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**Completely Operator Semi-Selfdecomposable
Distributions**

by

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COMPLETELY OPERATOR SEMI-SELFDECOMPOSABLE DISTRIBUTIONS

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Abstract. The class $L_\infty(b, Q)$ of completely operator semi-selfdecomposable distributions on \mathbf{R}^d for b and Q is studied. Here $0 < b < 1$ and Q is a $d \times d$ matrix whose eigenvalues have positive real parts. This is the limiting class of the decreasing sequence of classes $L_m(b, Q)$, $m = -1, 0, 1, \dots$, where $L_{-1}(b, Q)$ is the class of all infinitely divisible distributions on \mathbf{R}^d and $L_m(b, Q)$ is defined inductively as the class of distributions μ with characteristic function $\hat{\mu}(z)$ satisfying $\hat{\mu}(z) = \hat{\mu}(b^{Q'} z) \hat{\rho}(z)$ for some $\rho \in L_{m-1}(b, Q)$. Q' is the transpose of Q . Distributions in $L_\infty(b, Q)$ are characterized in terms of Gaussian covariance matrices and Lévy measures. The connection with the class $OSS(b, Q)$ of operator semi-stable distributions on \mathbf{R}^d for b and Q is established.

1. Introduction and a main result

In our previous paper [MSW98], we have introduced the class of operator semi-selfdecomposable distributions and its decreasing subclasses. To explain those classes, we start with the necessary notation. $\mathcal{P}(\mathbf{R}^d)$ is the class of all probability distributions on \mathbf{R}^d , $I(\mathbf{R}^d)$ is the class of all infinitely divisible distributions on \mathbf{R}^d , $M_+(\mathbf{R}^d)$ is the class of all $d \times d$ matrices all of whose eigenvalues have positive real parts, Q' is the transpose of $Q \in M_+(\mathbf{R}^d)$, I is the identity

matrix, $\hat{\mu}(z), z \in \mathbf{R}^d$, is the characteristic function of $\mu \in \mathcal{P}(\mathbf{R}^d)$, $\mu^{*t}, t \geq 0$, is the t -th convolution power of $\mu \in I(\mathbf{R}^d)$, $\mathcal{L}(X)$ is the law of X , $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in \mathbf{R}^d , and $|\cdot|$ is the norm induced by $\langle \cdot, \cdot \rangle$ in \mathbf{R}^d . For $b > 0$, $b^Q = \sum_{n=0}^{\infty} (n!)^{-1} (\log b)^n Q^n$. Convergence of probability distributions is always weak convergence.

Let $0 < b < 1, Q \in M_+(\mathbf{R}^d)$, m a nonnegative integer, and $L_{-1}(b, Q) = I(\mathbf{R}^d)$. A distribution $\mu \in I(\mathbf{R}^d)$ is said to belong to the class $L_m(b, Q)$ if there exists $\rho \in L_{m-1}(b, Q)$ such that $\hat{\mu}(z) = \hat{\mu}(b^{Q'} z) \hat{\rho}(z)$. Actually the classes $L_m(b, Q)$ have been defined in [MSW98] in a different way and it has been shown there that the definition above is a necessary and sufficient condition for the definition in [MSW98]. Define $L_\infty(b, Q)$ by $L_\infty(b, Q) = \bigcap_{m \geq 0} L_m(b, Q)$. We have called distributions in $L_0(b, Q)$ operator semi-selfdecomposable in [MSW98].

On the other hand, Jurek [J83] and Sato and Yamazato [SY85] introduced and studied the classes $L_m(Q)$ for m a nonnegative integer or ∞ . It has been proved in [MSW98] that $L_m(Q) = \bigcap_{0 < b < 1} L_m(b, Q)$, $0 \leq m \leq \infty$. Distributions in $L_\infty(Q)$ are called completely operator selfdecomposable and characterized in several ways in [SY85]. For this reason, we want to call distributions in $L_\infty(b, Q)$ completely operator semi-selfdecomposable. In [SY85], they studied the relationship between the class $L_\infty(Q)$ and that of operator stable distributions. The purpose of this paper is to give characterization of distributions in $L_\infty(b, Q)$ and to investigate the relationship between the class $L_\infty(b, Q)$ and that of operator semi-stable distributions.

Let $Q \in M_+(\mathbf{R}^d)$. A class $H \subset I(\mathbf{R}^d)$ is said to be Q -completely closed in the strong sense if H is closed under convergence, convolution, and Q -type equivalence, and is closed under going to the t -th convolution power for any $t > 0$. Here H is said to be closed under Q -type equivalence if $\mathcal{L}(X) \in H$, $a > 0$, and $c \in \mathbf{R}^d$ imply $\mathcal{L}(a^{-Q}X + c) \in H$. We can easily see from the definition that $L_m(b, Q), 0 \leq m \leq \infty$, are Q -completely closed in the strong sense, because

$L_{-1}(b, Q) = I(\mathbf{R}^d)$ is so. Furthermore let $OSS(b, Q)$ be the class of $\mu \in I(\mathbf{R}^d)$ such that $\widehat{\mu}(z)^a = \widehat{\mu}(bQ'z)e^{i\langle c, z \rangle}$ for some $0 < a < 1$ and $c \in \mathbf{R}^d$. Distributions in $OSS(b, Q)$ are called operator semi-stable. They are studied by Jajte [J77], Krakowiak [K80], Laha and Rohatgi [LR80], Łuczak [Ł81, Ł91], and others.

One of our main theorems is the following.

THEOREM 1.1. *Let $0 < b < 1$ and $Q \in M_+(\mathbf{R}^d)$. Then the class $L_\infty(b, Q)$ is the smallest Q -completely closed class in the strong sense containing the class $OSS(b, Q)$.*

This theorem is a “semi”-version of Theorem 7.3 in [SY85]. In Section 2, we state some results we need in the subsequent sections. In Section 3, we characterize Gaussian distributions in $L_\infty(b, Q)$, and in Section 4, we treat purely non-Gaussian distributions in $L_\infty(b, Q)$. The proof of Theorem 1.1 is given in Section 5. As our results are new even in case $Q = I$, we make some remarks on this case in Section 6.

2. Preliminary results

The following three propositions have recently been shown in [MSW98]. Since we need them in the subsequent sections, we state them below without proofs.

For a $d \times d$ matrix B we use the following notation: $BE = \{Bx : x \in E\}$ for $E \subset \mathbf{R}^d$ and $(T_B\nu)(E) = \nu(\{x : Bx \in E\})$ for a measure ν on \mathbf{R}^d . We use a mapping Ψ_B from the class of symmetric $d \times d$ matrices into itself defined by $\Psi_B(A) = A - BAB'$. Its iteration is $\Psi_B^\ell = \Psi_B \circ \Psi_B^{\ell-1}$ for $\ell = 2, 3, \dots$ with $\Psi_B^1 = \Psi_B$. Also let $\mathcal{B}_0(\mathbf{R}^d)$ be the class of Borel sets E in \mathbf{R}^d such that $E \subset \{|x| > \varepsilon\}$ for some $\varepsilon > 0$.

In what follows, we fix $0 < b < 1$ and $Q \in M_+(\mathbf{R}^d)$. We use $C_i, i = 1, 2, \dots$,

for positive constants. Following (3.4.3) in [JM93], we introduce a norm $|\cdot|_Q$ in \mathbf{R}^d depending on Q :

$$|x|_Q = \int_0^1 \frac{|u^Q x|}{u} du, \quad x \in \mathbf{R}^d.$$

Since $C_1 u^{C_2} |x| \leq |u^Q x| \leq C_3 u^{C_4} |x|$, $0 < u \leq 1$, $|x|_Q$ is well defined. The norm $|\cdot|_Q$ is comparable with the Euclidean norm $|\cdot|$, and has an advantage that, for any $x \in \mathbf{R}^d \setminus \{0\}$, $t \rightarrow |t^Q x|_Q$ ($t > 0$) is strictly increasing (Proposition 3.4.3 in [JM93]). Thus $\sup_{|x|_Q \leq 1} |b^Q x|_Q < 1$. Define

$$B = b^Q,$$

$$S_B = \{x \in \mathbf{R}^d : |x|_Q \leq 1 \quad \text{and} \quad |B^{-1}x|_Q > 1\},$$

and $\mathcal{B}(S_B)$ as the class of Borel sets in S_B . It might be better to write $S_{Q,b}$ instead of S_B , because it depends on Q and b .

We note that all our results in this paper remain true if S_B is defined by the usual norm in place of the norm $|\cdot|_Q$, provided that $|B| = \sup_{|x| \leq 1} |Bx|$, the operator norm of B , is less than 1. We also note that, since $|B^n x| \rightarrow 0$ for any $x \in \mathbf{R}^d$ as $n \rightarrow \infty$ and since the space is finite-dimensional, there is a positive integer n such that $|B^n| < 1$. Since $B^n = b^{nQ}$ and $L_m(b, Q) \subset L_m(b, nQ)$, $0 \leq m \leq \infty$, study of distributions in $L_m(b, Q)$ in the case $|B| < 1$ covers all cases in some sense. However, in characterization of the class $L_m(b, Q)$ itself, we cannot assume that $|B| < 1$. This is the reason that we use the norm $|\cdot|_Q$.

