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Operator Semi-Selfdecomposability, (C, Q)-Decomposability and Related Nested Classes

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OPERATOR SEMI-SELFDECOMPOSABILITY, (C, Q)-DECOMPOSABILITY AND RELATED NESTED CLASSES

(Abbreviated Title : Operator semi-selfdecomposability)

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SUMMARY. There are two types of generalizations of selfdecomposability of probability measures on $\mathbf{R}^d$, $d \geq 1$: the c-decomposability and the C-decomposability of Loève and Bunge on the one hand, and the semi-selfdecomposability of Maejima-Naito on the other. The latter implies infinite divisibility but the former does not in general. For $d \geq 2$ introduction of operator (matrix) normalizations yields four kinds of classes of distributions on $\mathbf{R}^d$: $L_0(b, Q)$, $\widetilde{L}_0(b, Q)$, $L_0(C, Q)$, and $\widetilde{L}_0(C, Q)$, where $0 < b < 1$, Q is a $d \times d$ matrix with eigenvalues having positive real parts, and C is a closed multiplicative subsemigroup of $[0, 1]$ containing 0 and 1. Further, each of these classes generates the Urbanik-Sato type decreasing sequence of its subclasses. Characterizations and relations of these classes and subclasses are established. They complement and generalize results of Bunge, Jurek, Maejima-Naito, and Sato-Yamazato.

1. Introduction and preliminaries. In Maejima and Naito (1998), the notion of semi-selfdecomposable distributions on $\mathbf{R}^d$ was introduced as an extension of selfdecomposable distributions, and the class of such distributions

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and its decreasing subclasses containing semistable distributions were studied. The distributions in those classes were defined as limiting distributions of the normalized partial sums, with scalar normalization, of independent infinitesimal $\mathbf{R}^d$-valued random variables, where each limit is taken through a subsequence. In this paper, we enlarge those classes by allowing the linear operator normalizations in the normalized partial sums. As a result, we extend the notion of semi-selfdecomposability to that of operator semi-selfdecomposability.

On the other hand, Bunge (1997) extended the notion of selfdecomposability to another direction by introducing the class of C-decomposable distributions. This is also an extension of c-decomposability in the earlier work of Loève (1945), (also see Loève (1977), page 312). While semi-selfdecomposable distributions are infinitely divisible, the distributions in the class by Bunge (1997) are not necessarily infinitely divisible. He studied the class and its decreasing subclasses. Here we also extend his notion to the linear operator setting ((C, Q)-decomposability), and compare two generalizations of operator selfdecomposable distributions. Operator selfdecomposable distributions were discussed in Jurek (1983) and Sato and Yamazato (1985).

We start with the notation we are going to use in this paper. $\mathcal{P}(\mathbf{R}^d)$ is the class of all probability distributions on $\mathbf{R}^d$, $I(\mathbf{R}^d)$ is the class of all infinitely divisible distributions on $\mathbf{R}^d$, $M_+(\mathbf{R}^d)$ is the class of all $d \times d$ matrices all of whose eigenvalues have positive real parts, Q' is the transposed matrix of $Q \in M_+(\mathbf{R}^d)$, I is the identity matrix, $\widehat{\mu}(z)$, $z \in \mathbf{R}^d$, is the characteristic function of $\mu \in \mathcal{P}(\mathbf{R}^d)$, $\mu^{*t}, t \geq 0$, is the t-th convolution power of $\mu \in \mathbf{R}^d$, $\mathcal{L}(X)$ is the law of X, $\langle \ , \ \rangle$ is the Euclidean inner product in $\mathbf{R}^d$, and $|\cdot|$ is the norm induced by $\langle \ , \ \rangle$ in $\mathbf{R}^d$. For $b > 0$, $b^Q = \sum_{n=0}^{\infty}(n!)^{-1}(\log b)^n Q^n$. If $b = 0$ and $Q \in M_+(\mathbf{R}^d)$, then b^Q is defined to be 0. Convergence of probability distributions is always weak convergence. Whenever we write $H \subset \mathcal{P}(\mathbf{R}^d)$, we mean that $\emptyset \neq H \subset \mathcal{P}(\mathbf{R}^d)$.

Let $0 < b < 1$ and $Q \in M_+(\mathbf{R}^d)$.

DEFINITION 1.1. Let $H \subset \mathcal{P}(\mathbf{R}^d)$. A distribution $\mu \in \mathcal{P}(\mathbf{R}^d)$ is said to belong to the class $\widetilde{K}(H,b,Q)$ if there exist independent $\mathbf{R}^d$-valued random variables $\{X_j\}, a_n > 0, \uparrow \infty, c_n \in \mathbf{R}^d, k_n \in \mathbf{N}, \uparrow \infty$, such that

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = b, \tag{1.1}$$

$$\mathcal{L}(X_j) \in H, \tag{1.2}$$

$$\mathcal{L}\left(a_n^{-Q}\sum_{j=1}^{k_n} X_j + c_n\right) \to \mu. \tag{1.3}$$

If, furthermore, the infinitesimal condition:

$$\lim_{n\to\infty} \max_{1\le j\le k_n} P\left\{\left|a_n^{-Q}X_j\right| > \varepsilon\right\} = 0, \quad \forall\varepsilon > 0, \tag{1.4}$$

is satisfied, we say that $\mu \in \mathcal{P}(\mathbf{R}^d)$ belongs to the class $K(H,b,Q)$.

From the definition, we see that

$$K(H,b,Q) \subset \widetilde{K}(H,b,Q), \tag{1.5}$$

$$K(H,b,Q) \subset I(\mathbf{R}^d), \tag{1.6}$$

$$K(H_1,b,Q) \subset K(H_2,b,Q) \quad \text{if } H_1 \subset H_2, \tag{1.7}$$

and

$$\widetilde{K}(H_1,b,Q) \subset \widetilde{K}(H_2,b,Q) \quad \text{if } H_1 \subset H_2. \tag{1.8}$$

The class $K(H,b,I)$ was introduced in Maejima and Naito (1998), and the class $\widetilde{K}(\mathcal{P}(\mathbf{R}^d),c,I)$ coincides with the class L_c on page 312 of Loève (1977).

DEFINITION 1.2. A class $H \subset \mathcal{P}(\mathbf{R}^d)$ is said to be Q-completely closed if H is closed under convergence, convolution, and Q-type equivalence. Here H is said to be closed under Q-type equivalence if $\mathcal{L}(X) \in H$, $a > 0$, and $c \in \mathbf{R}^d$, then

$\mathcal{L}(a^{-Q}X + c) \in H$. If, furthermore, $H \subset I(\mathbf{R}^d)$ and H is closed under going to the t-th convolution power for any $t > 0$, we say that H is Q-completely closed in the strong sense.

Our basic results are the following two statements. All theorems and propositions in this section will be proved in the next section.

THEOREM 1.1.

(i) *Suppose that* $H \subset \mathcal{P}(\mathbf{R}^d)$ *is* Q*-completely closed. If* $\mu \in K(H, b, Q)$, *then there exists* $\rho \in H \cap I(\mathbf{R}^d)$ *such that*

$$\widehat{\mu}(z) = \widehat{\mu}(b^{Q'}z)\widehat{\rho}(z), \quad \forall z \in \mathbf{R}^d. \tag{1.9}$$

(ii) *If* H *is completely closed in the strong sense, then the converse of* (i) *is also true.*

(iii) *If* H *is completely closed in the strong sense, then so is* $K(H, b, Q)$.

THEOREM 1.2.

(i) *Suppose that* $H \subset \mathcal{P}(\mathbf{R}^d)$ *is* Q*-completely closed. Then* $\mu \in \widetilde{K}(H, b, Q)$ *if and only if there exists* $\rho \in H$ *such that* (1.9) *is satisfied.*

(ii) *If* H *is* Q*-completely closed, then so is* $\widetilde{K}(H, b, Q)$.

In Theorem 1.1, the distribution ρ in (1.9) is uniquely determined by μ, b, and Q, since $\widehat{\mu}(z) \neq 0$ by (1.6). But, in Theorem 1.2, the ρ is not always unique. The problem of uniqueness is discussed in Loève (1945).

In view of these theorems, it is natural to introduce the following definition.

DEFINITION 1.3. Let $0 \le c \le 1, Q \in M_+(\mathbf{R}^d)$, and $H \subset \mathcal{P}(\mathbf{R}^d)$. A probability distribution $\mu \in \mathcal{P}(\mathbf{R}^d)$ is said to be (c, Q, H)-decomposable if $\widehat{\mu}(z) = \widehat{\mu}(c^{Q'}z)\widehat{\rho}(z)$ with some $\rho \in H$. Given $\mu \in \mathcal{P}(\mathbf{R}^d), Q \in M_+(\mathbf{R}^d)$, and $H \subset \mathcal{P}(\mathbf{R}^d)$, we denote by $D_{Q,H}(\mu)$ the set of $c \in [0, 1]$ such that μ is (c, Q, H)-decomposable.

PROPOSITION 1.1. *Suppose that* H *is* Q*-completely closed. If* $D_{Q,H}(\mu) \neq \{1\}$, *then* $D_{Q,H}(\mu)$ *is a closed multiplicative subsemigroup of* $[0, 1]$ *containing* 0 *and* 1.

The following proposition is a direct consequence of Theorems 1.1 and 1.2.

PROPOSITION 1.2.

