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Value Distribution of Painlevé Transcendents of the First and the Second Kind

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Abstract: We examine value distribution properties of the first and the second Painlevé transcendents. For every transcendental meromorphic solution $\phi(z)$ (resp. $\psi(z)$) of the first (resp. second) Painlevé equation, the deficiency $\delta(g,\phi)$ (resp. $\delta(g,\psi)$) of a small function $g(z)$ does not exceed 1/2. Furthermore, for $\phi(z)$, the ramification index satisfies $\vartheta(g,\phi) \le 5/12$.

1. Introduction

Consider the first and the second Painlevé equations

$$\text{(I)} \qquad w'' = 6w^2 + z,$$

$$\text{(II)} \qquad w'' = 2w^3 + zw + \alpha$$

($' = d/dz$), where $\alpha \in \mathbf{C}$. All the solutions of these equations are meromorphic in the whole complex plane ([1]). It is easy to see that every solution of (I) is transcendental. Equation (II) admits a unique rational solution, if and only if $\alpha \in \mathbf{Z}$ ([5]). For transcendental meromorphic solutions of these equations, the deficiency $\delta(a,w)$ and the ramification index $\vartheta(a,w)$ ($a \in \mathbf{C} \cup \{\infty\}$) were examined by [2], [6], [7], [10]. Throughout this paper we use the standard notation of the value distribution theory concerning a meromorphic function $f(z)$ such as $m(r,f), N(r,f), N_1(r,f), T(r,f), S(r,f), \delta(a,f), \vartheta(a,f)$ ($r > 0$, $a \in \mathbf{C} \cup \{\infty\}$) ([3]). Furthermore, for a meromorphic function $g(z)$, we say that $g(z)$ is *small with respect to* $f(z)$, if $T(r,g) = S(r,f)$. Define the deficiency and the ramification index of a small function $g(z)$ by

$$\delta(g,f) = \liminf_{r\to\infty} \frac{m(r,1/(f-g))}{T(r,f)}, \qquad \vartheta(g,f) = \liminf_{r\to\infty} \frac{N_1(r,1/(f-g))}{T(r,f)}.$$

The purpose of this paper is to estimate these quantities for transcendental meromorphic solutions of (I) and (II). Let $\phi(z)$ (resp. $\psi(z)$) be an arbitrary transcendental meromorphic solution of (I) (resp. of (II)). Our main results are stated as follows:

Theorem 1.1. *If $g(z)$ is small with respect to $\phi(z)$, then $\delta(g,\phi) \le 1/2$.*

Theorem 1.2. *If $g(z)$ is small with respect to $\psi(z)$, then $\delta(g,\psi) \le 1/2$.*

Theorem 1.3. *If $g(z)$ is small with respect to $\phi(z)$, then $\vartheta(g, \phi) \le 5/12$.*

Remark 1.1. In particular, for every complex number $a \in \mathbf{C}$, $\vartheta(a, \phi) \le 1/6$ (see [6], [7], [3]).

Our results are proved in Sections 3, 4 and 5. The auxiliary functions $H(z)$ and $K(z)$ in the proofs of Theorems 1.1 and 1.2 are due to [9]. In the proof of Theorem 1.3, we need to choose a different type of auxiliary function $\Omega(z)$. In these proofs, we use the additional notation as follows: for a set $A \subset \mathbf{C}$,

$$N(r, f)|_A = \int_0^r \Big(n(\rho, f)|_A - n(0, f)|_A \Big) \frac{d\rho}{\rho} + n(0, f)|_A \log r,$$
$$N_1(r, f)|_A = \int_0^r \Big(n_1(\rho, f)|_A - n_1(0, f)|_A \Big) \frac{d\rho}{\rho} + n_1(0, f)|_A \log r,$$
$$n(\rho, f)|_A = \sum_{\substack{\sigma \in A, |\sigma| \le \rho \\ f(\sigma) = \infty}} \mu(\sigma, f), \qquad n_1(\rho, f)|_A = \sum_{\substack{\sigma \in A, |\sigma| \le \rho \\ f(\sigma) = \infty}} (\mu(\sigma, f) - 1),$$

where $\mu(\sigma, f)$ denotes the multiplicity of the pole $z = \sigma$ of $f(z)$.