PROPOSITION 2.1. (Proposition 3.2 of [MSW98].)

(i) If ν is the Lévy measure of $\mu \in I(\mathbf{R}^d)$, then there exist a finite measure ν_0 on S_B and a Borel measurable function $g_n : S_B \rightarrow \mathbf{R}_+$ for each $n \in \mathbf{Z}$ satisfying the following conditions:

- (a) For $E \in \mathcal{B}(S_B)$, $\nu_0(E) = 0$ if and only if $\nu(B^n E) = 0, \forall n \in \mathbf{Z}$,
- (b) $\int_{S_B} \nu_0(dx) \sum_{n \in \mathbf{Z}} (|B^{-n}x|_Q^2 \wedge 1) g_n(x) < \infty$,
- (c) $\sum_{n \in \mathbf{Z}} g_n(x) > 0, \quad \nu_0$ -a.e.,

$$(d) \nu(E) = \int_{S_B} \nu_0(dx) \sum_{n \in \mathbf{Z}} g_n(x) 1_E(B^{-n}x), \quad \forall E \in \mathcal{B}(\mathbf{R}^d).$$

These $\{\nu_0, g_n, n \in \mathbf{Z}\}$ are uniquely determined in the following sense. If $\{\nu_0, g_n, n \in \mathbf{Z}\}$ and $\{\tilde{\nu}_0, \tilde{g}_n, n \in \mathbf{Z}\}$ satisfy the above conditions, then there exists a Borel measurable function $h(x)$ with $0 < h(x) < \infty$ such that

$$\tilde{\nu}_0(dx) = h(x)\nu_0(dx),$$

$$g_n(x) = h(x)\tilde{g}_n(x), \quad \nu_0\text{-a.e.}, \forall n \in \mathbf{Z}.$$

(ii) Conversely, if ν_0 , a finite measure on S_B , and $g_n, n \in \mathbf{Z}$, Borel measurable functions from S_B into $[0, \infty)$, are given, and satisfy (b) and (c), then ν defined by (d) is the Lévy measure of some $\mu \in I(\mathbf{R}^d)$ and (a) is also satisfied.

We call $\{\nu_0, g_n, n \in \mathbf{Z}\}$ determined uniquely from ν in (i) above the S_B -representation of ν . We may write $g(n, x)$ for $g_n(x)$ below. For $\{k(n), n \in \mathbf{Z}\}$, define $\Delta k(n) = k(n+1) - k(n)$ and $(\Delta^\ell k)(n) = \Delta(\Delta^{\ell-1}k)(n), \ell = 2, 3, \dots$. The sequence $\{k(n), n \in \mathbf{Z}\}$ is called completely monotone if

$$(-1)^\ell (\Delta^\ell k)(n) \geq 0, \quad \text{for } \forall \ell \geq 0, \forall n \in \mathbf{Z},$$

with $(\Delta^0 k)(n) = k(n)$.

PROPOSITION 2.2. (Lemma 4.3 and Corollary 4.1 of [MSW98].)

(i) If $\{k(n), n \in \mathbf{Z}\}$ is completely monotone, then

(a) there exists a unique measure ρ on $(0, 1]$ such that

$$(2.1) \quad k(n) = \int_{(0,1]} x^n \rho(dx), \quad n \in \mathbf{Z}$$

and

(b) for each $b \in (0, 1)$, there exists a unique measure Γ on $[0, \infty)$ such that

$$k(n) = \int_{[0, \infty)} b^{n\alpha} \Gamma(d\alpha).$$

(ii) Conversely, $\{k(n), n \in \mathbf{Z}\}$ having the representation (2.1) is completely monotone.

PROPOSITION 2.3. (Theorem 3.1 of [MSW98].) *Let $0 \leq m \leq \infty, \mu \in I(\mathbf{R}^d)$, A its Gaussian covariance matrix, ν its Lévy measure, and let $\{\nu_0, g(n, x), n \in \mathbf{Z}\}$ be the S_B -representation of ν . Then the following three statements are equivalent:*

- (i) $\mu \in L_m(b, Q)$,
 - (ii) $\Psi_B^\ell(A), 1 \leq \ell \leq m + 1$, are nonnegative definite, and $(I - T_B)^\ell \nu \geq 0, 1 \leq \ell \leq m + 1$, on $\mathcal{B}_0(\mathbf{R}^d)$,
 - (iii) $\Psi_B^\ell(A), 1 \leq \ell \leq m + 1$, are nonnegative definite, and $(-1)^\ell (\Delta^\ell g)(n, x) \geq 0, n \in \mathbf{Z}$, ν_0 -a.e.x for $1 \leq \ell \leq m + 1$.
- (In the above, when $m = \infty, 1 \leq \ell \leq m + 1$ should be read as $1 \leq \ell < \infty$.)

3. Gaussian distributions in $L_\infty(b, Q)$

The following are generalizations of some results in [SY85] to “semi”-version.

Let $\{\beta_1, \dots, \beta_p\}$ be the distinct eigenvalues of $B = b^Q$, and let $f(\zeta)$ be the minimal polynomial of B . Decompose it into linear factors

$$f(\zeta) = (\zeta - \beta_1)^{n_1} \cdots (\zeta - \beta_p)^{n_p},$$

where, for $1 \leq j \leq p, n_j$ is a positive integer not exceeding the multiplicity of β_j . Let

$$V_j = \text{Ker } (B - \beta_j I)^{n_j} \quad \text{in } \mathbf{C}^d \quad (1 \leq j \leq p).$$

Then

$$\mathbf{C}^d = V_1 \oplus \cdots \oplus V_p.$$

Let T_j be the projector of \mathbf{C}^d onto V_j in this direct sum decomposition. Similarly

we denote

$$V'_j = \text{Ker} (B' - \bar{\beta}_j I)^{n_j} \quad \text{in } \mathbf{C}^d \quad (1 \leq j \leq p),$$

and obtain

$$\mathbf{C}^d = V'_1 \oplus \cdots \oplus V'_p.$$

Then the projector of \mathbf{C}^d onto V'_j in this direct sum decomposition coincides with the adjoint operator T'_j of T_j . For $j \neq k$, V'_j and V'_k are orthogonal, where we use the Hermitian inner product denoted also by $\langle \cdot, \cdot \rangle$. The following is a characterization of Gaussian distribution in $L_\infty(b, Q)$.

THEOREM 3.1. *Let μ be a Gaussian distribution with covariance matrix A . Then the following three statements are equivalent:*

- (i) $\mu \in L_\infty(b, Q)$.
- (ii) $(B - \beta_j)AT'_j = 0$, for $1 \leq j \leq p$.
- (iii) (a) $A(B' - \bar{\beta}_j)T'_j = 0$, for $1 \leq j \leq p$, and (b) $T_k AT'_j = 0$ for $j \neq k$.

To prove the theorem, we need a lemma.

LEMMA 3.1. *Let μ be a Gaussian distribution with covariance matrix A . Then $\mu \in L_\infty(b, Q)$ if and only if for any $z \in \mathbf{R}^d$, $\langle AB'^n z, B'^n z \rangle, n \in \mathbf{Z}$, is completely monotone.*

Proof. Set $k_z(n) = \langle AB'^n z, B'^n z \rangle, n \in \mathbf{Z}, z \in \mathbf{R}^d$. Then observe that, for each $\ell \geq 1$, $(-1)^\ell (\Delta^\ell k_z)(n) \geq 0, \forall n \in \mathbf{Z}, \forall z \in \mathbf{R}^d$, if and only if $\Psi_B^\ell(A)$ is nonnegative definite. The nonnegative definiteness of $\Psi_B^\ell(A)$ for all $\ell \geq 0$ is a necessary and sufficient condition for that the Gaussian μ is in $L_\infty(b, Q)$ by Proposition 2.3. This concludes the lemma. \square

Proof of Theorem 3.1. We first show (i) \Rightarrow (iii). To show (iii) (a), it is enough to prove that, for any integer $k \geq 1$ and $z_0 \in \mathbf{C}^d$,

$$(3.1) \quad (B' - \bar{\beta}_j)^k z_0 = 0 \quad \text{implies} \quad A(B' - \bar{\beta}_j)z_0 = 0.$$

We prove this by induction in k . If $k = 1$, the assertion is trivial. Suppose that (3.1) is true for $k - 1$ in place of k , and assume $(B' - \bar{\beta}_j)^k z_0 = 0$. Since

$B'^n(B' - \bar{\beta}_j) = (B' - \bar{\beta}_j)B'^n$ for any $n \in \mathbf{Z}$, we have $(B' - \bar{\beta}_j)^{k-2+\ell}B'^n z_0 = 0$ for any $\ell \geq 2$ and $n \in \mathbf{Z}$. Hence by the induction hypothesis,

$$(3.2) \quad A(B' - \bar{\beta}_j)^\ell B'^n z_0 = 0 \quad \text{for } \ell \geq 2 \text{ and } n \in \mathbf{Z}.$$

Let

$$L(n) = \langle AB'^n z_0, B'^n z_0 \rangle \quad \text{for } n \in \mathbf{Z}.$$

We claim that

$$(3.3) \quad L(n) = |\beta_j|^{2n} \{ \langle Az_0, z_0 \rangle + 2n\Re \langle Az_0, z_1 \rangle + n^2 \langle Az_1, z_1 \rangle \}, \quad n \in \mathbf{Z},$$

where $z_1 = \bar{\beta}_j^{-1}(B' - \bar{\beta}_j)z_0$. If $n = 0$, this is trivial. We write $z_\ell = \bar{\beta}_j^{-\ell}(B' - \bar{\beta}_j)^\ell z_0$. If $n \geq 1$, then

$$B'^n z_0 = (\bar{\beta}_j + (B' - \bar{\beta}_j))^n z_0 = \bar{\beta}_j^n \sum_{\ell=1}^n \binom{n}{\ell} z_\ell$$

and, by (3.2),

$$L(n) = \langle \bar{\beta}_j^n A(z_0 + nz_1), B'^n z_0 \rangle = \langle \bar{\beta}_j^n (z_0 + nz_1), \bar{\beta}_j^n A(z_0 + nz_1) \rangle,$$

which is (3.3). Suppose $n \leq -1$ and write $n = -h$. Let $w = B'^n z_0 - \bar{\beta}_j^n (z_0 + nz_1)$.

