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Wavelet Transforms Associated to the Analytic Continuation of the Holomorphic Discrete Series of a Semisimple Lie Group

by

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Abstract

Grossmann, Morlet, and Paul pointed out that the wavelet theory can be interpreted in the scheme of square-integrable representations of locally compact groups. In this paper we shall obtain a wavelet transform outside this scheme; wavelet transforms associated to *non* square-integrable representations. Let G be a real connected semisimple Lie group with finite center and $\tilde{G}$ the universal covering group of G. We suppose that G/K is of hermitian type. Then G has a non empty family of square-integrable representations, which is called the holomorphic discrete series, and $\tilde{G}$ has its analytic continuation, which includes non square-integrable representations of $\tilde{G}$. We shall construct a wavelet transform associated to this analytic continuation.

1. Introduction.

Before stating our goal we shall recall the definition of the holomorphic discrete series of a semisimple Lie group and its analytic continuation. For most of the following facts we refer to [**6**], [**7**], and [**8**].

Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$. We suppose that $\mathfrak{k}$ has a non empty center $\mathfrak{z} = \boldsymbol{R}Z$ and the eigenvalues of the adjoint action adZ of Z on $\mathfrak{p}^C$ are $\pm i$. Let $\tilde{G}$ be the simply connected group with Lie algebra $\mathfrak{g}$ and $\tilde{K}$ the analytic subgroup of $\tilde{G}$ with Lie algebra $\mathfrak{k}$. Then $\tilde{G}/\tilde{K}$ is a hermitian symmetric space. The set of pairs $\Lambda = (\Lambda_0, \lambda)$, where Λ_0 is a dominant weight of $[\,\mathfrak{k}, \mathfrak{k}\,]$ and λ a real number, parametrizes the set of equivalence classes of irreducible unitary representations of $\tilde{K}$: let (U_Λ, V_Λ) denote the irreducible unitary representation of $\tilde{K}$ of highest weight Λ. We define

$$\begin{aligned} H_0(\Lambda) = \{\phi \in C^\infty(\tilde{G}, V_\Lambda); \phi(gk) = U_\Lambda(k)^{-1}\phi(g) \quad (g \in \tilde{G}, k \in \tilde{K}), \\ X\phi = 0 \quad (X \in \mathfrak{p}^-), \quad \text{and} \quad \int_{\tilde{G}/Z(\tilde{G})} \|\phi(g)\|^2_{V_\Lambda} d\dot{g} = \|\phi\|^2_{H_0(\Lambda)} < \infty\}, \end{aligned}$$

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where $\mathfrak{p}^{\pm}$ is the $\pm i$-eigensubspace of adZ on $\mathfrak{p}^C$, $Z(\tilde{G})$ the center of $\tilde{G}$, and $d\dot{g}$ a G-invariant measure on $\tilde{G}/\tilde{K}$. Then $H_0(\Lambda) \neq \{0\}$ if and only if Λ satisfies the Harish-Chandra condition (see (1) below). $\tilde{G}$ acts by left translations in $H_0(\Lambda)$: $T_\Lambda(x)\phi(g) = \phi(x^{-1}g)$ $(x, g \in \tilde{G})$, and $(T_\Lambda, H_0(\Lambda))$ is an irreducible unitary representation of $\tilde{G}$, which is said to belong to the holomorphic discrete series of $\tilde{G}$.

Let G_C be the simply connected group with Lie algebra $\mathfrak{g}^C$ and G, K, K_C, and $P_{\pm}$ the analytic subgroups of G_C with Lie algebra $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{k}^C$, and $\mathfrak{p}^{\pm}$ respectively. Then $\tilde{G}/\tilde{K}$ is isomorphic to G/K and $G \subset P_+K_CP_-$. For $g \in G$ we define $k(g) \in K_C$ by $g \in P_+k(g)P_-$ and we naturally lift the map $k : G \to K_C$ to $k : \tilde{G} \to \tilde{K}_C$, the universal covering group of K_C. We define

$$\psi_\Lambda(x) = \tilde{U}_\Lambda(k(x))^{-1}v_\Lambda \quad (x \in \tilde{G}),$$

where $\tilde{U}_\Lambda$ is the holomorphic representation of $\tilde{K}_C$ such that $\tilde{U}_\Lambda|_{\tilde{K}} = U_\Lambda$ and $v_\Lambda \in V_\Lambda$ is a highest weight vector of U_Λ. We suppose that ψ_Λ is of positive type (see [**6**, §1.4]) and we define $\mathcal{L}$ as the linear span of left translations of ψ_Λ. We here introduce a non zero Hilbert space $H(\Lambda)$ as the completion of $\mathcal{L}$ with the norm:

$$\|\sum_i c_iT_\Lambda(g_i)\psi_\Lambda\|^2_{H(\Lambda)} = \sum_i c_i\bar{c}_j\psi_\Lambda(g_i^{-1}g_j).$$

$(T_\Lambda, H(\Lambda))$ is an irreducible unitary representation of $\tilde{G}$. We call the family of $(T_\Lambda, H(\lambda))$ the analytic continuation of the holomorphic discrete series $(T_\Lambda, H_0(\lambda))$, in the sense that, if Λ satisfies the Harich-Chandra condition, then ψ_Λ is of positive type, $H_0(\Lambda) = H(\Lambda)$ with $\|\phi\|^2_{H_0(\Lambda)} = c_\Lambda\|\phi\|^2_{H(\Lambda)}$ where $c_\Lambda = \|\psi_\Lambda\|^2_{H(\Lambda)}$, and the characters of the analytically continued representations are holomorphic functions of Λ.