2. Lemmas

Substituting the Laurent series expansion of the solution $\phi(z)$ (or $\psi(z)$) into (I) (or (II)), we have the following.

Lemma 2.1. *Around a movable pole $z = z_\infty$ of $\phi(z)$,*

$$\phi(z) = \frac{1}{(z - z_\infty)^2} - \frac{z_\infty}{10}(z - z_\infty)^2 - \frac{1}{6}(z - z_\infty)^3 + \gamma(z - z_\infty)^4 + O((z - z_\infty)^5),$$

and, around a movable pole $z = z_\infty$ of $\psi(z)$,

$$\psi(z) = \frac{\pm 1}{z - z_\infty} \mp \frac{z_\infty}{6}(z - z_\infty) + \frac{(\mp 1 - \alpha)}{4}(z - z_\infty)^2 + \gamma(z - z_\infty)^3 + O((z - z_\infty)^4),$$

where γ is a parameter depending on the initial condition.

The following is a lemma of the Clunie type (cf. [3; Lemma 2.4.2 and Remark 1 in §2.3]).

Lemma 2.2. *Let f be a transcendental meromorphic function such that*

$$f^n P(z, f) = Q(z, f), \qquad n \in \mathbf{N},$$

where $P(z, u)$ and $Q(z, u)$ are polynomials in u and its derivatives with meromorphic coefficients $\{a_\mu(z) \mid \mu \in M\}$. If the total degree of $Q(z, u)$ as a polynomial in u and its derivatives is at most n, then

$$m(r, P(z, f)) = O\Big(\sum_{\mu \in M} m(r, a_\mu) \Big) + S(r, f).$$

The following lemma is due to [4; Theorem 6] (see also [3]).

Lemma 2.3. *Let $F(z,u)$ be a polynomial in u and its derivatives with meromorphic coefficients $\{b_\nu(z) \mid \nu \in N\}$. Assume that $u = f$ is a transcendental meromorphic solution of the differential equation $F(z,u) = 0$, and that $F(z,0) \not\equiv 0$. Then*

$$m(r, 1/f) = O\Big(\sum_{\nu \in N} T(r, b_\nu)\Big) + S(r, f).$$

3. Proof of Theorem 1.1

We put $\Phi(z) = \phi(z) - g(z)$. It is easy to see that

$$\Phi''(z) = 6\Phi(z)^2 + 12g(z)\Phi(z) + G(z), \tag{3.1}$$

where

$$G(z) = 6g(z)^2 + z - g''(z).$$

If $G(z) \not\equiv 0$, then, by Lemma 2.3, we have

$$m(r, 1/\Phi) = O(T(r, g) + \log r) + S(r, \Phi) = S(r, \phi),$$

which implies that $\delta(g, \phi) = 0$.

We may suppose that $G(z) \equiv 0$. Write (3.1) in the form

$$U'(z) = -2\Phi(z) \tag{3.2}$$

with

$$U(z) = \Phi'(z)^2 + 2g'(z)\Phi'(z) - 4\Phi(z)^3 - 12g(z)\Phi(z)^2 - 2(6g(z)^2 + z)\Phi(z). \tag{3.3}$$

Furthermore, by (3.2),

$$U(z) = U(c) - 2\int_c^z \Phi(t)dt \tag{3.4}$$

for c satisfying $U(c) \neq \infty$. Consider the function

$$H(z) = U(z)/\Phi(z). \tag{3.5}$$

From (3.3), (3.1) and Lemma 2.2, we have

$$\begin{aligned} m(r, H) &= O\big(m(r, \Phi'/\Phi) + m(r, \Phi) + T(r, g) + \log r\big) \\ &= S(r, \Phi) = S(r, \phi). \end{aligned} \tag{3.6}$$