(i) $\mu \in \widetilde{K}(H, b, Q)$ *if and only if* μ *is* (b, Q, H)*-decomposable, provided that* H *is* Q*-completely closed.*

(ii) $K(H, b, Q) = \widetilde{K}(H, b, Q)$*, whenever* H *is* Q*-completely closed in the strong sense.*

(iii) *Suppose that* H *is* Q*-completely closed and* $H \cap I(\mathbf{R}^d)$ *is* Q*-completely closed in the strong sense. Then, the following three conditions are equivalent:* $\mu \in K(H, b, Q)$*;* $\mu \in K(H \cap I(\mathbf{R}^d), b, Q)$*;* μ *is* $(b, Q, H \cap I(\mathbf{R}^d))$*-decomposable.*

In case $d = 1$, $Q = I$, and $H = \mathcal{P}(\mathbf{R}^d)$, a decisive study of the semigroup $D_{Q,H}(\mu)$ was made by Ilinskii (1978), and some examples of similarly defined multiplicative subsemigroups of $[-1, 1]$ were given by Urbanik (1976). For general d and $H = \mathcal{P}(\mathbf{R}^d)$, the class $\{c^Q : c \in D_{Q,H}(\mu)\}$ is a subsemigroup of the Urbanik decomposability semigroup $\mathbf{D}(\mu)$ in Jurek and Mason (1993).

Following Bunge (1997), let $\mathfrak{C}$ be the collection of all closed multiplicative subsemigroup C of $[0, 1]$ such that $C \supsetneqq \{0, 1\}$. Define, for $C \in \mathfrak{C}$,

$$K(H, C, Q) = \bigcap_{b \in C \setminus \{0,1\}} K(H, b, Q)$$

and

$$\widetilde{K}(H, C, Q) = \bigcap_{b \in C \setminus \{0,1\}} \widetilde{K}(H, b, Q).$$

Note that, by Proposition 1.2 (i), $\widetilde{K}(H, C, Q)$ is the class of μ such that $C \subset D_{Q,H}(\mu)$, provided that H is Q-completely closed. The class $\widetilde{K}(H, C, I)$ coincides with the class $\mathcal{L}^C(H)$ introduced by Bunge (1997). (For the detail of its proof, see Proposition 2.4 in the next section.)

Distributions in $K(\mathcal{P}(\mathbf{R}^d), [0, 1], I)$ are usually called selfdecomposable. Urbanik (1972) introduced the notion of a slowly varying sequence of random variables and found a decreasing sequence of subclasses of the class of selfdecomposable distributions. Sato (1980) defined an operation to make a new subclass from a subclass and showed that iteration of his operation generates the sequence of Urbanik. Our operation $K(\cdot, b, Q)$ is a development from his operation. We now define four kinds of classes of distributions and the Urbanik-Sato type decreasing sequences of those subclasses.

DEFINITION 1.4. For $0 < b < 1$ and $Q \in M_+(\mathbf{R}^d)$, define

$$L_0(b,Q) = K(\mathcal{P}(\mathbf{R}^d), b, Q),$$

$$L_m(b,Q) = K(L_{m-1}(b,Q), b, Q), \quad m = 1, 2, \cdots,$$

and

$$L_\infty(b,Q) = \bigcap_{m=0}^{\infty} L_m(b,Q).$$

Similarly define

$$\widetilde{L}_m(b,Q), \quad m = 0, 1, 2, \cdots, \infty,$$

using $\widetilde{K}$ instead of K. Furthermore, for $C \in \mathfrak{C}$, define

$$L_0(C,Q) = K(\mathcal{P}(\mathbf{R}^d), C, Q),$$

$$L_m(C,Q) = K(L_{m-1}(C,Q), C, Q). \quad m = 1, 2, \cdots,$$

$$L_\infty(C,Q) = \bigcap_{m=0}^{\infty} L_m(C,Q),$$

and define

$$\widetilde{L}_m(C,Q), \quad m = 0, 1, 2, \cdots, \infty,$$

using $\widetilde{K}$ instead of K.

In particular, we call $\mu \in \mathcal{P}(\mathbf{R}^d)$ operator semi-selfdecomposable if $\mu \in L_0(b,Q)$ for some $0 < b < 1$ and $Q \in M_+(\mathbf{R}^d)$, and (C,Q)-decomposable if $\mu \in \widetilde{L}_0(C,Q)$, respectively.

PROPOSITION 1.3. *Let $C \in \mathfrak{C}$. Then*

$$I(\mathbf{R}^d) \supset L_0(b,Q) \supset L_1(b,Q) \supset \cdots \supset L_\infty(b,Q),$$

$$I(\mathbf{R}^d) \supset L_0(C,Q) \supset L_1(C,Q) \supset \cdots \supset L_\infty(C,Q),$$

$$\widetilde{L}_0(b,Q) \supset \widetilde{L}_1(b,Q) \supset \cdots \supset \widetilde{L}_\infty(b,Q),$$

and

$$\widetilde{L}_0(C,Q) \supset \widetilde{L}_1(C,Q) \supset \cdots \supset \widetilde{L}_\infty(C,Q).$$

PROPOSITION 1.4. *Let $C_1, C_2 \in \mathfrak{C}$ and suppose $C_1 \subset C_2$. Then for any $0 \le m \le \infty$,*

$$L_m(C_1,Q) \supset L_m(C_2,Q)$$

and

$$\widetilde{L}_m(C_1,Q) \supset \widetilde{L}_m(C_2,Q).$$

Classes $L_m([0,1],Q)$ with C chosen as $[0,1]$ are the finite-dimensional case of the classes studied in Jurek's paper (Jurek (1983)). A one-parameter continuous multiplicative group $\{U_t, t>0\}$ with $U_t \to 0$ as $t \downarrow 0$ is used there in place of b^Q, but any such matrix group $\{U_t, t>0\}$ is expressed as $U_t = t^Q$ with $Q \in M_+(\mathbf{R}^d)$. Note that $Q \in M_+(\mathbf{R}^d)$ is equivalent to that $t^Q \to 0$ as $t \downarrow 0$ (Sato (1991), Lemma 2.6).

Operator selfdecomposable distributions are defined in the following way (Sato and Yamazato (1984, 1985)). Let $OL(Q)$ be the class of $\mu \in \mathcal{P}(\mathbf{R}^d)$ such that there exist independent $\mathbf{R}^d$-valued random variables $\{X_j\}, a_n > 0, \uparrow \infty$, and $c_n \in \mathbf{R}^d$ satisfying

$$\mathcal{L}\left(a_n^{-Q}\sum_{j=1}^n X_j + c_n\right) \to \mu, \quad n \to \infty$$

and

$$\lim_{n\to\infty} \max_{1\le j\le n} P\{|a_n^{-Q}X_j| > \varepsilon\} = 0, \quad \forall \varepsilon > 0.$$

It is known that $\mu \in OL(Q)$ if and only if μ is $(c, Q, \mathcal{P}(\mathbf{R}^d))$-decomposable for every $c \in [0,1]$. That is, $OL(Q) = L_0([0,1],Q)$ by our terminology in this paper. See Proposition 2.5 for details. Distributions $\mu \in OL(Q)$ are called

Q-selfdecomposable. A distribution μ is called operator selfdecomposable if it is Q-selfdecomposable for some $Q \in M_+(\mathbf{R}^d)$. Our terminology is different from that of Jurek and Mason (1993). The class of operator selfdecomposable distributions in the sense of Jurek and Mason (1993) is called the class OL in Sato and Yamazato (1984) and is strictly bigger than the class of operator selfdecomposable distributions in our sense (see Yamazato (1984)). But both definitions coincide as long as the distributions considered are full (that is, are not concentrated in any proper hyperplane in $\mathbf{R}^d$). This is a consequence of a theorem of Urbanik (1972).

In Section 2, we shall prove the results stated in this section and show that, for $0 \le m \le \infty$,

$$L_m([0,1],Q) = \bigcap_{b\in(0,1)} L_m(b,Q) = \widetilde{L}_m([0,1],Q) = \bigcap_{b\in(0,1)} \widetilde{L}_m(b,Q).$$

This should be compared with the fact that, if $C = \{b^n\}_{n=0}^{\infty} \cup \{0\}$ with some $b \in (0,1)$, then for $0 \le m < \infty$,

$$L_m(C,Q) \subsetneqq \widetilde{L}_m(C,Q),$$

since $\widetilde{L}_m(C,Q) \cap (I(\mathbf{R}^d))^c \ne \emptyset$ (Bunge (1997)).

In Section 3 we shall give characterization of $\mu \in L_m(b,Q)$, for $0 \le m \le \infty$, in properties of its Gaussian covariance matrix and Lévy measure. Examples will show that

$$L_m(b,Q) \supsetneqq L_{m+1}(b,Q)$$

for $0 \le m < \infty$.

