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Oscillation Results for n-th Order Linear Differential Equations with Meromorphic Periodic Coefficients

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OSCILLATION RESULTS FOR n-TH ORDER LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC PERIODIC COEFFICIENTS

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Abstract. Consider n-th order linear differential equations with meromorphic periodic coefficients of the form $w^{(n)} + R_{n-1}(e^z)w^{(n-1)} + \cdots + R_1(e^z)w' + R_0(e^z)w = 0$, $n \geq 2$, where $R_\nu(t)$ $(0 \leq \nu \leq n-1)$ are rational functions of t. Under certain assumptions, we prove oscillation theorems concerning meromorphic solutions, which contain necessary conditions for the existence of a meromorphic solution with finite exponent of convergence of the zero-sequence. We also discuss meromorphic or entire solutions whose zero-sequences have an infinite exponent of convergence, and give a new zero-density estimate for such solutions. In the proofs, we utilize asymptotic expressions of solutions of associate equations. Our theorems and their corollaries are extensions or improvements of previously known results concerning linear equations with entire periodic coefficients.

1. Introduction

Consider equations of the form

$$w^{(n)} + A_{n-1}(e^z)w^{(n-1)} + \cdots + A_1(e^z)w' + A_0(e^z)w = 0, \quad n \geq 2 \tag{1.1}$$

$('= d/dz)$, where $A_\nu(t)$ $(0 \leq \nu \leq n-1)$ are rational functions of t admitting poles at most at $t = 0, \infty$ only. The coefficients of (1.1) are entire periodic functions, and every solution is entire. In the case where $n = 2$, the zero distribution of solutions was first examined by [8]. Extension studies concerning (1.1) have been carried on by several authors, and various oscillation theorems have been obtained ([2], [4], [12], [14], [15], [16]).

In this paper we extend such results to meromorphic solutions of linear equations with meromorphic periodic coefficients. Some of our results, even in the case restricted to entire solutions, are also improvements of previously known ones concerning equations with entire periodic coefficients. We treat n-th order linear differential equations of the form

$$w^{(n)} + R_{n-1}(e^z)w^{(n-1)} + \cdots + R_1(e^z)w' + R_0(e^z)w = 0, \qquad n \geq 2. \tag{E}$$

Here $R_\nu(t)$ $(0 \leq \nu \leq n-1)$ are rational functions of t which may admit poles other than $t = 0$ or ∞, and hence the coefficients $R_\nu(e^z)$ are meromorphic on $\mathbf{C}$. Throughout this paper we suppose the following conditions on (E):

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(a) *around* $t = \infty$,

$$R_0(t) = t^q \sum_{k=0}^{\infty} a_k t^{-k}, \qquad q \in \mathbf{N}, \quad a_0 \neq 0, \tag{1.2}$$

$$R_\nu(t) = t^{q_\nu} \sum_{k=0}^{\infty} a_{\nu,k} t^{-k} \quad or \equiv 0, \qquad 1 \le \nu \le n-1, \tag{1.3}$$
$$q_\nu \in \mathbf{Z}, \quad a_{\nu,0} \neq 0,$$

where

$$q_\nu < q(n-1-\nu)/n \quad for\ 1 \le \nu \le n-2, \qquad q_{n-1} \le 0; \tag{1.4}$$

(b) *around* $t = 0$,

$$R_0(t) = t^{-p} \sum_{k=0}^{\infty} b_k t^k, \qquad p \in \mathbf{Z}, \quad b_0 \neq 0, \tag{1.5}$$

$$R_\nu(t) = t^{-p_\nu} \sum_{k=0}^{\infty} b_{\nu,k} t^k \quad or \equiv 0, \qquad 1 \le \nu \le n-1, \tag{1.6}$$
$$p_\nu \in \mathbf{Z}, \quad b_{\nu,0} \neq 0,$$

where

$$p_\nu < p(n-1-\nu)/n \quad for\ 1 \le \nu \le n-2, \quad p_{n-1} \le 0, \quad if\ p \ge 1, \tag{1.7}$$
$$p_\nu \le 0 \qquad for\ 1 \le \nu \le n-1, \qquad if\ p \le 0; \tag{1.8}$$

(c) *equation* (E) *possesses at least one solution which is nontrivial and meromorphic on the whole complex* z*-plane* $\mathbf{C}$.

We put

$$\mathcal{P} = \bigcup_{\nu=0}^{n-1} \mathcal{P}_\nu \subset \mathbf{C} - \{0\}, \tag{1.9}$$

where each $\mathcal{P}_\nu$ is the set of all the distinct poles of $R_\nu(t)$ other than $t = 0$ or ∞. Clearly $\mathcal{P}$ is a finite set. If (E) possesses a meromorphic solution with poles, then $\mathcal{P}$ is not empty. By the change of the variable $t = e^z$, (E) is taken into the equation

$$\vartheta^n w + R_{n-1}(t)\vartheta^{n-1} w + \cdots + R_1(t)\vartheta w + R_0(t) w = 0, \qquad \vartheta = t(d/dt), \tag{aE}$$

which is called *the associate equation of* (E). For an arbitrary solution $\phi(z)$ of (E), there exists a solution $\Phi(t)$ of (aE) such that $\phi(z) = \Phi(e^z)$ at least around a point $z = z_0$ at which $\phi(z)$ is analytic. Then, $\Phi(t)$ is continued meromorphically to $\mathcal{R}$, if and only if $\phi(z)$ is meromorphic on $\mathbf{C}$. Here $\mathcal{R}$ denotes the universal covering of $\mathbf{C} - \{0\}$, namely the Riemann surface of $\log t$. In general, solutions of equations of the form (aE) have a branch point at $t = \xi \in \mathcal{P}$. In the case which we are going to treat, the coefficients of (aE) need to satisfy suitable conditions under which (aE) possesses a nontrivial solution meromorphic on $\mathcal{R}$. For example, if every $\xi \in \mathcal{P}$ is an apparent singular point, namely a regular singular point at which all the characteristic exponents are integers and the series expansion of every solution does not contain a logarithmic term, then every solution of (aE) is meromorphic on $\mathcal{R}$. Such conditions for $n = 2$ are found in [9], [21; Chapter 6], [22]; see also examples in Section 3.1.2.

Our main results and their corollaries are stated in Sections 2 and 3. Theorem 2.1 is an extension of oscillation results for the entire periodic coefficients cases ([8; Theorem 1], [12; Theorem 2]), which gives necessary conditions for the existence of a meromorphic solution of (E) satisfying $\lambda(\phi) < +\infty$. Here $\lambda(f)$ denotes the exponent of convergence of the zero-sequence of a meromorphic function f, namely

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log N(r, 1/f)}{\log r},$$

in which $N(r, g)$ denotes the counting function (see [18], [20], [21]). Theorem 2.3 gives a zero-density estimate for every meromorphic solution of (E) satisfying $\lambda(\phi) = +\infty$. For (1.1) with entire periodic coefficients, it is known that a result corresponding to Theorem 2.1 is also valid under the condition $\log N(r, 1/w) = o(r)$ instead of $\lambda(w) < +\infty$ ([2], [4], [12]). Theorem 2.3 enables us to replace this by a weaker condition of the form $\liminf_{r\to\infty} r^{-1} \log N(r, 1/w) < C_0$ for some $C_0 > 0$ (Remark 2.3). Furthermore, combining this theorem with Corollary 3.3 which follows from Theorem 2.1, we estimate the zero-density of solutions of the Hill equation (Proposition 3.4). Theorem 2.2 or Corollary 3.5 contains the affirmative answer to the conjecture by Chiang and Wang [15] that every nontrivial solution of

$$w^{(n)} + (e^z + K_0)w = 0, \quad n \geq 3, \quad K_0 \in \mathbf{C}$$

satisfies $\lambda(\phi) = +\infty$ (Section 3.2). For (E) with entire periodic coefficients, Theorem 2.4 gives a sufficient condition under which arbitrary linearly independent solutions $\chi_0(z), \chi_1(z), ..., \chi_{n-1}(z)$ satisfy $\max\{\lambda(\chi_0), ..., \lambda(\chi_{n-1})\} = +\infty$; it is an extension of [7; Section 3, Fact (B)] (see also [10; Theorem 1], [11], [13; Theorem 4], [23]). In the proofs of these results, our main idea is to examine the asymptotic behaviour of solutions of (aE) near the singular points $t = \infty$ and $t = 0$. The asymptotic integration has been used in the study of the zero distribution of solutions of linear equations ([5], [6], [17], [19]). In Section 4, we give asymptotic solutions of (aE) and sectorial domains in which the expressions of them are valid. In Section 5, we define a zero-ample solution at $t = \infty$ (or at $t = 0$) of (aE), and show that it admits infinitely many zeros in some sectorial domain. Furthermore we give a characterisation of a solution which is not zero-ample. In Section 6, we prove Theorems 2.1 and 2.3. In the proof of Theorem 2.3, in addition to the zero-density estimate in Section 5, we employ the Wiman-Valiron theory. In Sections 7 and 8, observing the relation between solutions of (aE) near $t = \infty$ and near $t = 0$ carefully, we prove Theorems 2.2 and 2.4.

Throughout this paper, in addition to the standard notation of the Nevanlinna theory such as $T(r, f), N(r, f), \lambda(f)$, we use the notation below:

(1) We write $\varphi(r) \ll \psi(r)$ or $\psi(r) \gg \varphi(r)$ as $r \to \infty$, if $\varphi(r) = O(\psi(r))$ as $r \to \infty$.

(2) For a set A, $\sharp A$ denotes the cardinal number of A.

(3) For $\sigma \in \mathbf{C}$ and for $m \in \mathbf{Z} - \{0\}$, $O[t^\sigma]_{1/m}$ denotes a formal series expressed as $t^\sigma \sum_{k\ge 0} c_k t^{k/m}$ ($c_k \in \mathbf{C}$). When $f(t)$ admits a convergent series expression of the form $f(t) = t^\sigma \sum_{k\ge 0} c_k t^{k/m}$ around $t^{1/m} = 0$, we also write as $f(t) = O[t^\sigma]_{1/m}$.