The supposition $G(z) \equiv 0$ implies that $g(z)$ is also a solution of (I). By Lemma 2.1, every pole of $\Phi(z)$ is double, and, by (3.4), it is a zero of $H(z)$. Hence, using (3.6), we have

$$(1/2)N(r, \Phi) \le N(r, 1/H) \le T(r, H) + O(1) \le N(r, H) + S(r, \phi). \tag{3.7}$$

It is easy to see that every pole of $H(z)$ is a zero of $\Phi(z)$. By (3.5) and Lemma 2.2,

$$N(r, H) \le N(r, 1/\Phi) = T(r, \Phi) - m(r, 1/\Phi) + O(1) \le N(r, \Phi) - m(r, 1/\Phi) + S(r, \phi).$$

Substituting this into (3.7), and using $N(r, \Phi) \le N(r, \phi) + N(r, g) \le T(r, \phi) + S(r, \phi)$, we obtain

$$m(r, 1/\Phi) \le (1/2)T(r, \phi) + S(r, \phi),$$

which implies $\delta(g, \phi) \le 1/2$.

4. Proof of Theorem 1.2

The function $\Psi(z) = \psi(z) - g(z)$ satisfies

$$\Psi''(z) = 2\Psi(z)^3 + 6g(z)\Psi(z)^2 + (6g(z)^2 + z)\Psi(z) + \widetilde{G}(z), \tag{4.1}$$

where

$$\widetilde{G}(z) = 2g(z)^3 + zg(z) + \alpha - g''(z).$$

When $\widetilde{G}(z) \not\equiv 0$, by the same argument as in Section 3, we can show that $\delta(g, \psi) = 0$. Hence, in what follows, we suppose that $\widetilde{G}(z) \equiv 0$, namely, that $g(z)$ is a solution of (II). It is easy to see that (4.1) is equivalent to

$$V'(z) = -\Psi(z)^2 - 2g(z)\Psi(z) \tag{4.2}$$

or

$$V(z) = V(c) - \int_c^z \big(\Psi(t)^2 + 2g(t)\Psi(t)\big)dt, \qquad c \in \mathbf{C},\ V(c) \neq \infty, \tag{4.3}$$

where

$$\begin{aligned} V(z) = {} & \Psi'(z)^2 + 2g'(z)\Psi'(z) - \Psi(z)^4 - 4g(z)\Psi(z)^3 \\ & - (6g(z)^2 + z)\Psi(z)^2 - (4g(z)^3 + 2zg(z) + 2\alpha)\Psi(z). \end{aligned} \tag{4.4}$$

Put

$$K(z) = V(z)/\Psi(z).$$

From (4.4) and Lemma 2.2, it follows that

$$m(r, K) = S(r, \psi). \tag{4.5}$$

By Lemma 2.1, every pole of $\Psi(z) = \psi(z) - g(z)$ is simple and belongs to

$$\text{either} \quad P = \big\{\sigma \bigm| \psi(\sigma) = \infty,\, g(\sigma) \neq \infty\big\} \quad \text{or} \quad P' = \big\{\sigma \bigm| g(\sigma) = \infty\big\}.$$

Especially, around $z = z_\infty \in P$, $\Psi(z) = \pm(z - z_\infty)^{-1} + O(1)$. Using this expression and (4.3), we have

$$K(z_\infty) = \pm 1 \quad \text{for every} \quad z_\infty \in P. \tag{4.6}$$

As will be shown later, $K(z)$ satisfies

$$K(z) \not\equiv \pm 1. \tag{4.7}$$

4.1. Derivation of the conclusion under (4.7). Note that

$$\begin{aligned}N(r,\Psi) =&N(r,\Psi)|_P + N(r,\Psi)|_{P'} \le N(r,\Psi)|_P + N(r,g)\\ \le&N(r,\Psi)|_P + S(r,\psi).\end{aligned}$$

By (4.5), (4.6) and (4.7), we have

$$\begin{aligned}N(r,\Psi)|_P \le&N(r,1/(K+1)) + N(r,1/(K-1)) \le 2T(r,K) + O(1)\\ \le&2N(r,K) + S(r,\psi).\end{aligned}$$