Then

$$\begin{aligned} w &= B'^n \{ z_0 - \bar{\beta}_j^n B'^h (z_0 + nz_1) \} \\ &= B'^n \left\{ z_0 - \sum_{\ell=0}^h \binom{h}{\ell} (z_\ell + nz_{\ell+1}) \right\} \\ &= B'^n \left\{ n^2 z_2 - \sum_{\ell=2}^h \binom{h}{\ell} (z_\ell + nz_{\ell+1}) \right\}, \end{aligned}$$

where the sum over $2 \leq \ell \leq h$ is considered as zero if $h = 1$. Hence by (3.2),

$Aw = 0$. Thus

$$\begin{aligned} L(n) &= \langle AB'^n z_0 - Aw, B'^n z_0 \rangle \\ &= \langle \bar{\beta}_j^n A(z_0 + nz_1), B'^n z_0 \rangle \\ &= \langle \bar{\beta}_j^n (z_0 + nz_1), AB'^n z_0 - Aw \rangle \\ &= \langle \bar{\beta}_j^n (z_0 + nz_1), \bar{\beta}_j^n A(z_0 + nz_1) \rangle. \end{aligned}$$

Hence (3.3) is true for all $n \in \mathbf{Z}$. Since $L(n)$ is completely monotone in n by Lemma 3.1 under the assumption that $\mu \in L_\infty(b, Q)$, we have

$$(3.4) \quad L(n) = \int_{(0,1]} \beta^n \rho(d\beta)$$

for some measure ρ by Proposition 2.2. If we let

$$E_1 = \{\beta \in (0, 1] : \beta > |\beta_j|^2\}$$

and

$$E_2 = \{\beta \in (0, 1] : \beta < |\beta_j|^2\},$$

then from (3.3) and (3.4)

$$\begin{aligned} \rho(\{|\beta_j|^2\}) + \int_{E_1} \left(\frac{\beta}{|\beta_j|^2} \right)^n \rho(d\beta) + \int_{E_2} \left(\frac{\beta}{|\beta_j|^2} \right)^n \rho(d\beta) \\ = \langle Az_0, z_0 \rangle + 2n\Re\langle Az_0, z_1 \rangle + n^2\langle Az_1, z_1 \rangle = I, \end{aligned}$$

say. If $\rho(E_1) > 0$, then there exists $\varepsilon > 0$ such that $\rho((1 + \varepsilon)|\beta_j|^2 \leq \beta \leq 1) > 0$, and hence

$$I \geq (1 + \varepsilon)^n \rho((1 + \varepsilon)|\beta_j|^2 \leq \beta \leq 1), \quad n > 0.$$

Letting $n \rightarrow \infty$, we get a contradiction. Thus $\rho(E_1) = 0$. Similarly, if $\rho(E_2) > 0$, then there exists $\varepsilon > 0$ such that

$$I \geq (1 - \varepsilon)^n \rho(0 < \beta \leq (1 - \varepsilon)|\beta_j|^2), \quad n < 0,$$

and letting $n \rightarrow -\infty$ yields a contradiction. Thus $\rho(E_2) = 0$. Consequently,

$$\int_{(0,1]} \beta^n \rho(d\beta) = |\beta_j|^{2n} \rho(\{|\beta_j|^2\}) = |\beta_j|^{2n} \langle Az_0, z_0 \rangle,$$

and $\langle Az_1, z_1 \rangle = 0$. By Lemma 3.1 of [SY85], we conclude that $Az_1 = 0$. This proves (3.1).

Let us show (iii)(b). It is enough to show that

$$\langle Az_0, w_0 \rangle = 0 \quad \text{for any } z_0 \in V'_j \text{ and } w_0 \in V'_k \text{ with } j \neq k.$$

Since V'_j and V'_k are invariant under B'^h , $h \in \mathbf{Z}$, we have, by (iii)(a),

$$A(B' - \bar{\beta}_j)^\ell B'^h z_0 = A(B' - \bar{\beta}_k)^\ell B'^h w_0 = 0 \quad \text{for } \ell \geq 1 \text{ and } h \in \mathbf{Z}.$$

Hence, for $n \in \mathbf{Z}$,

$$\begin{aligned} (3.5) \quad L_\pm(n) &= \langle AB'^n(z_0 \pm w_0), B'^n(z_0 \pm w_0) \rangle \\ &= |\beta_j|^{2n} \langle Az_0, z_0 \rangle \pm 2\Re \bar{\beta}_j^n \beta_k^n \langle Az_0, w_0 \rangle + |\beta_k|^{2n} \langle Aw_0, w_0 \rangle. \end{aligned}$$

We consider two cases.

Case I ($|\beta_j| \neq |\beta_k|$). As before, there exist measures ρ_+ and ρ_- on $(0, 1]$ such that

$$(3.6) \quad L_\pm(n) = \int_{(0,1]} \beta^n \rho_\pm(d\beta).$$

Let us show

$$(3.7) \quad \rho_\pm(\{|\beta_j|^2\}) = \langle Az_0, z_0 \rangle$$

and

$$(3.8) \quad \rho_\pm(\{|\beta_k|^2\}) = \langle Aw_0, w_0 \rangle.$$

Without loss of generality, we assume that $|\beta_j| < |\beta_k|$. As in the case of $L(n)$, we observe

$$\rho_\pm(\beta < |\beta_j|^2) = \rho_\pm(\beta > |\beta_k|^2) = 0.$$

It follows from (3.5) and (3.6) that

$$\begin{aligned} \langle Aw_0, w_0 \rangle &= \lim_{n \rightarrow \infty} \frac{L_\pm(n)}{|\beta_k|^{2n}} \\ &= \lim_{n \rightarrow \infty} \int_{\beta < |\beta_k|^2} \left(\frac{\beta}{|\beta_k|^2} \right)^n \rho_\pm(d\beta) + \rho_\pm(\{|\beta_k|^2\}) \\ &= \rho_\pm(\{|\beta_k|^2\}). \end{aligned}$$

This shows (3.8). (3.7) can be shown similarly, by considering $L_{\pm}(n)/|\beta_j|^{2n}$ and letting $n \rightarrow -\infty$. Thus we have

$$\pm 2\Re \bar{\beta}_j^n \beta_k^n \langle Az_0, w_0 \rangle = \int_{(|\beta_j|^2, |\beta_k|^2)} \beta^n \rho_{\pm}(d\beta) \geq 0,$$

concluding $\Re \langle Az_0, w_0 \rangle = 0$. Since $iz_0 \in V_j$, we also have $\Im \langle Az_0, w_0 \rangle = 0$.

Case II ($|\beta_j| = |\beta_k|$). We have

$$L_+(n) = |\beta_j|^{2n} \left\{ \langle Az_0, z_0 \rangle + \langle Aw_0, w_0 \rangle + 2\Re \left(\frac{\beta_k}{\beta_j} \right)^n \langle Az_0, w_0 \rangle \right\}.$$

As in the case of $L(n)$, we see that

$$L_+(n) = \int_{(0,1]} \beta^n \rho_+(d\beta) = \text{const.} \times |\beta_j|^{2n}.$$

Therefore, $\Re(\beta_k/\beta_j)^n \langle Az_0, w_0 \rangle$ is independent of n . Since $\beta_k/\beta_j \neq 1$, we have $\langle Az_0, w_0 \rangle = 0$. This concludes (iii)(b).