2. Main theorems

We define α_k ($k \ge 0$) by

$$\left[-t^q \sum_{0\le k\le q/n} a_k t^{-k}\right]^{1/n} = t^{q/n} \sum_{k\ge 0} \alpha_k t^{-k}, \qquad \alpha_0 = (-a_0)^{1/n},$$

near $t = \infty$. When $p \in \mathbf{N}$, we define β_k ($k \ge 0$) by

$$\left[(-1)^{n+1} t^{-p} \sum_{0\le k\le p/n} b_k t^k\right]^{1/n} = t^{-p/n} \sum_{k\ge 0} \beta_k t^k, \qquad \beta_0 = -(-b_0)^{1/n},$$

near $t = 0$. Here a_k, b_k ($k \ge 0$) are the coefficients of (1.2) and (1.5). When $q/n \in \mathbf{N}$, $t^{q/n} \sum_{0\le k\le q/n} \alpha_k t^{-k}$ is the approximate n-the root of [1], [7]. Put

$$V_\infty(t) = t^{q/n} \sum_{0\le k< q/n} \frac{\alpha_k}{q/n - k} t^{-k}, \tag{2.1}$$

$$V_0(t) = t^{-p/n} \sum_{0\le k< p/n} \frac{\beta_k}{p/n - k} t^{k} \qquad (p \in \mathbf{N}). \tag{2.2}$$

If $p \le 0$, then we put $V_0(t) \equiv 0$.

Theorem 2.1. *Suppose that* (E) *possesses a meromorphic solution* $w = \phi(z)$ ($\not\equiv 0$) *satisfying* $\lambda(\phi) < +\infty$. *Then* $\phi(z)$ *is expressible in the form*

$$\phi(z) = \Phi(e^z),$$

$$\Phi(t) = \Big(\prod_{\xi\in\mathcal{P}} (t-\xi)^{-\iota(\xi)}\Big) P(t^{1/n}) t^\kappa \exp\big(\omega_\infty V_\infty(t) + \omega_0 V_0(t)\big), \tag{2.3}$$

and one of the following must hold:

(i) $q/n \notin \mathbf{N}$, $p/n \notin \mathbf{N}$, $p \ge 1$, *and*

$$nI_\phi(\mathcal{P}) - (n-1)(q+p)/2 - (R_{n-1}(\infty) - R_{n-1}(0)) \in \mathbf{N} \cup \{0\};$$

(ii) $q/n \in \mathbf{N}$, $p/n \in \mathbf{N}$, *and*

$$I_\phi(\mathcal{P}) - (n-1)(q+p)/(2n) - (R_{n-1}(\infty) - R_{n-1}(0))/n + \omega_\infty \alpha_{q/n} + \omega_0 \beta_{p/n} \in \mathbf{N} \cup \{0\};$$

(iii) $q/n \notin \mathbf{N}$, $p \leq 0$, *and, for some* $m \in \mathbf{Z}$ *satisfying* $m \leq nI_\phi(\mathcal{P})$,

$$\rho_m = (2m - (n-1)q - 2R_{n-1}(\infty))/(2n)$$

is a root of the equation

$$\rho^n + \sum_{\nu=0}^{n-1} R_\nu(0)\rho^\nu = 0; \tag{2.4}$$

(iv) $q/n \in \mathbf{N}$, $p \leq 0$, *and, for some* $m \in \mathbf{Z}$ *satisfying* $m \leq I_\phi(\mathcal{P})$,

$$\tilde{\rho}_m = m - (n-1)q/(2n) - R_{n-1}(\infty)/n + \omega_\infty \alpha_{q/n}$$

is a root of (2.4).
Here,

(a) $\iota(\xi) \in \{0\} \cup \mathbf{N}$, $I_\phi(\mathcal{P}) = \sum_{\xi \in \mathcal{P}} \iota(\xi)$;
(b) $(\omega_\infty, \omega_0)$ *is some pair of n-th roots of* 1;
(c) κ *is a constant given by*

$$\kappa = \begin{cases} (n-1)p/(2n) - R_{n-1}(0)/n & \textit{in case (i)}, \\ (n-1)p/(2n) - R_{n-1}(0)/n - \omega_0\beta_{p/n} & \textit{in case (ii)}, \\ \rho_m & \textit{in case (iii)}, \\ \tilde{\rho}_m & \textit{in case (iv)}; \end{cases}$$

(d) $P(\tau)$ *is a polynomial in* τ *which satisfies* $P(0) \neq 0$ *and is not divisible by* $\tau^n - \xi$ *for every* $\xi \in \mathcal{P}$ *satisfying* $\iota(\xi) \in \mathbf{N}$, *and in particular, when* $q/n \in \mathbf{N}$, $P(t^{1/n})$ *is a polynomial in t such that* $P(\xi^{1/n}) \neq 0$ *for every* $\xi \in \mathcal{P}$ *satisfying* $\iota(\xi) \in \mathbf{N}$.

Remark 2.1. In the theorem above, for each solution $\phi(z)$ such that $\lambda(\phi) < +\infty$, the integer $I_\phi(\mathcal{P})$ is uniquely determined. If $\mathcal{P} = \emptyset$, then every solution is entire, and hence $I_\phi(\mathcal{P}) = 0$. There exists a case where $\mathcal{P} \neq \emptyset$ and every solution is entire (see Section 3.1.2).

Remark 2.2. When $\mathcal{P} = \emptyset$, by (1.4) and (1.7) (or (1.8)), we have $R_{n-1}(t) \equiv C \in \mathbf{C}$. Then, by the transformation $w = e^{-Cz/n}v$, our problem is reduced to one concerning (E) with $R_{n-1}(t) \equiv 0$.

In the special case where $q/n \notin \mathbf{N}$, $p \leq 0$, we have the following:

Theorem 2.2. *Suppose that* $q/n \notin \mathbf{N}$ *and that* $p \leq 0$. *Put* $n = n_0 d_0$, $q = q_0 d_0$, *where* d_0 *is the greatest common divisor of n and q. If there exists a meromorphic solution* $w = \phi(z)$ $(\not\equiv 0)$ *of* (E) *satisfying* $\lambda(\phi) < +\infty$, *then*

(1) $\phi_j(z) = \phi(z + 2j\pi i)$ $(j = 0, 1, ..., n_0 - 1)$ *are linearly independent solutions of* (E) *satisfying* $\lambda(\phi_j) < +\infty$;

(2) *the equation*

$$\rho^n + \sum_{\nu=0}^{n-1} n_0^{n-\nu} R_\nu(0)\rho^\nu = 0 \tag{2.5}$$

admits n_0 distinct roots expressed as $-(n-1)q_0/2 - R_{n-1}(\infty)/d_0 + m_j$ *with* $m_j \in \mathbf{Z}$ $(j = 0, 1, ..., n_0 - 1)$ *satisfying* $m_0 \leq n_0 I_\phi(\mathcal{P})$, $m_0 < m_1 < \cdots < m_{n_0-1}$.

For the zero-density of solutions, we have the following:

Theorem 2.3. *Let $\phi(z)$ be an arbitrary meromorphic solution of* (E) *satisfying* $\lambda(\phi) = +\infty$. *Then we have*

$$\log N(r, 1/\phi) = O(r) \tag{2.6}$$

and

$$\log N(r, 1/\phi) \geq (m_0(p,q)/n)r + O(\log r) \tag{2.7}$$

as $r \to \infty$, *where*

$$m_0(p,q) = \begin{cases} \min\{p,q\} & \text{if } p \geq 1, \\ q & \text{if } p \leq 0. \end{cases}$$

Remark 2.3. This theorem implies that the condition $\lambda(\phi) < +\infty$ of Theorem 2.1 or 2.2 can be replaced by

$$\liminf_{r\to\infty} r^{-1} \log N(r, 1/\phi) < m_0(p,q)/n.$$

Theorem 2.4. *Suppose that* $\mathcal{P} = \emptyset$, $R_{n-1}(t) \equiv 0$, *and that either of the following holds*:

(i) $p \geq 1$;

(ii) $p \leq 0$, $q/n \in \mathbf{N}$, *and* $\alpha_{q/n}(\omega^j - \omega^{j'}) \notin \mathbf{Z}$ *for every pair* (j, j') *of integers satisfying* $0 \leq j < j' \leq n-1$, *where* $\omega = \exp(2\pi i/n)$.

Then, for arbitrary linearly independent solutions $\chi_0(z), \chi_1(z), ..., \chi_{n-1}(z)$ *of* (E), *we have* $\max\{\lambda(\chi_0), \lambda(\chi_1), ..., \lambda(\chi_{n-1})\} = +\infty$.

3. Corollaries and examples

3.1. Corollaries of Theorem 2.1. From Theorem 2.1, we can derive sufficient conditions under which a meromorphic solution of (E) satisfies $\lambda(\phi) = +\infty$.

Corollary 3.1. *Let* $\phi(z) = \Phi(e^z)$ $(\not\equiv 0)$ *be a meromorphic solution of* (E) *such that* $\Phi(t)\prod_{\xi\in\mathcal{P}}(t-\xi)^{\iota^*(\xi)}$ *is analytic on* $\mathcal{R}$, *where* $\iota^*(\xi) \in \mathbf{N} \cup \{0\}$, $\xi \in \mathcal{P}$. *Suppose that, for* $I^*_\phi(\mathcal{P}) = \sum_{\xi\in\mathcal{P}} \iota^*(\xi)$, *one of the following holds*:

(i) $q/n \notin \mathbf{N}$, $p/n \notin \mathbf{N}$, $p \geq 1$, *and*

$$nI^*_\phi(\mathcal{P}) - (n-1)(q+p)/2 - (R_{n-1}(\infty) - R_{n-1}(0)) \notin \mathbf{N} \cup \{0\};$$

(ii) $q/n \in \mathbf{N}$, $p/n \in \mathbf{N}$, *and, for every* $(j_1, j_2) \in \mathbf{Z}^2$,

$$I^*_\phi(\mathcal{P}) - (n-1)(q+p)/(2n) - (R_{n-1}(\infty) - R_{n-1}(0))/n + \omega^{j_1}\alpha_{q/n} + \omega^{j_2}\beta_{p/n} \notin \mathbf{N} \cup \{0\},$$

where $\omega = \exp(2\pi i/n)$.