Hence

$$N(r,\Psi) \le 2N(r,K) + S(r,\psi). \tag{4.8}$$

By the definition of $K(z)$, each pole of $K(z)$ belongs to either $Z = \{\sigma \mid \Psi(\sigma) = 0\}$ or P'. By (4.3) and Lemma 2.1, if $\sigma \in P' \setminus Z$, then σ is a pole of $\Psi(z)$ and is not a pole of $K(z)$. Hence every pole of $K(z)$ belongs to Z and is not a pole of $V(z)$. Using this fact and Lemma 2.2, we have

$$\begin{aligned}N(r,K) \le&N(r,1/\Psi) \le T(r,\Psi) - m(r,1/\Psi) + O(1)\\ \le&N(r,\Psi) - m(r,1/\Psi) + S(r,\psi).\end{aligned} \tag{4.9}$$

From (4.8), (4.9) and $N(r,\Psi) \le N(r,\psi) + N(r,g) \le T(r,\psi) + S(r,\psi)$, it follows that

$$m(r,1/\Psi) \le (1/2)T(r,\psi) + S(r,\psi),$$

which implies $\delta(g,\psi) \le 1/2$.

4.2. Verification of (4.7). It remains to verify (4.7). To do so, we suppose the contrary

$$K(z) \equiv \pm 1. \tag{4.10}$$

Then we have the following.

Lemma 4.1. *The functions $g(z)$ and $\psi(z)$ are solutions of the Riccati differential equation*

$$w' = \mp(w^2 + z/2). \tag{4.11}$$

Proof. From (4.3) and (4.10), we have

$$V(z) = V(c) - \int_c^z \big(\Psi(t)^2 + 2g(t)\Psi(t)\big)dt = \pm\Psi(z),$$

which implies

$$\Psi'(z) = \mp(\Psi(z)^2 + 2g(z)\Psi(z)). \tag{4.12}$$

On the other hand, by (4.4),

$$\begin{aligned}\Psi'(z)^2 + 2g'(z)\Psi'(z) - \Psi(z)^4 - 4g(z)\Psi(z)^3 \\ -(6g(z)^2 + z)\Psi(z)^2 - (4g(z)^3 + 2zg(z) + 2\alpha)\Psi(z) = \pm\Psi(z).\end{aligned} \tag{4.13}$$

Substitution of (4.12) into (4.13) yields

$$F_\mp(z)\Psi(z) \pm F'_\mp(z) = 0, \qquad F_\mp(z) = -2g(z)^2 \mp 2g'(z) - z. \tag{4.14}$$

If $F_\mp(z) \not\equiv 0$, then $T(r,\psi) = O(T(r,g))$, which contradicts the condition $T(r,g) = S(r,\psi)$. Hence, $F_\mp(z) \equiv 0$, which implies that $g(z)$ satisfies (4.11). Putting $\Psi(z) = \psi(z) - g(z)$ in (4.12), we can verify that $\psi(z)$ is also a solution of (4.11). □

Simple computation shows that every solution of (4.11) is expressible in the form

$$\chi(z) = \pm h'(z)/h(z), \qquad h(z) = A(-z/\sqrt[3]{2}), \tag{4.15}$$

where $A(s)$ is a solution of the Airy equation

$$\frac{d^2y}{ds^2} - sy = 0. \tag{4.16}$$

Consider the sectors defined by

$$S_0: \ -\pi/3 < \arg s < \pi, \qquad S_1: \ \pi/3 < \arg s < 5\pi/3, \qquad S_2: \ \pi < \arg s < 7\pi/3.$$