We next show (iii) \Rightarrow (i). By the Jordan decomposition of B' , we can find a basis $\{z_{j\ell k} = \bar{\beta}_j^{-k} (B' - \bar{\beta}_j)^k z_{j\ell}, 1 \leq j \leq p, 1 \leq \ell \leq \ell_j, 0 \leq k \leq k(j, \ell)\}$ of \mathbf{C}^d for some integers ℓ_j and $k(j, \ell)$. Here $z_{j\ell} \in V'_j$ ($1 \leq \ell \leq \ell_j$) and $(B' - \bar{\beta}_j)^{k(j, \ell)+1} z_{j\ell} = 0$. Thus, for any $z \in \mathbf{C}^d$,

$$z = \sum_{j, \ell, k} c_{j\ell k} z_{j\ell k} \quad \text{with some } c_{j\ell k} \in \mathbf{C}$$

and, hence,

$$B'^n z = \sum_{j, \ell, k} \bar{\beta}_j^n c_{j\ell k} \sum_{m=0}^{(n(j, \ell) - k) \wedge n} \binom{n}{m} z_{j, \ell, k+m}$$

for all $n \in \mathbf{Z}_+$. Therefore, by (iii) for any $n \in \mathbf{Z}$,

$$A(B'^n z - \sum_{j, \ell} \bar{\beta}_j^n c_{j\ell 0} z_{j\ell 0}) = 0.$$

Hence, for $n \in \mathbf{Z}$,

$$\begin{aligned} & \langle AB'^n z, B'^n z \rangle \\ &= \sum_{j=1}^p |\beta_j|^{2n} \langle A \sum_{\ell} c_{j\ell 0} z_{j\ell 0}, \sum_{\ell} c_{j\ell 0} z_{j\ell 0} \rangle \\ &= \int_{(0,1]} \beta^n \rho(d\beta), \end{aligned}$$

which is completely monotone, if we define ρ by $\rho(\{|\beta_j|^2\}) = \langle A \sum_{\ell} c_{j\ell 0} z_{j\ell 0}, \sum_{\ell} c_{j\ell 0} z_{j\ell 0} \rangle$ and $\rho((0, 1] \setminus \{|\beta_1|^2, \dots, |\beta_p|^2\}) = 0$. It follows from Lemma 3.1 that $\mu \in L_{\infty}(b, Q)$.

We finally show (ii) \Leftrightarrow (iii). (iii) \Rightarrow (ii) is easily seen, because we have, for any $z, w \in \mathbf{C}^d$,

$$\langle (B - \beta_j)AT'_j z, w \rangle = \sum_{k=1}^p \langle T'_j z, A(B' - \bar{\beta}_j)T'_k w \rangle = 0,$$

using (iii)(a) for $k = j$ and (iii)(b) for $k \neq j$. As to (ii) \Rightarrow (iii), we have

$$\langle A(B' - \bar{\beta}_j)T'_j z, (B' - \bar{\beta}_j)T'_j z \rangle = \langle (B - \beta_j)A(B' - \bar{\beta}_j)T'_j z, T'_j z \rangle = 0,$$

which together with Lemma 3.1 of [SY85] implies that $A(B' - \bar{\beta}_j)T'_j z = 0$, namely (iii)(a). Also we have (iii)(b), since (ii) implies that AT'_j has its range in V_j . The proof of Theorem 3.1 is thus complete. \square

Theorem 3.1 uses the direct sum decomposition of \mathbf{C}^d . Let us give to it an expression using a decomposition of \mathbf{R}^d , and then prove a decomposition theorem of Gaussian distributions in $L_{\infty}(b, Q)$. For this purpose we arrange the distinct eigenvalues of $B = b^Q$ in such a way that β_1, \dots, β_q are real and $\beta_{q+1}, \dots, \beta_p$ are not real, $\beta_j = \bar{\beta}_{j+r}$ ($q+1 \leq j \leq q+r$), and $q+2r = p$. Here q or r may possibly be zero. Let γ_j and δ_j be the real and the imaginary part of β_j , respectively. The real factorization of the minimal polynomial $f(\zeta)$ of B' is

$$f(\zeta) = f_1(\zeta)^{n_1} \cdots f_{q+r}(\zeta)^{n_{q+r}},$$

where $f_j(\zeta) = \zeta - \beta_j = \zeta - \gamma_j$, $1 \leq j \leq q$, and $f_j(\zeta) = (\zeta - \gamma_j)^2 + \delta_j^2$, $q+1 \leq j \leq q+r$. Let

$$(3.9) \quad W'_j = \text{Ker } f_j(B')^{n_j} \quad \text{in } \mathbf{R}^d, \quad 1 \leq j \leq q+r.$$

Then

$$(3.10) \quad \mathbf{R}^d = W'_1 \oplus \cdots \oplus W'_{q+r}.$$

As in the proof (iii) \Rightarrow (i) of Theorem 3.1, let

$$\{z_{j\ell k} = \bar{\beta}_j^{-k}(B' - \bar{\beta}_j)^k z_{j\ell} : 1 \leq j \leq p, 1 \leq \ell \leq \ell_j, 0 \leq k \leq k(j, \ell)\}$$

be a basis of \mathbf{C}^d , where $z_{j\ell} \in V'_j$ and $(B' - \bar{\beta}_j)^{k(j, \ell)+1} z_{j\ell} = 0$. For $1 \leq j \leq q$, we can choose $z_{j\ell}$ real so that $\{z_{j\ell k} : 1 \leq \ell \leq \ell_j, 0 \leq k \leq k(j, \ell)\}$ is a basis of W'_j . For $q+1 \leq j \leq q+r$, we have $\ell_j = \ell_{j+r}$ and $k(j, \ell) = k(j+r, \ell)$ and we can choose $z_{j\ell}$ and $z_{j+r, \ell}$ in such a way that $z_{j\ell} = \bar{z}_{j+r, \ell}$. Let $\xi_{j\ell k}$ and $\eta_{j\ell k}$ be the real and the imaginary part of $z_{j\ell k}$, respectively, for $q+1 \leq j \leq q+r$. Here complex conjugates, real parts, and imaginary parts of vectors in \mathbf{C}^d are taken component-wise. The system $\{\xi_{j\ell k}, \eta_{j\ell k} : 1 \leq \ell \leq \ell_j, 0 \leq k \leq k(j, \ell)\}$ is then a basis of W'_j . The following theorem gives a matrix representation when these bases are used.

THEOREM 3.2. *Let μ be Gaussian with covariance matrix A . Then $\mu \in L_\infty(b, Q)$ if and only if the following four conditions are satisfied:*

- (i) $\langle Az_{j\ell k}, z_{j\ell k} \rangle = 0$ for $1 \leq j \leq q, 1 \leq \ell \leq \ell_j, k \geq 1$,
- (ii) $\langle A\xi_{j\ell k}, \xi_{j\ell k} \rangle = \langle A\eta_{j\ell k}, \eta_{j\ell k} \rangle = 0$ for $q+1 \leq j \leq q+r, 1 \leq \ell \leq \ell_j, k \geq 1$,
- (iii) $\langle A\xi_{j\ell 0}, \xi_{jm 0} \rangle = \langle A\eta_{j\ell 0}, \eta_{jm 0} \rangle$ and $\langle A\xi_{j\ell 0}, \eta_{jm 0} \rangle = -\langle A\eta_{j\ell 0}, \xi_{jm 0} \rangle$ for $q+1 \leq j \leq q+r, 1 \leq \ell \leq \ell_j, 1 \leq m \leq \ell_j$ with $\ell = m$ inclusive,
- (iv) $\langle Az, w \rangle = 0$ for $z \in W'_j, w \in W'_k$, for $1 \leq j \leq q+r, 1 \leq k \leq q+r$ with $j \neq k$.

Proof. The proof is exactly the same as that of Theorem 4.1 of [SY85]. So we omit it here. \square

Let us consider the direct sum decomposition of \mathbf{R}^d associated with $B = b^Q$.
Let

$$W_j = \text{Ker } f_j(B)^{n_j} \quad \text{in } \mathbf{R}^d, 1 \leq j \leq q+r.$$

Then

$$(3.11) \quad \mathbf{R}^d = W_1 \oplus \cdots \oplus W_{q+r}.$$

This is the decomposition dual to (3.10). Let U_j be the projector of \mathbf{R}^d onto W_j in the decomposition (3.11). The transposed matrix U_j' of U_j is the projector onto W_j' in the decomposition (3.10). For $q+1 \leq j \leq q+r$, we have $V_j = \overline{V}_{j+r}$ and $T_j x = \overline{T_{j+r} x}$ for $x \in \mathbf{R}^d$. Thus $U_j x = T_j x + T_{j+r} x$ for $x \in \mathbf{R}^d$ for $q+1 \leq j \leq q+r$. For $1 \leq j \leq q$, we have $U_j x = T_j x$ for $x \in \mathbf{R}^d$. Let

$$N_j = \text{Ker } f_j(B) \quad \text{in } \mathbf{R}^d, \quad 1 \leq j \leq q+r.$$

THEOREM 3.3. *Suppose that μ is a centered Gaussian distribution in $L_\infty(b, Q)$. Then, the support of μ is a B -invariant linear subspace of \mathbf{R}^d and the minimal polynomial of the restriction of B to the support of μ does not have double roots. There exists a unique decomposition $\mu = \mu_1 * \cdots * \mu_{q+r}$, where each μ_j is a centered Gaussian distribution such that $\mu_j \in L_\infty(b, Q)$ and the support of μ_j is contained N_j and hence in W_j .*

Proof. Again, the proof is exactly the same as that of Theorem 4.2 of [SY85]. So we omit it here. \square

REMARK 3.1. *The μ_j in Theorem 3.3 is centered Gaussian with covariance matrix $A_j = U_j A U_j'$. Thus, since $\mu_j \in L_\infty(b, Q)$, by Theorem 3.1,*

$$\begin{aligned} \widehat{\mu}_j(B'z) &= \exp\left\{-\frac{1}{2}\langle A_j B'z, B'z \rangle\right\} \\ &= \exp\left\{-\frac{1}{2}\langle A_j B'U_j'z, B'U_j'z \rangle\right\} \\ &= \exp\left\{-\frac{1}{2}|\beta_j|^2 \langle A_j U_j'z, U_j'z \rangle\right\} \\ &= \exp\left\{-\frac{1}{2}|\beta_j|^2 \langle A_j z, z \rangle\right\} \\ &= \widehat{\mu}_j(z)^{|\beta_j|^2}, \end{aligned}$$

which means that $\mu_j \in \text{OSS}(b, Q)$.