Then $\lambda(\phi) = +\infty$.

Corollary 3.2. *Suppose that* $q/n \notin \mathbf{N}$, $p/n \in \mathbf{N}$, *or that* $q/n \in \mathbf{N}$, $p/n \notin \mathbf{N}$, $p \geq 1$. *Then every nontrivial meromorphic solution of* (E) *satisfies* $\lambda(\phi) = +\infty$.

Observing that every entire solution $\phi(z)$ of (E) satisfies $I^*_\phi(\mathcal{P}) = 0$ in Corollary 3.1, we have the following:

Corollary 3.3. *Suppose that* $p \geq 1$, $R_{n-1}(\infty) - R_{n-1}(0) \geq 0$, *and that either of the following holds*:

(i) $q/n \notin \mathbf{N}$ *or* $p/n \notin \mathbf{N}$;

(ii) $q/n \in \mathbf{N}$, $p/n \in \mathbf{N}$, *and, for every* $(j_1, j_2) \in \mathbf{Z}^2$,

$$-(n-1)(q+p)/(2n) - (R_{n-1}(\infty) - R_{n-1}(0))/n + \omega^{j_1}\alpha_{q/n} + \omega^{j_2}\beta_{p/n} \notin \mathbf{N} \cup \{0\}.$$

Then every nontrivial entire solution of (E) *satisfies* $\lambda(\phi) = +\infty$; *under the additional conditions* $\mathcal{P} = \emptyset$, $R_{n-1}(t) \equiv 0$, *every nontrivial solution of* (E) *is entire and satisfies* $\lambda(\phi) = +\infty$.

3.1.1. Hill equation. Consider the Hill equation

$$\text{(HE)} \qquad \frac{d^2w}{dz^2} + \Big(\Theta_0 + 2\Theta_1\cos(2z) + \cdots + 2\Theta_q\cos(2qz)\Big)w = 0, \qquad \Theta_q \neq 0.$$

By the change of the variable $s = 2iz$, (HE) is taken into

$$\frac{d^2w}{ds^2} + R_0(e^s)w = 0,$$
$$R_0(t) = -\frac{1}{4}\Big(\Theta_0 + \Theta_1(t + t^{-1}) + \cdots + \Theta_q(t^q + t^{-q})\Big).$$

For every odd integer q, it is known that every solution $\phi(z)$ of (HE) satisfies $\lambda(\phi) = +\infty$ ([8]). When q is even, we put

$$\left[\frac{t^q}{4}\sum_{0\leq k\leq q/2}\Theta_{q-k}t^{-k}\right]^{1/2} = t^{q/2}\sum_{k\geq 0}\alpha_k t^{-k}, \qquad \alpha_0 = \Theta_q^{1/2}/2.$$

Note that $\alpha_{q/2} = \beta_{q/2}$. For example, if $\Theta_{q-1} = \Theta_{q-2} = \cdots = \Theta_{q-l} = 0$, $l \geq q/4$, then $\alpha_{q/2} = \beta_{q/2} = \Theta_{q/2}\Theta_q^{-1/2}/4$. By Corollary 3.3 and Theorem 2.3, we have the following:

Proposition 3.4. *Suppose that* (HE) *has either of the following properties*:

(i) q *is odd*;

(ii) q *is even, and* $\pm 2\alpha_{q/2} - q/2 \notin \mathbf{N} \cup \{0\}$.

Then every solution $\phi(z)$ *of* (HE) *satisfies*

$$\log N(r, 1/\phi) \geq qr + O(\log r), \qquad \log N(r, 1/\phi) = O(r)$$

as $r \to \infty$.

3.1.2. Meromorphic coefficients cases. Consider equations of the form

$$(\text{E}_2) \qquad w'' + R_1(e^z)w' + R_0(e^z)w = 0$$

with

$$(3.1) \qquad R_1(t) = 0, \qquad R_0(t) = \frac{-2}{(t-1)^2} + \frac{-3}{t-1} - \frac{1}{2}\Big(t^3 + \frac{1}{t^3}\Big),$$

with

$$R_1(t) = -1, \qquad R_0(t) = -\frac{t}{t-1} - t, \tag{3.2}$$

and with

$$R_1(t) = \frac{-t}{t-1}, \qquad R_0(t) = 2 - t^2 - \frac{1}{t}. \tag{3.3}$$

The associate equations of (E_2) with (3.1), (3.2), (3.3) possess linearly independent solutions given by

$$\Phi_1^1(t) = T^{-1}\big(1 + O[T^4]_1\big), \quad \Phi_1^2(t) = T^2\big(1 + O[T]_1\big), \tag{3.4}$$

$$\Phi_2^1(t) = T\big(1 + O[T]_1\big), \qquad \Phi_2^2(t) = 1 + O[T]_1 + \Phi_2^1(t)\log T, \tag{3.5}$$

$$\Phi_3^1(t) = 1 + O[T^3]_1, \qquad \Phi_3^2(t) = T^2\big(1 + O[T]_1\big) \tag{3.6}$$

($T = t - 1$), respectively, around the regular singular point $t = 1$. By (3.4), every solution $\phi(z)$ ($\not\equiv 0$) of (E_2) with (3.1) is meromorphic. By Corollary 3.1 with $I_\phi^*(\mathcal{P}) = 1$, we have $\lambda(\phi) = +\infty$. Equation (E_2) with (3.2) possesses a one-parameter family of entire solutions $\big\{\phi_C(z) = C\Phi(e^z) \bigm| C \in \mathbf{C}\big\}$. (Note that every solution of the associate equation is analytic around $t = 0$.) By Theorem 2.1, (iii), we have $\lambda(\phi_C) = +\infty$ for every $C \in \mathbf{C} - \{0\}$. Although $R_1(e^z)$ with (3.3) is meromorphic, every solution of (E_2) with (3.3) is entire. By Corollary 3.2, it satisfies $\lambda(\phi) = +\infty$.

3.2. Corollaries of Theorem 2.2. From Theorem 2.2, we immediately have the following:

Corollary 3.5. *Suppose that q and n are relatively prime, and that $p \le 0$. If the characteristic equation*

$$\rho^n + \sum_{\nu=0}^{n-1} n^{n-\nu} R_\nu(0)\rho^\nu = 0 \tag{3.7}$$

has a multiple root or has a root ρ_* such that $\rho_* + (n-1)q/2 + R_{n-1}(\infty) \notin \mathbf{Z}$, then every meromorphic solution $\phi(z)$ of (E) *satisfies $\lambda(\phi) = +\infty$.*

Corollary 3.6. *Under the same supposition as in Corollary* 3.5, *if* (E) *possesses a meromorphic solution $\phi(z)$ ($\not\equiv 0$) such that $\lambda(\phi) < +\infty$, then $\phi_j(z) = \phi(z + 2j\pi i)$ ($j = 0, 1, ..., n-1$) are linearly independent solutions of* (E) *satisfying $\lambda(\phi_j) < +\infty$.*

Consider an equation of the form

$$\begin{aligned} &w^{(n)} + K_1 w' + R_0(e^z)w = 0, \qquad n \ge 3, \\ &R_0(t) = L_q t^q + \cdots + L_1 t + L_0, \quad L_q \ne 0, \quad L_k \in \mathbf{C} \quad (0 \le k \le q), \\ &K_1 \in \mathbf{C} - \big\{x \bigm| x < 0\big\}, \end{aligned} \tag{3.8}$$

where n and q are relatively prime. Equation (3.7) is written in the form

$$\rho^n + n^{n-1}K_1\rho + n^n L_0 = 0. \tag{3.9}$$

Since $n \geq 3$, (3.9) has a root $\rho = \rho_*$ such that $\operatorname{Im} \rho_* \neq 0$, or has a multiple root $\rho = 0$. Hence every solution of (3.8) satisfies $\lambda(\phi) = +\infty$. This result is an extension of [15, Theorem 3.2].

In the case where $n = 2$, $R_1(t) \equiv 0$, $R_0(t) = t + K_0$ or in the case where $n = 3$, $R_2(t) \equiv 0$, $R_1(t) \equiv K_1$, $R_0(t) = t + K_0$, a result corresponding to Corollary 3.6 is known ([10], [15; Theorem 3.1]). For example, as is shown in [15], when $K_1 = -7/9$, the equation

$$(3.10) \qquad w^{(3)} + K_1 w' + (e^z - 2/9)w = 0, \qquad K_1 \in \mathbf{C}$$

has the linearly independent solutions

$$\phi_0(z) = (1 + (3/2)e^{z/3}) \exp(-3e^{z/3} - (2/3)z),$$
$$\phi_1(z) = \phi_0(z + 2\pi i), \qquad \phi_2(z) = \phi_0(z + 4\pi i)$$

satisfying $\lambda(\phi_j) < +\infty$ $(j = 0, 1, 2)$. The characteristic equation corresponding to (3.10) is given by

$$(3.11) \qquad \rho^3 + 9K_1\rho - 6 = 0.$$

When $K_1 = -7/9$, (3.11) has the roots $-2, -1, 3 \in \mathbf{Z}$. For every $K_1 \in \mathbf{C} - \{-7/9\}$, (3.11) has a root $\rho = \rho_* \notin \mathbf{Z}$. Hence by Corollary 3.5, every solution of (3.10) with $K_1 \neq -7/9$ satisfies $\lambda(\phi) = +\infty$.