Now we shall state our goal. We assume that $\Lambda_0 = 0$ and we denote $\Lambda = (0, \lambda)$ by λ for simplicity. Let (HC) be the set of $\lambda \in \boldsymbol{R}$ satisfying the Harish-Chandra condition and let (P) be the one of $\lambda \in \boldsymbol{R}$ for which ψ_λ is of positive type. These two sets are explicitly described as follows.

$$(HC) = \{\lambda; \lambda \ < \ - < \rho, H >\}, \tag{1}$$

$$(P) = \{\lambda; \lambda < -\frac{(r-1)p}{2}\} \ \cup \ \{-\frac{(e-1)p}{2}; 1 \leq e \leq r\} \tag{2}$$

(see [**6**, §4.6], [**7**, Theorem 5.10], and §2 for the definition of ρ, H, r, p). We denote by (PC) the continuous part of (P): $(PC) = \{\lambda; \lambda < -(r-1)p/2\}$ and we note that

$$(HC) \subset (PC) \subset (P).$$

If $\lambda \in (HC)$, the matrix coefficients of $(T_\lambda, H(\lambda))$ are square-integrable over $\tilde{G}/Z(\tilde{G})$, so $(T_\lambda, H(\lambda))$ is a square-integrable representation of $\tilde{G}$ *mod* $Z(\tilde{G})$. More precisely,

$$\int_{\tilde{G}/Z(\tilde{G})} | < \phi, T_\lambda(g)\psi >_{H(\lambda)} |^2 d\dot{g} = c_\lambda\|\phi\|^2_{H(\lambda)}\|\psi\|^2_{H(\lambda)} \tag{3}$$

for all $\phi, \psi \in H(\lambda)$ and therefore,

$$\phi(x) = c_\lambda^{-1} \|\psi\|_{H(\lambda)}^{-2} \int_{\tilde{G}/Z(\tilde{G})} < \phi, T_\lambda(g)\psi >_{H(\lambda)} T_\lambda(g)\psi(x) d\dot{g} \tag{4}$$

(cf. [**3**, §1]). As pointed out by Grossmann, Morlet, and Paul [**2**], these formulas give a group theoretical interpretation of continuous wavelet transforms of $H(\lambda)$: in the wavelet theory ψ in (4) is called a mother wavelet and $< \phi, T_\lambda(g)\psi >_{H(\lambda)}$ the wavelet transform of ϕ. Here we rewrite (3) by decomposing $\tilde{G}$ as $\tilde{G} = B\tilde{K}$, where B is a Borel subgroup of G. Since $Z(\tilde{G}) \subset \tilde{K}$ and $T_\lambda(k)\psi_\lambda = U_\lambda(k)\psi_\lambda$ for all $k \in \tilde{K}$, letting $\psi = \psi_\lambda$ in (3), we have for all $\phi \in H(\lambda)$

$$\int_B | < \phi, T_\lambda(b)\psi_\lambda >_{H(\lambda)} |^2 db = c_\lambda \|\phi\|_{H(\lambda)}^2 \|\psi_\lambda\|_{H(\lambda)}^2, \tag{5}$$

where db is a left invariant measure on B. Clearly, (5) is equivalent to (3), so it can be regarded as the wavelet transform associated to the holomorphic discrete series $(T_\lambda, H(\lambda))$ $(\lambda \in (HC))$. We note that this formula does not make sense if $\lambda \notin (HC)$, because the both sides diverge. When $\lambda \in (P) \cap (HC)^c$ corresponds to the limit of the holomorphic discrete series of $\tilde{G}$, we obtained a generalization of (5) in the previous papers [**3**] and [**4**].

Our goal of this paper is to generalize the integral formula (5) for all $\lambda \in (PC)$ and to obtain a wavelet transform associated to non square-integrable representations of $\tilde{G}$ *mod* $Z(\tilde{G})$. Since the both sides of (5) diverge for $\lambda \in (PC)$, to generalize the formula for $\lambda \in (PC)$ we need to replace ψ_λ with a suitable function ψ on $\tilde{G}$, which is said to be admissible. The program to carry out this process is as follows. First we shall consider a combination of the Fourier transforms $\phi \to \hat{\phi} \to (\hat{\phi})^\sim$ of $H(\lambda)$ (see 3.2 and 3.3) and we identify $H(\lambda)$ with $\tilde{H}(\lambda) = L^2(\Omega^* \times W, J_\lambda(\xi)d\xi d\eta)$. Basic properties about these transforms are summalized in §4. Especially, the Fourier transform $f \to \tilde{f}$ separates a convolution $f \tilde{*} \psi$ into the multiplication of each Fourier transforms (see Lemma 4.1) and an L^2-norm of $f \tilde{*} \psi$ coinsides with an L^2-norm of the matrix coefficient of $\hat{T}_\lambda$, where $\hat{T}_\lambda(g)\hat{\phi}$ $= (T_\lambda(g)\phi)^\wedge$ (see Lemma 4.2). Puting together these properties, we can extend (5) for all $\lambda \in (PC)$: the desired function ψ are defined in Definition 5.1 and the main theorem is given by Theorem 5.2. In §6 we shall take an abelian subgroup T_0 of G_C outside of $\tilde{G}$ and consider a dilation on W induced from T_0. This dilation yields a large class of ψ satisfying a generalized formula (see Theorem 6.2).

2. Notation.

We keep the notations in §1 and we also refer [**5**], [**6**], [**7**], and [**8**] for the contents in this section.

Let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{k}$. Then $\mathfrak{h}^C$ is a Cartan subalgebra of $\mathfrak{g}^C$, so the set Δ of roots for $(\mathfrak{g}^C, \mathfrak{h}^C)$ is defined. Let Δ^+ be the set of positive roots in Δ, $\Delta_\mathfrak{p}^+$ the set of noncompact roots in Δ^+, and Ψ the maximal set of orthogonal roots in $\Delta_\mathfrak{p}^+$ chosen as in [**6**, §2.1]. For each $\gamma \in \Delta$ let $(\mathfrak{g}^C)^\gamma$ denote the root space of γ and H_γ the unique element in $i\mathfrak{h} \cap [(\mathfrak{g}^C)^\gamma, (\mathfrak{g}^C)^{-\gamma}]$ such that $\gamma(H_\gamma) = 2$. If γ is the highest root in Ψ, we put $H = H_\gamma$. For each $\gamma \in \Delta_\mathfrak{p}^+$ let E_γ be an element in $(\mathfrak{g}^C)^\gamma$ such that $[E_\gamma, \bar{E}_\gamma] = H_\gamma$. We put $E_{-\gamma} = \bar{E}_\gamma$ and $X_\gamma = E_\gamma + E_{-\gamma}$. Then $\mathfrak{a} = \sum_{\gamma \in \Psi} \boldsymbol{R} X_\gamma$ is a maximal abelian subalgebra of $\mathfrak{p}$. Let r be the real rank of $\mathfrak{g}$, that is, $r = \dim \mathfrak{a}$. Let Σ be the set of restricted roots for $(\mathfrak{g}, \mathfrak{a})$,