For each S_j $(j = 0, 1, 2)$, $A(s)$ admits an asymptotic expression

(4.17,j)
$$A(s) = c_j s^{-1/4}(1 + o(1))\exp(-2s^{3/2}/3) + c'_j s^{-1/4}(1 + o(1))\exp(2s^{3/2}/3),$$
$$(c_j, c'_j) \in \mathbf{C}^2 \setminus \{(0,0)\}$$

as $s \to \infty$ through the sector S_j ([8; §§8, 21]). Then, we have

$$T(r,\chi) = O(T(r,h)) = O(r^{3/2}).$$

Furthermore, by (4.17,0), if $c_0 c'_0 \neq 0$, then

$$A(s) = c'_0 s^{-1/4}(1 + o(1))\exp(-2s^{3/2}/3)\Big((c_0/c'_0) + o(1) + \exp(4s^{3/2}/3)\Big)$$

as $s \to \infty$ through S_0. This implies that $A(s)$ possesses a sequence of simple zeros

$$s_m = (3\pi/2)^{2/3} m^{2/3}(1 + O(m^{-1}))e^{\pi i/3}, \quad m = 1, 2, \ldots \tag{4.18}$$

in S_0. Hence

$$N(r,\chi) \geq Cr^{3/2}$$

for some $C > 0$. In the case where $c_0 = 0$ or $c'_0 = 0$, we obtain the same estimate. For example, if $c_0 \neq 0$, $c'_0 = 0$, then $c_1 c'_1 \neq 0$, and there exists a sequence analogous to (4.18) in S_1 (see [8; §22]). Thus we have the following.

Lemma 4.2. *For every solution $\chi(z)$ of* (4.11)*, we have*

$$C_1 r^{3/2} \leq T(r, \chi) \leq C_2 r^{3/2},$$

where C_1, C_2 are some positive constants.

From this lemma, it immediately follows that $T(r, \psi) = O(T(r, g))$, which is a contradiction. Thus the proof of the theorem is completed.

Remark 4.1. Every solution of (4.11) satisfies (II) with $\alpha = \mp 1/2$. Let η_1, η_2 be solutions of (4.11) such that $\eta_1 - \eta_2 \not\equiv 0$. Since they are expressible in the form $\eta_1 = \pm h_1'/h_1$, $\eta_2 = \pm h_2'/h_2$, where h_1, h_2 are linearly independent solutions of (4.16), we have $N(r, 1/(\eta_1 - \eta_2)) = 0$. Therefore, equation (II) with $\alpha = \mp 1/2$ admits a one-parameter family of solutions $\{\chi_c(z) \mid c \in \mathbf{C} \cup \{\infty\}\}$ with the properties:
(1) $C_1 r^{3/2} \leq T(r, \chi_c) \leq C_2 r^{3/2}$ for some positive constants $C_1 = C_1(c), C_2 = C_2(c)$;
(2) if $c_1 \neq c_2$, then $N(r, 1/(\chi_{c_1} - \chi_{c_2})) = 0$.

5. Proof of Theorem 1.3

We may suppose that $g'(z) \not\equiv 0$ (cf. Remark 1.1). Recall that $\Phi(z) = \phi(z) - g(z)$ ($\not\equiv 0$) satisfies (3.1).

5.1. Case I. First consider the case where

$$G(z) \equiv 0. \tag{5.1}$$

Then we have the following.

Lemma 5.1. *Suppose that $\Phi(z_0) = 0$. If $g(z_0) \neq \infty$, then $z = z_0$ is a simple zero of $\Phi(z)$. If $g(z_0) = \infty$, then it is a quadruple zero of $\Phi(z)$.*

Proof. Suppose that $g(z_0) \neq \infty$. If $\Phi(z_0) = \Phi'(z_0) = 0$, then, by (3.1) with $G(z) \equiv 0$, we have $\Phi(z) \equiv 0$, which is a contradiction. Hence $z = z_0$ is a simple zero of $\Phi(z)$. Suppose that $g(z_0) = \infty$. Since $g(z)$ satisfies (I), by Lemma 2.1, we have $\Phi(z) = \gamma'(z - z_0)^4 + O((z - z_0)^5)$, $\gamma' \neq 0$, which implies the second assertion. $\square$

Put $Z_0 = \{\sigma \mid \Phi(\sigma) = 0,\ g(\sigma) = \infty\}$. By Lemmas 2.1 and 5.1,

$$N_1(r, 1/\Phi) = N_1(r, 1/\Phi)|_{Z_0} \leq 2N(r, g) = S(r, \phi),$$

which implies $\vartheta(g, \phi) = 0$.

