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Anomalous Exponent for Convergent Star-product on Fréchet-Poisson Algebras

by

H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka

Hideki Omori
Department of Mathematics
Science University of Tokyo

Yoshiaki Maeda
Department of Mathematics
Keio University

Naoya Miyazaki
Yokohama City University

Akira Yoshioka
Department of Mathematics
Science University of Tokyo

Department of Mathematics
Faculty of Science and Technology
Keio University

3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522 Japan

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Hideki Omori *
Science University of Tokyo

Yoshiaki Maeda †
Keio University

Naoya Miyazaki ‡
Yokohama City University

Akira Yoshioka §
Science University of Tokyo

Dedicated to the memory of Professor Moshé Flato

1 Introduction

Beyond formal deformation quantization; deformation quantizations of Poisson algebras with the formal deformation parameter (cf.[BF]), one can ask whether the formal parameter converges. For this direction, Rieffel [R] presented a notion of strict deformation quantization; deformation quantization with the convergence product in the C^*-algebras. This suggests us opportunities of finding the various kind of notions of deformation quantizations with the convergence product for suitable categories of algebras.

The purpose of this paper is to give a notion of deformation quantization corresponding to the Rieffel's work [R] in the Fréchet categories and to show that C^*-framework does not work well, if we want to exponentiate quadratic forms.

Let $\mathcal{F}$ be a commutative, associative Fréchet algebra over $\mathbb{C}$, i.e., $\mathcal{F}$ has a metrizable complete topology defined by a family of semi-norms, and a product operation denoted by dott $\cdot$ which is smooth.

$\mathcal{F}$ is called a *Fréchet Poisson algebra* if $\mathcal{F}$ has a bilinear operation $\{\ ,\ \}$: $\mathcal{F}\times\mathcal{F}\to\mathcal{F}$ (called a *Poisson bracket* on $\mathcal{F}$) such that for any $f,g,h\in\mathcal{F}$,

P1 $\{f,g\}=-\{g,f\}$ (skew symmetric)

*Department of Mathematics, Faculty of Science and Technology, Science University of Tokyo, Noda, Chiba, 278, Japan, email: omori@ma.noda.sut.ac.jp

†Department of Mathematics, Faculty of Science and Technology, Keio University, Hiyoshi, Yokohama, 223, Japan, email: maeda@math.keio.ac.jp

‡Department of Mathematics, Faculty of Science, Yokohama City University, Kanazawa-ku, Yokohama, 236, Japan, email: Naoya.Miyazaki@math.yokohama-cu.ac.jp

§Department of Mathematics, Faculty of Engineering, Scinece University of Tokyo, Kagurazaka, Shinjyuku-ku, Tokyo 162-8601, Japan, email: yoshioka@rs.kagu.sut.ac.jp

P2 $\sum_{\text{cyclic sum}}\{f,\{g,h\}\}=0$ (Jacobi identity)

P3 $\{f,gh\}=\{f,g\}h+g\{f,h\}$ (bi-derivation)

Generally, if $\mathcal{F}$ is a Fréchet space and has an associative product $*$ on $\mathcal{F}$ such that the operation $*:\mathcal{F}\times\mathcal{F}\to\mathcal{F}$ is continuous, we call $(\mathcal{F},*)$ a *Fréchet algebra.*

We now give a notion of *deformation quantization of Fréchet-Poisson algebra* as a family of associative product $*_\hbar$ on $\mathcal{F}$ parametrized by $\hbar\in\mathbb{R}$.

Definition 1.1 *Let $\hbar\in\mathbb{R}$. Let $\mathcal{F}$ be a Fréchet Poisson algebra. $(\mathcal{F},*_\hbar)$ is called a deformation quantization of Fréchet-Poisson algebra $\mathcal{F}$ if the following conditions hold:*

(FD1) *For any $\hbar$, there exists an associative product $*_\hbar$ on $\mathcal{F}$ so that $(\mathcal{F},*_\hbar)$ is a Fréchet algebra.*

(FD2) *$f*_\hbar g\to f\cdot g$ as $\hbar\to 0$ for every $f,g\in\mathcal{F}$ independent from $\hbar$.*

(FD3) *$\frac{1}{\hbar i}\{f*_\hbar g-f\cdot g\}\to\frac{1}{2}\{f,g\}$ as $\hbar\to 0$ for every $f,g\in\mathcal{F}$ independent from $\hbar$.*

We now give an example of the above notion, which is based on the Moyal product for functions as follows: Let $\mathbb{C}^2$ be the complex 2-plane, and x,y the coordinates on $\mathbb{C}^2$.

Denote by $\mathcal{P}(\mathbb{C}^2)$ the set of all polynomials of x and y, and denote by $\mathcal{E}(\mathbb{C}^2)$ the set of all entire functions on $\mathbb{C}^2$. $\mathcal{E}(\mathbb{C}^2)$ is a complete topological vector space under the compact open topology.