We shall study $L_\infty(C,Q)$ and $\widetilde{L}_\infty(C,Q)$ in Section 4. It will be shown that

$$L_\infty(C,Q) = \widetilde{L}_\infty(C,Q)$$

for any $C \in \mathfrak{C}$. The main result in Section 4 is the following. Define, for $C \in \mathfrak{C}$,

$$\Xi(C) = \{b \in (0,1) : C \subset \{b^n\}_{n=0}^{\infty} \cup \{0\}\}.$$

When $\Xi(C)$ is nonempty, let b_0 be its infimum. Clearly $b_0 \in \Xi(C)$ in this case. Then

$$L_\infty(C,Q) = L_\infty(b_0,Q) \quad \text{if } \Xi(C) \neq \emptyset,$$

$$L_\infty(C,Q) = L_\infty([0,1],Q) \quad \text{if } \Xi(C) = \emptyset.$$

We shall examine, in Section 5, the relationship between $L_m(C,I)$ and $\bigcap_{b \in C\setminus\{0,1\}} L_m(b,I)$ when $d = 1$ and $C \neq [0,1]$. It will be shown that, for $1 \leq m < \infty$, there exists $C \in \mathfrak{C}$ such that

$$L_m(C,1) \subsetneqq \bigcap_{b \in C\setminus\{0,1\}} L_m(b,1)$$

and

$$\widetilde{L}_m(C,1) \subsetneqq \bigcap_{b \in C\setminus\{0,1\}} \widetilde{L}_m(b,1).$$

We conclude this section with other related problems which are not dealt with in this paper. The distribution in the class $L_\infty([0,1],Q)$ is called completely operator selfdecomposable in Sato and Yamazato (1985), where the relationship between the class $L_\infty([0,1],Q)$ and that of operator stable distributions was studied. A natural question is how the class $L_\infty(b,Q)$ is related to that of operator semi-stable distributions. This problem will be discussed in the forthcoming paper by the authors of the present paper.

Another important problem is the continuity properties (the absolute continuity and the smoothness for instance) of distributions in the classes we are discussing here. This will be studied in another forthcoming paper by Watanabe.

2. Basic results on $K(H,b,Q)$ and $\widetilde{K}(H,b,Q)$. Throughout this paper, let $0 < b < 1$, $Q \in M_+(\mathbf{R}^d)$, and $C \in \mathfrak{C}$. The following two propositions can be proved in the same way as for Proposition 2.3 in Maejima and Naito (1998).

PROPOSITION 2.1. *If $H \subset \mathcal{P}(\mathbf{R}^d)$ is Q-completely closed, then $K(H,b,Q) \subset H$ and $\widetilde{K}(H,b,Q) \subset H$.*

PROPOSITION 2.2. *We have*

$$K(H,b,Q) \subset K(H,b^n,Q) \quad \textit{for } \forall n \in \mathbf{N}$$

and

$$\widetilde{K}(H,b,Q) \subset \widetilde{K}(H,b^n,Q) \quad \textit{for } \forall n \in \mathbf{N}.$$

The following proposition is also obvious from the definition and Proposition 2.2.

PROPOSITION 2.3. *If*

$$C = \{b^n\}_{n=0}^{\infty} \cup \{0\} \quad \textit{for some } b \in (0,1),$$

then

$$K(H,C,Q) = K(H,b,Q)$$

and

$$\widetilde{K}(H,C,Q) = \widetilde{K}(H,b,Q).$$

Bunge (1997) introduced the following class when $d = 1$.

DEFINITION 2.1. Suppose that $H \subset \mathcal{P}(\mathbf{R}^d)$ is I-completely closed. A distribution $\mu \in \mathcal{P}(\mathbf{R}^d)$ is said to belong to the class $\mathcal{L}^C(H)$, if, for every $b \in C \setminus \{0,1\}$, there exist independent $\mathbf{R}^d$-valued random variables $\{X_j^b\}$, $a_n^b > 0$, $c_n^b \in \mathbf{R}^d$ such that

$$\lim_{n\to\infty} \frac{a_n^b}{a_{n+1}^b} = b, \tag{2.1}$$

$$\mathcal{L}(X_j^b) \in H, \tag{2.2}$$

$$\mathcal{L}((a_n^b)^{-1} \sum_{j=1}^{n} X_j^b + c_n^b) \to \mu. \tag{2.3}$$

Then we have the following.

PROPOSITION 2.4. $\mathcal{L}^C(H) = \widetilde{K}(H, C, I)$.

PROOF. The differences in the definitions of two classes are that $a_n \uparrow \infty$ and $\sum_{j=1}^{k_n} X_j$ in Definition 1.1. If $\mu \in \widetilde{K}(H, C, I)$, then $X_j^b := \sum_{\ell=k_{j-1}+1}^{k_j} X_\ell$ satisfies (2.3) and hence $\mu \in \mathcal{L}^C(H)$. Conversely if $\mu \in \mathcal{L}^C(H)$, then $a_n^b < a_{n+1}^b \to \infty$ for large n, since $0 < b < 1$, and hence $\mu \in \widetilde{K}(H, C, I)$. □

We are now going to prove the statements mentioned in Sections 1 and 2 up to now.

Proof of Theorem 1.1.

(i) The same as for Theorem 2.1 (i) of Maejima and Naito (1998).

(ii) We first show that (1.9) implies that $\widehat{\mu}(z) \neq 0$, $\forall z \in \mathbf{R}^d$. If not, there exists $z_0 \in \mathbf{R}^d$ such that $\widehat{\mu}(z_0) = 0$ and $\widehat{\mu}(z) \neq 0$ when $|z| < |z_0|$. We have $\lim_{t \downarrow 0} t^Q x = 0$ for every $x \in \mathbf{R}^d$, since $Q \in M_+(\mathbf{R}^d)$. Hence for large n,

$$|b^{nQ'} z_0| < |z_0|. \tag{2.4}$$

It follows from (1.9) that, for every $n = 1, 2, \ldots$, there exists $\rho_n \in H \cap I(\mathbf{R}^d)$ such that

$$0 = \widehat{\mu}(z_0) = \widehat{\mu}(b^{nQ'} z_0)\widehat{\rho}_n(z_0).$$

By (2.4), we have $\widehat{\mu}(b^{nQ'} z_0) \neq 0$, which implies $\widehat{\rho}_n(z_0) = 0$, contradicting that $\rho_n \in I(\mathbf{R}^d)$. Thus $\widehat{\mu}(z) \neq 0$, $\forall z \in \mathbf{R}^d$. The rest of the proof is the same as for Theorem 2.1 (ii) of Maejima and Naito (1998).

(iii) The same as for Theorem 2.1 (iii) of Maejima and Naito (1998). □

Proof of Theorem 1.2. (i) "If part." By the repeated use of (1.9), we have for any $n \geq 1$

$$\widehat{\mu}(z) = \widehat{\mu}(b^{(n+1)Q'} z) \prod_{j=0}^{n} \widehat{\rho}(b^{jQ'} z).$$

Since $b^{(n+1)Q'} z \to 0$ as $n \to \infty$, we have

$$\widehat{\mu}(z) = \lim_{n\to\infty} \prod_{j=0}^{n} \widehat{\rho}(b^{jQ'} z) = \lim_{n\to\infty} \prod_{j=0}^{n} \widehat{\rho}(b^{(n-j)Q'} z).$$

If we define independent random variables $\{X_j\}$ by

$$\widehat{\mathcal{L}(X_j)}(z) = \widehat{\rho}(b^{-jQ'}z),$$

then $\mathcal{L}(X_j) \in H$ and

$$\mathcal{L}\left(b^{nQ}\sum_{j=0}^{n} X_j\right) \to \mu.$$

Therefore, (1.3) holds with $a_n = b^{-n}$ and $c_n = 0$.

(i) "Only if part." We need the following lemma.

LEMMA 2.1. (Loève (1945).) *Suppose $\mu_n, \sigma_n, \rho_n \in \mathcal{P}(\mathbf{R}^d)$, $\mu_n \to \mu$, $\sigma_n \to \sigma$, and $\mu_n = \sigma_n * \rho_n$. Then ρ_n converges through a subsequence of n.*

Now suppose $\mu \in \widetilde{K}(H, b, Q)$. Then there exist $\{X_j\}$, $\{a_n\}$, $\{c_n\}$ and $\{k_n\}$ satisfying (1.1)–(1.3) in Definition 1.1. We have

(2.5)
$$a_n^{-Q}\sum_{j=1}^{k_n} X_j + c_n = a_n^{-Q}a_{n-1}^{Q}\left(a_{n-1}^{-Q}\sum_{j=1}^{k_{n-1}} X_j + c_{n-1}\right) + \left(a_n^{-Q}\sum_{j=k_{n-1}+1}^{k_n} X_j + c_n - c_{n-1}a_n^{-Q}a_{n-1}^{Q}\right)$$

and denote the distributions of the left hand side of (2.5) and of the first and the second terms on the right hand side of (2.5) by μ_n, σ_n, and ρ_n, respectively. By (1.3), $\mu_n \to \mu$, and by (1.3) and (1.1), $\widehat{\sigma}_n(z) \to \widehat{\mu}(b^{Q'}z)$. Thus by Lemma 2.1, ρ_n converges, through a subsequence of n, to ρ (say) in $\mathcal{P}(\mathbf{R}^d)$. However, since $\mathcal{L}(X_j) \in H$ and H is Q-completely closed, we conclude that $\rho \in H$ and $\widehat{\mu}(z) = \widehat{\mu}(b^{Q'}z)\widehat{\rho}(z)$ in the limit.