5.2. Case II. In what follows we suppose that

$$G(z) \not\equiv 0. \tag{5.2}$$

We write (3.1) in the form

$$W'(z) = -12g'(z)\Phi(z)^2 - 2G'(z)\Phi(z) \tag{5.3}$$

or

$$(5.4)\qquad W(z) = W(c) - \int_c^z \big(12g'(s)\Phi(s)^2 + 2G'(s)\Phi(s)\big)ds, \qquad W(c) \neq \infty,$$

where

$$(5.5)\qquad W(z) = \Phi'(z)^2 - 4\Phi(z)^3 - 12g(z)\Phi(z)^2 - 2G(z)\Phi(z).$$

We put

$$(5.6)\qquad \Omega(z) = \frac{W(z)^2}{g'(z)^2\Phi(z)^3}.$$

5.2.1. The following lemmas are used in the proof.

Lemma 5.2. *Suppose that, for some* $k \geq 2$, $\Phi(z_0) = \Phi'(z_0) = \cdots = \Phi^{(k)}(z_0) = 0$. *Then* $G(z_0) = G'(z_0) = \cdots = G^{(k-2)}(z_0) = 0$.

Proof. Consider the case where $g(z_0) = \infty$. Comparing both sides of

$$(3.1)\qquad \Phi''(z) = 6\Phi(z)^2 + 12g(z)\Phi(z) + G(z), \quad G(z) = 6g(z)^2 + z - g''(z)$$

around $z = z_0$, we see that $z = z_0$ is a double pole of $g(z)$. By supposition, $\Phi(z) = O((z - z_0)^{k+1})$ around $z = z_0$. Differentiating (3.1), we have

$$(5.7,j)\qquad \Phi^{(j+2)}(z) = 6(\Phi(z)^2)^{(j)} + 12(g(z)\Phi(z))^{(j)} + G^{(j)}(z)$$

for $j = 0, 1, ..., k-2$. Substitution of $z = z_0$ into (5.7,j) yields $G(z_0) = G'(z_0) = \cdots = G^{(k-2)}(z_0) = 0$. In the case where $g(z_0) \neq \infty$, we can derive the same conclusion from (5.7,j) by the same argument. □

Lemma 5.3. *If* $\Phi(z_0) = \Phi'(z_0) = 0$ *and* $g(z_0) \neq \infty$, *then* $W(z_0) = W'(z_0) = W''(z_0) = 0$.

Proof. This lemma immediately follows from (5.5), (5.3) and

$$W''(z) = -12g''(z)\Phi(z)^2 - 24g'(z)\Phi(z)\Phi'(z) - 2G''(z)\Phi(z) - 2G'(z)\Phi'(z). \quad □$$

Lemma 5.4. (i) *If* $z = z_\infty$ *is a pole of* $\Phi(z)$ *satisfying* $g(z_\infty) \neq \infty$ *and* $g'(z_\infty) \neq 0$, *then* $\Omega(z_\infty) = 16$.

(ii) $\Omega(z) \not\equiv 16$.

Proof. By assumption and Lemma 2.1, $\Phi(z) = (z - z_\infty)^{-2} + O(1)$ around $z = z_\infty$. Substituting this into (5.6) and using (5.4), we obtain $\Omega(z_\infty) = 16$. To show the second assertion, suppose that $\Omega(z) \equiv 16$, namely $W(z)^2 = 16g'(z)^2\Phi(z)^3$. By (5.5),

$$(5.8)\qquad \big(\Phi'(z)^2 - 4\Phi(z)^3 - 12g(z)\Phi(z)^2 - 2G(z)\Phi(z)\big)^2 = 16g'(z)^2\Phi(z)^3.$$

