Lambda\phi + \nabla\cdot\big((1+\eta-b)\nabla\phi\big) &= \int_0^1 \nabla\cdot\Big((1+\partial_{n+1}\theta_{n+1})\int_{x_{n+1}}^1 \nabla\partial_{n+1}\tilde\Phi(\cdot,y)dy\Big)dx_{n+1}\\ &\quad -\delta^2\int_0^1 \nabla\cdot P_{11}\nabla\tilde\Phi dx_{n+1} - \int_0^1 \nabla\cdot(\mathbf{p}_{12}\partial_{n+1}\tilde\Phi)dx_{n+1}.\end{aligned}$$

Therefore, we obtain

$$\|\Lambda\phi + \nabla\cdot((1+\eta-b)\nabla\phi)\|_s \le C(\delta^2\|J^{s+1}\nabla\tilde\Phi\|_{L^2(\Omega_0)} + \|J^{s+1}\partial_{n+1}\tilde\Phi\|_{L^2(\Omega_0)}).$$

On the other hand, it follows from (5.11) that

$$\|J^{s+1}\partial_{n+1}\tilde{\Phi}\|_{L^2(\Omega_0)} \le C\delta^2 \|J^{s+2}\nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}.$$

These estimates together with Lemmas 5.6 and 5.7 imply the desired estimate. □

By the uniform estimate obtained in Theorem 2.1, Lemma 9.1, and the standard compactness argument, we see that as $\delta \to +0$

$$(\eta^\delta, \phi^\delta) \to (\eta^0, \phi^0) \quad \text{weakly* in} \quad L^\infty(0,T;H^{s+3}),$$

where $(\eta^0, \phi^0)$ is a unique solution of the shallow water equations (3.3) with initial conditions $(\eta^0, \phi^0)|_{t=0} = (\eta_0^0, \phi_0^0)$. Next, we will show the strong convergence. It follows from (2.14) and (3.3) that

$$\text{(9.5)} \qquad \left\{ \begin{array}{l} (\eta^\delta - \eta^0)_t + \nabla\cdot\big((1+\eta^\delta - b)\nabla(\phi^\delta - \phi^0) + (\eta^\delta - \eta^0)\nabla\phi^0\big) = \delta^2 f_3^\delta, \\ (\phi^\delta - \phi^0)_t + (\eta^\delta - \eta^0) + \frac{1}{2}\nabla(\phi^\delta + \phi^0)\cdot\nabla(\phi^\delta - \phi^0) = \delta^2 f_4^\delta, \end{array} \right.$$

where

$$\begin{aligned} f_3^\delta &= \delta^{-2}\big(\Lambda\phi^\delta + \nabla\cdot((1+\eta^\delta - b)\nabla\phi^\delta)\big), \\ f_4^\delta &= \tfrac{1}{2}(1+\delta^2|\nabla\eta^\delta|^2)^{-1}(\Lambda\phi^\delta + \nabla\eta^\delta\cdot\nabla\phi^\delta) \quad \big(= \tfrac{1}{2}(1+\delta^2|\nabla\eta^\delta|^2)(Z^\delta)^2\big). \end{aligned}$$

By (9.3) and Lemma 9.1, we easily have $\|f_3^\delta(t)\|_s + \|f_4^\delta(t)\|_{s+1} \le C$ for $0 \le t \le T$ and $0 < \delta \le \delta_0$. Taking these equations into account, we will consider the following system of linear equations

$$\text{(9.6)} \qquad \left\{ \begin{array}{l} \zeta_t + \nabla\cdot(a\nabla\psi + b_1\zeta) = f_1, \\ \psi_t + \zeta + b_2\cdot\nabla\psi = f_2, \end{array} \right.$$

where $a$, $b_1 = (b_{11}, \ldots, b_{1n})$, $b_2 = (b_{21}, \ldots, b_{2n})$, $f_1$, and $f_2$ are given function of $x$ and $t$.

**Lemma 9.2.** *Let $s > n/2$ and suppose that*

$$M^{-1} \le a(x,t) \le M, \quad \|(a_t, \nabla a)\|_s + \|(b_1, b_2)\|_{s+1} \le M.$$

*Then, there exists a constant $C = C(M,s)$ such that for any smooth solution $(\zeta,\psi)$ of (9.6) we have*

$$\|\zeta(t)\|_s^2 + \|\psi(t)\|_{s+1}^2 \le C\mathrm{e}^{Ct}(\|\zeta(0)\|_s^2 + \|\psi(0)\|_{s+1}^2) + C\int_0^t \mathrm{e}^{C(t-\tau)}(\|f_1(\tau)\|_s^2 + \|f_2(\tau)\|_{s+1}^2)d\tau.$$

**Proof**. We define an energy function $E_s(t)$ by

$$E_s(t) := \|\zeta(t)\|_s^2 + (a\nabla J^s\psi(t), \nabla J^s\psi(t)) + \|\psi(t)\|_s^2,$$

which is equivalent to $\|\zeta(t)\|_s^2 + \|\psi(t)\|_{s+1}^2$. Let $(\zeta,\psi)$ be a solution of (9.6). Then, we see that

$$\begin{aligned} \frac{d}{dt}E_s(t) &= 2(J^s\zeta_t, J^s\zeta) + 2(a\nabla J^s\psi_t, \nabla J^s\psi) + (a_t\nabla J^s\psi, \nabla J^s\psi) + 2(J^s\psi_t, J^s\psi) \\ &= -2(\nabla\cdot[J^s,a]\nabla\psi, J^s\zeta) - 2(\nabla\cdot[J^s,b_1]\zeta, J^s\zeta) + ((\nabla\cdot b_1)J^s\zeta, J^s\zeta) + 2(J^s f_1, J^s\zeta) \\ &\quad -2(a\nabla([J^s,b_2]\cdot\nabla\psi), \nabla J^s\psi) + ((\nabla\cdot(ab_2))\nabla J^s\psi, \nabla J^s\psi) \\ &\quad -2\sum_{j=1}^n (a(\partial_j b_2)\cdot\nabla J^s\psi, \partial_j J^s\psi) + 2(a\nabla J^s f_2, \nabla J^s\psi) + (a_t\nabla J^s\psi, \nabla J^s\psi) \\ &\quad +2(J^s(f_2 - \zeta - b_2\cdot\nabla\psi), J^s\psi) \\ &\le CE_s(t) + \|f_1(t)\|_s^2 + \|f_2(t)\|_{s+1}^2. \end{aligned}$$

Therefore, the desired energy estimate comes from Gronwall's inequality. □

Applying the energy estimate to (9.5) we obtain

$$\|\eta^\delta(t)-\eta^0(t)\|_s+\|\phi^\delta(t)-\phi^0(t)\|_{s+1}\le C(\|\eta_0^\delta-\eta_0^0\|_s+\|\phi_0^\delta-\phi_0^0\|_{s+1}+\delta^2)$$

for $0\le t\le T$ and $0<\delta\le\delta_0$ with a constant $C$ independent of $\delta$ and $t$. This shows the strong convergence of the solution $(\eta^\delta,\phi^\delta)$ in $C([0,T];H^s\times H^{s+1})$. Since we have a uniform bound of the solution in $C([0,T];H^{s+3})$, by the interpolation inequality we obtain the strong convergence of the solution in $C([0,T];H^{s+3-\varepsilon})$ for each $\varepsilon>0$. The latter part of the theorem comes from directly the above estimate. The proof of Theorem 3.1 is complete.

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