Σ^+ the set of positive roots in Σ, and $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ the restrictions to $\mathfrak{a}$ of the Cayley transforms of Ψ. Let $\rho = \sum_{\alpha\in\Sigma^+} \alpha/2$ and $\mathfrak{n} = \sum_{\alpha\in\Sigma^+} \mathfrak{g}^\alpha$ where $\mathfrak{g}^\alpha$ is the root space of α. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$ and $p = \dim \mathfrak{g}^{(\alpha_i+\alpha_j)/2}$ is independent of i, j. Let $\mathfrak{b} = \mathfrak{a} + \mathfrak{n}$ and $\mathfrak{b}^C = \mathfrak{b}^+ + \mathfrak{b}^-$ where $\mathfrak{b}^\pm = \mathfrak{b}^C \cap (\mathfrak{k}^C + \mathfrak{p}^\pm)$. We define $J : \mathfrak{b} \to \mathfrak{b}$ such as $\mathfrak{b}^\pm = \{X \mp iJX; X \in \mathfrak{b}\}$ and we put $s = \sum_{\gamma\in\Psi} U_\gamma = \sum_{\gamma\in\Psi} i(H_\gamma - E_\gamma + E_{-\gamma})/2$. Then Js has eigenvalues $0, \pm 1/2, \pm 1$, and it yields the Js-eigenspace decompositions of $\mathfrak{g}$ and $\mathfrak{b}$:

$$\mathfrak{g} = \mathfrak{g}(-1) + \mathfrak{g}(-\frac{1}{2}) + \mathfrak{g}(0) + \mathfrak{g}(\frac{1}{2}) + \mathfrak{g}(1) \quad \text{and} \quad \mathfrak{b} = \mathcal{H}_0 + \mathcal{H}_{1/2} + \mathcal{H}_1,$$

where $\mathcal{H}_i \subset \mathfrak{g}(i)$ $(i = 0, 1/2, 1)$. We note that $\dim \mathcal{H}_0 = \dim \mathcal{H}_1 = r + r(r-1)p/2 > 0$. We put $\tau(X) = (X - iJX)/2$ for $X \in \mathcal{H}_{1/2}$ and we let $\mathcal{H}_{1/2}^+ = \mathfrak{b}^+ \cap \mathcal{H}_{1/2}^C$. Then τ is a complex isomorphism of $\mathcal{H}_{1/2}$ to $\mathcal{H}_{1/2}^+$ (see [**6**, p.17]).

Let B, A, N, and H_i $(i = 0, 1/2, 1)$ be the analytic subgroups of G with Lie algebras $\mathfrak{b}$, $\mathfrak{a}$, $\mathfrak{n}$, and $\mathcal{H}_i$ respectively. Then $B = H_0 \cdot H_{1/2}H_1$ (semidirect), solvable and isomorphic to $\tilde{G}/\tilde{K} \cong G/K$. We put $G(0) = \{g \in G; g \cdot Js = s\}$, where $g\cdot$ implies the adjoint action of g, and we shall define an open convex cone Ω in $\mathcal{H}_1$ and its dual cone Ω^* as follows.

$$\Omega = G(0) \cdot s \quad \text{and} \quad \Omega^* = \{\xi \in \mathcal{H}_1^*; <\xi, X> > 0 \quad \text{for all } X \in \bar{\Omega} - \{0\}\}.$$

The map $h \to h \cdot s$ is a diffeomorphism of H_0 onto Ω and, if we put $\xi_0 = \sum_{\gamma\in\Psi} U_\gamma^*$, the map $h \to h \cdot \xi_0$ is one of H_0 onto Ω^*. For $\xi \in \Omega^*$ we define $h(\xi) \in H_0$ by $\xi = h(\xi) \cdot \xi_0$. Let $\tilde{Q} : \mathcal{H}_{1/2}^+ \times \mathcal{H}_{1/2}^+ \to \mathcal{H}_1^C$ denote the hermitian form on $\mathcal{H}_{1/2}^+$ defined by $\tilde{Q}(u, v) = i[u, \bar{v}]/2$. We also define the hermitian form Q on $\mathcal{H}_{1/2}$ by $Q(u, v) = \tilde{Q}(\tau(u), \tau(v))$ for $u, v \in \mathcal{H}_{1/2}$. Then $\tilde{G}/\tilde{K}$ is isomorphic to the Siegel domain:

$$D(\Omega, \tilde{Q}) = \{(z, \tilde{u}) \in \mathcal{H}_1^C \times \mathcal{H}_{1/2}^+; \Im z - Q(\tilde{u}, \tilde{u}) \in \Omega\}.$$

Actually, if we define $\alpha : G \to \mathcal{H}_1^C \times \mathcal{H}_{1/2}^+$ by $\alpha(g) = c \cdot \zeta(c^{-1}g)$, where c is the Cayley transform and $\zeta(g)$ is the P_+-component of $g \in P_+K_CP_-$, then the map α induces a G-invariant isomorphism: $\alpha : \tilde{G}/\tilde{K} \to D(\Omega, \tilde{Q})$ (see [**6**, p.17]).

Let $d\tilde{u}$, du, dx, dt, and $d\xi$ be Lebesgue measures on $\mathcal{H}_{1/2}^+$, $\mathcal{H}_{1/2}$, $\mathcal{H}_1$, Ω, and Ω^* respectively, and dh a left-invariant Haar measure on H_0. Then $db = dhdxdu$ is a left invariant measure on B: for a compactly supported function F on B

$$\int_B F(b)db = \int_{\mathcal{H}_0\times\mathcal{H}_{1/2}\times H_1} F(h \exp x \exp u) dhdxdu. \tag{6}$$

3. Fourier transform of $H(\lambda)$.

We retain the references [**5**] and [**6**] for the contents in this section and we suppose that $\lambda \in (PC)$ for simplicity (see Remark 3.3 below). In what follows we shall give a characterization of $H(\lambda)$ as an L^2-space $\tilde{H}(\lambda)$ on $\Omega^* \times W$, where W is a real form of $\mathcal{H}_{1/2}$. This follows from a combination of the identifications $\alpha : \tilde{G}/\tilde{K} \to D(\Omega, \tilde{Q})$, $\tau : \mathcal{H}_{1/2} \to \mathcal{H}_{1/2}^+$, and Fourier transforms $\hat{f}$ and $\tilde{F}$ on $\mathcal{H}_1$ and W respectively.