4. Asymptotic solutions of (aE)

4.1. Propositions. Formal solutions of (aE) are given by the following:

Proposition 4.1. *Near $t = \infty$, equation* (aE) *possesses formal solutions of the form*

$$W_j(t) = Y_j(t) \exp(\omega^j V_\infty(t) + \kappa_j \log t), \qquad j = 0, 1, ..., n-1,$$
$$\omega = \exp(2\pi i/n),$$
$$\kappa_j = \begin{cases} -(n-1)q/(2n) - R_{n-1}(\infty)/n & \text{if } q/n \notin \mathbf{N}, \\ -(n-1)q/(2n) - R_{n-1}(\infty)/n + \omega^j \alpha_{q/n} & \text{if } q/n \in \mathbf{N}. \end{cases}$$

Here $V_\infty(t)$ is the function given by (2.1) *and $Y_j(t)$ $(0 \leq j \leq n-1)$ are formal power series of the form*

$$Y_j(t) = \sum_{h \geq 0} c_j(h) t^{-h/n}, \qquad c_j(0) = 1.$$

In particular, when $q/n \in \mathbf{N}$,

$$Y_j(t) = \sum_{h \geq 0} c_j(nh) t^{-h}.$$

Let M_∞ be a sufficiently large positive constant and δ a sufficiently small positive constant. For each $\mu \in \mathbf{Z}$, in the universal covering $\mathcal{R}$ of $\mathbf{C} - \{0\}$, we define the sector S_μ by

$$S_\mu = \{t \in \mathcal{R} \mid \mu\pi - \delta < (q/n) \arg t < (\mu + 1)\pi, \ |t| > M_\infty\}.$$

Then $\bigcup_{\mu \in \mathbf{Z}} S_\mu = \mathcal{R}_\infty = \{t \in \mathcal{R} \mid |t| > M_\infty\}$.

Proposition 4.2. *For each sector* S_μ ($\mu \in \mathbf{Z}$), *equation* (aE) *possesses linearly independent solutions* $\varphi_{\mu,0}(t), ..., \varphi_{\mu,n-1}(t)$ *which admit the asymptotic representations*

$$\varphi_{\mu,j}(t) \sim W_j(t), \qquad j = 0, 1, .., n-1, \tag{4.1}$$

as $t \to \infty$ *through the sector* S_μ. *Furthermore these solutions are uniquely determined by* (4.1).

When $p \in \mathbf{N}$, we also have the following:

Proposition 4.3. *Suppose that* $p \in \mathbf{N}$. *Near* $t = 0$, *equation* (aE) *possesses formal solutions of the form*

$$W_j^{(0)}(t) = Y_j^{(0)}(t) \exp(\omega^j V_0(t) - \kappa_j^0 \log t), \qquad j = 0, 1, ..., n-1,$$

$$\kappa_j^0 = \begin{cases} -(n-1)p/(2n) + R_{n-1}(0)/n & \text{if } p/n \notin \mathbf{N}, \\ -(n-1)p/(2n) + R_{n-1}(0)/n + \omega^j \beta_{p/n} & \text{if } p/n \in \mathbf{N}. \end{cases}$$

Here $V_0(t)$ *is the function given by* (2.2) *and* $Y_j^{(0)}(t)$ $(0 \le j \le n-1)$ *are formal power series of the form*

$$Y_j^{(0)}(t) = \sum_{h \ge 0} c_j^0(h) t^{h/n}, \qquad c_j^0(0) = 1.$$

In particular, when $p/n \in \mathbf{N}$,

$$Y_j^{(0)}(t) = \sum_{h \ge 0} c_j^0(nh) t^h.$$

For each $\mu \in \mathbf{Z}$, denote by S_μ^0 the sector given by

$$S_\mu^0 = \big\{ t \in \mathcal{R} \bigm| \mu\pi < (p/n) \arg t < (\mu+1)\pi + \delta, \ \ |t| < \varepsilon_0 \big\},$$

where δ and ε_0 are sufficiently small positive constants.

Proposition 4.4. *Under the supposition* $p \in \mathbf{N}$, *for each sector* S_μ^0 ($\mu \in \mathbf{Z}$), *equation* (aE) *possesses linearly independent solutions* $\varphi_{\mu,0}^{(0)}(t), ..., \varphi_{\mu,n-1}^{(0)}(t)$ *which admit the asymptotic representations*

$$\varphi_{\mu,j}^{(0)}(t) \sim W_j^{(0)}(t), \qquad j = 0, 1, ..., n-1, \tag{4.2}$$

as $t \to 0$ *through the sector* S_μ^0. *Furthermore these solutions are uniquely determined by* (4.2).

Propositions 4.3 and 4.4 are obtained from Propositions 4.1 and 4.2 by putting $t = 1/\tau$ and using (1.5).

4.2. Proofs of Propositions 4.1 and 4.2. Let w be an arbitrary solution of (aE). Then the column vector function

$$\mathbf{w} = D(t)\begin{pmatrix} w \\ \vartheta w \\ \vdots \\ \vartheta^{n-1} w \end{pmatrix}, \qquad D(t) = \mathrm{diag}[1, t^{-q/n}, ..., t^{-q(n-1)/n}]$$

satisfies a system of the form

$$\text{(S)} \qquad \begin{aligned} &\vartheta \mathbf{w} = A(t)\mathbf{w}, \\ &A(t) = D(t)\Xi(t)D(t)^{-1} - D(t)\vartheta(D(t)^{-1}) \\ &\Xi(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & & \ddots & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ -R_0 & -R_1 & \cdots & \cdots & -R_{n-2} & -R_{n-1} \end{pmatrix}. \end{aligned}$$

Observing (1.4), we can verify that

$$A(t) = t^{q/n}\sum_{k\ge 0} A_k t^{-k} + \sum_{\nu=0}^{n-2} t^{-q\nu/n} \sum_{k\ge -q\nu/n} A_k^{(\nu)} t^{-k} - D_0,$$

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & 0 & \cdots & \cdots & 0 \end{pmatrix}, \qquad D_0 = (q/n)\,\mathrm{diag}[0, 1, ..., n-1].$$

Note that A_0 has the distinct eigenvalues $\omega^j(-a_0)^{1/n} = \omega^j \alpha_0$ $(j = 0, 1, ..., n-1)$. When $q/n \in \mathbf{N}$, system (S) admits a formal fundamental matrix solution of the form

$$U(t)\exp\Big(t^{q/n}\sum_{0\le k<q/n} \Delta_k t^{-k} + \Delta_* \log t\Big), \qquad U(t) = \sum_{k\ge 0} U_k t^{-k}.$$

Here U_k $(k \ge 0)$ are n by n matrices, and Δ_k $(0 \le k < q/n)$, Δ_* are diagonal matrices; in particular $\Delta_0 = \big(\alpha_0/(q/n)\big)\,\mathrm{diag}[1, \omega, ..., \omega^{n-1}]$, and $U_0 \in GL(n, \mathbf{C})$ satisfies $U_0^{-1} A_0 U_0 = \alpha_0\,\mathrm{diag}[1, \omega, ..., \omega^{n-1}]$ (see [26; Sections 10, 11]). Hence equation (aE) has formal solutions of the form

$$\text{(4.3)} \qquad \tilde{W}_j(t) = \tilde{Y}_j(t)\exp\Big(t^{q/n}\sum_{0\le k<q/n} \tilde{\alpha}_{j,k} t^{-k} + \tilde{\kappa}_j \log t\Big),$$

$$\tilde{Y}_j(t) = \sum_{h\ge 0} \tilde{c}_{j,h} t^{-h}, \quad \tilde{c}_{j,0} = 1, \quad \tilde{\alpha}_{j,0} = \frac{\omega^j \alpha_0}{q/n} \quad (0 \le j \le n-1).$$

In the case where $q/n \notin \mathbf{N}$, putting $\tau = t^{1/n}$ in (S), we have

$$\tilde{\vartheta}\mathbf{w} = nA(\tau^n)\mathbf{w}, \qquad \tilde{\vartheta} = \tau(d/d\tau)$$

with

$$nA(\tau^n) = \tau^q \sum_{0 \le k < q/n} nA_k \tau^{-nk} + \sum_{k \ge 0} A'_k \tau^{-k}.$$

From this we obtain formal solutions of (aE) expressed as

$$(4.4) \qquad \tilde{\tilde{W}}_j(t) = \tilde{\tilde{Y}}_j(t) \exp\Big(t^{q/n} \sum_{0 \le k < q/n} \tilde{\tilde{\alpha}}_{j,k} t^{-k} + \tilde{\tilde{\kappa}}_j \log t\Big),$$

$$\tilde{\tilde{Y}}_j(t) = \sum_{h \ge 0} \tilde{\tilde{c}}_{j,h} t^{-h/n}, \quad \tilde{\tilde{c}}_{j,0} = 1, \quad \tilde{\tilde{\alpha}}_{j,0} = \frac{\omega^j \alpha_0}{q/n} \quad (0 \le j \le n-1).$$

It is known that, for each sector S_μ, there exist uniquely determined linearly independent solutions $\varphi_{\mu,0}(t), ..., \varphi_{\mu,n-1}(t)$ of (aE) admitting the asymptotic representations

$$\varphi_{\mu,j}(t) \sim \tilde{W}_j(t) \quad (\text{or} \ \sim \tilde{\tilde{W}}_j(t)), \quad j = 0, 1, ..., n-1$$

as $t \to \infty$ through the sector S_μ ([3; Theorem A], see also [24], [25]).