For entire functions $f=f(x,y)$ and $g=g(x,y)$ on $\mathbb{C}^2$, we set

$$\{f,g\}=\partial_x f\cdot\partial_y g-\partial_y f\cdot\partial_x g. \tag{1.1}$$

This gives a Poisson bracket on $\mathbb{C}^2$, which is canonical in the sense that the coordinates x,y turn to be a Darboux coordinates.

The Moyal product formula is given as follow (cf.[OMY]):

$$f*_\hbar g=\sum_k\left(\frac{\hbar i}{2}\right)^k\frac{1}{k!}f\,(\overleftarrow{\partial}_y\cdot\overrightarrow{\partial}_x-\overleftarrow{\partial}_x\cdot\overrightarrow{\partial}_y)^k\,g. \tag{1.2}$$

In general, $f*_\hbar g$ is not always defined for any $f,g\in\mathcal{E}(\mathbb{C}^2)$, but the following properties hold:

- $*_\hbar$ defines an associative on the space $\mathcal{P}(\mathbb{C}^2)$.
- For every fixed $p\in\mathcal{P}(\mathbb{C}^2)$, the left- and the right-multiplication $p*_\hbar,\ *_\hbar p$ extend to continuous linear mappings

$$p*_\hbar,\ *_\hbar p:\ \mathcal{E}(\mathbb{C}^2)\to\mathcal{E}(\mathbb{C}^2).$$

Using polynomial approximation, we see easily that the associativity

$$f * (g * h) = (f * g) * h$$

holds, if two of f, g, h are polynomials. We call such a system $(\mathcal{E}(\mathbb{C}^2), \mathcal{P}(\mathbb{C}^2); *_\hbar)$ a $\mathcal{P}(\mathbb{C}^2)$-*bimodule.*

We consider the following subspace of $\mathcal{E}(\mathbb{C}^2)$: For every positive $p > 0$, set

$$(1.3) \qquad \mathcal{E}_p(\mathbb{C}^2) = \{f \in \mathcal{E}(\mathbb{C}^2) \mid \|f\|_{p,s} = \sup |f|\, e^{(-s|\xi|^p)} < \infty, \forall s > 0\}$$

where $|\xi| = (|x|^2 + |y|^2)^{1/2}$. The family $\{\| \ \|_{p,s}\}_{s>0}$ induces a topology on $\mathcal{E}_p(\mathbb{C}^2)$ and $(\mathcal{E}_p(\mathbb{C}^2), \cdot)$ is an associative commutative Fréchet algebra, where the dott $\cdot$ is the ordinary multiplication for functions in $\mathcal{E}_p(\mathbb{C}^2)$. It is easily seen that for $0 < p < p'$, we have a continuous embedding

$$(1.4) \qquad \mathcal{E}_p(\mathbb{C}^2) \subset \mathcal{E}_{p'}(\mathbb{C}^2)$$

as a commutative Fréchet algebra (cf.[GS]).

It is obvious that every polynomial is contained in $\mathcal{E}_p(\mathbb{C}^2)$ and $\mathcal{P}(\mathbb{C}^2)$ is dense in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 0$. The Poisson bracket (??) is also well-defined on $\mathcal{E}_p(\mathbb{C}^2)$, and $(\mathcal{P}(\mathbb{C}^2), \{,\}, \cdot)$ is a dense Poisson subalgebra of $(\mathcal{E}_p(\mathbb{C}^2), \{,\}, \cdot)$. We remark that every exponential function $e^{\alpha x+\beta y}$ is contained in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 1$, but not in $\mathcal{E}_1(\mathbb{C}^2)$, and functions such as $e^{ax^2+by^2+2cxy}$ are contained in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 2$, but not in $\mathcal{E}_2(\mathbb{C}^2)$.

Our main result in this paper is as follows :

Theorem 1.2 *The Moyal product formula* (??) *gives the following:*

(i) *For* $0 < p \leq 2$, *the space* $(\mathcal{E}_p(\mathbb{C}^2), *_\hbar)$ *is a deformation quantization of* $(\mathcal{E}_p(\mathbb{C}^2), \cdot, \{,\})$.

(ii) *For* $p > 2$ *and a fixed* $\hbar \in \mathbb{R}$, *the Moyal product formula gives a continuous bi-liner mapping of*

$$(1.5) \qquad \begin{array}{l} \mathcal{E}_p(\mathbb{C}^2) \times \mathcal{E}_{p'}(\mathbb{C}^2) \to \mathcal{E}_p(\mathbb{C}^2), \\ \mathcal{E}_{p'}(\mathbb{C}^2) \times \mathcal{E}_p(\mathbb{C}^2) \to \mathcal{E}_p(\mathbb{C}^2), \end{array}$$

for every p' *such that* $\frac{1}{p} + \frac{1}{p'} \geq 1$.

We remark here about the statement (ii). Since $p > 2$, p' must be $p' \leq 2$, hence the statement (i) gives that $(\mathcal{E}_{p'}(\mathbb{C}^2); *_\hbar$ is a Fréchet algebra. So the statement (ii) means that every $\mathcal{E}_p(\mathbb{C}^2)$, $p > 2$, is a topological 2-sided $\mathcal{E}_{p'}(\mathbb{C}^2)$-module.