(ii) We only show that $\widetilde{K}(H, b, Q)$ is closed under convergence. Let $\mu_n \in \widetilde{K}(H, b, Q)$ and suppose $\mu_n \to \mu_\infty$. By (i), for each $n \geq 1$,

$$\widehat{\mu}_n(z) = \widehat{\mu}_n(b^{Q'}z)\widehat{\rho}_n(z), \qquad \rho_n \in H.$$

Again by Lemma 2.1, ρ_n converges, through a subsequence of n, to ρ_∞ (say) in $\mathcal{P}(\mathbf{R}^d)$. Since H is closed under convergence, we see that

$$\widehat{\mu}_\infty(z) = \widehat{\mu}_\infty(b^{Q'}z)\widehat{\rho}_\infty(z), \qquad \rho_\infty \in H.$$

Thus by (i) again, we conclude that $\mu_\infty \in \widetilde{K}(H, b, Q)$. Closedness under convolution and Q-type equivalence can be shown similarly. □

Proof of Proposition 1.1. Denote by δ_x the unit mass at x. First note that $\delta_0 \in H$. In fact, $\mathcal{L}(X) \in H$ implies that $\delta_0 = \lim_{n\to\infty} \mathcal{L}(b_n^Q X) \in H$ when $b_n \downarrow 0$. Next note that $1 \in D_{Q,H}(\mu)$, which follows from that $\delta_0 \in H$. Let b_1 and b_2 be in $D_{Q,H}(\mu)$. Then, for $j = 1, 2$, $\widehat{\mu}(z) = \widehat{\mu}(b_j^{Q'}z)\widehat{\rho}_j(z)$ with some $\rho_j \in H$. Hence

$$\widehat{\mu}(z) = \widehat{\mu}(b_2^{Q'}b_1^{Q'}z)\widehat{\rho}_2(b_1^{Q'}z)\widehat{\rho}_1(z).$$

Note that $b_2^{Q'}b_1^{Q'} = (b_2b_1)^{Q'}$ and that $\widehat{\rho}_2(b_1^{Q'}z)\widehat{\rho}_1(z)$ is the characteristic function of a distribution in H. Hence $b_1b_2 \in D_{Q,H}(\mu)$. Now we can show that $0 \in D_{Q,H}(\mu)$. In fact, we can choose $c \neq 1$ in $D_{Q,H}(\mu)$. Then $c^n \in D_{Q,H}(\mu)$ for $n = 1, 2, \ldots$ and hence $\widehat{\mu}(z) = \widehat{\mu}(c^{nQ'}z)\widehat{\rho}_n(z)$ for some $\rho_n \in H$, which implies that $\rho_n \to \mu$, $\mu \in H$, and $0 \in D_{Q,H}(\mu)$. Closedness of $D_{Q,H}(\mu)$ is proved from Lemma 2.1. □

Proof of Proposition 1.2. (i) Restatement of Theorem 1.2 (i).

(ii) If H is Q-completely closed in the strong sense, then, by its definition, $H = H \cap I(\mathbf{R}^d)$. Thus the assertion follows from Theorems 1.1 and 1.2.

(iii) If $\mu \in K(H, b, Q)$, then it is $(b, Q, H \cap I(\mathbf{R}^d))$-decomposable by Theorem 1.1. If μ is $(b, Q, H \cap I(\mathbf{R}^d))$-decomposable, then $\mu \in \widetilde{K}(H \cap I(\mathbf{R}^d), b, Q) = K(H \cap I(\mathbf{R}^d), b, Q)$ by (i) and (ii). Finally, $K(H \cap I(\mathbf{R}^d), b, Q) \subset K(H, b, Q)$ by (1.7). □

Proof of Proposition 1.3. We have $I(\mathbf{R}^d) \supset L_0(b, Q)$ by (1.6). It follows from (1.7) and the definition that $L_0(b, Q) \supset L_1(b, Q)$. Hence $L_m(b, Q) \supset L_{m+1}(b, Q)$ by induction and (1.7). The other assertions are proved similarly. □

Proof of Proposition 1.4. For $m = 0$, we have

$$
\begin{aligned}
L_0(C_1, Q) = K(\mathcal{P}(\mathbf{R}^d), C_1, Q) &= \bigcap_{b \in C_1 \setminus \{0,1\}} K(\mathcal{P}(\mathbf{R}^d), b, Q) \\
&\supset \bigcap_{b \in C_2 \setminus \{0,1\}} K(\mathcal{P}(\mathbf{R}^d), b, Q) = K(\mathcal{P}(\mathbf{R}^d), C_2, Q) \\
&= L_0(C_2, Q).
\end{aligned}
$$

If $L_m(C_1, Q) \supset L_m(C_2, Q)$, then

$$
\begin{aligned}
L_{m+1}(C_1, Q) = K(L_m(C_1, Q), C_1, Q) &= \bigcap_{b \in C_1 \setminus \{0,1\}} K(L_m(C_1, Q), b, Q) \\
&\supset \bigcap_{b \in C_1 \setminus \{0,1\}} K(L_m(C_2, Q), b, Q) \quad \text{(by (1.7))} \\
&\supset \bigcap_{b \in C_2 \setminus \{0,1\}} K(L_m(C_2, Q), b, Q) = K(L_m(C_2, Q), C_2, Q) \\
&= L_{m+1}(C_2, Q).
\end{aligned}
$$

The assertion for $\widetilde{L}$ can be proved in exactly the same way if we use (1.8) instead of (1.7) above. □

THEOREM 2.1. *Let* $0 \leq m \leq \infty$. $\mu \in L_m(b, Q)$ *if and only if there exists* $\rho_m \in L_{m-1}(b, Q)$ *such that* $\widehat{\mu}(z) = \widehat{\mu}(b^{Q'} z)\widehat{\rho}_m(z)$, *where* $L_{-1}(b, Q)$ *and* $L_{\infty-1}(b, Q)$ *are understood as* $I(\mathbf{R}^d)$ *and* $L_\infty(b, Q)$, *respectively. Furthermore,* $L_m(b, Q)$ *is* Q*-completely closed in the strong sense.*

PROOF. Obviously $I(\mathbf{R}^d)$ is Q-completely closed in the strong sense. Thus the assertion for $m = 0$ comes from Proposition 1.2 (iii). Then we can prove the assertion for $1 \leq m < \infty$ by induction, using Theorem 1.1 (i), (ii), and (iii). Let $\mu \in L_0(b, Q)$. Then, since $\widehat{\mu}(z) \neq 0$, ρ_0 is determined uniquely, and $\mu \in L_\infty(b, Q)$ if and only if $\rho_0 \in L_\infty(b, Q)$. The assertion for $m = \infty$ thus follows. □

REMARK 2.1. The statement of Theorem 2.1 remains valid for $\widetilde{L}$ in place of L, if $\widetilde{L}_{-1}(b, Q)$ is understood as $\mathcal{P}(\mathbf{R}^d)$ and if we delete "in the strong sense" at the end. Proof is straightforward from Theorem 1.2 in case $0 \leq m < \infty$. If $\mu \in \widetilde{L}_\infty(b, Q)$, then for every $m < \infty$, $\mu \in \widetilde{L}_m(b, Q)$ and $\widehat{\mu}(z) = \widehat{\mu}(b^{Q'} z)\widehat{\rho}_m(z)$

with some $\rho_m \in \widetilde{L}_{m-1}(b,Q)$. By Lemma 2.1, ρ_m tends to some ρ_∞ as $m \to \infty$ through a subsequence, and $\widehat{\mu}(z) = \widehat{\mu}(b^{Q'}z)\widehat{\rho}_\infty(z)$. Since $\rho_{m'} \in \widetilde{L}_m(b,Q)$ for $m' > m$, we see $\rho_\infty \in \widetilde{L}_m(b,Q)$ and hence $\rho_\infty \in \widetilde{L}_\infty(b,Q)$. Conversely, if μ is $(b,Q,\widetilde{L}_\infty(b,Q))$-decomposable, then it is $(b,Q,\widetilde{L}_m(b,Q))$-decomposable for all $m < \infty$, and hence $\mu \in \widetilde{L}_\infty(b,Q)$.

REMARK 2.2. The class $L_\infty(b,Q)$ is the largest class that is invariant under the operation $K(\cdot,b,Q)$. The class $L_\infty(C,Q)$ is the largest class that is invariant under the operation $K(\cdot,C,Q)$. These statements remain valid if we replace L and K by $\widetilde{L}$ and $\widetilde{K}$, respectively. Proof is as follows. $L_\infty(b,Q) = K(L_\infty(b,Q),b,Q)$ by Theorem 2.1 and Proposition 1.2. If H satisfies $K(H,b,Q) = H$, then $L_\infty(b,Q) \supset H$ since, by the repeated use of (1.7), $L_m(b,Q) \supset H$ for $0 \le m < \infty$. For $C \in \mathfrak{C}$, $L_\infty(C,Q) \supset K(L_\infty(C,Q),b,Q)$ by Proposition 2.1, and hence $L_\infty(C,Q) \supset K(L_\infty(C,Q),C,Q)$. On the other hand,

$$\begin{aligned} L_\infty(C,Q) &= \bigcap_{m<\infty} L_m(C,Q) = \bigcap_{m<\infty} K(L_{m-1}(C,Q),C,Q) \\ &= \bigcap_{m<\infty} \bigcap_{b\in C\setminus\{0,1\}} K(L_{m-1}(C,Q),b,Q), \end{aligned}$$

and hence $\mu \in L_\infty(C,Q)$ implies that $\widehat{\mu}(z) = \widehat{\mu}(b^{Q'}z)\widehat{\rho}_b(z)$ for $b \in C\setminus\{0,1\}$ with $\rho_b \in L_{m-1}(C,Q)$. This ρ_b does not depend on m and thus $\rho_b \in L_\infty(C,Q)$. Hence $L_\infty(C,Q) \subset K(L_\infty(C,Q),b,Q)$ for $b \in C \setminus \{0,1\}$. It follows that $L_\infty(C,Q) \subset K(L_\infty(C,Q),C,Q)$. Hence the equality holds. If $K(H,C,Q) = H$, then we have $L_0(C,Q) \supset H$ from that $K(\mathcal{P}(\mathbf{R}^d),b,Q) \supset K(H,b,Q)$ and, similarly $L_m(C,Q) \supset H$ for all m, that is, $L_\infty(C,Q) \supset H$. The assertion for $\widetilde{L}$ and $\widetilde{K}$ is proved similarly by use of Lemma 2.1.