By the facts above, it is sufficient to show that (4.3) (or (4.4)) coincides with the formal solution $W_j(t)$ of Proposition 4.1. We write $\tilde{W}_j(t)$ in the form

$$\tilde{W}_j(t) = \exp(\Omega_j(t)), \qquad \Omega_j(t) = t^{q/n} \sum_{0 \le k < q/n} \tilde{\alpha}_{j,k} t^{-k} + \tilde{\kappa}_j \log t + O[t^{-1}]_{-1}.$$

By induction on $\nu \in \mathbf{N}$, we can verify that

$$\vartheta^\nu = t^\nu \frac{d^\nu}{dt^\nu} + N^\nu_{\nu-1} t^{\nu-1} \frac{d^{\nu-1}}{dt^{\nu-1}} + \cdots + N^\nu_1 t \frac{d}{dt}, \qquad N^\nu_{\nu-1} = \nu(\nu-1)/2,$$

and that

$$\big(\exp(\Omega_j(t))\big)^{(\nu)} / \exp(\Omega_j(t)) = \Omega'_j(t)^\nu + N^\nu_{\nu-1} \Omega'_j(t)^{\nu-2} \Omega''_j(t) + O[t^{(\nu-2)q/n-\nu}]_{-1}.$$

Using (1.4) and observing $q/n \in \mathbf{N}$, we have

$$\begin{aligned}(4.5) \qquad & \tilde{W}_j(t)^{-1} \Big(\vartheta^n + R_{n-1}(t)\vartheta^{n-1} + \cdots + R_1(t)\vartheta + R_0(t)\Big) \tilde{W}_j(t) \\ =& \tilde{W}_j(t)^{-1} (\vartheta^n + R_{n-1}(t)\vartheta^{n-1}) \tilde{W}_j(t) + R_0(t) + O[t^{(n-1)q/n-1}]_{-1} \\ =& (t\Omega'_j(t))^n + \frac{1}{2} n(n-1) \Big((t\Omega'_j(t))^{n-2} (t^2 \Omega''_j(t)) + (t\Omega'_j(t))^{n-1}\Big) \\ & + R_{n-1}(\infty)(t\Omega'_j(t))^{n-1} + R_0(t) + O[t^{(n-1)q/n-1}]_{-1} \\ =& 0.\end{aligned}$$

Note that $t\Omega_j'(t) = \omega^j \alpha_0 t^{q/n}(1 + O[t^{-1}]_{-1})$, and that

$$(4.6)\qquad (t\Omega_j'(t))^{n-2}(t^2\Omega_j''(t)) + (t\Omega_j'(t))^{n-1} = \omega^{-j}(q/n)\alpha_0^{n-1}t^{(n-1)q/n} + O[t^{(n-1)q/n-1}]_{-1}.$$

By the definition of $V_\infty(t)$,

$$(4.7)\qquad -R_0(t) = (tV_\infty'(t))^n + n\alpha_0^{n-1}\alpha_{q/n}t^{(n-1)q/n} + O[t^{(n-1)q/n-1}]_{-1}.$$

Substitution of these into (4.5) yields

$$(t\Omega_j'(t))^n - (tV_\infty'(t))^n - n\omega^{-j}\Big(\omega^j\alpha_{q/n} - (n-1)q/(2n) - R_{n-1}(\infty)/n\Big)\alpha_0^{n-1}t^{(n-1)q/n} + O[t^{(n-1)q/n-1}]_{-1} = 0,$$

from which we obtain

$$t\Omega_j'(t) = \omega^j tV_\infty'(t) + \Big(\omega^j\alpha_{q/n} - (n-1)q/(2n) - R_{n-1}(\infty)/n\Big) + O[t^{-1}]_{-1}.$$

This implies that $\tilde{W}_j(t)$ coincides with $W_j(t)$. In case $q/n \notin \mathbf{N}$, replacing $O[t^{-1}]_{-1}$, $O[t^{(n-1)q/n-1}]_{-1}$ by $O[t^{-1/n}]_{-1/n}$, $O[t^{(n-1)q/n-1/n}]_{-1/n}$, respectively, in the argument above, and using

$$(t\Omega_j'(t))^{n-2}(t^2\Omega_j''(t)) + (t\Omega_j'(t))^{n-1} = \omega^{-j}(q/n)\alpha_0^{n-1}t^{(n-1)q/n} + O[t^{(n-1)q/n-1/n}]_{-1/n},$$

$$-R_0(t) = (tV_\infty'(t))^n + O[t^{(n-1)q/n-1/n}]_{-1/n}$$

instead of (4.6), (4.7), respectively, we can verify that $\tilde{\tilde{W}}_j(t) = W_j(t)$. Thus the propositions are proved.

5. Zero-ample solutions of (aE)

Recall the sector S_μ and the corresponding linearly independent solutions $\varphi_{\mu,0}(t), ..., \varphi_{\mu,n-1}(t)$ of (aE) given by Proposition 4.2. Let $\chi(t)$ be an arbitrary nontrivial solution of (aE). In each sector S_μ, it is uniquely expressed as

$$(5.1)\qquad \chi(t) = \gamma_{\mu,0}\varphi_{\mu,0}(t) + \cdots + \gamma_{\mu,n-1}\varphi_{\mu,n-1}(t), \qquad \gamma_{\mu,j} \in \mathbf{C}.$$

We call $\chi(t)$ *a zero-ample solution at* $t = \infty$, if, for some μ ($\in \mathbf{Z}$), there exist at least two distinct indices j, j' $(0 \le j < j' \le n-1)$ such that $\gamma_{\mu,j}\gamma_{\mu,j'} \neq 0$.

Proposition 5.1. *Let* $\chi(t)$ *be a zero-ample solution at* $t = \infty$. *Then, for some sector* S_μ,

$$\sharp\{t \in S_\mu \mid \chi(t) = 0,\ \ M_\infty < |t| < r\} \gg r^{q/n}$$

as $r \to \infty$.

Proof. There exists a sector S_μ such that expression (5.1) of $\chi(t)$ contains at least two non-vanishing coefficients. Since the opening of S_μ is larger than $n\pi/q$, there exist a pair (j_1, j_2) $(j_1 \neq j_2)$ of indices and the direction $\arg t = \theta_0 = \theta_0(j_1, j_2)$ in the interior of S_μ with the properties:

(1) $\gamma_{\mu,j_1}\gamma_{\mu,j_2} \neq 0$;

(2) $\mathrm{Re}\big((\omega^{j_2} - \omega^{j_1})\alpha_0 t^{q/n}\big) = 0$ on the ray $\arg t = \theta_0$;

(3) for every j satisfying $j \neq j_1, j_2$ and $\gamma_{\mu,j} \neq 0$, $\mathrm{Re}\big((\omega^{j} - \omega^{j_1})\alpha_0 t^{q/n}\big) < 0$ on the ray $\arg t = \theta_0$.

Then, for a sufficiently small positive constant ε, we have

$$\begin{aligned}\chi(t) =& \gamma_{\mu,j_2}\varphi_{\mu,j_1}(t)\big[\gamma_{\mu,j_1}/\gamma_{\mu,j_2} + \varphi_{\mu,j_2}(t)/\varphi_{\mu,j_1}(t) + o(1)\big]\\ =& \gamma_{\mu,j_2}\varphi_{\mu,j_1}(t)\Big[\gamma_{\mu,j_1}/\gamma_{\mu,j_2} + \exp\big((\omega^{j_2} - \omega^{j_1})(n\alpha_0/q)t^{q/n}(1+o(1))\big) + o(1)\Big]\end{aligned}$$

as $t \to \infty$ through $S(\theta_0, \varepsilon) = \big\{t \in S_\mu \ \big|\ |\arg t - \theta_0| < \varepsilon\big\}$. This yields

$$\sharp\big\{t \in S(\theta_0, \varepsilon) \ \big|\ \chi(t) = 0,\ \ M_\infty < |t| < r\big\} \gg r^{q/n},$$

from which the desired estimate follows. □

Proposition 5.2. *Suppose that* $\psi(t)$ $(\not\equiv 0)$ *is not a zero-ample solution at* $t = \infty$. *Then, in every sector* S_μ, *we have*

$$\psi(t) = \gamma_0\varphi_{\mu,j_*}(t),$$

where the constant γ_0 $(\neq 0)$ *and the index* j_* *are independent of* μ.

Proof. By definition, $\psi(t)$ is expressed as $\psi(t) = \tilde{\gamma}_{\mu,j(\mu)}\varphi_{\mu,j(\mu)}(t)$ $(\tilde{\gamma}_{\mu,j(\mu)} \neq 0)$ in each S_μ. By Proposition 4.2,

$$\psi(t) \sim \tilde{\gamma}_{\mu,j(\mu)}W_{j(\mu)}(t), \quad \text{and} \quad \psi(t) \sim \tilde{\gamma}_{\mu+1,j(\mu+1)}W_{j(\mu+1)}(t)$$

as $t \to \infty$ through $S_\mu \cap S_{\mu+1} \neq \emptyset$. Viewing the asymptotic behaviour, we have $j(\mu) = j(\mu+1)$ and $\tilde{\gamma}_{\mu,j(\mu)} = \tilde{\gamma}_{\mu+1,j(\mu+1)}$. Repeating this procedure, we can verify the assertion. □

Proposition 5.3. *For each* (μ, j) $(\mu \in \mathbf{Z},\ 0 \le j \le n-1)$, *the solution* $\varphi_{\mu,j}(t)$ *of Proposition* 4.2 *is not zero-ample at* $t = \infty$, *if and only if the formal series* $Y_j(t)$ *is convergent around* $t = \infty$.