On the other hand, by (5.4)

$$W(z) = W(c) - \int_c^z \big(12g'(s)\Phi(s)^2 + 2G'(s)\Phi(s)\big)ds = \pm 4g'(z)\Phi(z)^{3/2}. \tag{5.9}$$

Differentiating (5.9), we have

$$\Phi'(z)^2 = 4\bigg(\Phi(z) \pm \frac{g''(z)}{3g'(z)}\Phi(z)^{1/2} + \frac{G'(z)}{6g'(z)}\bigg)^2\Phi(z).$$

Substituting this into (5.8), we see that $\Phi(z)$ satisfies an algebraic equation whose coefficients are rational functions of z and $g^{(j)}(z)$ $(0 \le j \le 3)$. By Valiron and Mohon'ko's theorem ([3; Theorem 2.2.5]), $T(r,\Phi) = O(T(r,g))$, which is a contradiction. □

5.2.2. We put $P_0 = \{\sigma \mid \Phi(\sigma) = \infty\}$, $Z_* = \{\sigma \mid g'(\sigma) = 0\}$, $P_* = \{\sigma \mid g(\sigma) = \infty\}$. By Lemma 2.1, we have

$$\begin{aligned} N(r,\Phi) &\le N(r,\Phi)|_{P_1} + N(r,\Phi)|_{Z_*\cup P_*} \\ &\le N(r,\Phi)|_{P_1} + 2(N(r,1/g') + N(r,g)) \\ &\le N(r,\Phi)|_{P_1} + S(r,\phi), \end{aligned} \tag{5.10}$$

where

$$P_1 = P_0 \setminus (Z_* \cup P_*).$$

Applying Lemmas 2.2 and 2.3 to (3.1) with (5.2), we have $m(r,\Phi) = S(r,\phi)$, $m(r,1/\Phi) = S(r,\phi)$, and hence

$$m(r,\Omega) = O\big(m(r,\Phi'/\Phi) + m(r,\Phi) + m(r,1/\Phi) + T(r,g)\big) = S(r,\phi). \tag{5.11}$$

Note that every pole belonging to P_1 is double. By Lemma 5.4 and (5.11),

$$N(r,\Phi)|_{P_1} \le 2N(r,1/(\Omega-16)) \le 2T(r,\Omega) + O(1) \le 2N(r,\Omega) + S(r,\phi).$$

Substitution of this into (5.10) yields

$$N(r,\Phi) \le 2N(r,\Omega) + S(r,\phi). \tag{5.12}$$

5.2.3. The set of all the zeros of $\Phi(z)$ is expressible in the form

$$Z = Z_1 \cup Z_2 \cup Z_3$$

with

$$Z_1 = \{\sigma \mid \Phi(\sigma) = 0,\ \Phi'(\sigma) \ne 0\}, \quad Z_2 = \{\sigma \mid \Phi(\sigma) = \Phi'(\sigma) = 0,\ \Phi''(\sigma) \ne 0\},$$
$$Z_3 = \{\sigma \mid \Phi(\sigma) = \Phi'(\sigma) = \Phi''(\sigma) = 0\}.$$

To estimate $N(r,\Omega)$, we note that every pole of $\Omega(z)$ belongs to the set $Z \cup P_0 \cup Z_* \cup P_*$, and that

$$(5.13)\qquad N(r,\Omega) \le N(r,\Omega)|_{Z\setminus(Z_*\cup P_*)} + N(r,\Omega)|_{Z\cap(Z_*\cup P_*)} + N(r,\Omega)|_{(Z_*\cup P_*)\setminus(Z\cup P_0)} + N(r,\Omega)|_{P_0}.$$