PROPOSITION 2.5. $I(\mathbf{R}^d) \supset OL(Q) = L_0([0,1],Q) = \widetilde{L}_0([0,1],Q)$.

PROOF. The following three statements are equivalent: $\mu \in OL(Q)$; μ is $(b,Q,\mathcal{P}(\mathbf{R}^d))$-decomposable for all $b \in (0,1)$; μ is $(b,Q,I(\mathbf{R}^d))$-decomposable for all $b \in (0,1)$. This is essentially Theorem 2.1 and Corollary 2.4 of Sato (1980). See also Theorem 3.3.5 of Jurek and Mason (1993). On the other hand, μ is

$(b, Q, \mathcal{P}(\mathbf{R}^d))$-decomposable for all b if and only if $\mu \in \widetilde{L}_0([0,1], Q)$, by Theorem 1.2. Also, μ is $(b, Q, I(\mathbf{R}^d))$-decomposable for all b if and only if $\mu \in L_0([0,1], Q)$, by Proposition 1.2 (iii). Thus the proposition is proved. □

Now, let us compare $L_m(C, Q)$ and $\bigcap_{b \in C \setminus \{0,1\}} L_m(b, Q)$. In general, we see that, for $1 \le m < \infty$,

$$
\begin{aligned}
(2.6) \qquad L_m(C, Q) &= K(L_{m-1}(C, Q), C, Q) \\
&= \bigcap_{b \in C \setminus \{0,1\}} K(L_{m-1}(C, Q), b, Q) \\
&\subset \bigcap_{b_1, b_2 \in C \setminus \{0,1\}} K(L_{m-1}(b_1, Q), b_2, Q) \\
&\subset \bigcap_{b \in C \setminus \{0,1\}} K(L_{m-1}(b, Q), b, Q) \\
&= \bigcap_{b \in C \setminus \{0,1\}} L_m(b, Q).
\end{aligned}
$$

The resulting inclusion for $m = \infty$ follows from the case $m < \infty$. Then a natural question is when two sides are equal. The answer is the following.

THEOREM 2.2. *For* $0 \le m \le \infty$,

(i) $L_m([0,1], Q) = \bigcap_{b \in (0,1)} L_m(b, Q)$,

(ii) $\widetilde{L}_m([0,1], Q) = \bigcap_{b \in (0,1)} \widetilde{L}_m(b, Q)$,

(iii) $L_m([0,1], Q) = \widetilde{L}_m([0,1], Q)$.

REMARK 2.3. If $C \neq [0,1]$, the statement of Theorem 2.2 is not necessarily true. A counterexample for (i) and (ii) will be given in Section 5.

Proof of Theorem 2.2. We first show (i). Let $0 \le m < \infty$. Let $b_n = 2^{-2^{-n}}$. Then $b_n = b_{n+1}^2$ and $\lim_{n \to \infty} b_n = 1$. Thus by Proposition 2.2,

$$
(2.7) \qquad L_m(b_{n+1}, Q) \subset L_m(b_n, Q).
$$

Let

$$
(2.8) \qquad K_m = \bigcap_{n=1}^{\infty} L_m(b_n, Q).
$$

If we could show

$$K_m = L_m([0,1], Q), \tag{2.9}$$

then, by (2.6), we would get (i) for $1 \le m < \infty$. On the other hand, (i) for $m = 0$ is true by Definition 1.4.

For any $b \in (0,1)$, there exist $N(k) \uparrow \infty$ and $n(k) \uparrow \infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty} b_{N(k)}^{n(k)} = \lim_{k\to\infty} 2^{-n(k)2^{-N(k)}} = b. \tag{2.10}$$

It follows from (2.7) and (2.8) that

$$K_m = \bigcap_{k=1}^{\infty} L_m(b_{N(k)}, Q).$$

Let $\mu \in K_m$. Then, by Theorem 2.1, there exists $\rho \in L_{m-1}(b_{N(k)}, Q)$ for each $k \ge 1$ and

$$\widehat{\mu}(z) = \widehat{\mu}(b_{N(k)}^{Q'} z)\widehat{\rho}(z) = \widehat{\mu}(b_{N(k)}^{n(k)Q'} z) \prod_{j=0}^{n(k)-1} \widehat{\rho}(b_{N(k)}^{jQ'} z) = \widehat{\mu}(b_{N(k)}^{n(k)Q'} z)\widehat{\rho}_k(z),$$

where $\rho_k \in L_{m-1}(b_{N(k)}, Q)$. Here we understand $L_{-1}(b_{N(k)}, Q) = I(\mathbf{R}^d)$ as in Theorem 2.1. Let $k \to \infty$. Then we have from (2.10)

$$\widehat{\mu}(z) = \widehat{\mu}(b^{Q'} z) \lim_{k\to\infty} \widehat{\rho}_k(z), \tag{2.11}$$

and $\rho_\infty := \lim_{k\to\infty} \rho_k \in \bigcap_{k=1}^{\infty} L_{m-1}(b_{N(k)}, Q) = K_{m-1}$, with the understanding that $K_{-1} = I(\mathbf{R}^d)$. If $m = 0$, then (2.11) means that $\widehat{\mu}(z) = \widehat{\mu}(b^{Q'} z)\widehat{\rho}_\infty(z)$, $\rho_\infty \in I(\mathbf{R}^d)$, for any $b \in (0,1)$, and hence $\mu \in L_0([0,1], Q)$. Thus we have (2.9) for $m = 0$. This together with (2.6) and (2.11) implies (2.9) for $0 \le m < \infty$ by induction. For $m = \infty$, we have

$$\bigcap_{b\in(0,1)} L_\infty(b, Q) = \bigcap_{m=0}^{\infty} \bigcap_{b\in(0,1)} L_m(b, Q) = \bigcap_{m=0}^{\infty} L_m([0,1], Q) = L_\infty([0,1], Q).$$

This shows (i).

To show (ii), use Remark 2.1 instead of Theorem 2.1. Note that (2.6) holds with $\widetilde{L}$ and $\widetilde{K}$ in place of L and K, respectively. Then the proof of (i) works with replacement of L by $\widetilde{L}$. The only place we must be careful is the convergence of $\widehat{\rho}_k(z)$ in (2.11), as $\widehat{\rho}_k(z)$ can possibly vanish for some $z \in \mathbf{R}^d$. But here we can use Lemma 2.1, to see convergence through a subsequence.

We finally show (iii). We have, by their definitions,

$$L_1([0,1],Q) = K(L_0([0,1],Q),[0,1],Q)$$

and

$$\widetilde{L}_1([0,1],Q) = \widetilde{K}(\widetilde{L}_0([0,1],Q),[0,1],Q).$$

Recall that $L_0([0,1],Q) = \widetilde{L}_0([0,1],Q)$ by Proposition 2.5. Since $L_m(b,Q)$ is Q-completely closed in the strong sense, so is $L_m([0,1],Q)$ by (i). Hence, by Proposition 1.2 (ii), $L_1([0,1],Q) = \widetilde{L}_1([0,1],Q)$. Repeating this argument, we conclude that, for any $m \geq 1$, $L_m([0,1],Q) = \widetilde{L}_m([0,1],Q)$. This completes the proof. □

3. Characterization for $L_m(b,Q), 0 \leq m \leq \infty$. We are going to characterize the classes $L_m(b,Q), 0 \leq m \leq \infty$, in terms of Gaussian covariance matrices and Lévy measures in the Lévy representation of their characteristic functions.

The characteristic function of any $\mu \in I(\mathbf{R}^d)$ is uniquely expressed in the form

$$\widehat{\mu}(z) = \exp\left[i\langle\gamma,z\rangle - \frac{1}{2}\langle z,Az\rangle + \int_{\mathbf{R}^d} r(z,x)\nu(dx)\right], \tag{3.1}$$

$$r(z,x) = e^{i\langle z,x\rangle} - 1 - \frac{\langle z,x\rangle}{1+|x|^2},$$

where $\gamma \in \mathbf{R}^d$, A (called the Gaussian covariance matrix of μ) is a symmetric nonnegative definite $d \times d$ matrix, and ν (called the Lévy measure of μ) is a measure on $\mathbf{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbf{R}^d} |x|^2(1+|x|^2)^{-1}\nu(dx) < \infty$. We call (γ, A, ν) the generating triplet of $\mu \in I(\mathbf{R}^d)$.