We remark also that if $\hbar > 0$, then $e^{\pm\frac{2}{\hbar}xy} \in \mathcal{E}_p(\mathbb{C}^2)$ for every $p > 2$, but $e^{\frac{2}{\hbar}xy} * e^{-\frac{2}{\hbar}xy}$ diverges. This means in a sense the above result is the best possible.

It is obvious that $\mathcal{E}_p(\mathbb{C}^2)$ is closed under the complex conjugation, and in general this looks very nice property. However, this property causes some trouble for $p > 2$. This is because that if $\hbar > 0$ is fixed, the coordinate function y has a right inverse $\frac{1}{y}(1 - e^{\frac{2i}{\hbar}xy})$, and a left inverse $\frac{1}{y}(1 - e^{-\frac{2i}{\hbar}xy})$ as the *complex conjugate* of the right inverse given above. Here the complex conjugate means the anti-homomorphism generated by $\bar{x} = x, \bar{y} = y, \bar{i} = -i$ etc.

If the associativity holds, then these should be the same genuine inverse. Hence we must set

$$\frac{1}{y}\sin\frac{2}{\hbar}xy = 0$$

if $(\mathcal{E}_p(\mathbb{C}^2); *_\hbar)$ were an associative algebra. We discuss such phenomena more closely in the last section.

2 Completion of free tensor algebra

Let $\mathcal{T}$ be the free tensor algebra generated by X_1, X_2. We introduce a topology into $\mathcal{T}$ so that $\mathcal{T}$ becomes a topological algebra. By definition, an element of $\mathcal{T}$ is written in the form

$$T = \sum c_\alpha X_\alpha \quad \text{(finite sum)}$$

where $X_\alpha = X_{\alpha_1} \otimes X_{\alpha_2} \otimes \cdots \otimes X_{\alpha_n}$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$. We call X_α a *word* and we denote by $|\alpha|$ the *length* n of the word X_α. We define a system of semi-norms $\|\ \|_{\tau,s}$ $(\tau \geq 0, s > 0)$ by

$$\|T\|_{\tau,s} = \sum |c_\alpha| |\alpha|^{\tau|\alpha|} s^{\tau|\alpha|}. \tag{2.1}$$

For a fixed $\tau > 0$, we consider a topology introduced by a system of semi-norms $\{\|\cdot\|_{\tau,s}\}_{s>0}$. We set $\mathcal{T}_\tau$ the completion of $\mathcal{T}$ with respect to this topology. Then one easily sees

$$\mathcal{T}_\tau = \{T = \sum_\alpha c_\alpha X_\alpha \ \text{(infinite sum)} \ ; \ \|T\|_{\tau,s} < \infty \ \text{ for any } \ s > 0\} \tag{2.2}$$

Then we have

Lemma 2.1 *For $\tau > 0$, $\mathcal{T}_\tau$ becomes a Fréchet algebra satisfying*

$$\|T \otimes T'\|_{\tau,s} \leq \|T\|_{\tau,2s} \|T'\|_{\tau,2s}$$

Proof. It follows easily from the inequality

$$m^{\tau m} n^{\tau n} \leq (m+n)^{\tau(m+n)} \leq 2^{\tau(m+n)} m^{\tau m} n^{\tau n} \tag{2.3}$$

for non-negative integers m, n. □

Further, we have

Lemma 2.2 *Let $T = \sum c_\alpha X_\alpha$ be an element of $\mathcal{T}$ and let N be the maximum length of words X_α of T. Then $e^T = \sum \frac{1}{k!} T^{\otimes k}$ is an element of $\mathcal{T}_\tau$ for $0 < \tau < \frac{1}{N}$.*

Proof . It suffices to see the convergence of $\sum_{k=0}^{\infty} \frac{1}{k!}(kN)^{k\tau N} s^{k\tau N}$ under the condition $\tau N < 1$. Since

$$\begin{aligned} R_k &= \frac{1}{(k+1)!}((k+1)N)^{(k+1)\tau N} s^{(k+1)\tau N} / \frac{1}{k!}(kN)^{k\tau N} s^{k\tau N} \\ &= s^{\tau N} \frac{((k+1)N)^{\tau N}}{k+1} (1 + \frac{1}{k})^{k\tau N} \end{aligned}$$

one sees easily $\lim_{k \to \infty} R_k = 0$ if $\tau N < 1$. □

3 Subspaces of symmetric elements

Now, we consider a subspace $\mathcal{S}_\tau \subset \mathcal{T}_\tau$ equipped with the symmetric product.