Proof. If $Y_j(t)$ is convergent, then, clearly, $\varphi_{\mu,j}(t) = W_j(t)$ $(t \in \mathcal{R}_\infty)$ is not zero-ample. Suppose that $\varphi_{\mu,j}(t)$ is not zero-ample, and that $t \in S_\mu$. Note that $e^{2n\pi i}t \in S_{\mu+2q}$. By Propositions 5.2 and 4.2,

$$\varphi_{\mu,j}(e^{2n\pi i}t) = \varphi_{\mu+2q,j}(e^{2n\pi i}t) \sim W_j(e^{2n\pi i}t) = e^{2n\kappa_j\pi i}W_j(t) \tag{5.2}$$

as $e^{2n\pi i}t \to \infty$ through $S_{\mu+2q}$, namely as $t \to \infty$ through S_μ. On the other hand, by the monodromic property, there exist constants $C_0, ..., C_{n-1}$ such that

$$\varphi_{\mu,j}(e^{2n\pi i}t) = C_0\varphi_{\mu,0}(t) + \cdots + C_{n-1}\varphi_{\mu,n-1}(t), \tag{5.3}$$

and hence, by Proposition 4.2,

$$\varphi_{\mu,j}(e^{2n\pi i}t) \sim C_0W_0(t) + \cdots + C_{n-1}W_{n-1}(t) \tag{5.4}$$

as $t \to \infty$ through S_μ. Since the opening of S_μ is larger than $n\pi/q$, from (5.2) and (5.4) it follows that $C_j = e^{2n\kappa_j\pi i}$, $C_l = 0$ $(l \neq j)$. Hence, by (5.3), we have

$$\varphi_{\mu,j}(e^{2n\pi i}t) = e^{2n\kappa_j\pi i}\varphi_{\mu,j}(t),$$

which implies that $\varphi_*(\tau) = \tau^{-n\kappa_j}\varphi_{\mu,j}(\tau^n)$ satisfies $\varphi_*(e^{2\pi i}\tau) = \varphi_*(\tau)$, and that

$$\varphi_*(\tau) \sim \tau^{-n\kappa_j}W_j(\tau^n) = Y_j(\tau^n)\exp\bigl(\omega^jV_\infty(\tau^n)\bigr)$$

around $\tau = \infty$. Therefore $Y_j(t)$ converges around $t = \infty$. Thus the proof is completed. □

Remark 5.1. In the case where $p \in \mathbf{N}$, we call a solution $\chi^{(0)}(t)$ of (aE) *a zero-ample solution at* $t = 0$, if $\chi^{(0)}(1/\tau)$ is zero-ample at $\tau = \infty$. By Propositions 5.1 and 4.4, there exists a sector $S^0_{\mu_0}$ such that

$$\sharp\bigl\{t \in S^0_{\mu_0} \bigm| \chi^{(0)}(t) = 0,\ \ 1/r < |t| < \varepsilon_0\bigr\} \gg r^{p/n}$$

as $r \to \infty$. Furthermore, by Propositions 4.4, 5.2 and 5.3, we have the following:

(1) if $\psi^{(0)}(t)$ $(\not\equiv 0)$ is not zero-ample at $t = 0$, then, for every $\mu \in \mathbf{Z}$,

$$\psi^{(0)}(t) = \gamma^0_0\varphi^{(0)}_{\mu,j_{**}}(t)$$

in S^0_μ, where γ^0_0 $(\neq 0)$ and the index j_{**} are independent of μ;

(2) for each (μ, j), the solution $\varphi^{(0)}_{\mu,j}(t)$ is not zero-ample at $t = 0$, if and only if $Y^{(0)}_j(t)$ is convergent around $t = 0$.

Remark 5.2. Suppose that $p \leq 0$. Then $t = 0$ is at most a regular singular point of (aE). For convenience' sake, in this case, we regard an arbitrary solution $\chi^{(0)}(t)$ $(\not\equiv 0)$ of (aE) as non-zero-ample at $t = 0$.

6. Proofs of Theorems 2.1 and 2.3

6.1. Solution of (aE) which is zero-ample neither at ∞ nor at 0. Suppose that (aE) possesses a meromorphic solution $\Phi(t)$ $(\not\equiv 0)$ which is zero-ample neither at $t = \infty$ nor at $t = 0$. Then, by Propositions 4.1, 5.2 and 5.3, around $t = \infty$,

$$\Phi(t) = Y^{(\infty)}(t)t^{\kappa(\infty)}\exp(\omega_\infty V_\infty(t)), \tag{6.1}$$
$$\kappa(\infty) = \begin{cases} -(n-1)q/(2n) - R_{n-1}(\infty)/n & \text{if } q/n \notin \mathbf{N}, \\ -(n-1)q/(2n) - R_{n-1}(\infty)/n + \omega_\infty\alpha_{q/n} & \text{if } q/n \in \mathbf{N}, \end{cases}$$

in which $Y^{(\infty)}(t) = O[1]_{-1/n}$ (if $q/n \notin \mathbf{N}$), $= O[1]_{-1}$ (if $q/n \in \mathbf{N}$) satisfies $Y^{(\infty)}(\infty) \neq 0$ and converges near $t = \infty$, and ω_∞ is an n-th root of 1. Furthermore, when $p \geq 1$, by Proposition 4.3 and Remark 5.1, around $t = 0$,

$$\Phi(t) = Y^{(0)}(t)t^{\kappa(0)}\exp(\omega_0V_0(t)), \tag{6.2}$$
$$\kappa(0) = \begin{cases} (n-1)p/(2n) - R_{n-1}(0)/n & \text{if } p/n \notin \mathbf{N}, \\ (n-1)p/(2n) - R_{n-1}(0)/n - \omega_0\beta_{p/n} & \text{if } p/n \in \mathbf{N}, \end{cases}$$

in which $Y^{(0)}(t) = O[1]_{1/n}$ (if $p/n \notin \mathbf{N}$), $= O[1]_1$ (if $p/n \in \mathbf{N}$) satisfies $Y^{(0)}(0) \neq 0$ and converges near $t = 0$, and ω_0 is an n-th root of 1. There exists an integer $\iota_0 \in \mathbf{N}$ such that the multiplicity of every pole of $\Phi(t)$ in $\mathcal{R}$ does not exceed ι_0. This fact is verified by substituting a Laurent series expansion of $\Phi(t)$ into (aE) around each pole. We can choose non-negative integers $\iota(\xi)$ ($\xi \in \mathcal{P}$) as small as possible in such a way that $\Phi(t)\prod_{\xi\in\mathcal{P}}(t-\xi)^{\iota(\xi)}$ is analytic on $\mathcal{R}$.

6.1.1. Case $p \geq 1$. Consider the function

$$F(t) = \Phi(t)\Big(\prod_{\xi\in\mathcal{P}}(t-\xi)^{\iota(\xi)}\Big)t^{-\kappa(0)}\exp\big(-\omega_0 V_0(t) - \omega_\infty V_\infty(t)\big), \tag{6.3}$$

which is analytic on $\mathcal{R}$. Then we have

$$\begin{aligned} F(t) &= Y^{(0)}(t)\Big(\prod_{\xi\in\mathcal{P}}(t-\xi)^{\iota(\xi)}\Big)\exp(-\omega_\infty V_\infty(t)) \\ &= Y^{(0)}(0)\prod_{\xi\in\mathcal{P}}(-\xi)^{\iota(\xi)} + O[t^{1/n}]_{1/n} \end{aligned} \tag{6.4}$$

near $t = 0$, and

$$\begin{aligned} F(t) &= Y^{(\infty)}(t)\Big(\prod_{\xi\in\mathcal{P}}(t-\xi)^{\iota(\xi)}\Big)t^{\kappa(\infty)-\kappa(0)}\exp(-\omega_0 V_0(t)) \\ &= Y^{(\infty)}(\infty)t^{\kappa^*}(1 + O[t^{-1/n}]_{-1/n}), \\ \kappa^* &= I(\Phi,\mathcal{P}) + \kappa(\infty) - \kappa(0), \qquad I(\Phi,\mathcal{P}) = \sum_{\xi\in\mathcal{P}}\iota(\xi), \end{aligned} \tag{6.5}$$

near $t = \infty$. By (6.4), $F(\tau^n)$ is entire with respect to τ and satisfies $F(0) \neq 0$. Hence, by (6.5), $F(\tau^n) = P(\tau)$ is a polynomial in τ, and $n\kappa^* = n\big(I(\Phi,\mathcal{P}) + \kappa(\infty) - \kappa(0)\big) \in \mathbf{N}\cup\{0\}$. This implies that $\Phi(t)$ is written in the form (2.3) with $\kappa = \kappa(0)$. By the definition of $\iota(\xi)$, $P(\tau)$ is not divisible by $\tau^n - \xi$ for every $\xi \in \mathcal{P}$ satisfying $\iota(\xi) \in \mathbf{N}$. In particular, if $p/n \in \mathbf{N}$, then we see that $F(t)$ is a polynomial in t, and that $\kappa^* = I(\Phi,\mathcal{P}) + \kappa(\infty) - \kappa(0) \in \mathbf{N}\cup\{0\}$. Suppose that $q/n \notin \mathbf{N}$, $p/n \in \mathbf{N}$. Then, by (6.2), $\Psi(t) = \Phi(t)t^{-\kappa(0)}$ is single-valued on $\mathbf{C} - \{0\}$. On the other hand, around $t = \infty$,

$$\Psi(t) = Y^{(\infty)}(t)t^{-\kappa(0)+\kappa(\infty)}\exp\big((\omega_\infty n\alpha_0/q)t^{q/n}(1 + O(t^{-1}))\big).$$

Since $q/n \notin \mathbf{N}$, $\Psi(t)$ is not single-valued around $t = \infty$, which is a contradiction. In a similar way, we can show that $q/n \in \mathbf{N}$ and $p/n \notin \mathbf{N}$ do not hold simultaneously. Thus we have proved that either of the following cases occurs:

(a) $q/n \notin \mathbf{N}$, $p/n \notin \mathbf{N}$, $p \geq 1$, $-(n-1)(q+p)/2 - (R_{n-1}(\infty) - R_{n-1}(0)) + nI(\Phi,\mathcal{P}) \in \mathbf{N}\cup\{0\}$;

(b) $q/n \in \mathbf{N}$, $p/n \in \mathbf{N}$, $p \geq 1$, $-(n-1)(q+p)/(2n) - (R_{n-1}(\infty) - R_{n-1}(0))/n + \omega_\infty\alpha_{q/n} + \omega_0\beta_{p/n} + I(\Phi,\mathcal{P}) \in \mathbf{N}\cup\{0\}$.