For a $d \times d$ matrix B we use the following notation: $BE = \{Bx : x \in E\}$ for $E \subset \mathbf{R}^d$, $(T_B\nu)(E) = \nu(\{x : Bx \in E\})$ for a measure ν on $\mathbf{R}^d$. We use a mapping Ψ_B from the class of symmetric $d \times d$ matrices into itself defined by $\Psi_B(A) = A - BAB'$. Its iteration is $\Psi_B^\ell = \Psi_B \circ \Psi_B^{\ell-1}$ for $\ell = 2, 3, \cdots$ with $\Psi_B^1 = \Psi_B$. Also let $\mathcal{B}_0(\mathbf{R}^d)$ be the class of Borel sets E in $\mathbf{R}^d$ such that $E \subset \{|x| > \varepsilon\}$ for some $\varepsilon > 0$.

Following (3.4.3) in Jurek and Mason (1993), we introduce a norm $|\cdot|_Q$ in $\mathbf{R}^d$ depending on Q:

$$|x|_Q = \int_0^1 \frac{|u^Q x|}{u} du, \quad x \in \mathbf{R}^d.$$

Since, for $Q \in M_+(\mathbf{R}^d)$, there exist $c_j > 0$ $(1 \le j \le 4)$ such that $c_1 u^{c_2}|x| \le |u^Q x| \le c_3 u^{c_4}|x|, 0 < u \le 1$, $|x|_Q$ is well defined. The norm $|\cdot|_Q$ is comparable with the Euclidean norm $|\cdot|$. An advantage of the norm $|\cdot|_Q$ is that for any $x \in \mathbf{R}^d \setminus \{0\}$, $t \to |t^Q x|_Q$ $(t > 0)$ is strictly increasing (Proposition 3.4.3 in Jurek and Mason ((1993)). Thus for any $b \in (0,1)$ and $Q \in M_+(\mathbf{R}^d)$, $\sup_{|x|_Q \le 1} |b^Q x|_Q < 1$. We write $B = b^Q$ and define

$$S_B = \{x \in \mathbf{R}^d : |x|_Q \le 1 \quad \text{and} \quad |B^{-1}x|_Q > 1\}.$$

PROPOSITION 3.1. ([Łuczak (1981), pp.289-290.) *For each* $x \in \mathbf{R}^d \setminus \{0\}$, *let* $\tau_x = \{B^n x : n \in \mathbf{Z}\}$. *Then*

(i) $\{x \in \mathbf{R}^d : |x|_Q = 1\} \subset S_B$,

(ii) *if* $x, y \in S_B, x \ne y$, *then* $\tau_x \cap \tau_y = \emptyset$.

(iii) *if* $x \in S_B$, *then* $\tau_x \cap S_B = \{x\}$,

(iv) *for any* $x \in \mathbf{R}^d \setminus \{0\}, \tau_x \cap S_B \ne \emptyset$,

(v) $B^n S_B \cap B^m S_B = \emptyset$ *for* $n \ne m$,

(vi) $\{x \in \mathbf{R}^d : 0 < |x|_Q \le 1\} = \bigcup_{n \ge 0} B^n S_B$ *and* $\{x \in \mathbf{R}^d : |x|_Q > 1\} = \bigcup_{n<0} B^n S_B$.

PROPOSITION 3.2.

(i) *If* ν *is the Lévy measure of* $\mu \in I(\mathbf{R}^d)$, *then there exist a finite measure* ν_0 *on* S_B *and a Borel measurable function* $g_n : S_B \to [0,\infty)$ *for each* $n \in \mathbf{Z}$ *satisfying the following conditions.*

(a) *For* $E \in \mathcal{B}(S_B), \nu_0(E) = 0$ *if and only if* $\nu(B^n E) = 0, \forall n \in \mathbf{Z}$,

(b) $\int_{S_B} \nu_0(dx) \sum_{n\in\mathbf{Z}} (|B^{-n}x|_Q^2 \wedge 1) g_n(x) < \infty$,

(c) $\sum_{n\in\mathbf{Z}} g_n(x) > 0, \quad \nu_0$*-a.e.*,

(d) $\nu(E) = \int_{S_B} \nu_0(dx) \sum_{n\in\mathbf{Z}} g_n(x) 1_E(B^{-n}x), \quad \forall E \in \mathcal{B}(\mathbf{R}^d)$.

These $\{\nu_0, g_n, n \in \mathbf{Z}\}$ *are uniquely determined in the following sense. If* $\{\nu_0, g_n, n \in \mathbf{Z}\}$ *and* $\{\widetilde{\nu}_0, \widetilde{g}_n, n \in \mathbf{Z}\}$ *satisfy the above conditions, then there exists a Borel measurable function* $h(x)$ *with* $0 < h(x) < \infty$ *such that*

$$\widetilde{\nu}_0(dx) = h(x)\nu_0(dx),$$

$$g_n(x) = h(x)\widetilde{g}_n(x), \quad \nu_0\text{-a.e.}, \forall n \in \mathbf{Z}.$$

(ii) *Conversely, if* ν_0, *a finite measure on* S_B, *and* $g_n, n \in \mathbf{Z}$, *Borel measurable functions from* S_B *into* $[0,\infty)$, *are given, and satisfy* (b) *and* (c), *then* ν *defined by* (d) *is the Lévy measure of some* $\mu \in I(\mathbf{R}^d)$ *and* (a) *is also satisfied.*

We call $\{\nu_0, g_n, n \in \mathbf{Z}\}$ determined uniquely from ν in (i) above the S_B-representation of ν. In the following, we may write $g(n,x)$ for $g_n(x)$ occasionally.

PROOF. (i) Define

$$\nu_0(E) = \sum_{n\in\mathbf{Z}} 2^{-|n|} \frac{\nu(B^n E)}{\nu(B^n S_B)}, \quad E \in \mathcal{B}(S_B),$$

with the convention that $\nu(B^n E)/\nu(B^n S_B) = 0$ when $\nu(B^n S_B) = 0$. It is obvious that ν_0 is a finite measure and satisfies (a).

Let $[\nu]_{B^n S_B}$ be the restriction of ν to $B^n S_B$. Then $[\nu]_{B^n S_B}$ is absolutely continuous with respect to $T_{B^n}\nu_0$. Then by the Radon-Nikodym theorem, there exists $h_n(x)$ on $B^n S_B$ such that

$$\nu(dx) = h_n(x)(T_{B^n}\nu_0)(dx) \quad \text{on } B^n S_B.$$

Hence

$$\begin{aligned}\nu(E) &= \sum_{n\in\mathbf{Z}} \nu(E \cap B^n S_B) \\ &= \sum_{n\in\mathbf{Z}} \int_{E\cap B^n S_B} h_n(x)(T_{B^n}\nu_0)(dx) \\ &= \sum_{n\in\mathbf{Z}} \int_{S_B} 1_E(B^n x) h_n(B^n x)\nu_0(dx).\end{aligned}$$

Thus, if we define $g_{-n}(x) = h_n(B^n x)$, then (d) holds.

As to (b), we have, by using (d),

$$\int_{S_B} \nu_0(dx) \sum_{n\in\mathbf{Z}} (|B^{-n}x|_Q^2 \wedge 1) g_n(x) = \int_{\mathbf{R}^d} (|x|_Q^2 \wedge 1)\nu(dx) < \infty. \tag{3.2}$$

As to (c), if $E \in \mathcal{B}(S_B)$, then by (d),

$$\nu(B^{-n}E) = \int_{S_B} \nu_0(dx) g_n(x) 1_E(x) = \int_E \nu_0(dx) g_n(x). \tag{3.3}$$

Thus, if $\nu_0(E) > 0$, then

$$\int_E \sum_{n\in\mathbf{Z}} g_n(x)\nu_0(dx) = \sum_{n\in\mathbf{Z}} \nu(B^{-n}E) > 0$$

by (a).

We are going to show the uniqueness of $\{\nu_0, g_n, n \in \mathbf{Z}\}$. Suppose that $\{\nu_0, g_n, n \in \mathbf{Z}\}$ and $\{\widetilde{\nu}_0, \widetilde{g}_n, n \in \mathbf{Z}\}$ satisfy (a)–(d). If follows from (a) that ν_0 and $\widetilde{\nu}_0$ are absolutely continuous each other so that $\nu_0(dx) = h(x)\nu_0(dx)$ for some Borel measurable function h with $0 < h(x) < \infty$. For any $E \in \mathcal{B}(S_B)$, by (3.3),

$$\nu(B^{-n}E) = \int_E \nu_0(dx) g_n(x)$$

and

$$\nu(B^{-n}E) = \int_E \widetilde{\nu}_0(dx)\widetilde{g}_n(x) = \int_E \nu_0(dx)h(x)\widetilde{g}_n(x).$$

Hence we conclude that $g_n(x) = h(x)\widetilde{g}_n(x), \nu_0$-a.e.

(ii) For given $\{\nu_0, g_n, n \in \mathbf{Z}\}$ satisfying (b) and (c), define ν by (d). Then by (3.2) and (b), we have $\int_{\mathbf{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$, and by (d), $\nu(\{0\}) = 0$. Hence ν is the Lévy measure of some $\mu \in I(\mathbf{R}^d)$. (a) also follows from (c) by (3.3). This completes the proof. □

PROPOSITION 3.3. *Suppose that* $\mu \in I(\mathbf{R}^d)$ *has the generating triplet* (γ, A, ν). *A necessary and sufficient condition for that* $\mu \in L_0(b, Q)$ *is that*

(i) $\Psi_B(A)$ *is nonnegative definite, and*

(ii) $\nu(E) - \nu(B^{-1}E) \geq 0$ *for any* $E \in \mathcal{B}_0(\mathbf{R}^d)$.