Combining Theorem 3.3 and Remark 3.1, we have

THEOREM 3.4. *Let s be the number of distinct absolute values of eigenvalues of $B = b^Q$. If μ is a Gaussian distribution in $L_\infty(b, Q)$, then μ can be*

expressed as the convolution of at most s Gaussian distributions in $OSS(b, Q)$.

EXAMPLE. For $d = 2, 3, 4$, explicit forms of the covariance matrices of Gaussian distributions in $L_\infty(Q)$ are given in [SY85]. Let $d = 2$ and let I_G be the class of all Gaussian distributions on \mathbf{R}^2 . Let $\mu \in I_G$ with covariance matrix A .

First consider the case $Q = \begin{pmatrix} \gamma & 1 \\ 0 & \gamma \end{pmatrix}$ with $\gamma > 0$. For $0 < b < 1$, $b^Q = b^\gamma \begin{pmatrix} 1 & \log b \\ 0 & 1 \end{pmatrix}$ and hence b^Q has the Jordan form $\begin{pmatrix} b^\gamma & 1 \\ 0 & b^\gamma \end{pmatrix}$. As is shown in [SY85], $\mu \in L_\infty(Q)$ if and only if $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $a \geq 0$. As our Theorem 3.2 is formally the same as Theorem 4.1 of [SY85], we see that $\mu \in L_\infty(b, Q)$ if and only if $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $a \geq 0$. Thus $L_\infty(b, Q) \cap I_G = L_\infty(Q) \cap I_G$ for any $0 < b < 1$ in this case.

Next consider the case $Q = \begin{pmatrix} \gamma & -\delta \\ \delta & \gamma \end{pmatrix}$ with $\gamma > 0$ and $\delta \in \mathbf{R} \setminus \{0\}$. Then, for $0 < b < 1$, $b^Q = b^\gamma \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ with $\theta = \delta \log b$. In this case, $\mu \in L_\infty(Q)$ if and only if $A = aI$, $a \geq 0$, as is shown in [SY85]. If $b = e^{n\pi/\delta}$ with $n \in \mathbf{Z}$ and $n/\delta < 0$, then $b^Q = b^\gamma (-1)^n I$ and hence $L_\infty(b, Q) \cap I_G = I_G$. Otherwise, b^Q is of the same type as Q and $L_\infty(b, Q) \cap I_G = L_\infty(Q) \cap I_G$. Thus, $L_\infty(b, Q) \cap I_G$ truly depends on b and, for some b , $L_\infty(b, Q) \cap I_G$ is strictly larger than $L_\infty(Q) \cap I_G$.

4. Purely non-Gaussian distributions in $L_\infty(b, Q)$

Now we give a representation of the Lévy measure of purely non-Gaussian $\mu \in L_\infty(b, Q)$. For each $x \in \mathbf{R}^d \setminus \{0\}$, let

$$\beta(x) = \max\{|\beta_j| : 1 \leq j \leq q + 2r, T_j x \neq 0\},$$

$$n(x, j) = \max\{n : n \geq 0, (B - \beta_j)^n T_j x \neq 0 \text{ for } T_j x \neq 0\},$$

$$n(x) = \max\{n(x, j) : 1 \leq j \leq q + 2r, T_j x \neq 0, |\beta_j| = \beta(x)\},$$

and

$$\gamma(x) = \frac{\log \beta(x)}{\log b}.$$

We show the following. Given two measurable spaces $(\Theta_1, \mathcal{B}_1)$ and $(\Theta_2, \mathcal{B}_2)$, we say that $\{\Gamma_{\theta_1}, \theta_1 \in \Theta_1\}$, a system of measures on $(\Theta_2, \mathcal{B}_2)$, is measurable in θ_1 if $\Gamma_{\theta_1}(E)$ is measurable in θ_1 for every $E \in \mathcal{B}_2$.

THEOREM 4.1. (i) *Suppose that μ is a purely non-Gaussian distribution in $I(\mathbf{R}^d)$ with nonzero Lévy measure ν . Then $\mu \in L_\infty(b, Q)$ if and only if ν is expressed as*

$$(4.1) \quad \nu(E) = \int_{S_B} \nu_0(dx) \int_{(0, 2\gamma(x))} \Gamma_x(d\alpha) \sum_{n \in \mathbf{Z}} b^{n\alpha} 1_E(B^{-n}x), \quad E \in \mathcal{B}(\mathbf{R}^d),$$

where ν_0 is a nonzero finite measure on S_B and $\Gamma_x, x \in S_B$, are nonzero finite measures on $(0, \infty)$ measurable in x , each Γ_x is concentrated on $(0, 2\gamma(x))$ and

$$(4.2) \quad \int_{S_B} \nu_0(dx) \int_{(0, 2\gamma(x))} \Gamma_x(d\alpha) \sum_{n \in \mathbf{Z}} b^{n\alpha} (|B^{-n}x|_Q^2 \wedge 1) < \infty.$$

(ii) *If a nonzero finite measure ν_0 on S_B and nonzero finite measures $\Gamma_x, x \in S_B$, on $(0, \infty)$ measurable in x are given and if each Γ_x is concentrated on $(0, 2\gamma(x))$ and (4.2) is satisfied, then the measure ν defined by (4.1) is the Lévy measure of some $\mu \in I(\mathbf{R}^d)$.*

(iii) *If $\mu \in L_\infty(b, Q)$ has nonzero Lévy measure ν and if ν is expressed by ν_0 and Γ_x as in (i), then ν_0 and Γ_x are unique in the following sense: if $\tilde{\nu}$ and $\tilde{\Gamma}_x$ give another expression of ν , then there exists a Borel measurable function $h(x)$ with $0 < h(x) < \infty$ such that $\tilde{\nu}_0(dx) = h(x)\nu_0(dx)$ and $\Gamma_x(d\alpha) = h(x)\tilde{\Gamma}_x(d\alpha)$ for ν_0 -a.e. x . The measures Γ_x necessarily satisfy*

$$(4.3) \quad \int_{(0, 2\gamma(x))} (\alpha^{-1} + (2\gamma(x) - \alpha)^{-2n(x)-1}) \Gamma_x(d\alpha) < \infty, \quad \nu_0\text{-a.e. } x.$$

LEMMA 4.1. *There exist positive constants C_j ($j = 5, 6, 7$) and $b_j(x)$ ($j = 1, 2, 3$) such that, for $x \in S_B$,*

$$(4.4) \quad |B^k x|_Q \leq C_5 \beta(x)^k k^{n(x)} \quad \text{for } k \geq 1,$$

$$(4.5) \quad |B^k x|_Q \geq b_2(x) \beta(x)^k k^{n(x)} \quad \text{for } k \geq b_1(x),$$

$$(4.6) \quad \begin{aligned} C_6 \alpha^{-1} + b_3(x) (2\gamma(x) - \alpha)^{-2n(x)-1} &\leq \sum_{n \in \mathbf{Z}} (|B^n x|_Q^2 \wedge 1) b^{-n\alpha} \\ &\leq C_7 (\alpha^{-1} + (2\gamma(x) - \alpha)^{-2n(x)-1}) \quad \text{for } 0 < \alpha < 2\gamma(x). \end{aligned}$$

If $\alpha \geq 2\gamma(x)$, then

$$(4.7) \quad \sum_{n \in \mathbf{Z}} (|B^n x|_Q^2 \wedge 1) b^{-n\alpha} = \infty.$$

Proof. Let $|x|_Q$ be defined by $|x|_Q = \int_0^1 \frac{|u^Q x|}{u} du$ aslo to $x \in \mathbf{C}^d$. We have

$$(4.8) \quad \begin{aligned} B^k T_j x &= (\beta_j + (B - \beta_j))^k T_j x \\ &= \beta_j^k \sum_{\ell=0}^{n(x,j) \wedge k} \binom{k}{\ell} \beta_j^{-\ell} (B - \beta_j)^\ell T_j x. \end{aligned}$$