3.1. Let c be the element in G_C that induces the Cayley transform in $\tilde{G}$ (see [**6**, §2]). Since $c^{-1}G \subset P_+K^C P_-$, it is easy to see that $k(c^{-1}g)$ is well-defined for $g \in \tilde{G}$ and let

$$\phi_\lambda(g) = U_\lambda(k(c^{-1}))^{-1}U_\lambda(k(c^{-1}g)) \quad (g \in \tilde{G}).$$

ϕ_λ is the character of $G(0)$ such that $\phi_\lambda(\exp tX_\gamma) = e^{t\lambda}$ for $\gamma \in \Psi$ and $\phi_\lambda \equiv 1$ on $H_{1/2}H_1$. Here for each function $f(g\tilde{K})$ on $\tilde{G}/\tilde{K}$ we define a function on $\tilde{G}$ as

$$P_\lambda f(g) = \phi_\lambda(g)^{-1} f(g\tilde{K}).$$

Then, by considering $(P_\lambda^{-1}\phi)(\alpha^{-1}(p))$ for $\phi \in H(\lambda)$ and $p \in D(\Omega, \tilde{Q})$, we can regard $(T_\lambda, H(\lambda))$ as a representation realized on $D(\Omega, \tilde{Q})$, which we denote by the same symbol. Especially, for a function $f(p)$ on $D(\Omega, \tilde{Q})$ G acts by the left translation:

$$T_\lambda(g)f(p) = \phi_\lambda(g)f(g^{-1}\cdot p) \quad (p \in D(\Omega, \tilde{Q}), g \in \tilde{G}).$$

3.2. Let $f(z, \tilde{u})$ be a holomorphic function on $D(\Omega, \tilde{Q})$. If $f(z, \tilde{u})$ is of Schwartz class with respect to $\Re z$ for a fixed $\Im z$, the Fourier transform of $f(z, \tilde{u})$ is defined by

$$\hat{f}(\xi, \tilde{u}) = \int_{\mathcal{H}_1} f(z, \tilde{u}) e^{-2\pi i <z,\xi>} dx \quad (\xi \in \mathcal{H}_1^*).$$

Here the integral is independent of $\Im z$. Conversely, if a function $\phi(\xi, \tilde{u})$ on $\Omega^* \times \mathcal{H}_{1/2}^+$ is of Schwartz class with respect to ξ for a fixed $\tilde{u}$, the inverse Fourier transform of ϕ is given by

$$\check{\phi}(z, \tilde{u}) = \int_{\Omega^*} \phi(\xi, \tilde{u}) e^{2\pi i <\xi,z>} d\xi \quad ((z, \tilde{u}) \in D(\Omega, \tilde{Q})).$$

We here introduce the Hilbert space $\hat{H}(\lambda)$ as the space of measurable functions $\phi(\xi, \tilde{u})$ on $\Omega^* \times \mathcal{H}_{1/2}^+$ satisfying

$$\begin{aligned} &(i) \qquad \phi(\xi, \tilde{u}) \text{ is holomorphic in } \tilde{u} \text{ for almost all } \xi \in \Omega^*, \\ &(ii) \qquad \int_{\Omega^* \times \mathcal{H}_{1/2}^+} |\phi(\xi, \tilde{u})|^2 e^{-4\pi \tilde{Q}_\xi(\tilde{u}, \tilde{u})} J_\lambda(\xi) d\xi d\tilde{u} = \|\phi\|^2_{\hat{H}(\lambda)} < \infty, \end{aligned}$$

where $\tilde{Q}_\xi(\tilde{u}, \tilde{u}) =< \xi, \tilde{Q}(\tilde{u}, \tilde{u}) >$ and $J_\lambda(\xi) = \phi_\lambda(h)^{-2}(\det_{\mathcal{H}_1} h)^{-1}$ $(\det_{\mathcal{H}_{1/2}} h)^{-1}$ for $\xi = h \cdot \xi_0 \in \Omega^*$. If we define the representation $(\hat{T}_\lambda, \hat{H}(\lambda))$ of $\tilde{G}$ by

$$\hat{T}_\lambda(g)\phi = (T_\lambda(g)\check{\phi})^\wedge,$$

then it follows from [**6**, §4.6] that

Theorem 3.1. *If $\lambda \in (PC)$, then the map $\phi \to \hat{\phi}$ gives an isomorphism of $H(\lambda)$ onto $\hat{H}(\lambda)$ with $\|\phi\|^2_{H(\lambda)} = d_\lambda \|\hat{\phi}\|^2_{\hat{H}(\lambda)}$, where $d_\lambda = \Pi_{i=1}^r \frac{1}{2}(2\pi)^{\frac{1}{2}(\lambda+(i-1)p)}\Gamma(-\frac{1}{2}(\lambda + (i-1)p))$. In particular, $(T_\lambda, H(\lambda))$ is unitary equivalent to $(\hat{T}_\lambda, \hat{H}(\lambda))$.*

3.3. Let $\hat{H}(\xi, Q)$ $(\xi \in \Omega^*)$ denote the space of holomorphic functions F on $\mathcal{H}_{1/2}$ such that

$$\|F\|_\xi^2 = \int_{\mathcal{H}_{1/2}} |F(u)|^2 e^{-4\pi Q_\xi(u,u)} du < \infty,$$

where $Q_\xi(u,v) =< \xi, Q(u,v) >$. Let W be a real form of $\mathcal{H}_{1/2}$ such that $Q(W,W) \subset \mathcal{H}_1$ and θ the conjugation of $\mathcal{H}_{1/2}$ with respect to W. Then it follows from [**5**, Theorem 2.24] that $\hat{H}(\xi, Q) \cong L^2(W)$ for a fixed $\xi \in \Omega^*$. More precisely, if we define the quadratic form $P(u,u)$ on $\mathcal{H}_{1/2}$ by $P(u,u) = Q(u, \theta(u))$ and put $P_\xi(u,u) =< \xi, P(u,u) >$, the isometry $F \to \tilde{F}$ of $\hat{H}(\xi, Q)$ onto $L^2(W)$ is given by

$$\tilde{F}(\eta) = c_0^{1/2} e^{4\pi Q_\xi(\eta,\eta)} \int_W F(u) e^{-2\pi P_\xi(u,u)} e^{8\pi i Q_\xi(u,\eta)} dw, \tag{7}$$

where $du = dw dv_\xi$ if $u = w + iv$ $(w, v \in W)$ and dv_ξ is the product measure with respect to orthogonal coordinates for $Q_\xi(u,v)$. Moreover, c_0 is given by (12) below and the integral is independent of v.