PROOF. By Theorem 2.1, $\mu \in L_0(b, Q)$ if and only if $\widehat{\rho}(z) = \widehat{\mu}(z)/\widehat{\mu}(B'z)$ is infinitely divisible characteristic function. We have, form (3.1),

$$\widehat{\rho}(z) = \exp\{i\langle(I-B)\gamma + c, z\rangle - \frac{1}{2}\langle(A - BAB')z, z\rangle + \int_{\mathbf{R}^d\setminus\{0\}} r(z,x)(\nu(dx) - \nu(d(B^{-1}x)))\},$$

where

$$c = \int_{\mathbf{R}^d\setminus\{0\}} \left(\frac{1}{1+|B^{-1}x|^2} - \frac{1}{1+|x|^2}\right) x\nu(dx).$$

Therefore, for that $\rho \in I(\mathbf{R}^d)$, it is necessary and sufficient that $A - BAB'$ is nonnegative definite (which is (i)) and

$$\nu(E) - \nu(B^{-1}E) \geq 0, \quad \forall E \in \mathcal{B}_0(\mathbf{R}^d),$$

(which is (ii)). □

The following theorem is an extension of the results of Sato (1980), Jurek ((1983), and Maejima and Naito (1998).

THEOREM 3.1. *Let* $0 \leq m \leq \infty, \mu \in I(\mathbf{R}^d), A$ *its Gaussian covariance matrix,* ν *its Lévy measure, and let* $\{\nu_0, g(n,x), n \in \mathbf{Z}\}$ *be the* S_B*-representation of* ν. *Then the following three statements are equivalent.*

(i) $\mu \in L_m(b,Q)$,

(ii) $\Psi_B^\ell(A), 1 \le \ell \le m+1$, *are nonnegative definite, and* $(I-T_B)^\ell \nu \ge 0, 1 \le \ell \le m+1$, *on* $\mathcal{B}_0(\mathbf{R}^d)$,

(iii) $\Psi_B^\ell(A), 1 \le \ell \le m+1$, *are nonnegative definite, and* $(-1)^\ell(\Delta^\ell g)(n,x) \ge 0, n \in \mathbf{Z}$, ν_0*-a.e.x for* $1 \le \ell \le m+1$, *where for* $k(n), n \in \mathbf{Z}, (\Delta k)(n) = k(n+1) - k(n)$.

(In the above, when $m = \infty, 1 \le \ell \le m+1$ *should be read as* $1 \le \ell < \infty$.*)*

PROOF. Note that the condition that $(-1)^\ell(\Delta^\ell g)(n,x)$ for ν_0-a.e.x does not depend on the choice of our S_B-representation of ν, as the representation has uniqueness in the sense described in Proposition 3.2 (i). Also note that, since $E \in \mathcal{B}_0(\mathbf{R}^d)$ implies that $B^{-1}E \in \mathcal{B}_0(\mathbf{R}^d)$, $(I-T_B)^\ell \nu_0$ is well-defined on $\mathcal{B}_0(\mathbf{R}^d)$ for $\ell \ge 1$.

We first show the equivalence of (i) and (ii). Since

$$((I-T_B)\nu)(E) = \nu(E) - \nu(B^{-1}E),$$

we have (i) $\Leftrightarrow$ (ii) for $m=0$ by Proposition 3.3. Next we suppose (i) $\Leftrightarrow$ (ii) for m and will show it for $m+1$. By Theorem 2.1, $\mu \in L_{m+1}(b,Q)$ if and only if $\widehat{\mu}(z) = \widehat{\mu}(b^{Q'}z)\widehat{\rho}(z)$ for some $\rho \in L_m(b,Q)$. The Gaussian covariance matrix A_ρ of ρ is $A_\rho = \Psi_B(A)$. Hence

$$\Psi_B^\ell(A_\rho) = \Psi_B^{\ell+1}(A),$$

and therefore $\Psi_B^\ell(A_\rho)$ are nonnegative definite for $1 \le \ell \le m+1$ if and only if $\Psi_B^\ell(A)$ are nonnegative definite for $2 \le \ell \le m+2$. Similarly, the Lévy measure ν_ρ of ρ satisfies $\nu_\rho = \nu - T_B\nu$ on $\mathcal{B}_0(\mathbf{R}^d)$. Hence

$$(I-T_B)^\ell \nu_\rho = (I-T_B)^{\ell+1}\nu \quad \text{on } \mathcal{B}_0(\mathbf{R}^d)$$

and therefore

$$(I-T_B)^\ell \nu_\rho \ge 0, \quad 1 \le \ell \le m+1, \quad \text{on } \mathcal{B}_0(\mathbf{R}^d)$$

if and only if

$$(I-T_B)^\ell \nu \geq 0, \quad 2 \leq \ell \leq m+2, \quad \text{on } \mathcal{B}_0(\mathbf{R}^d).$$

This proves the case for $m+1$.

We next show the equivalence of (ii) and (iii). We have, for $E \in \mathcal{B}_0(\mathbf{R}^d)$,

$$\begin{aligned}
((I-T_B)\nu)(E) &= \nu(E) - \nu(B^{-1}E) \\
&= \int_{S_B} \nu_0(dx) \left(\sum_{n\in\mathbf{Z}} g_n(x) 1_E(B^{-n}x) - \sum_{n\in\mathbf{Z}} g_n(x) 1_{B^{-1}E}(B^{-n}x) \right) \\
&= \int_{S_B} \nu_0(dx) \sum_{n\in\mathbf{Z}} (g_n(x) - g_{n+1}(x)) 1_E(B^{-n}x) \\
&= \int_{S_B} \nu_0(dx) \sum_{n\in\mathbf{Z}} (-\Delta g(n,x)) 1_E(B^{-n}x).
\end{aligned}$$

Hence for $E \in \mathcal{B}(S_B)$,

$$\begin{aligned}
&((I-T_B)\nu)(B^{-n}E) \\
&\quad = \int_{S_B} \nu_0(dx) \sum_{n\in\mathbf{Z}} (g_n(x) - g_{n+1}(x)) 1_{B^{-n}E}(B^{-n}x) \\
&\quad = \int_E (-\Delta g(n,x)) \nu_0(dx),
\end{aligned}$$

and thus

$$(I-T_B)\nu \geq 0 \quad \text{on } \mathcal{B}_0(\mathbf{R}^d)$$

if and only if

$$(-\Delta g)(n,x) \geq 0, \quad \forall n \in \mathbf{Z}, \nu_0\text{-a.e. } x.$$

Using the above expression of $(I-T_B)\nu$ in place of (d) of Proposition 3.2, we get

$$((I-T_B)^2\nu)(E) = \int_{S_B} \nu_0(dx) \sum_{n\in\mathbf{Z}} (-\Delta)^2 g(n,x) 1_E(B^{-n}x).$$

Repeating this argument, we conclude, for each $\ell \geq 1$, that

$$(I-T_B)^\ell \nu \geq 0 \quad \text{on } \mathcal{B}_0(\mathbf{R}^d)$$

if and only if

$$((-\Delta)^{\ell} g)(n,x) \geq 0, \quad \forall n \in \mathbf{Z}, \nu_0\text{-a.e. } x.$$

This completes the proof. □

REMARK 3.1. In Proposition 1.3, we have seen that

$$L_m(b,Q) \supset L_{m+1}(b,Q), \quad 0 \leq m < \infty, \tag{3.4}$$

$$L_m(C,Q) \supset L_{m+1}(C,Q), \quad 0 \leq m < \infty. \tag{3.5}$$

The inclusion (3.4) is proper, and the inclusion (3.5) is also proper for any $C \in \mathfrak{C}$. We show below that the inclusion in (3.4) is proper by giving an example of $\mu \in L_m(b,Q) \setminus L_{m+1}(b,Q)$. To show that the inclusion in (3.5) is proper, we need Theorem 4.4 in the next section, and thus we shall show it in Remark 4.2 right after Theorem 4.4 in the next section.

To show that the inclusion in (3.4) is proper, fix $x_0 \in \mathbf{R}^d \setminus \{0\}$. For $0 \leq m < \infty$, define

$$\nu_m(dx) = \sum_{n \in \mathbf{Z}} k_m(n) \delta_{B^{-n}x_0}(dx).$$

If we assume that $k_m(n) \geq k_m(n+1) \geq 0, \forall n \in \mathbf{Z}$, then

$$\begin{aligned}
(I - T_B)&\nu_m(\{B^{-n}x_0\}) \\
&= \nu_m(\{B^{-n}x_0\}) - \nu_m(\{B^{-(n+1)}x_0\}) \\
&= k_m(n) - k_m(n+1) \geq 0.
\end{aligned}$$

Hence, by Theorem 3.1, ν_m is the Lévy measure of some $\mu \in L_0(b,Q)$, provided that

$$\sum_{n \geq 0} k_m(n) < \infty \quad \text{and} \quad \sum_{n<0} |B^{-n}x_0|^2 k_m(n) < \infty. \tag{3.6}$$

Fix $0 < c < 1$ and let

$$k_0(n) = \begin{cases} c^n, & n \geq 0, \\ 1, & n < 0. \end{cases} \tag{3.7}$$

Then

$$\begin{aligned}(I-T_B)\nu_0(\{B^{-n}x_0\}) &= k_0(n)-k_0(n+1)\\ &= \begin{cases}(1-c)c^n, & n\ge 0\\ 0, & n<0\end{cases}\\ &\ge 0, \quad n\in\mathbf{Z},\end{aligned}$$

and

$$(I-T_B)^2\nu_0(\{B^{-n}x_0\})<0 \quad \text{if } n=-1.$$

Thus, by Theorem 3.1, ν_0 is the Lévy measure of a $\mu\in L_0(b,Q)\setminus L_1(b,Q)$.