Thus

$$|B^k x|_Q \leq C_8 \beta(x)^k \sum_{j=1}^p k^{n(x,j)} \quad k \geq 1.$$

Hence we have (4.4). It follows from (4.8) that there are $b_4(x)$ and $b_5(x)$ such that, for $k \geq b_4(x)$,

$$\begin{aligned} |B^k T_j x|_Q &\geq 2^{-1} |\beta_j|^k k^{n(x,j)} (n(x,j)!)^{-1} |(B - \beta_j)^{n(x,j)} T_j x|_Q \\ &\geq b_5(x) |\beta_j|^k k^{n(x,j)} \end{aligned}$$

for all j satisfying $T_j x \neq 0$. Choose a norm $\|\cdot\|$ in \mathbf{C}^d as $\|x\| = \sum_{j=1}^p |T_j x|_Q$. Since arbitrary two norms are equivalent, we have $C_9|x|_Q \leq \|x\| \leq C_{10}|x|_Q$. Choosing j such that $\beta(x) = \beta_j$ and $n(x) = n(x, j)$, we obtain

$$|B^k x|_Q \geq C_{10}^{-1} \|B^k x\| \geq C_{10}^{-1} |B^k T_j x|_Q.$$

Hence (4.5) follows. Let $0 < \alpha < 2\gamma(x)$. Note that $\sum_{k=-\infty}^0 b^{-k\alpha} = (1 - b^\alpha)^{-1}$ and $C_{11}\alpha^{-1} \leq (1 - b^\alpha)^{-1} \leq C_{12}\alpha^{-1}$. We see from (4.4) that

$$\begin{aligned} \sum_{k=1}^{\infty} |B^k x|_Q^2 b^{-k\alpha} &\leq C_5^2 \sum_{k=1}^{\infty} \beta(x)^{2k} k^{2n(x)} b^{-k\alpha} \\ &= C_5^2 \sum_{k=1}^{\infty} b^{k(2\gamma(x)-\alpha)} k^{2n(x)} \leq C_{13} (2\gamma(x) - \alpha)^{-2n(x)-1}. \end{aligned}$$

This proves the second inequality in (4.6). The first inequality is obtained from (4.5) as follows. We have

$$\begin{aligned} \sum_{k \geq b_1(x)} |B^k x|^2 b^{-k\alpha} &\geq b_2(x)^2 \sum_{k \geq b_1(x)} b^{k(2\gamma(x)-\alpha)} k^{2n(x)} \\ &\geq b_6(x) (2\gamma(x) - \alpha)^{-2n(x)-1}, \end{aligned}$$

for some $b_6(x) > 0$. Hence the first inequality in (4.6) is obtained. The proof of (4.7) for $\alpha \geq 2\gamma(x)$ is similar. \square

Proof of Theorem 4.1. In the following the conditions (a) – (d) refer to those in Proposition 2.1.

(i) Let $\{\nu_0, g_n, n \in \mathbf{Z}\}$ be the S_B -representation of ν . Suppose that $\mu \in L_\infty(b, Q)$. It follows from Propositions 2.2 and 2.3 that, for ν_0 -a.e. x , there exists a measure Γ_x such that

$$g_n(x) = \int_{[0, \infty)} b^{n\alpha} \Gamma_x(d\alpha).$$

By (c), Γ_x is nonzero. By (d), for any $E \in \mathcal{B}(\mathbf{R}^d)$,

$$\nu(E) = \int_{S_B} \nu_0(dx) \sum_{n \in \mathbf{Z}} \int_{[0, \infty)} 1_E(B^{-n}x) b^{n\alpha} \Gamma_x(d\alpha).$$

Note that for any nonnegative measurable function $h(\alpha)$ on $[0, \infty)$, $\int h(\alpha)\Gamma_x(d\alpha)$ is measurable in x . By (b),

$$\infty > \sum_{n \geq 0} \int_{S_B} g_n(x) \nu_0(dx) = \int_{S_B} \nu_0(dx) \int_{[0, \infty)} \Gamma_x(d\alpha) \sum_{n \geq 0} b^{n\alpha}.$$

When $\alpha = 0$, $\sum_{n \geq 0} b^{n\alpha} = \infty$. Hence

$$\int_{S_B} \nu_0(dx) \Gamma_x(\{0\}) = 0.$$

Next we have, by (d),

$$\begin{aligned} \infty &> \int_{\mathbf{R}^d} (|y|_Q^2 \wedge 1) \nu(dy) \\ &= \sum_{n \in \mathbf{Z}} \int_{S_B} ((|B^{-n}x|_Q^2 \wedge 1) \nu_0(dx) \int_{(0, \infty)} b^{n\alpha} \Gamma_x(d\alpha) \\ &= \int_{S_B} \nu_0(dx) \int_{(0, \infty)} \Gamma_x(d\alpha) \sum_{n \in \mathbf{Z}} b^{n\alpha} (|B^{-n}x|_Q^2 \wedge 1). \end{aligned}$$

Thus by Lemma 4.1, we have

$$\int_{S_B} \nu_0(dx) \int_{[2\gamma(x), \infty)} \Gamma_x(d\alpha) = 0,$$

concluding (4.1), and the integrability condition (4.2) is also proved.

Conversely suppose that ν has the representation (4.1) with ν_0 and Γ_x satisfying (4.2). Set

$$(4.9) \quad g_n(x) = \int_{(0, 2\gamma(x))} b^{n\alpha} \Gamma_x(d\alpha), \quad n \in \mathbf{Z}.$$

Then (4.1) and (4.9) imply (d). We observe that $g_n(x)$ in (4.9) satisfies (a), (b), and (c). As to (b),

$$\begin{aligned} &\int_{S_B} \nu_0(dx) \sum_{n \in \mathbf{Z}} (|B^{-n}x|_Q^2 \wedge 1) g_n(x) \\ &= \int_{S_B} \nu_0(dx) \sum_{n \in \mathbf{Z}} (|B^{-n}x|_Q^2 \wedge 1) \int_{(0, 2\gamma(x))} b^{n\alpha} \Gamma_x(d\alpha) \\ &= \int_{S_B} \nu_0(dx) \int_{(0, 2\gamma(x))} \Gamma_x(d\alpha) \sum_{n \in \mathbf{Z}} b^{n\alpha} (|B^{-n}x|_Q^2 \wedge 1) < \infty \end{aligned}$$

by (4.2). (a) and (c) are obvious because $\Gamma_x(d\alpha)$ is nonzero for each x . Therefore $\{\nu_0, g_n, n \in \mathbf{Z}\}$ is the S_B -representation of ν . It follows from (4.9) and Proposition 2.2 that $g_n(x)$ is completely monotone in $n \in \mathbf{Z}$. Thus, by Proposition 2.3, $\mu \in L_\infty(b, Q)$.

(ii) In order to see that ν is the Lévy measure of some μ , it is enough to show that $\nu(\{0\}) = 0$ and that $\int_{\mathbf{R}^d} (|x|^2 \wedge 1) \nu(dx) < \infty$. The former is obvious from (4.1). The latter follows (4.2) since $|x| \leq \text{const.} \times |x|_Q$.

(iii) To show the uniqueness, suppose that both $\{\nu_0, \Gamma_x\}$ and $\{\tilde{\nu}_0, \tilde{\Gamma}_x\}$ represent ν . Let

$$(4.10) \quad g_n(x) = \int_{(0, 2\gamma(x))} b^{n\alpha} \Gamma_x(d\alpha), \quad \tilde{g}_n(x) = \int_{(0, 2\tilde{\gamma}(x))} b^{n\alpha} \tilde{\Gamma}_x(d\alpha).$$

By the proof above, $\{\nu_0, g_n\}$ and $\{\tilde{\nu}_0, \tilde{g}_n\}$ are S_B -representations of ν . Thus by the uniqueness of them in Proposition 2.1, there exists a Borel measurable function $h(x)$ with $0 < h(x) < \infty$ such that $\tilde{\nu}_0(dx) = h(x)\nu_0(dx)$ and $g_n(x) = h(x)\tilde{g}_n(x)$, ν_0 -a.e. x for any $n \in \mathbf{Z}$. Thus by the uniqueness assertion in Proposition 2.2 (i)(b) and by (4.10), we conclude that $\Gamma_x(d\alpha) = h(x)\tilde{\Gamma}_x(d\alpha)$. The assertion (4.3) for Γ_x follows from (4.2) and (4.6). \square

5. Proof of Theorem 1.1

We first show that $OSS(b, Q) \subset L_\infty(b, Q)$. Let $\mu \in OSS(b, Q)$. That is, for some $0 < a < 1$ and $c \in \mathbf{R}^d$,

$$(5.1) \quad \hat{\mu}(z)^a = \hat{\mu}(b^{Q'} z) e^{i\langle c, z \rangle}.$$

Then

$$(5.2) \quad \hat{\mu}(z) = \hat{\mu}(b^{Q'} z) \hat{\rho}(z),$$

with

$$(5.3) \quad \widehat{\rho}(z) = \widehat{\mu}(z)^{1-a} e^{i\langle c, z \rangle}.$$

To show that $\mu \in L_\infty(b, Q)$, by the definition, it is enough to show that $\rho \in L_\infty(b, Q)$. Since $\mu \in I(\mathbf{R}^d)$, $\rho \in I(\mathbf{R}^d)$. Hence by (5.2), $\mu \in L_0(b, Q)$. Since $L_0(b, Q)$ is Q -completely closed in the strong sense as mentioned in Section 1, ρ in (5.3) is in $L_0(b, Q)$. Thus by the definition, (5.2) implies that $\mu \in L_1(b, Q)$. Repeating this argument, we conclude that $\mu \in L_m(b, Q)$ for any $1 \leq m < \infty$ and therefore $\mu \in L_\infty(b, Q)$. Hence $OSS(b, Q) \subset L_\infty(b, Q)$.