For each function $\phi \in \hat{H}(\lambda)$ we define a function ϕ_τ on $\Omega^* \times \mathcal{H}_{1/2}$ as

$$\phi_\tau(\xi, u) = \phi(\xi, \tau(u)).$$

Then $\phi_\tau(\xi, u) \in \hat{H}(\xi, Q)$ and the isometry of $\hat{H}(\xi, Q)$ onto $L^2(W)$ yields that

$$\|\phi\|^2_{\hat{H}(\lambda)} = \int_{\Omega^*} \|\phi_\tau(\xi, \cdot)\|_\xi^2 J_\lambda(\xi) d\xi = \int_{\Omega^* \times W} |\tilde{\phi}_\tau(\xi, \eta)|^2 J_\lambda(\xi) d\xi d\eta = \|\tilde{\phi}_\tau\|^2_{\tilde{H}(\lambda)}, \tag{8}$$

where

$$\tilde{H}(\lambda) = L^2(\Omega^* \times W, J_\lambda(\xi) d\xi d\eta).$$

Therefore, if we define the representation $(\tilde{T}_\lambda, \tilde{H}(\lambda))$ of $\tilde{G}$ by

$$\tilde{T}_\lambda(g)\tilde{\phi}_\tau = (\hat{T}_\lambda(g)\phi)_\tau^\sim,$$

we can obtain the following.

Theorem 3.2. *If $\lambda \in (PC)$, then the map $\phi \to \tilde{\phi}_\tau$ gives an isomorphism of $\hat{H}(\lambda)$ onto $\tilde{H}(\lambda)$ with $\|\phi\|^2_{\hat{H}(\lambda)} = \|\tilde{\phi}_\tau\|^2_{\tilde{H}(\lambda)}$. In particular, $(T_\lambda, H(\lambda))$ is unitary equivalent to $(\tilde{T}_\lambda$, $\tilde{H}(\lambda))$.*

3.4. Let $\mathcal{S}(W)$ be the Schwartz space on W. We introduce $\tilde{\mathcal{S}}(\lambda)$ as a subspace of $\tilde{H}(\lambda)$ consisting of functions $\phi \in \tilde{H}(\lambda)$ such that $\phi(\xi, \eta)$ belongs to $\mathcal{S}(W)$ for each fixed $\xi \in \Omega^*$. We define $\tilde{\mathcal{S}}'(\lambda)$ as the dual space of $\tilde{\mathcal{S}}(\lambda)$ with respect to the inner product of $\tilde{H}(\lambda)$. Furthermore, we define $\hat{\mathcal{S}}(\lambda)$ (resp. $\hat{\mathcal{S}}'(\lambda)$) by $(\hat{\mathcal{S}}(\lambda))_\tau^\sim = \tilde{\mathcal{S}}(\lambda)$ (resp. $(\hat{\mathcal{S}}'(\lambda))_\tau^\sim = \tilde{\mathcal{S}}'(\lambda)$).

Remark 3.3 We can similarly treat the case of $\lambda \in (P) \cap (PC)^c$: $\lambda = -(e-1)p/2$ $(1 \leq e \leq r)$. In this case $\Omega^* \times W$ is replaced with $O_e^* \times W_e$, where O_e^* is a $G(0)$-orbit of $\xi_e = \sum_{i=1}^{e-1} U_i^*$ and W_e is the kernel of $\tilde{Q}_{\xi_e}$. Especially, O_e^* is contained in the boundary of Ω^* and the measure $J_\lambda(\xi) d\xi$ on Ω^* is replaced with a positive measure $d\mu_e^*$ on O_e^*.

4. Key lemmas.

We recall that each $b \in B$ acts on $D(\Omega, \tilde{Q})$ as an affine transformation: if $b = h \exp u \exp x \in B = H_0 H_{1/2} H_1$ and $p = (z, \tilde{u}) \in D(\Omega, \tilde{Q})$, then $b \cdot p$ is given by

$$(h \cdot x + h \cdot z + 2i\tilde{Q}(h \cdot \tilde{u}, h \cdot \tau(u)) + i\tilde{Q}(h \cdot \tau(u), h \cdot \tau(u)), h \cdot \tilde{u} + h \cdot \tau(u))$$

(cf. [**8**, §2]). Thereby, according to the isomorphism in Theorem 3.1, we have for $\phi \in \hat{H}(\lambda)$

$$\begin{aligned} \hat{T}_\lambda(h)\phi(\xi, \tilde{u}) &= \phi_\lambda(h)(\det_{\mathcal{H}_1} h)\phi(h^{-1}\xi, h^{-1}\tilde{u}), \\ \hat{T}_\lambda(\exp u)\phi(\xi, \tilde{u}) &= e^{4\pi\tilde{Q}_\xi(\tilde{u}, \tau(u)) - 2\pi\tilde{Q}_\xi(\tau(u), \tau(u))}\phi(\xi, \tilde{u} - \tau(u)), \\ \hat{T}_\lambda(\exp x)\phi(\xi, \tilde{u}) &= e^{-2\pi i <\xi, x>}\phi(\xi, \tilde{u}). \end{aligned} \tag{9}$$

Now we suppose that $\lambda \in (PC)$ and $f, \psi \in \hat{\mathcal{S}}(\lambda)$ (see 3.4). We define a convolution on $\Omega^* \times \mathcal{H}_{1/2}$ as follows.

$$f \tilde{*} \psi(\xi, u) = \int_{\mathcal{H}_{1/2}} f_\tau(\xi, u')\bar{\psi}_\tau(\xi, u' - u)e^{-4\pi\bar{Q}_\xi(u', u' - u)}du'. \tag{10}$$

Lemma 4.1. *For $f, \psi \in \hat{\mathcal{S}}(\lambda)$*

$$[f \tilde{*} \psi(\xi, \cdot)]^\sim(\eta) = c_0^{1/2}\tilde{f}_\tau(\xi, \eta)\tilde{\bar{\psi}}_\tau(\xi, \eta)e^{4\pi Q_\xi(\eta, \eta)},$$

where c_0 is independent of f, ψ.