First we introduce a product $a \circ b = \frac{1}{2}(a \otimes b + b \otimes a)$ in $\mathcal{T}$. We set (cf.[OMY])

$$(a\circ)^p \cdot c = a \circ (a \circ (\cdots (a \circ c) \cdots))$$

and

$$(a\circ)^p \cdot (b\circ)^q \cdot c = a \circ (a \circ (\cdots (a \circ (b\circ)^q \cdot c) \cdots)).$$

Using this notation, we define a linear subspace $\mathcal{S}$ as

$$\mathcal{S} = \{\sum c_{\alpha,\beta} (X_1\circ)^\alpha \cdot (X_2\circ)^\beta \cdot 1\}.$$

We also introduce a commutative product $\odot$ into $\mathcal{S}$ by

$$\begin{aligned} &((X_1\circ)^k \cdot (X_2\circ)^l \cdot 1) \odot ((X_1\circ)^m \cdot (X_2\circ)^n \cdot 1) \\ &= (X_1\circ)^{k+m} \cdot (X_2\circ)^{l+n} \cdot 1. \end{aligned}$$

For simplicity, we write

$$X_1^\alpha \odot X_2^\beta = (X_1\circ)^\alpha \cdot (X_2\circ)^\beta \cdot 1. \tag{3.1}$$

Then the set $\mathcal{S}$ with the multiplication $\odot$ becomes a commutative associative algebra.

Let $\mathcal{S}_\tau$ be the closure of $\mathcal{S}$ in $\mathcal{T}_\tau$. It is easy to see the following:

Proposition 3.1 *For every $\tau > 0$, the bilinear product $\odot$ is continuous and $\mathcal{S}_\tau$ is a commutative Fréchet algebra.*

On the space $\mathcal{S}$ we consider a non-commutative product $*_\hbar$ by using the Moyal product formula with the parameter $\hbar$.

To obtain the norm estimate of the $*_\hbar$-product, we remark first the following

Lemma 3.2 *For every $\tau > 0$ and every $s >> 0$, we have an inequality*

$$||\partial^k_{X_1}\partial^l_{X_2}X_1^m\odot X_2^n||_{\tau,s} \leq ||X_1^m\odot X_2^n||_{\tau,s}m^{k(1-\tau)}n^{l(1-\tau)}$$

Proof Obviously $\partial^k_{X_1}\partial^l_{X_2}X_1^m\odot X_2^n = \frac{m!}{(m-k)!}\frac{n!}{(n-k)!}X_1^{m-k}\odot X_2^{n-l}$ By the definition of semi-norms, we see

$$\begin{aligned}
&||\partial^k_{X_1}\partial^l_{X_2}X_1^m\odot X_2^n||_{\tau,s}\\
&=\frac{m!}{(m-k)!}\frac{n!}{(n-l)!}s^{\tau(m+n-k-l)}(m+n-k-l)^{\tau(m+n-k-l)}\\
&<m^{\tau k}m^{k(1-\tau)}n^{\tau l}n^{l(1-\tau)}s^{\tau(m+n-k-l)}(m+n)^{\tau(m+n-k-l)}\\
&<m^{k(1-\tau)}n^{l(1-\tau)}s^{\tau(m+n-k-l)}(m+n)^{\tau(m+n)}
\end{aligned}$$

The desired estimate follows easily. □

Using this estimate we have the following:

Theorem 3.3 *The product $*_\hbar$ extends for $\mathcal{S}_\tau$ for $\frac{1}{2} \leq \tau$ to define $(\mathcal{S}_\tau, *_\hbar)$ a non-commutative Fréchet algebra.*

In this algebra we have also the inequality

$$||f * g||_{\tau,s} \leq ||f\odot g||_{\tau,se^{|\hbar|/\tau}}. \tag{3.2}$$

Proof. For $f = \sum a_{k,l}X_1^k\odot X_2^l$, $g = \sum b_{m,n}X_1^m\odot X_2^n$, a rough estimate gives

$$\begin{aligned}
||f * g||_{\tau,s} \leq \sum_{k,l,m,n} &|a_{k,l}||b_{m,n}|s^{\tau(k+l+m+n)}(k+l)^{\tau(k+l)}(m+n)^{\tau(m+n)}\\
&\times\sum_p \frac{1}{p!}|\hbar|^p(k+l)^{p(1-\tau)}(m+n)^{p(1-\tau)}
\end{aligned}$$

Remarking

$$(k+l)^{p(1-\tau)}(m+n)^{p(1-\tau)} < (k+l+m+n)^{2p(1-\tau)}$$

and

$$\sum_p \frac{1}{p!}|\hbar|^p(k+l+m+n)^{2p(1-\tau)} = e^{|\hbar|(k+l+m+n)^{2(1-\tau)}},$$

we see that $\frac{1}{2} \leq \tau$ gives $2(1-\tau) \leq 1$ and hence

$$||f * g||_{\tau,s} \leq ||f \odot g||_{\tau, se^{|\hbar|/\tau}}$$

□

In the above estimate, we have no need to sum up p for all non-negative integers, but p is restricted in fact in the domain $0 \leq p \leq \min(k+l, m+n)$. Thus, if $0 < \tau < \tau'$ then we can set

$$(k+l)^{\tau(k+l)}(k+l)^{p(1-\tau)} \leq (k+l)^{\tau'(k+l)}(k+l)^{p(1-\tau')}.$$

Now suppose $0 < \tau < \frac{1}{2}$. Then we have

Lemma 3.4 *For τ' such that $0 < \tau < \tau'$ and $\tau + \tau' \geq 1$, it holds*

$$||f * g||_{\tau,s} \leq ||f||_{\tau, se^{|\hbar|/\tau}} ||g||_{\tau', se^{|\hbar|/\tau'}}.$$

Thus, we have

Theorem 3.5 *If $0 < \tau < \frac{1}{2}$, the product $*_\hbar$ gives continuous bilinear products*

$$*_\hbar : \mathcal{S}_\tau \times \mathcal{S}_{\tau'} \to \mathcal{S}_\tau, \quad *_\hbar : \mathcal{S}_{\tau'} \times \mathcal{S}_\tau \to \mathcal{S}_\tau$$

for every τ' with $\tau < \tau'$ and $\tau + \tau' \geq 1$.