Since $L_\infty(b, Q)$ is Q -completely closed in the strong sense, it only remains to prove “the smallest”. Let K be any Q -completely closed class in the strong sense containing $OSS(b, Q)$. First, notice the following fact. Let $\alpha > 0$ and $r(z, x) = e^{i\langle z, x \rangle} - 1 - \frac{i\langle z, x \rangle}{1+|x|^2}$. If ν_0 is a finite measure concentrated on $S_B \cap \{x : 2\gamma(x) > \alpha\}$ satisfying

$$\int_{S_B} \nu_0(dx) \sum_{n < 0} b^{n\alpha} |b^{-nQ} x|^2 < \infty$$

and if μ is a distribution with

$$\widehat{\mu}(z) = \exp \left\{ \int_{S_B} \nu_0(dx) \sum_{n \in \mathbf{Z}} b^{n\alpha} r(z, b^{-nQ} x) \right\},$$

then $\mu \in OSS(b, Q)$, since (5.1) holds with $a = b^\alpha$. Now let μ be a purely non-Gaussian distribution in $L_\infty(b, Q)$. Then its Lévy measure is represented as in Theorem 4.1, and we have

$$\widehat{\mu}(z) = \exp \left\{ i\langle \gamma, z \rangle + \int_{S_B} \nu_0(dx) \int_{(0, 2\gamma(x))} \Gamma_x(d\alpha) \sum_{n \in \mathbf{Z}} b^{n\alpha} r(z, b^{-nQ} x) \right\}$$

with some $\gamma \in \mathbf{R}^d$. For $\varepsilon > 0$, define μ_ε by

$$\widehat{\mu}_\varepsilon(z) = \exp \left\{ i\langle \gamma, z \rangle + \int_{S_B} \nu_0(dx) \int_{(\varepsilon, 2\gamma(x) - \varepsilon)} \Gamma_x(d\alpha) \sum_{n \in \mathbf{Z}} b^{n\alpha} r(z, b^{-nQ} x) \right\}.$$

Since the number of the possible values of $\gamma(x)$ is finite, $2\gamma(x) - \varepsilon > \varepsilon$ for all x , if ε is small enough. Then, for fixed $\varepsilon > 0$, we can choose measures $\Gamma_x^{(n)}(d\alpha)$ satisfying the following conditions: $\Gamma_x^{(n)}(d\alpha)$ is concentrated on the points $\{k2^{-n}, k = 1, 2, \dots\} \cap (\varepsilon, 2\gamma(x) - \varepsilon)$, $\Gamma_x^{(n)}(d\alpha)$ converges to $\Gamma_x(d\alpha)$ for each $x \in S_B$ as $n \rightarrow \infty$, the total mass of $\Gamma_x^{(n)}(d\alpha)$ does not exceed that of $\Gamma_x(d\alpha)$ for each $x \in S_B$, and $\{\Gamma_x^{(n)}\}$ is measurable in x . Define $\mu_\varepsilon^{(n)}$ by

$$\widehat{\mu}_\varepsilon^{(n)}(z) = \exp \left\{ i \langle \gamma, z \rangle + \int_{S_B} \nu_0(dx) \int_{(\varepsilon, 2\gamma(x) - \varepsilon)} \Gamma_x^{(n)}(d\alpha) \sum_{n \in \mathbf{Z}} b^{n\alpha} r(z, b^{-nQ}x) \right\}.$$

We see that $\mu_\varepsilon^{(n)}$ is a convolution of a finite number of purely non-Gaussian distributions in $OSS(b, Q)$. Hence $\mu_\varepsilon^{(n)} \in K$. We see from (4.6) that, for any fixed $\varepsilon > 0$,

$$(5.4) \quad C_{14} \leq \sum_{n \in \mathbf{Z}} b^{n\alpha} (|b^{-nQ}x|_Q^2 \wedge 1) \leq C_{15} \quad \text{for } \alpha \in (\varepsilon, 2\gamma(x) - \varepsilon) \text{ and } x \in S_B.$$

Hence, by (4.2),

$$(5.5) \quad \int_{S_B} \nu_0(dx) \Gamma_x((\varepsilon, 2\gamma(x) - \varepsilon)) < \infty.$$

We show that, for fixed $z \in \mathbf{R}^d$, $\sum_{n \in \mathbf{Z}} b^{n\alpha} r(z, b^{-nQ}x)$ is bounded in $\alpha \in (\varepsilon, 2\gamma(x) - \varepsilon)$ and $x \in S_B$, and continuous in α . Since $|r(z, x)| \leq C_{16}(|x|_Q^2 \wedge 1)$, we see the boundedness from (5.4). The continuity is obvious. Thus by (5.5) and Lebesgue's dominated convergence theorem, we have that as $n \rightarrow \infty$ $\widehat{\mu}_\varepsilon^{(n)}(z) \rightarrow \widehat{\mu}_\varepsilon(z)$, and that $\mu_\varepsilon \in K$. Finally, $\widehat{\mu}_\varepsilon \rightarrow \widehat{\mu}$ as $\varepsilon \downarrow 0$. Thus $\mu \in K$. This proves that if $\mu \in L_\infty(b, Q)$ is purely non-Gaussian, then $\mu \in K$.

If $\mu \in L_\infty(b, Q)$ is Gaussian, then by Theorem 3.4, μ is a convolution of finite number of Gaussian distributions in $OSS(b, Q) \subset K$, and thus $\mu \in K$. As Proposition 2.3 shows, any $\mu \in L_\infty(b, Q)$ is decomposable in $L_\infty(b, Q)$ as the convolution of a Gaussian and a purely non-Gaussian. Hence $L_\infty(b, Q) \subset K$ and the proof is complete. \square

6. Remarks on the case $Q = I$

Let us consider the case $Q = I$. Let $0 < b < 1$. The classes $L_m(b, I)$, $0 \leq m \leq \infty$, were introduced by Maejima and Naito [MN98], of which our paper [MSW98] was a matrix generalization on \mathbf{R}^d . Distributions in $L_0(b, I)$ are called semi-selfdecomposable. Distributions in $L_\infty(b, I)$ should be called completely semi-selfdecomposable. The class $OSS(b, I)$ consists of $\mu \in \mathcal{P}(\mathbf{R}^d)$ that satisfies $\hat{\mu}(z)^a = \hat{\mu}(bz)e^{i\langle c, z \rangle}$ for some $0 < a < 1$ and $c \in \mathbf{R}^d$; namely it is the class of $\mu \in \mathcal{P}(\mathbf{R}^d)$ that satisfies $\hat{\mu}(z)^{b^\alpha} = \hat{\mu}(bz)e^{i\langle c, z \rangle}$ for some $0 < \alpha \leq 2$ and $c \in \mathbf{R}^d$. Thus distributions in $OSS(b, I)$ are exactly semi-stable distributions studied by many authors beginning with Lévy [L37]. Now we have $B = bI$, $|x|_Q = |x|$, and $S_B = \{x \in \mathbf{R}^d : b < |x| \leq 1\}$. We write S_B as S_b . Further we have $p = q = 1$, $b_1 = b$, $n_1 = 1$, and $f(\zeta) = \zeta - b$. Since $\Psi_B(A) = (1 - b^2)A$, Proposition 2.3 shows that all Gaussian distributions are in $L_\infty(b, I)$. Since $\beta(x) = b$ and $\gamma(x) = 1$ for all $x \in \mathbf{R}^d$, the following result is obtained from Theorem 4.1.

THEOREM 6.1. *(i) Suppose that μ is in $I(\mathbf{R}^d)$ with nonzero Lévy measure ν . Then $\mu \in L_\infty(b, I)$ if and only if ν is expressed as*

$$(6.1) \quad \nu(E) = \int_{S_b} \nu_0(dx) \int_{(0,2)} \Gamma_x(d\alpha) \sum_{n \in \mathbf{Z}} b^{n\alpha} 1_E(b^{-n}x), \quad E \in \mathcal{B}(\mathbf{R}^d),$$

where ν_0 is a nonzero finite measure on S_b and Γ_x , $x \in S_b$, are measures on $(0, 2)$ measurable in x satisfying

$$(6.2) \quad \int_{(0,2)} \left(\frac{1}{\alpha} + \frac{1}{2-\alpha} \right) \Gamma_x(d\alpha) = 1.$$

(ii) If a nonzero finite measure ν_0 on S_b and measures Γ_x , $x \in S_b$, on $(0, 2)$ measurable in x satisfying (6.2) are given, then the measure ν defined by (6.1) is the Lévy measure of some $\mu \in I(\mathbf{R}^d)$.

(iii) If $\mu \in L_\infty(b, I)$ with nonzero Lévy measure ν , then the measure ν_0 in (i) is uniquely determined by ν and the measures Γ_x , $x \in S_b$, are unique in the sense

that $\tilde{\Gamma}_x = \Gamma_x$ for ν_0 -a.e. x for any $\tilde{\Gamma}_x$ which expresses ν by (6.1) in place of Γ_x .