Proof. It follows from (7) and (10) that $[f \tilde{*} \psi(\xi, \cdot)]^\sim(\eta)$ is eqaul to

$$c_0^{1/2}e^{4\pi Q_\xi(\eta, \eta)} \int_W \int_{\mathcal{H}_{1/2}} f_\tau(\xi, u')\bar{\psi}_\tau(\xi, u' - u)e^{-4\pi\bar{Q}_\xi(u', u' - u)}du' e^{-2\pi P_\xi(u, u)}e^{8\pi i Q_\xi(u, \eta)}dw. \tag{11}$$

Let $u' = w' + iv'$ and $u = w$. We change the variable w to $-w + w'$ and put $\tilde{u}' = w' + iv'$ and $\tilde{u} = w + iv'$ respectively. Then we see that

$$\begin{aligned} Q_\xi(u', u' - u) &= Q_\xi(\tilde{u}', \tilde{u}) = Q_\xi(w', w) + Q_\xi(v', v') - iQ_\xi(w', v') + iQ_\xi(v', w), \\ Q_\xi(u, \eta) &= Q_\xi(\tilde{u}' - \tilde{u}, \eta) = -\bar{Q}_\xi(\tilde{u}, \eta) + Q_\xi(\tilde{u}', \eta) - 2iQ_\xi(v', \eta), \end{aligned}$$

and moreover, $P_\xi(u, u) = P_\xi(\tilde{u}' - \tilde{u}, \tilde{u}' - \tilde{u})$ is given by

$$\bar{P}_\xi(\tilde{u}, \tilde{u}) + P_\xi(\tilde{u}', \tilde{u}') + 2(-Q_\xi(v', v') - Q_\xi(w', w) + iQ_\xi(w, v') - iQ_\xi(v', w')).$$

By substituting these relations with $-4\pi\bar{Q}_\xi(u', u' - u) - 2\pi P_\xi(u, u) + 8\pi i Q_\xi(u, \eta)$, we can deduce that the sum of power of the exponents in (11) is equal to

$$-2\pi\bar{P}_\xi(\tilde{u}, \tilde{u}) - 8\pi i\bar{Q}_\xi(\tilde{u}, \eta) - 2\pi P_\xi(\tilde{u}', \tilde{u}') + 8\pi i Q_\xi(\tilde{u}', \eta) - 8\pi Q_\xi(v', v') + 16\pi Q_\xi(v', \eta).$$

We here recall that $du' = d\tilde{u}' = dw'dv'_\xi$ and we apply the notation of the Fourier transform (7) to the integrals with respect to dw and dw' respectively. Thereby, we see that (11) is equal to

$$c_0^{-1/2}\tilde{f}_\tau(\xi,\eta)\tilde{\bar{\psi}}_\tau(\xi,\eta)e^{-4\pi Q_\xi(\eta,\eta)}\int_W e^{-8\pi Q_\xi(v',v')}e^{16\pi Q_\xi(v',\eta)}dv'_\xi.$$

Then the desired result follows from the following integral formula:

$$\int_W e^{-8\pi Q_\xi(v',v')}e^{16\pi Q_\xi(v',\eta)}dv'_\xi = c_0 e^{8\pi Q_\xi(\eta,\eta)} \tag{12}$$

(see [**5**, p.345]). ∎

Lemma 4.2. *For $f,\psi \in \hat{\mathcal{S}}(\lambda)$ such that $f\tilde{*}\psi(\xi,u)J_\lambda(\xi)^{1/2} \in \hat{H}(\lambda)$*

$$\int_{\mathcal{H}_{1/2}\times\mathcal{H}_1} |< f, \hat{T}_\lambda(\exp x \exp u)\psi >_{\hat{H}(\lambda)}|^2 dx du = \int_{\Omega^*}\|f\tilde{*}\psi(\xi,\cdot)\|_\xi^2 J_\lambda^2(\xi)d\xi.$$

Proof. It follows from (9) that

$$< f, \hat{T}_\lambda(\exp x \exp u)\psi >_{\hat{H}(\lambda)} = \int_{\Omega^*} I(\xi,u)e^{-2\pi\tilde{Q}_\xi(\tau(u),\tau(u))}e^{2\pi i<\xi,x>}J_\lambda(\xi)d\xi,$$

where

$$\begin{aligned} I(\xi,u) &= \int_{\mathcal{H}^+_{1/2}} f(\xi,\tilde{u})\bar{\psi}(\xi,\tilde{u}-\tau(u))e^{4\pi\bar{\tilde{Q}}_\xi(\tilde{u},\tau(u))}e^{-4\pi\tilde{Q}_\xi(\tilde{u},\tilde{u})}d\tilde{u} \\ &= \int_{\mathcal{H}_{1/2}} f_\tau(\xi,u')\bar{\psi}_\tau(\xi,u'-u)e^{-4\pi\bar{Q}_\xi(u',u'-u)}du' \\ &= f\tilde{*}\psi(\xi,u). \end{aligned}$$

The lemma follows from the Plancherel formula for the Euclidean Fourier transform of $L^2(\mathcal{H}_1)$. ∎

Lemma 4.3. *Let $\psi \in \hat{\mathcal{S}}(\lambda)$ and $h \in H_0$. Then*

$$(\hat{T}_\lambda(h)\psi)^\sim_\tau(\xi,\eta) = \phi_\lambda(h)(\det_{\mathcal{H}_1}h)(\det_{\mathcal{H}_{1/2}}h)\tilde{\psi}_\tau(h^{-1}\xi,h^{-1}\eta).$$

Proof. It follows from (9) and (7) that $(\hat{T}_\lambda(h)\psi)^\sim_\tau(\xi,\eta)$ is given by

$$c_0^{1/2}e^{4\pi Q_\xi(\eta,\eta)}\phi_\lambda(h)(\det_{\mathcal{H}_1}h)\int_W \psi_\tau(h^{-1}\xi,h^{-1}w)e^{-2\pi P_\xi(w,w)}e^{8\pi i Q_\xi(w,\eta)}dw.$$

Since $Q_\xi(hw,h\eta) = Q_{h^{-1}\xi}(w,\eta)$ for all $w,\eta \in \mathcal{H}_{1/2}$ and $\xi \in \Omega^*$, the change of the variable w to hw yields the lemma. ∎

5. Dilation on Ω^*.

We retain the notations in the previous sections and the assumption that $\lambda \in (PC)$.