4 Two-sided ideal of the relations

Suppose $\hbar \in \mathbb{C}$. Let $\mathcal{R}$ be a two-sided ideal generated by the fundamental relation of the Weyl algebra:

$$X_1 \otimes X_2 - X_2 \otimes X_1 + \hbar i$$

Definition 4.1 *We call the quotient algebra $\mathcal{T}/\mathcal{R} = \mathcal{W}$ the Weyl algebra.*

An element of $\mathcal{S}$ is written as $T = \sum c_{\alpha,\beta}(X_1)^\alpha \odot (X_2)^\beta$. ¿From the identity (??), we have the natural embedding

$$\iota : \mathcal{S} \to \mathcal{T}.$$

We consider now the replacement $\otimes$ sign in every word of an element of $\mathcal{T}$ by $*$ sign in $\mathcal{S}$. For instance;

$$X_2 \otimes X_1 \otimes X_2 \to X_2 * X_1 * X_2 = X_2 \odot (X_1 * X_2) + \frac{\hbar i}{2} X_2 = X_1 \odot X_2^2 - \frac{\hbar i}{2} X_2 + \frac{\hbar i}{2} X_2$$

Obviously X_1, X_2 are contained in $\mathcal{S}$, and this replacement gives a linear mapping φ from $\mathcal{T}$ onto $\mathcal{S}$. Since the product $*$ is defined by using the

Moyal product formula, it is not hard to see φ is an algebra homomorphism of $(\mathcal{T},\otimes)$ onto $(\mathcal{S},*)$. Remarking that $\varphi\circ\iota(X_1^m\odot X_2^n) = X_1^m\odot X_2^n$, we see that $\varphi\circ\iota =$ identity.

Recall now that Weyl algebra can be naturally identified with the space of all polynomials with the Moyal product formula (cf [OMY]). Hence, we see that the kernel of φ must be the ideal $\mathcal{R}$ of the relations of Weyl algebra.

Thus, we have a linear splitting

$$\mathcal{T} = \mathcal{S} \oplus \mathcal{R}.$$

Now, we consider the closures of these spaces $\mathcal{T}_\tau$, $\mathcal{S}_\tau$ and $\mathcal{R}_\tau$. By Proposition**??** and the inequality (**??**), we see the following:

Theorem 4.2 *Suppose* $\tau \geq \frac{1}{2}$. *Then,* φ *and* ι *extend to continuous mappings* $\varphi : \mathcal{T}_\tau \to \mathcal{S}_\tau$ *and* $\iota : \mathcal{S}_\tau \to \mathcal{T}_\tau$. *Hence if* $\tau \geq \frac{1}{2}$, *then* $\mathcal{T}_\tau = \mathcal{S}_\tau \oplus \mathcal{R}_\tau$.

5 Collapsing of algebras

In this section, we show the following:

Theorem 5.1 *If* $0 < \tau < \frac{1}{2}$, *then* $\mathcal{T}_\tau = \mathcal{R}_\tau$.

The above theorem is obtained by showing $\mathcal{R}_\tau \ni 1$ for $0 < \tau < \frac{1}{2}$.

Recall that for any $\tau > 0$, $\mathcal{R}_\tau$ is a closed two-sided ideal of $\mathcal{T}_\tau$ and the quotient algebra $\mathcal{T}_\tau/\mathcal{R}_\tau$ is a topological associative algebra.

Suppose $\mathcal{T}_\tau/\mathcal{R}_\tau$ is not a trivial algebra. $\mathcal{T}_\tau/\mathcal{R}_\tau$ is then an associative Fréchet algebra. Since the Weyl algebra $\mathcal{W}$ is simple and is identified with $\mathcal{T}/\mathcal{R}$, $\mathcal{W}$ is densely embedded in $\mathcal{T}_\tau/\mathcal{R}_\tau$.

Recall that W is also identified with the space of all polynomials with the product given by Moyal product formula. Thus, it is identified with $\mathcal{S}$ with the Moyal product $*_\hbar$.

Hence $(\mathcal{S}, *_\hbar)$ can be naturally embedded in $\mathcal{T}_\tau/\mathcal{R}_\tau$. If $a *_\hbar b = c$ in $\mathcal{T}_\tau/\mathcal{R}_\tau$, then this implies that $a\otimes b = c \bmod \mathcal{R}_\tau$ in the space $\mathcal{T}_\tau$. We also note that the product $*_\hbar$ is continuous in the topology τ.