6.1.2. Case $p \le 0$. We put

$$G(t) = \Phi(t)\Big(\prod_{\xi\in\mathcal{P}}(t-\xi)^{\iota(\xi)}\Big)t^{-\kappa(\infty)}\exp(-\omega_\infty V_\infty(t)).$$

Then $G(t)$ is analytic on $\mathcal{R}$. Consider the case where $q/n \notin \mathbf{N}$. Since

$$G(t) = Y^{(\infty)}(t)\prod_{\xi\in\mathcal{P}}(t-\xi)^{\iota(\xi)} = t^{I(\Phi,\mathcal{P})}O[1]_{-1/n} \tag{6.6}$$

converges near $t=\infty$, the function $G(\tau^n)$ is analytic on $\mathbf{C}-\{0\}$, and $\tau=\infty$ is at most a pole of $G(\tau^n)$. Hence, observing that $t=0$ is a regular singular point, we have

$$\begin{aligned} t^{-\kappa(\infty)}\Phi(t) &= G(t)\Big(\prod_{\xi\in\mathcal{P}}(t-\xi)^{-\iota(\xi)}\Big)\exp(\omega_\infty V_\infty(t)) \\ &= t^{m/n}(c_0 + O[t]_{1/n}), \quad c_0 \neq 0 \end{aligned} \tag{6.7}$$

near $t=0$ for some $m\in\mathbf{Z}$. This implies that, for the solution $\Phi(t)$, $\rho_m = m/n + \kappa(\infty) = (2m-(n-1)q)/(2n) - R_{n-1}(\infty)/n$ is a characteristic exponent at $t=0$, and hence ρ_m is a root of (2.4). By (6.7), $t^{-m/n}G(t) = P(t^{1/n}) = O[1]_{1/n}$, $P(0)\neq 0$, and hence $P(\tau)$ is a polynomial in τ. Furthermore, by (6.6), $m/n \le I(\Phi,\mathcal{P})$. When $q/n\in\mathbf{N}$, the function $G(t)$ is analytic on $\mathbf{C}-\{0\}$, and $t=\infty$ is at most a pole of $G(t)$. By an analogous argument, we verify that, for some $m\in\mathbf{Z}$ satisfying $m\le I(\Phi,\mathcal{P})$, $\tilde\rho_m = m+\kappa(\infty) = m-(n-1)q/(2n) - R_{n-1}(\infty)/n + \omega_\infty\alpha_{q/n}$ is a root of (2.4), and that $t^{-m}G(t)$ is a polynomial in t.

Summing up the facts above, we have the following:

Proposition 6.1. *Suppose that there exists a meromorphic solution $\Phi(t)$ ($\not\equiv 0$) of* (aE) *which is zero-ample neither at $t=\infty$ nor at $t=0$. Then $\Phi(t)$ is expressible in the form* (2.3)*, and one of the cases* (i), (ii), (iii), (iv) *of Theorem* 2.1 *with $I(\Phi,\mathcal{P})$ in place of $I_\phi(\mathcal{P})$ occurs.*

6.2. Proof of Theorem 2.1. Concerning the zero-density we have the following:

Lemma 6.2. *Let $\phi(z)=\Phi(e^z)$ be a meromorphic solution of* (E)*. If $\Phi(t)$ is zero-ample at $t=\infty$, then $N(r,1/\phi) \gg r^{-1}e^{(q/n)r}$. If $p\ge 1$, and if $\Phi(t)$ is zero-ample at $t=0$, then $N(r,1/\phi)\gg r^{-1}e^{(p/n)r}$.*

Proof. Suppose that $\Phi(t)$ is zero-ample at $t=\infty$. Let S_μ be a sector such that Proposition 5.1 is valid for $\chi(t)=\Phi(t)$. Note that, by $t=e^z$, the strip $\log M_\infty < \operatorname{Re} z < r'$, $(n/q)(\mu\pi-\delta) < \operatorname{Im} z < (n/q)(\mu+1)\pi$ is conformally mapped onto the region $\{t\in S_\mu \mid M_\infty < |t| < e^{r'}\}$. By Proposition 5.1 the number of zeros of $\phi(z)=\Phi(e^z)$ in $|z|<r$ is estimated as $n(r,1/\phi)\gg e^{(q/n)r+O(1/r)}$, so that

$$N(r,1/\phi) \gg \int_1^r \frac{1}{\sigma}\big(n(\sigma,1/\phi)-n(0,1/\phi)\big)d\sigma \gg r^{-1}e^{(q/n)r}.$$

The second assertion is verified in a similar way. □

Suppose that $\phi(z)=\Phi(e^z)$ ($\not\equiv 0$) is a meromorphic solution of (E) satisfying $\lambda(\phi)<+\infty$. Then, by Lemma 6.2, $\Phi(t)$ is zero-ample neither at $t=\infty$ nor at $t=0$. Combining this fact with Proposition 6.1, we obtain Theorem 2.1.

6.3. Proof of Theorem 2.3. Suppose that a meromorphic solution $\phi(z) = \Phi(e^z)$ of (E) satisfies $\lambda(\phi) = +\infty$. Then, $\Phi(t)$ is zero-ample at $t = 0$ or $t = \infty$; otherwise, by Proposition 6.1, we have $\lambda(\phi) < +\infty$. For example consider the case where $\Phi(t)$ is zero-ample at $t = \infty$. Then, by Lemma 6.2, $N(r, 1/\phi) \gg r^{-1}e^{(q/n)r}$. The other cases are treated in a similar way. Thus we obtain (2.7).

Take a polynomial of the form $\Pi(t) = \prod_{\xi\in\mathcal{P}}(t-\xi)^{\delta(\xi)}$, $\delta(\xi) \in \mathbf{N}\cup\{0\}$ in such a way that $\eta(z) = \phi(z)\Pi(e^z) = \Phi(e^z)\Pi(e^z)$ is entire. It is easy to see that $\eta(z)$ satisfies an equation of the form

$$\text{(E}'\text{)} \qquad \eta^{(n)} + Q_{n-1}(e^z)\eta^{(n-1)} + \cdots + Q_1(e^z)\eta' + Q_0(e^z)\eta = 0.$$

Here $Q_h(t)$ $(h = 0, 1, ..., n-1)$ are rational functions of t whose poles belong to $\mathcal{P}$. All the poles of the coefficients of (E$'$) are written in the form $\zeta_{d,l} = z_d + 2l\pi i$, $e^{z_d} \in \mathcal{P}$ $(d = 1, ..., d_0 \le \sharp\mathcal{P},\ l \in \mathbf{Z})$. Consider the domain

$$\Delta = \mathbf{C} - \bigcup_{d=1}^{d_0}\bigcup_{l\in\mathbf{Z}}\{z \mid |z-\zeta_{d,l}| \le (|l|+1)^{-2}\}.$$

All the radiuses of the circles $\Gamma_r : |z| = r$ satisfying $\Gamma_r \not\subset \Delta$ constitute the set $E_0 \subset \mathbf{R}_+ = \{r | r > 0\}$ of finite linear measure. If $r \in \mathbf{R}_+ - E_0$, then $\Gamma_r \subset \Delta$. Note that $|l| \ll \zeta_{d,l}$ as $|l| \to \infty$. Hence

$$\text{(6.8)} \qquad \log|Q_h(e^z)| \ll r, \qquad h = 0, 1, ..., n-1,$$

as $r \to \infty$, $r \notin E_0$. For the entire function $\eta(z) = \sum_{k\ge 0} c_k z^k$, we put

$$\mu(r,\eta) = \max\{|c_k|r^k \mid k \ge 0\}, \qquad \nu(r,\eta) = \max\{k \mid \mu(r,\eta) = |c_k|r^k\}.$$

Then, by the Wiman-Valiron theory ([20], [21]),

$$\text{(6.9)} \qquad \eta^{(h)}(z) = \left(\frac{\nu(r,\eta)}{z}\right)^h (1+o(1))\eta(z), \quad r = |z|, \quad h = 1, ..., n$$

for z satisfying $|\eta(z)| = M(r,\eta) = \max\{|\eta(\zeta)| \mid |\zeta| = r\}$, $|z| \notin E_1$, where E_1 is a set of finite logarithmic measure. Substituting (6.9) into (E$'$), and using (6.8), we have $\log\nu(r,\eta) \ll r$ as $r \to \infty$, $r \notin F = E_0 \cup E_1$. By [20; Satz 4.4],

$$\log M(r,\eta) \le \log\Big(\mu(r,\eta)(\nu(2r,\eta)+2)\Big) \ll \nu(r,\eta)\log r + \log\nu(2r,\eta)$$

as $r \to \infty$, and hence $\log T(r,\eta) \ll r$, as $r \to \infty$, $r \notin F$. Note that $\int_F dx/x = u_0 < +\infty$, and that, for every $r > 0$, $\int_r^{U_0 r} dx/x = 2u_0$ $(U_0 = \exp(2u_0))$. There exists $r' = r'(r)$ satisfying $r < r' < U_0 r$ and $r' \notin F$. Observing that $\log T(r,\eta)$ is monotone increasing, we have $\log T(r,\eta) \le \log T(r',\eta) \ll r' < U_0 r$ for $r \ge r_0$, where r_0 is a sufficiently large positive constant. Therefore

$$\log N(r, 1/\phi) \ll \log T(r,\eta) + \log T(r, \Pi(e^z)) \ll r$$

for $r \ge r_0$, which implies (2.6). Thus the proof is completed.