For $f \in \tilde{\mathcal{S}}(\lambda)$ and $\psi \in \tilde{\mathcal{S}}'(\lambda)$ we choose $F \in \hat{\mathcal{S}}(\lambda)$ and $\Psi \in \hat{\mathcal{S}}'(\lambda)$ such that $f = \tilde{F}_\tau$ and $\psi = \tilde{\Psi}_\tau$ respectively. We suppose that $\psi(\xi,\eta)e^{4\pi Q_\xi(\eta,\eta)}\ J_\lambda(\xi)$ belongs to $\tilde{\mathcal{S}}'(\lambda)$. Since $(F\tilde{*}\Psi)^\sim = c_0^{1/2} f\bar{\psi}\ e^{4\pi Q_\xi(\eta,\eta)} \in \tilde{\mathcal{S}}(\lambda)$, it follows from (6), (8), Lemmas 4.1, 4.2, and 4.3 that

$$\begin{aligned}
&\int_B |< f, \tilde{T}_\lambda(b)\psi >_{\tilde{H}(\lambda)}|^2 db \\
=\ & \int_{H_0\times\mathcal{H}_{1/2}\times\mathcal{H}_1} |< \hat{T}_\lambda(h^{-1})F, \hat{T}_\lambda(\exp x \exp u)\Psi >_{\hat{H}(\lambda)}|^2 dh dx du \\
=\ & c_0 \int |f(h\xi, h\eta)|^2 |\psi(\xi,\eta)|^2 e^{8\pi Q_\xi(\eta,\eta)} \phi_\lambda(h)^{-2} (\det_{\mathcal{H}_1} h)^{-2} (\det_{\mathcal{H}_{1/2}} h)^{-2} J_\lambda^2(\xi) dh d\xi d\eta,
\end{aligned}$$

where the last integral is taken over $H_0 \times \Omega^* \times W$. Then, changing the variables ξ and η to $h^{-1}\xi$ and $h^{-1}\eta$ respectively, and using the relation: $J_\lambda(h^{-1}\xi) = \phi_\lambda^2(h)\ (\det_{\mathcal{H}_1} h)\ (\det_{\mathcal{H}_{1/2}} h)\ J_\lambda(\xi)$ (see [**8**, p.33]), we see that the integral is equal to

$$c_0 \int |f(\xi,\eta)|^2 |\psi(h^{-1}\xi, h^{-1}\eta)|^2 e^{8\pi Q_\xi(\eta,\eta)} \phi_\lambda(h)^{-2} (\det_{\mathcal{H}_1} h)^{-1} (\det_{\mathcal{H}_{1/2}} h)^{-1} J_\lambda^2(h^{-1}\xi) dh d\xi d\eta$$

$$= c_0 \int_{H_0\times\Omega^*\times W} |f(\xi,\eta)|^2 |\psi(h^{-1}\xi, h^{-1}\eta)|^2 e^{8\pi Q_\xi(\eta,\eta)} J_\lambda(\xi) J_\lambda(h^{-1}\xi) d\xi d\eta dh. \quad (13)$$

Definition 5.1. *$\psi \in \tilde{\mathcal{S}}'(\lambda)$ is said to be B-admissible if*

$$\int_{H_0} |\psi(h^{-1}\xi, h^{-1}\eta)|^2 e^{8\pi Q_\xi(\eta,\eta)} J_\lambda(h^{-1}\xi) dh = c_\psi < \infty \quad (14)$$

for all $\xi \in \Omega^*$ and $\eta \in W$, where c_ψ is independent of ξ and η. Especially, $|\psi(\xi,\eta)|$ is of the form $|\psi(\xi,\eta)| = \Psi(\xi)e^{-4\pi Q_\xi(\eta,\eta)}$ and

$$c_\psi = \int_{H_0} |\Psi(h^{-1}\xi)|^2 J_\lambda(h^{-1}\xi) dh = \int_{\Omega^*} |\Psi(\xi)|^2 J_\lambda(\xi) (\det_{\mathcal{H}_1^*} h(\xi))^{-1} d\xi < \infty.$$

Let $\psi \in \tilde{\mathcal{S}}'(\lambda)$ be B-admissible. Then the integral (13) makes sense for $f \in \tilde{\mathcal{S}}(\lambda)$ and it coincides with

$$c_0 c_\psi \int_{\Omega^*\times W} |f(\xi,\eta)|^2 J_\lambda(\xi) d\xi d\eta = c_0 c_\psi \|f\|^2_{\tilde{H}(\lambda)}.$$

Therefore, we can obtain a generalization of (5):

Theorem 5.2. *Let $\lambda \in (PC)$ and let $\psi \in \tilde{\mathcal{S}}'(\lambda)$ be B-admissible. Then*

$$\int_B |< f, \tilde{T}_\lambda(b)\psi >_{\tilde{H}(\lambda)}|^2 db = c_0 c_\psi \|f\|^2_{\tilde{H}(\lambda)}.$$

for all $f \in \tilde{\mathcal{S}}(\lambda)$.

Remark 5.3. (1) Let $\psi \in \tilde{\mathcal{S}}'(\lambda)$ be B-admissible. Then ψ is in $\tilde{H}(\lambda)$ if and only if

$$\|\psi\|^2_{\tilde{H}(\lambda)} = \int_{\Omega^*} |\Psi(\xi)|^2 J_\lambda(\xi) \int_W e^{-8\pi Q_\xi(\eta,\eta)} d\eta d\xi = c_0 \int_{\Omega^*} |\Psi(\xi)|^2 J_\lambda(\xi)(\det Q_\xi)^{-1} d\xi < \infty.$$

(2) If $\lambda = -<\rho, H>$, then $\lambda \in (PC)$ (see §1) and $J_\lambda(\xi) \equiv 1$. The corresponding representation $(T_\lambda, H(\lambda))$ is called the limit of the holomorphic discrete series of $\tilde{G}$ and it can be realized as a reducible component of a unitary principal series representation $(\pi_\lambda, L^2(N))$ of $\tilde{G}$. Thereby, it is quite natural that the B-admissible condition in Definition 5.1 has the similar form of the admissible condition of a wavelet transform associated to the principal series representation (see [**6**, (9) in I] and cf. [**9**]).

6. Dilation on W.

In order to find a large class of ψ in $\tilde{\mathcal{S}}'(\lambda)$ satisfying a formula similar to Theorem 5.2 we shall consider a dilation on W induced by a subgroup T_0 of G_C.

We put $\mathfrak{t}_0 = \{X \in \mathfrak{h}; c(X) = X\}$, where c is the Cayley transform, and $T_0 = \exp i\mathfrak{t}_0 \subset G_C$. Since $T_0 P_+ K_C P_- \subset P_+ K_C P_-$, we can extend the map α (see 3.1) as $\alpha : T_0 G \to \mathcal{H}_1^C \times \mathcal{H}^+_{1/2}$ and then we can define the action of T_0 on $\alpha(T_0 G)$. Especially, if $(z, \tilde{u}) \in D(\Omega, \tilde{Q})$,

$$t \cdot (z, \tilde{u}) = (z, t \cdot \tilde{u}) \quad (t \in T_0),$$

where $t \cdot \tilde{u} = Ad(t)\tilde{u}$ for $\tilde{u} \in \mathcal{H}^+_{1/2}$. Thereby, T_0 acts on $\mathcal{H}_{1/2}$ as $\tau(t \cdot u) = t \cdot \tau(u)$ and then, on W. We note that on the Eucledian space $\boldsymbol{R}$ the dilation $f_t(x)$ of f $(t > 0)$ is defined by $t^{-1} f(t^{-1}x)$ and its Fourier transform is equal to $\hat{f}(t\lambda)$. So, we shall define the dilation on W as