Proof. (i) We apply Theorem 4.1. Note that we do not assume that μ is purely non-Gaussian, since all Gaussians are in $L_\infty(b, I)$. Suppose that $\mu \in L_\infty(b, I)$. Then we get (6.1) with some ν_0 and Γ_x satisfying

$$\int_{S_b} \nu_0(dx) \int_{(0,2)} \Gamma_x(d\alpha) \sum_{n \in \mathbf{Z}} b^{n\alpha} ((b^{-2n}|x|^2) \wedge 1) < \infty.$$

This is equivalent to

$$\int_{S_b} \nu_0(dx) \int_{(0,2)} \Gamma_x(d\alpha) \left(\sum_{n \geq 0} b^{n\alpha} + \sum_{n < 0} b^{-n(2-\alpha)} \right) < \infty.$$

Since $\sum_{n \geq 0} b^{n\alpha} \sim \frac{C_{17}}{\alpha}$ as $\alpha \downarrow 0$ and $\sum_{n < 0} b^{-n(2-\alpha)} \sim \frac{C_{18}}{2-\alpha}$ as $\alpha \uparrow 2$, the condition is equivalent to

$$\int_{S_B} \nu_0(dx) \int_{(0,2)} \left(\frac{1}{\alpha} + \frac{1}{2-\alpha} \right) \Gamma_x(d\alpha) < \infty.$$

Let $h(x) = \int_{(0,2)} \left(\frac{1}{\alpha} + \frac{1}{2-\alpha} \right) \Gamma_x(d\alpha)$ and use $h(x)\nu_0(dx)$ and $\frac{1}{h(x)}\Gamma_x(d\alpha)$ in place of ν_0 and Γ_x to obtain (6.2). The converse is similarly proved. (ii) and (iii) follow from Theorem 4.1 (ii) and (iii), respectively. \square

Another form of the theorem above is as follows.

THEOREM 6.2. (i) Suppose that $\mu \in I(\mathbf{R}^d)$ with nonzero Lévy measure ν . Then $\mu \in L_\infty(b, I)$ if and only if

$$(6.3) \quad \nu(E) = \int_{(0,2)} \Gamma(d\alpha) \int_{S_b} \nu_\alpha(dx) \sum_{n \in \mathbf{Z}} b^{n\alpha} 1_E(b^{-n}x), \quad E \in \mathcal{B}(\mathbf{R}^d),$$

where Γ is a nonzero measure on $(0, 2)$ with

$$(6.4) \quad \int_{(0,2)} \left(\frac{1}{\alpha} + \frac{1}{2-\alpha} \right) \Gamma(d\alpha) < \infty$$

and $\nu_\alpha, \alpha \in (0, 2)$, are probability measures on S_b , measurable in α .

(ii) If a nonzero measure Γ on $(0, 2)$ satisfying (6.4) and probability measures ν_α , $\alpha \in (0, 2)$, measurable in α are given, then the measure ν defined by (6.3) is the Lévy measure of some $\mu \in I(\mathbf{R}^d)$.

(iii) If $\mu \in L_\infty(b, I)$ with nonzero Lévy measure ν , then the measure Γ in (i) is uniquely determined by ν and the probability measures ν_α , $\alpha \in (0, 2)$, are unique in the sense that $\tilde{\nu}_\alpha = \nu_\alpha$ for Γ -a.e. α for any $\tilde{\nu}_\alpha$ that expresses ν by (6.3) in place of ν_α .

Proof. In order to go to the representation (6.3) from (6.1), consider the probability measure $\frac{1}{\nu_0(S_b)}\nu_0(dx)(\frac{1}{\alpha} + \frac{1}{2-\alpha})\Gamma_x(d\alpha)$ on $S_b \times (0, 2)$ and apply the existence theorem for conditional distributions. Transfer in the reverse direction is similar. \square

Finiteness and infiniteness of the moments of distributions in $L_\infty(b, I)$ are determined only by the measure Γ . This is an application of Theorem 6.2.

THEOREM 6.3. *Let μ be a distribution in $L_\infty(b, I)$ with nonzero Lévy measure ν . Let $\Gamma(d\alpha)$ be the nonzero measure on $(0, 2)$ uniquely determined by ν in Theorem 6.2. Let $\alpha_0 \in [0, 2)$ be the infimum of the support of Γ . Then, finiteness and infiniteness of $M_\eta = \int_{\mathbf{R}^d} |x|^\eta \mu(dx)$ are as follows.*

(i) If $\eta > \alpha_0$, then $M_\eta = \infty$.

(ii) If $\alpha_0 > 0$ and $0 < \eta < \alpha_0$, then $M_\eta < \infty$.

(iii) If $\alpha_0 > 0$ and $\Gamma(\{\alpha_0\}) > 0$, then $M_{\alpha_0} = \infty$.

(iv) If $\alpha_0 > 0$ and $\Gamma(\{\alpha_0\}) = 0$, then M_{α_0} is finite or infinite according as $\int_{(\alpha_0, 2)} \frac{1}{\alpha - \alpha_0} \Gamma(d\alpha)$ is finite or infinite.

Proof. It is known that $M_\eta < \infty$ if and only if $\int_{|x|>1} |x|^\eta \nu(dx) < \infty$

(Kruglov [K70]). We have, from (6.3),

$$\begin{aligned} \int_{|x|>1} |x|^\eta \nu(dx) &= \int_{(0,2)} \Gamma(d\alpha) \int_{S_b} \nu_\alpha(dx) \sum_{n \in \mathbf{Z}} b^{n\alpha} |b^{-n}x|^\eta \mathbf{1}_E(b^{-n}x) \\ &= \int_{(0,2)} \Gamma(d\alpha) \int_{S_b} \nu_\alpha(dx) |x|^\eta \sum_{n \geq 1} b^{n(\alpha-\eta)}, \end{aligned}$$

where $E = \{x : |x| > 1\}$. If $\eta > \alpha_0$, then $\sum_{n \geq 1} b^{n(\alpha-\eta)} = \infty$ for $\alpha \in [\alpha_0, \eta)$ and $\int_{|x|>1} |x|^\eta \nu(dx) = \infty$. If $\alpha_0 > 0$ and $0 < \eta < \alpha_0$, then

$$\int_{|x|>1} |x|^\eta \nu(dx) \leq \frac{1}{1 - b^{\alpha_0 - \eta}} \int_{[\alpha_0, 2)} \Gamma(d\alpha) \int_{S_b} |x|^\eta \nu_\alpha(dx) < \infty.$$

If $\alpha_0 > 0$ and $\Gamma(\{\alpha_0\}) > 0$, then

$$\int_{|x|>1} |x|^{\alpha_0} \nu(dx) \geq \Gamma(\{\alpha_0\}) \int_{S_b} \nu_\alpha(dx) |x|^{\alpha_0} \sum_{n \geq 1} 1 = \infty.$$

Hence we obtain (i), (ii), and (iii). Consider the final case, $\alpha_0 > 0$ and $\Gamma(\{\alpha_0\}) = 0$. Since $\sum_{n \geq 1} b^{n(\alpha-\alpha_0)} \sim \frac{C_{19}}{\alpha-\alpha_0}$ as $\alpha \downarrow \alpha_0$, we have

$$C_{20} \int_{(\alpha_0, 2)} \frac{1}{\alpha - \alpha_0} \Gamma(d\alpha) \leq \int_{|x|>1} |x|^{\alpha_0} \nu(dx) \leq C_{21} \int_{(\alpha_0, 2)} \frac{1}{\alpha - \alpha_0} \Gamma(d\alpha).$$

Hence the assertion (iv) follows. \square

REFERENCES

- [J77] Jajte, R.(1977). Semi-stable probability measures on \mathbf{R}^N , *Studia Math.* **61** 29-39.
- [J83] Jurek, Z.J. (1983). Limit distributions and one-parameter groups of linear operators on Banach spaces. *J. Multivar. Anal.* **13** 578-604.
- [JM93] Jurek, Z.J. and Mason, J.D. (1993). *Operator-Limit Distributions in Probability Theory*. Wiley, New York.
- [K80] Krakowiak, W.(1980). Operator semi-stable probability measures on Banach spaces. *Colloq. Math.* **43** 351-363.
- [K70] Kruglov, V.M.(1970). A note on infinitely divisible distributions. *Theory Prob. Appl.* **15** 319-324.
- [LR80] Laha, R.G. and Rohatgi, V.K.(1980). Operator semi-stable probability measures on Hilbert space. *Bull. Austral. Math. Soc.* **22** 397-406. (Corrigenda, **22** 479-480.)
- [L37] Lévy, P.(1937). *Théorie de l'addition des variables aléatoires* Gauthier-Villars, Paris.
- [Ł81] Łuczak, A. (1981). Operator semi-stable probability measures on \mathbf{R}^N . *Colloq. Math.* **45** 287-300. (Corrigenda, **52** (1987) 167-169.)
- [Ł91] Łuczak, A. (1991). Operator-semi-stable operator Lévy's measures on finite dimensional vector spaces. *Probab. Theory Rel. Fields* **90** 317-340.
- [MN98] Maejima, M. and Naito, Y.(1998). Semi-selfdecomposable distributions and a new class of limit theorems. To appear in *Probab. Theory Rel. Fields*
- [MSW98] Maejima, M., Sato, K. and Watanabe, T. (1998). Operator semi-selfdecomposability, (C, Q) -decomposability and related nested subclasses. Preprint.

[SY85] Sato, K. and Yamazato, M. (1985). Completely operator-selfdecomposable distributions and operator-stable distributions. *Nagoya Math. J.* **97** 71-94.

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