Remark that for every $\hbar \neq 0$, we see that

$$e_\odot^{\frac{t}{\hbar}X_1\odot X_2} = \sum_k \frac{1}{k!}(\frac{t}{\hbar})^k X_1^k\odot X_2^k$$

is an element of $\mathcal{S}^\tau$ for every $0 < \tau < \frac{1}{2}$ and similarly,

$$X_2^\circ = \frac{1}{X_2}(1 - e_\odot^{-\frac{2}{\hbar}X_1\odot X_2}) \ \in \mathcal{S}^\tau, \quad 0 < \tau < \frac{1}{2},$$

where the right hand side stands for

$$\sum_{k=1}^{\infty}\frac{1}{k!}(\frac{2i}{\hbar})^k X_1^k \odot X_2^{k-1}.$$

Since $X_i \in \mathcal{S}_{\tau'}$, $\frac{1}{2} < \tau'$, $X_2 *_\hbar X_2^\circ$, $X_2^\circ *_\hbar X_2$ are well-defined, a direct calculation using the Moyal product formula gives that $X_2 *_\hbar X_2^\circ = 1$, but $X_2 *_\hbar X_2^\circ = 1 - 2e_\odot^{-\frac{2}{\hbar}X_1\odot X_2}$.

Considering a polynomial approximation of X_2° in $\mathcal{T}_\tau$ and using the continuity of $X_2 *_\hbar$ in $\mathcal{S}^\tau$, we see that $X_2 \circledast X_2^\circ = 1 \bmod \mathcal{R}_\tau$.

Taking the complex conjugate, we have $\overline{X_2^\circ}\circledast X_2 = 1 \bmod \mathcal{R}_\tau$.

Hence, we have

$$\frac{1}{X_2}\sin_\odot(\frac{2}{\hbar}\hbar X_1 \odot X_2),$$

and

$$X_2 \circledast (\frac{1}{X_2}\sin_\odot^{\frac{2}{\hbar}X_1\odot X_2}) - (\sin_\odot^{\frac{2}{\hbar}X_1\odot X_2} + i\cos_\odot^{\frac{2}{\hbar}X_1\odot X_2}) \in \mathcal{R}_\tau.$$

It follows $\sin_\odot^{\frac{2i}{\hbar}X_1\odot X_2}$, $\cos_\odot^{\frac{2i}{\hbar}X_1\odot X_2} \in \mathcal{R}_\tau$. Hence we see $e_\odot^{\pm\frac{2i}{\hbar}X_1\odot X_2} \in \mathcal{R}_\tau$.

Then we have

Lemma 5.2 $1 = e_\odot^{\frac{2i}{\hbar}X_1\odot X_2} \odot e_\odot^{-\frac{2i}{\hbar}X_1\odot X_2} \in \mathcal{R}_\tau$.

Proof. We first remark

$$(X_1^m \odot X_2^n)\circledast(\partial_{X_1}^k \partial_{X_2}^l e_\odot^{\pm\frac{2i}{\hbar}X_1\odot X_2})\circledast(X_1^{m'}\odot X_2^{n'})$$

is an element of $\mathcal{R}_\tau$. This is because that $\partial_{X_i} f$ can be written by using the commutator bracket.

Hence, $(X_1^m \odot X_2^n)\odot e_\odot^{-\frac{2i}{\hbar}X_1\odot X_2} \in \mathcal{R}_\tau$. This implies

$$(\sum_{k=0}^{n}\frac{1}{k!}X_1^k \odot X_2^k)\odot e_\odot^{-\frac{2i}{\hbar}X_1\odot X_2} \in \mathcal{R}_\tau.$$

Taking the limit of the above quantity as $n \to \infty$, we see that $1 \in \mathcal{R}_\tau$, under the assumption $\mathcal{T}_\tau/\mathcal{R}_\tau$ is not a trivial algebra. This is a contradiction, and hence we see that $\mathcal{R}_\tau \ni 1$. □

6 $\mathcal{E}_p(\mathbb{C}^2)$ and $\mathcal{S}_\tau$

Let us consider an entire function f with the growth condition $|f(\xi)| \leq C_s e^{s|\xi|^p}$ where $\xi \in \mathbb{C}^2$ and $s, p > 0$ and a constant $C_s > 0$. For a positive $p > 0$, we set a space of such functions

$$\mathcal{E}_p(\mathbb{C}^2) = \{f \in \mathcal{E}(\mathbb{C}^2)\,;\,|f(\xi)| \leq C_s e^{s|\xi|^p}, \forall s > 0\}.$$

On $\mathcal{E}_p(\mathbb{C}^2)$ we consider a family of semi-norms $\|f\|_{p,s}$ given by

$$(6.1) \qquad \|f\|_{p,s} = \sup_{\xi \in \mathbb{C}^2} |f(\xi)| e^{-s|\xi|^p}.$$

Then, the space $\mathcal{E}_p(\mathbb{C}^2)$ becomes a Fréchet space with the system of semi-norms $\{\| \ \|_{p,s}\}_{s>0}$.