$$\tilde{D}(t) f(\xi, \eta) = f(\xi, t \cdot \eta) \quad (t \in T_0).$$

Definition 6.1. $\psi \in \tilde{\mathcal{S}}'(\lambda)$ *is said to be* BT_0*-admissible if*

$$\int_{T_0} \int_{H_0} |\psi(h^{-1}\xi, h^{-1}t^{-1} \cdot \eta)|^2 e^{8\pi Q_\xi(t^{-1}\cdot\eta, t^{-1}\cdot\eta)} J_\lambda(h^{-1}\xi) dt dh = c_\psi < \infty, \tag{15}$$

where dt *is a Haar measure on* T_0 *and* c_ψ *is independent of* $\xi \in \Omega^*$ *and* $\eta \in W$.

For each BT_0-admissible $\psi \in \tilde{\mathcal{S}}'(\lambda)$, the same process used in §5 yields that

$$\begin{aligned} & \int_{T_0} \int_B | < f, \tilde{D}(t^{-1}) \tilde{T}_\lambda(b) \psi >_{\tilde{H}(\lambda)} |^2 dt db \\ = \ & c_0 \int_{H_0 \times \Omega^* \times W \times T_0} |f(\xi,\eta)|^2 |\psi(h^{-1}\xi, h^{-1}t^{-1} \cdot \eta)|^2 e^{8\pi Q_\xi(t^{-1}\cdot\eta, t^{-1}\cdot\eta)} J_\lambda(\xi) J_\lambda(h^{-1}\xi) d\xi d\eta dh dt \\ = \ & c_0 c_\psi \|f\|^2_{\tilde{H}(\lambda)}. \end{aligned}$$

for $f \in \tilde{\mathcal{S}}(\lambda)$. Therefore, we can obtain the following.

Theorem 6.2. *Let $\lambda \in (PC)$ and let $\psi \in \tilde{\mathcal{S}}'(\lambda)$ be BT_0-admissible. Then*

$$\int_{T_0}\int_B |< f, \tilde{D}(t^{-1})\tilde{T}_\lambda(b)\psi >_{\tilde{H}(\lambda)}|^2 dtdb = c_0 c_\psi \|f\|^2_{\tilde{H}(\lambda)}$$

for all $f \in \tilde{\mathcal{S}}(\lambda)$.

Example 6.3. Let us suppose that $G = SU(n,1)$. In this case the subalgebras $\mathcal{H}_0$, $\mathcal{H}_{1/2}$, and $\mathcal{H}_1$ in §2 consist of matrices being of the form:

$$H_{t_0} = \begin{pmatrix} & & t_0 \\ & & \\ t_0 & & \end{pmatrix}, \quad X_u = \begin{pmatrix} & u & \\ -u^* & & u^* \\ & u & \end{pmatrix}, \quad \text{and} \quad X_x = \begin{pmatrix} ix & & -ix \\ & & \\ ix & & -ix \end{pmatrix}$$

respectively, where $t_0, x \in \boldsymbol{R}$ and $u \in \boldsymbol{C}^{n-1}$. Moreover, Ω and Ω^* can be identified with $\boldsymbol{R}_+$ and $J_\lambda(\xi) = \xi^{-(\lambda+n)}$. Therefore, (HC) and (PC) are respectively given by $\lambda < -n$ and $\lambda < 0$ (see [**6**, Corollary 4.4.7 and Corollary 4.4.8] and [**7**, Lemma 4.1 and Corollary 4.2]). We choose a real form of $\mathcal{H}_{1/2}$ as $W = \{X_u; u \in \boldsymbol{R}^{n-1}\}$ and we parametrize each element $t \in T_0$ as follows.

$$t = \exp \begin{pmatrix} t_n & & & & & \\ & t_1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & t_{n-1} & \\ & & & & & t_n \end{pmatrix},$$

where $t_i \in \boldsymbol{R}$ and $2t_n = -\sum_{j=1}^{n-1} t_j$. Then $h\xi = e^{2t_0}\xi$ if $h = \exp H_{t_0}$ and $t \cdot \eta = (e^{t_n - t_i}\eta_i)$ if $\eta = (\eta_i)$ and t as above. We can rewrite the admissible condition in (14) as follows. Let $\mathcal{E}^m$ be the set of $\varepsilon = (\varepsilon_i) \in \boldsymbol{R}^m$ such that $\varepsilon_i = \pm 1$ $(1 \le i \le m)$, and for each $\varepsilon \in \mathcal{E}^m$ let $\boldsymbol{R}^m_+(\varepsilon) = \{x = (x_i) \in \boldsymbol{R}^m; \mathrm{sgn} x_i = \mathrm{sgn}\varepsilon_i (1 \le i \le m)\}$. Then $\psi \in \tilde{\mathcal{S}}'(\lambda)$ is BT_0-admissible if and only if

$$\int_{\boldsymbol{R}_+}\int_{\boldsymbol{R}^{n-1}_+(\varepsilon)} |\psi(x,y)|^2 x^{-(\lambda+n)} e^{8\pi x\|y\|^2} \frac{dx}{x}\frac{dy}{|y|} = c_\psi < \infty \tag{16}$$

for all $\varepsilon \in \mathcal{E}^{n-1}$, where dx and dy are Lebesgue measures on $\boldsymbol{R}_+$ and $\boldsymbol{R}^{n-1}$ respectively, $|y| = \prod_{i=1}^{n-1}|y_i|$ if $y = (y_i)$, and c_ψ is independent of ε.

Remark 6.4. When $G = SU(1,1)$ $(n = 1)$, $\mathcal{H}_{1/2} = W = \{0\}$ and the integral over $\boldsymbol{R}^{n-1}_+(\varepsilon)$ in (16) is not necessary. In this case, if $\lambda = -1$, the whole contents in the above example are same as the ones obtained by Grossmann and Morlet in the case of the one-dimensional affine group $ax + b$ (see [**1**]). They use a square-integrable representation of $ax + b$ that corresponds to the limit of holomorphic discrete series of $SU(1,1)$ in our scheme. If $-1 < \lambda < 0$, the wavelet transform in Theorem 6.2 associates to a representation that goes past the limit of holomorphic discrete series of the universal covering group of $SU(1,1)$ (see [**7**, §4]).

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