7. Proof of Theorem 2.2

By definition, $n_0 = n/d_0$ and $q_0 = q/d_0$ are relatively prime. Since $q/n \notin \mathbf{N}$, we have $d_0 < n$, so that $n_0 > 1$. By the change of the variable $e^{z/n_0} = s$, equation (E) is transformed into

$$\text{(aE}^*\text{)} \qquad \check{\vartheta}^n w + n_0 R_{n-1}(s^{n_0})\check{\vartheta}^{n-1} w + \cdots + n_0^{n-1} R_1(s^{n_0})\check{\vartheta} w + n_0^n R_0(s^{n_0}) w = 0,$$

$\check{\vartheta} = s(d/ds)$, where

$$n_0^n R_0(s^{n_0}) = n_0^n (a_0 s^{qn_0} + a_1 s^{(q-1)n_0} + \cdots + a_k s^{(q-k)n_0} + \cdots).$$

Observing that $qn_0/n = q_0 \in \mathbf{N}$, we write

$$\Bigg[-n_0^n s^{qn_0} \sum_{0 \le n_0 k \le q_0} a_k s^{-n_0 k}\Bigg]^{1/n} = s^{q_0} \sum_{k=0}^{\infty} \tilde{a}_k s^{-n_0 k} = s^{q_0} \sum_{l=0}^{\infty} A_l s^{-l}.$$

Since $q_0/n_0 \notin \mathbf{N}$,

$$\text{(7.1)} \qquad A_{qn_0/n} = A_{q_0} = 0.$$

Suppose that (E) possesses a meromorphic solution $\phi(z) = \Phi(e^z)$ ($\not\equiv 0$) satisfying $\lambda(\phi) < +\infty$, where $\Phi(t)$ is a meromorphic solution of (aE). Note that $\tilde{\phi}(\zeta) = \phi(n_0\zeta) = \Phi_0(e^\zeta)$ ($\zeta = z/n_0$) also satisfies $\lambda(\tilde{\phi}) < +\infty$ as a function of ζ, where $\Phi_0(s)$ is a solution of (aE*). Then, by Theorem 2.1 with qn_0 (instead of q), $\Phi_0(s)$ is written in the form

$$\text{(7.2)} \qquad \Phi_0(s) = \Bigg(\prod_{\xi' \in \mathcal{P}'} (s - \xi')^{-\iota'(\xi')}\Bigg) P_*(s) s^{\kappa_0} \exp(\omega_\infty V_\infty^*(s)),$$

$$V_\infty^*(s) = s^{q_0} \sum_{0 \le l < q_0} \frac{A_l}{q_0 - l} s^{-l},$$

where ω_∞ is some n-th root of 1, $\mathcal{P}' = \{\xi' \in \mathbf{C} - \{0\} \mid \xi'^{n_0} \in \mathcal{P}\}$, and $\iota'(\xi') \in \mathbf{N} \cup \{0\}$. Note that case (iv) of Theorem 2.1 with qn_0 occurs. For some $m_0 \in \mathbf{Z}$ satisfying $m_0 \le I'_\phi(\mathcal{P}') = \sum_{\xi' \in \mathcal{P}'} \iota'(\xi')$,

$$\kappa_0 = m_0 - (n-1)qn_0/(2n) - R_{n-1}(\infty)n_0/n = m_0 - (n-1)q_0/2 - R_{n-1}(\infty)/d_0$$

(cf. (7.1)), and $P_*(s)$ is a polynomial in s satisfying $P_*(0) \neq 0$ and $P_*(\xi') \neq 0$ for every $\xi' \in \mathcal{P}'$ such that $\iota'(\xi') \in \mathbf{N}$. Observing that $\Phi_0(s) = \Phi(s^{n_0})$ (cf. (2.3)), and that, for each $\xi \in \mathcal{P}$,

$$(s^{n_0} - \xi)^{-\iota(\xi)} = \prod_{j=0}^{n_0-1} (s - \tilde{\omega}^j \xi^{1/n_0})^{-\iota(\xi)}, \qquad \tilde{\omega} = \exp(2\pi i/n_0),$$

we have $\iota(\xi) \geq \iota'(\xi')$, if $\xi'^{n_0} = \xi$, $\xi' \in \mathcal{P}'$. Hence $m_0 \leq I'_\phi(\mathcal{P}') \leq n_0 I_\phi(\mathcal{P})$. By Theorem 2.1, (iv) (with (c)), κ_0 is a characteristic exponent of $\Phi_0(s)$ at $s = 0$, and $\Phi_0(s)$ is expressed as

$$\Phi_0(s) = s^{\kappa_0} \sum_{k \geq 0} c_0(k) s^k, \qquad c_0(0) \neq 0 \tag{7.3}$$

near $s = 0$. To derive other characteristic exponents, we note the fact that equation (aE*) remains invariant under the replacement of s by $\tilde{\omega}^j s$ ($j \in \mathbf{Z}$). Hence, $\Phi_0(\tilde{\omega}^j s)$ ($j = 0, 1, ..., n_0 - 1$) are solutions of (aE*). Furthermore these solutions are linearly independent, because the leading terms of $V^*_\infty(\tilde{\omega}^j s)$ and $V^*_\infty(\tilde{\omega}^h s)$ coincide with each other, only when $(h - j)q_0/n_0 \in \mathbf{Z}$, which is equivalent to $(h - j)/n_0 \in \mathbf{Z}$. It is easy to see that $\phi_j(z) = \phi(z + 2j\pi i) = \Phi_0(e^{(z+2j\pi i)/n_0})$ ($j = 0, 1, ..., n_0 - 1$) satisfy $\lambda(\phi_j) < +\infty$. By (7.3), we have, for $j = 0, 1, ..., n_0 - 1$,

$$\psi_j = \Phi_0(\tilde{\omega}^j s) = s^{\kappa_0} \sum_{k \geq 0} c_j(k) s^k, \qquad c_j(0) = \tilde{\omega}^{\kappa_0 j} c_0(0) \neq 0.$$

From these solutions, we derive the linearly independent solutions

$$\begin{aligned} &\psi_0 = s^{\kappa_0}(c_0(0) + O(s)), \\ &\psi_j - \tilde{\omega}^{\kappa_0 j}\psi_0 = s^{\kappa_0 + l(j)}(c^1_j(0) + O(s)), \quad l(j) \in \mathbf{N}, \quad c^1_j(0) \neq 0, \quad 1 \leq j \leq n_0 - 1. \end{aligned}$$

Thus we obtain the sequence

$$\kappa_0 < \kappa_1 = \kappa_0 + l(j_1) \leq \kappa_0 + l(j_2) \leq \cdots \leq \kappa_0 + l(j_{n_0-1}),$$

which contains at least two distinct characteristic exponents κ_0, κ_1. Repeating this procedure within $(n_0 - 1)$ times, we obtain n_0 distinct characteristic exponents $\kappa_j = \kappa_0 + l_j$ $(0 \leq j \leq n_0 - 1)$, $l_j \in \mathbf{Z}$, $l_0 = 0 < l_1 < \cdots < l_{n_0-1}$. Hence they satisfy equation (2.5). Thus the proof is completed.

8. Proof of Theorem 2.4

Suppose that there exist linearly independent entire solutions $\chi_j(z) = \Phi_j(e^z)$ $(j = 0, 1, ..., n - 1)$ satisfying $\lambda(\chi_j) < +\infty$.

8.1. Case $p \geq 1$**.** Under the assumptions $\mathcal{P} = \emptyset$, $R_{n-1}(t) \equiv 0$, the case (i) of Theorem 2.1 does not occur. It is sufficient to treat the case where $q/n \in \mathbf{N}$, $p/n \in \mathbf{N}$. Then, each $\Phi_j(t)$ is zero-ample neither at $t = \infty$ nor at $t = 0$. Using Propositions 4.1, 4.3, 5.2 and 5.3, we have, for $j = 0, 1, ..., n - 1$,

$$\Phi_j(t) = t^{\kappa_j} Y^{(\infty)}_j(t) \exp(\omega^j V_\infty(t)), \quad \omega = \exp(2\pi i/n)$$

(the indices of $\Phi_0(t), ..., \Phi_{n-1}(t)$ are suitably rearranged if necessary), and

$$\Phi_j(t) = t^{\kappa_j(0)} Y^{(0)}_j(t) \exp(\omega^{\varepsilon(j)} V_0(t)).$$

Here $Y_j^{(\infty)}(t) = O[1]_{-1}$ and $Y_j^{(0)}(t) = O[1]_1$ converge near $t = \infty$ and near $t = 0$, respectively, $\varepsilon(j)$ $(j = 0, 1, ..., n-1)$ are integers satisfying $0 \leq \varepsilon(j) \leq n-1$, and

$$\kappa_j = -(n-1)q/(2n) + \omega^j \alpha_{q/n}, \quad \kappa_j(0) = (n-1)p/(2n) - \omega^{\varepsilon(j)} \beta_{p/n}.$$

By the same argument as in the proof of Theorem 2.1 (cf. Section 6.1.1), we have, for $j = 0, 1, ..., n-1$,

$$(n-1)(p+q) = 2n(-m'_j + \omega^j \alpha_{q/n} + \omega^{\varepsilon(j)} \beta_{p/n}), \tag{8.1.j}$$

where $m'_j \in \mathbf{N} \cup \{0\}$ $(I_{\chi_j}(\mathcal{P}) = 0)$. Note that $\varepsilon(j) \neq \varepsilon(j')$ for $j \neq j'$, because $\Phi_0(t), ..., \Phi_{n-1}(t)$ are linearly independent solutions of (aE). Summing (8.1.j) over $0 \leq j \leq n-1$, we have $n(n-1)(p+q) = -2n \sum_{j=0}^{n-1} m'_j \leq 0$, which is a contradiction.

8.2. Case $p \leq 0$. By Theorem 2.1 with $\mathcal{P} = \emptyset$,

$$\Phi_j(t) = t^{\kappa_j} P_j(t) \exp(\omega^j V_\infty(t)), \quad j = 0, 1, ..., n-1,$$

where $P_j(t)$ $(j = 0, ..., n-1)$ are polynomials in t and

$$\kappa_j = m''_j - (n-1)q/(2n) + \omega^j \alpha_{q/n}, \quad m''_j \leq 0.$$

Hence

$$\sum_{j=0}^{n-1} \kappa_j = -(n-1)q/2 + \sum_{j=0}^{n-1} m''_j < 0. \tag{8.2}$$

On the other hand, by assumption, κ_j $(0 \leq j \leq n-1)$ are n distinct characteristic exponents at $t = 0$. Then, from (2.4), we have $\sum_{j=0}^{n-1} \kappa_j = 0$, which contradicts (8.2). Thus the proof is completed.

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