Now, we identify a polynomial on $\mathbb{C}^2$ and an element of the symmetric algebra $\mathcal{S}$ by

$$x^k y^l \longleftrightarrow X_1^k \odot X_2^l.$$

To complete the proof of Theorem??, it sufficies to show:

Proposition 6.1 *For every $p > 0$, $\mathcal{S}_{\frac{1}{p}}$ is linearly isomorphic to $\mathcal{E}_p(\mathbb{C}^2)$.*

Proof Though the well-known estimates for entire functions gives the proposition (cf.[GS]), we will repeat the proof. First, we remark the following relation between semi-norms of polynomials and the symmetric algebras:

$$(6.2) \qquad \|x^k y^l\|_{p,s} \leq \|X_1^k \odot X_2^l\|_{\frac{1}{p},(sp)^{-1}} \leq \|x^k y^l\|_{p,sp\delta} \quad \text{for every} \quad s >> 0,$$

where $\delta = e^{-(1+\frac{p}{2})}$. To prove these inequalities, we remark the maximum point of

$$|x^k y^l| e^{-s|\xi|^p}, \quad |\xi|^2 = |x|^2 + |y|^2$$

satisfies $k + l = sp|\xi|^p$ and the maximum value is

$$k^{\frac{k}{2}} l^{\frac{l}{2}} e^{-\frac{1}{p}(k+l)} (sp)^{-\frac{1}{p}(k+l)} (k+l)^{(\frac{1}{p}-\frac{1}{2})(k+l)}.$$

It follows that

$$\|x^k y^l\|_{p,s} \leq (sp)^{-\frac{1}{p}(k+l)} (k+l)^{\frac{1}{p}(k+l)}$$

and hence $\|x^k y^l\|_{p,s} \leq \|X_1^k \odot X_2^l\|_{\frac{1}{p},(sp)^{-1}}$.

To obtain the next inequality, we remark $2 < e$ and recall the inequality (??). Then, we have

$$(sp)^{-\frac{1}{p}(k+l)} (k+l)^{\frac{1}{p}(k+l)} \leq k^{\frac{k}{2}} l^{\frac{l}{2}} e^{-\frac{1}{p}(k+l)} (sp\delta)^{-\frac{1}{p}(k+l)} (k+l)^{(\frac{1}{p}-\frac{1}{2})(k+l)},$$

where $\delta = e^{-(1+\frac{p}{2})}$. It follows $\|X_1^k \odot X_2^l\|_{\frac{1}{p},(sp)^{-1}} \leq \|x^k y^l\|_{p,sp\delta}$.

The inequality (??) naturally yields the embedding

$$\mathcal{S}_{\frac{1}{p}} \hookrightarrow \mathcal{E}_p(\mathbb{C}^2),$$

since $T = \sum a_{k,l} X_1^k \odot X_2^l$ and the corresponding $t = \sum a_{k,l} x^k y^l$ satisfy the inequality

$$||t||_{p,s} \leq ||T||_{\frac{1}{p},(sp)^{-1}}.$$

Next, we show that $\mathcal{E}_p(\mathbb{C}^2) \subset \mathcal{S}_{\frac{1}{p}}$.

Since every $f \in \mathcal{E}_p(\mathbb{C}^2)$ is an entire function, f is expressed as a converging power series $f = \sum c_{m,n} x^m y^n$. Let $M(r) = \max_{|\xi|=r} |f(x,y)|$. By Cauchy's integration formula, there is a constant $C_\delta > 0$ such that

$$|c_{m,n}| \leq \frac{M(r)}{r^{m+n}} \leq C_\delta \frac{e^{\delta r^p}}{r^{m+n}}$$

hold for every $\delta > 0$, since every $f \in \mathcal{E}_p(\mathbb{C}^2)$ satisfies by definition that $|f(x,y)| \leq Ce^{\delta|\xi|^p}$.

Taking the minimal value of the right hand side with respect to r for a fixed δ, we have

$$|c_{m,n}| \leq C_\delta (\delta ep)^{\frac{1}{p}(m+n)} (\frac{1}{m+n})^{\frac{1}{p}(m+n)} \tag{6.3}$$

Using (??), we easily see that $f \in \mathcal{S}_{\frac{1}{p}}$, if δ is sufficiently small. This gives also that every $f \in \mathcal{E}_p(\mathbb{C}^2)$ is also an element of $\mathcal{S}_{\frac{1}{p}}$. Hence we see that $\mathcal{E}_p(\mathbb{C}^2)$ and $\mathcal{S}_{\frac{1}{p}}$ are linearly isomorphic. □

7 The product formula for e^{isxy}

As mentioned in the introduction, y has a right-inverse y° and a left-inverse $y^\bullet$ at the same time in the space $\mathcal{E}_p(\mathbb{C}^2)$ for $p > 2$. Hence the associativity of the product $*_\hbar$ fails if $\hbar > 0$:

$$y^\bullet *_\hbar (y *_\hbar y^\circ) \neq (y^\bullet *_\hbar y) *_\hbar y^\circ.$$

In this section, we show that if $p > 2$, and $\hbar > 0$, then $\mathcal{E}_p(\mathbb{C}^2)$ is not closed under the product $*_\hbar$. In fact, we see that $y^\bullet * y^\circ$ diverges. See also [OMMY] for details.