Furthermore,

$$(5.14)\qquad N(r,\Phi)|_{Z_*\cup P_*} \le 2(N(r,1/g') + N(r,g)) = S(r,\phi),$$

$$(5.15)\qquad N(r,1/\Phi)|_{(Z_1\cup Z_2)\cap(Z_*\cup P_*)} \le 2(N(r,1/g') + N(r,g)) = S(r,\phi),$$

$$(5.16)\qquad N_1(r,1/\Phi)|_{(Z_1\cup Z_2)\cap(Z_*\cup P_*)} \le N(r,1/g') + N(r,g) = S(r,\phi),$$

$$(5.17)\qquad N(r,1/\Phi)|_{Z_3} \le 3N(r,1/G) = S(r,\phi),$$

$$(5.18)\qquad N_1(r,1/\Phi)|_{Z_3} \le 2N(r,1/G) = S(r,\phi),$$

where (5.17), (5.18) are derived from Lemma 5.2. By Lemma 5.3, $\Omega(z)$ is analytic at $z = \sigma \in Z_2 \setminus (Z_* \cup P_*)$. This implies

$$(5.19)\qquad \begin{aligned} &N(r,\Omega)|_{Z\setminus(Z_*\cup P_*)} \\ \le &3N(r,1/\Phi)|_{Z\setminus(Z_*\cup P_*)} - 6N_1(r,1/\Phi)|_{Z_1\cup Z_2\setminus(Z_*\cup P_*)} \\ \le &3T(r,\Phi) - 6N_1(r,1/\Phi)|_{Z_1\cup Z_2\setminus(Z_*\cup P_*)} + O(1). \end{aligned}$$

Using (5.16) and (5.18), we have

$$\begin{aligned} &N_1(r,1/\Phi)|_{Z_1\cup Z_2\setminus(Z_*\cup P_*)} \\ =&N_1(r,1/\Phi) - N_1(r,1/\Phi)|_{Z_3} - N_1(r,1/\Phi)|_{(Z_1\cup Z_2)\cap(Z_*\cup P_*)} \\ =&N_1(r,1/\Phi) + S(r,\phi). \end{aligned}$$

Combining this with (5.19), we have

$$(5.20)\qquad N(r,\Omega)|_{Z\setminus(Z_*\cup P_*)} \le 3T(r,\Phi) - 6N_1(r,1/\Phi) + S(r,\phi).$$

Futhermore, by (5.15) and (5.17),

$$(5.21)\qquad \begin{aligned} &N(r,\Omega)|_{Z\cap(Z_*\cup P_*)} \\ &= O\big(N(r,1/\Phi)|_{(Z_1\cup Z_2)\cap(Z_*\cup P_*)} + N(r,1/\Phi)|_{Z_3} + N(r,g) + N(r,1/g')\big) \\ &=S(r,\phi), \end{aligned}$$

$$(5.22)\qquad N(r,\Omega)|_{Z_*\cup P_*\setminus(Z\cup P_0)} = O\big(N(r,1/g') + N(r,g)\big) = S(r,\phi).$$

Since $\Omega(\sigma) = 16$ at $\sigma \in P_0 \setminus (Z_* \cup P_*)$ (cf. Lemma 5.4), by (5.5)

$$(5.23)\qquad \begin{aligned} N(r,\Omega)|_{P_0} &=N(r,\Omega)|_{P_0\cap(Z_*\cup P_*)} \\ &=O\big(N(r,W)|_{P_0\cap(Z_*\cup P_*)} + N(r,1/g')\big) \\ &=O\big(N(r,\Phi)|_{Z_*\cup P_*} + T(r,g)\big) = S(r,\phi). \end{aligned}$$

Substituting (5.20), (5.21), (5.22) and (5.23) into (5.13), we have

$$(5.24)\qquad N(r,\Omega) \le 3T(r,\Phi) - 6N_1(r,1/\Phi) + S(r,\phi).$$

5.2.4. From (5.24)and (5.12), it follows that

$$N(r,\Phi) \leq 6T(r,\Phi) - 12N_1(r,1/\Phi) + S(r,\phi).$$

Hence,

$$\begin{aligned} 12N_1(r,1/\Phi) &\leq 6T(r,\Phi) - N(r,\Phi) + S(r,\phi) \\ &= 5T(r,\Phi) + m(r,\Phi) + S(r,\phi) = 5T(r,\Phi) + S(r,\phi), \end{aligned}$$

which implies $\vartheta(g,\phi) \leq 5/12$. Thus the proof is completed.

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