Since xy is a polynomial, the product $(ixy) * f(x,y)$ is well-defined for every $\mathcal{E}_p(\mathbb{C}^2)$. Thus, we consider a differential equation

$$\frac{d}{dt} f_t = (ixy) * f_t(x,y), \quad f_0(x,y) = f(x,y).$$

By the Moyal product formula (??), this is written as the ordinary evolution equation

$$\frac{d}{dt} f_t = (ixy) f_t - \frac{\hbar}{2}(x\partial_x f_t - y\partial_y f_t) + i\frac{\hbar^2}{4}\partial_y\partial_x f_t.$$

The existence of the solution does not hold in general, but the uniqueness holds in the real analytic category in t. Hence if the initial function f_0 is a function $F(xy)$ of xy, we may suppose that the solution is also a function of xy.

Under this assumption, we consider the differential equation for $z = xy$:

$$(7.1) \qquad \frac{d}{dt}f_t(z) = izf_t(z) + i\frac{\hbar^2}{4}(f_t'(z) + zf_t''(z)).$$

The solution with initial condition 1 may be written as the $*_\hbar$-exponential function e_*^{itz}, which is given in fact by

$$(7.2) \qquad e_*^{itz} = \frac{1}{\cosh\frac{\hbar t}{2}}e^{i(\frac{2}{\hbar}\tanh\frac{\hbar t}{2})z}, \quad t \in \mathbb{C} - \{(m+\tfrac{1}{2})\frac{i}{\hbar};\ m \in \mathbb{Z}\}.$$

The exponential law $e_*^{isz} * e_*^{itz} = e_*^{i(s+t)z}$ holds by the uniquness whenever $s, t, s+t \in \mathbb{C} - \{(m+\frac{1}{2})\frac{i}{\hbar};\ m \in \mathbb{Z}\}$.

Rewriting the exponential law, we obtain the product formula

$$(7.3) \qquad ae^{isz} * be^{itz} = \frac{4ab}{4+\hbar^2 st}e^{\frac{4i(s+t)}{4+\hbar^2 st}z}.$$

It is easy to check the associativity where products are defined, and the inverse of ae^{isz} is $\frac{a^{-1}}{4}(4-\hbar^2s^2)e^{-isz}$ for $4-\hbar^2s^2 \neq 0$. If $s = \pm\frac{2}{\hbar}$, then $e^{\pm\frac{2i}{\hbar}z}$ are not invertible but have the idempotent property:

$$(7.4) \qquad 2e^{\pm\frac{2i}{\hbar}z} * 2e^{\pm\frac{2i}{\hbar}z} = 2e^{\pm\frac{2i}{\hbar}z}.$$

However, $e^{\frac{2i}{\hbar}z} * e^{-\frac{2i}{\hbar}z}$ is not defined. We refer elements $2e^{\pm\frac{2i}{\hbar}z}$ as *vacuums*.

By the above argument, we see

$$e_*^{itxy} = \frac{1}{\cosh\frac{\hbar t}{2}}e^{(\frac{2i}{\hbar}\tanh\frac{\hbar t}{2})xy}$$

is an element of $\mathcal{E}_p(\mathbb{C}^2)$, $p > 2$, for $t \in \mathbb{R}$. If $\hbar$ is fixed in a positive real, then

$$\int_0^\infty e_*^{itxy}dt, \qquad \int_{-\infty}^0 e_*^{itxy}dt$$

are also elements of $\mathcal{E}^2(\mathbb{C}^2)$, $p > 2$ and $-i\int_0^\infty e_*^{itxy}dt$, and $i\int_{-\infty}^0 e_*^{itxy}dt$ are two different inverse elements of xy, that is

$$(xy) * (-i\int_0^\infty e_*^{itxy}dt) = (xy) * (i\int_{-\infty}^0 e_*^{itxy}dt) = 1$$

Thus the associativity fails again: By denoting

$$(xy)_{0i+}^{-1} = -i\int_0^\infty e_*^{itxy}dt, \quad (xy)_{0i-}^{-1} = (i\int_{-\infty}^0 e_*^{itxy}dt)$$

we have

$$((xy)^{-1}_{0i+} * (xy)) * (xy)^{-1}_{0i-} \neq (xy)^{-1}_{0i+} * ((xy) * (xy)^{-1}_{0i-})$$

This is because

$$(xy)^{-1}_{0i+} - (xy)^{-1}_{0i-} = -i\int_{-\infty}^{\infty} e_*^{itxy}dt \neq 0$$

The right hand side may be written as $-i\delta_*(xy)$ and the above identity corresponds to a relation of hyper functions (cf. [M]).

If $\hbar > 0$, then by the well-known Hansen-Bessel formula(cf.[E], p.1802), we see that

$$\int_{-\infty}^{\infty} e_*^{itxy}dt = \int_{-\infty}^{\infty} \frac{1}{\cosh\frac{\hbar t}{2}} e^{(\frac{2i}{\hbar}\tanh\frac{\hbar t}{2})xy}dt = \frac{\pi}{2}J_0(\frac{2}{\hbar}xy).$$

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