Starting with $\{k_0(n), n\in\mathbf{Z}\}$ above, define $\{k_m(n), n\in\mathbf{Z}\}, m\ge 1$, inductively as follows:

$$k_m(n)=\begin{cases}c^n, & n\ge 0\\ 1+(1-c)\sum_{j=n}^{-1}k_{m-1}(j), & n<0.\end{cases} \tag{3.8}$$

Then

$$\begin{aligned}(I-T_B)\nu_m(\{B^{-n}x_0\}) &= k_m(n)-k_m(n+1)\\ &= \begin{cases}(1-c)c^n, & n\ge 0\\ (1-c)k_{m-1}(n), & n<0\end{cases}\\ &= (1-c)\nu_{m-1}(\{B^{-n}x_0\}).\end{aligned}$$

Thus

$$(I-T_B)^{\ell}\nu_m(\{B^{-n}x_0\})\ge 0 \quad \text{for } \ell=1,2,\cdots,m+1, \quad \text{for any } n\in\mathbf{Z}$$

and

$$(I-T_B)^{m+2}\nu_m(\{B^{-n}x_0\})<0, \quad \text{for some } n\in\mathbf{Z}.$$

Thus, ν_m is the Lévy measure of some $\mu\in L_m(b,Q)\setminus L_{m+1}(b,Q)$, once again by Theorem 3.1. In the above, it is easily seen that (3.7) and (3.8) satisfy the condition (3.6). If one wants to construct a full measure in $L_m(b,Q)\setminus L_{m+1}(b,Q)$, then it is enough to choose a set of linearly independent d vectors $\{x_1,\cdots,x_d\}$ in $\mathbf{R}^d$, and to define ν_m by

$$\nu_m(dx)=\sum_{n\in\mathbf{Z}}k_m(n)\sum_{j=1}^{d}\delta_{b^{-nQ}x_j}(dx).$$

4. The class $L_\infty(C,Q)$. Define

$$\mathcal{P}_{\log^m}(\mathbf{R}^d) = \left\{\mu \in \mathcal{P}(\mathbf{R}^d) : \int_{-\infty}^{\infty} (\log(1+|x|))^m \mu(dx) < \infty\right\}.$$

The following two theorems are extensions of some results in Bunge (1997). In the following, $C \in \mathfrak{C}$ and $Q \in M_+(\mathbf{R}^d)$.

THEOREM 4.1. *Let* $0 \le m < \infty$.

(i) *If* $\mu \in \widetilde{L}_m(C,Q)$, *then for any* $b \in C\backslash\{0,1\}$, *there exists* $\eta_b \in \mathcal{P}_{\log^{m+1}}(\mathbf{R}^d)$ *such that*

$$\widehat{\mu}(z) = \prod_{n=0}^{\infty} \widehat{\eta}_b(b^{nQ'}z)^{\binom{m+n}{m}}, \tag{4.1}$$

where $\binom{m+n}{m}$ *are the binomial coefficients.*

(ii) *Let* $0 < b < 1$ *and take* $\mu \in \mathcal{P}(\mathbf{R}^d)$. *If there exists* $\eta_b \in \mathcal{P}_{\log^{m+1}}(\mathbf{R}^d)$ *satisfying* (4.1), *then* $\mu \in \widetilde{L}_m(C,Q)$ *with* $C = \{b^n\}_{n=0}^{\infty} \cup \{0\}$.

THEOREM 4.2. $\widetilde{L}_\infty(C,Q) \subset I(\mathbf{R}^d)$.

Bunge (1997) showed the above theorems when $d = 1$. However a slight modification of his proofs concludes Theorems 4.1 and 4.2, and thus we omit their proofs here.

We see, from (1.5), (1.7), (1.8), and the definitions, that

$$L_m(C,Q) \subset \widetilde{L}_m(C,Q), \qquad 0 \le m \le \infty. \tag{4.2}$$

On the other hand, Bunge (1997) also showed that $\widetilde{L}_m(C,Q) \cap (I(\mathbf{R}^d))^c \ne \emptyset$ if $0 \le m < \infty$ and if $C = \{b^n\}_{n=0}^{\infty} \cup \{0\}$. Hence for $0 \le m < \infty$ and such a C,

$$L_m(C,Q) \subsetneqq \widetilde{L}_m(C,Q).$$

However, because of Theorem 4.2, it is worthwhile to compare $L_\infty(C,Q)$ and $\widetilde{L}_\infty(C,Q)$. (This problem was already proposed in Maejima and Naito (1998).) Actually we have the following.

THEOREM 4.3. $L_\infty(C,Q) = \widetilde{L}_\infty(C,Q)$.

PROOF. Because of (4.2), it is enough to show that $\widetilde{L}_\infty(C,Q) \subset L_\infty(C,Q)$. We have by Remark 2.2, Theorem 4.2 and (1.8), and Proposition 1.2 (iii) that $\widetilde{L}_\infty(C,Q) = \widetilde{K}(\widetilde{L}_\infty(C,Q),C,Q) \subset \widetilde{K}(I(\mathbf{R}^d),C,Q) = K(\mathcal{P}(\mathbf{R}^d),C,Q) = L_0(C,Q)$. Repeating this argument, we conclude that $\widetilde{L}_\infty(C,Q) \subset \bigcap_{m<\infty} L_m(C,Q) = L_\infty(C,Q)$. $\square$

As we have announced in Section 1, our main theorem in this section is the following. Recall the definition of $\Xi(C)$ stated in Section 1.

THEOREM 4.4.

(i) *When* $\Xi(C) \neq \emptyset$, $L_\infty(C,Q) = L_\infty(b_0,Q)$, *where* $b_0 = \inf \Xi(C)$.

(ii) *When* $\Xi(C) = \emptyset$, $L_\infty(C,Q) = L_\infty([0,\infty],Q)$.

We need several lemmas.

LEMMA 4.1. *Let* $0 \le m < \infty$. *A necessary and sufficient condition for that* $\mu \in L_m(C,Q)$ *is that* $\mu \in I(\mathbf{R}^d)$ *with the Gaussian covariance matrix* A *and Lévy measure* ν *of* μ *satisfying the following. For any* $n \le m+1$, *for any not necessarily distinct* $b_j \in C\backslash\{0,1\}$ $(j = 1,2,\dots,n)$, *with* $B_j = b_j^Q$,

$$(I - T_{B_1})\cdots(I - T_{B_n})\nu \ge 0 \quad \text{on } \mathcal{B}_0(\mathbf{R}^d) \tag{4.3}$$

and

$$\Psi_{B_1} \circ \cdots \circ \Psi_{B_n}(A) \text{ is nonnegative definite.} \tag{4.4}$$

PROOF. If $m = 0$, then the assertion is just Proposition 3.3. We assume that the assertion is true for m, and will show for $m+1$. Recall the definition of $L_{m+1}(C,Q)$, that is,

$$L_{m+1}(C,Q) = K(L_m(C,Q),C,Q) = \bigcap_{b\in C\backslash\{0,1\}} K(L_m(C,Q),b,Q).$$

Thus $\mu \in L_{m+1}(C,Q)$ if and only if, for any $b \in C \setminus \{0,1\}$, ρ_b satisfying $\widehat{\mu}(z) = \widehat{\mu}(b^{Q'}z)\widehat{\rho}_b(z)$ is in $L_m(C,Q)$. By Proposition 3.3, the Lévy measure ν_{ρ_b} and the

Gaussian covariance matrix A_{ρ_b} of ρ_b satisfy $\nu_{\rho_b} = (I - T_{bQ})\nu$ on $\mathcal{B}_0(\mathbf{R}^d)$ and $A_{\rho_b} = \Psi_{bQ}(A)$. By the assumption of the induction, $\rho_b \in L_m(C, Q)$ if and only if (4.3) and (4.4) hold for ν_{ρ_b} and A_{ρ_b} in place of ν and A. Then it is equivalent to (4.3) and (4.4) for all $n \leq m+2$. This is the assertion for $m+1$. □

Lemma 4.1 yields

LEMMA 4.2. *$\mu \in L_\infty(C, Q)$ if and only if* (4.3) *and* (4.4) *hold for any* $n \geq 1$.

REMARK 4.1. If $C = \{b^n\}_{n=0}^{\infty} \cup \{0\}$, then, by Proposition 2.3, $L_m(C, Q) = L_m(b, Q)$ and Lemma 4.1 is reduced to Theorem 3.1 (i) $\Leftrightarrow$ (ii). Note that if $\nu - T_{bQ}\nu \geq 0$ on $\mathcal{B}_0(\mathbf{R}^d)$ and $\Psi_{bQ}(A)$ is nonnegative definite, then $\nu - T_{b^nQ}\nu \geq 0$ on $\mathcal{B}_0(\mathbf{R}^d)$ and $\Psi_{b^nQ}(A)$ is nonnegative definite for any $n \geq 1$.

In Theorem 3.1 with $m = \infty$, the condition for $\{g(n,x), n \in \mathbf{Z}\}$ is called complete monotonicity. Namely, in general, $\{k(n), n \in \mathbf{Z}\}$ is called a completely monotone sequence if

$$(-1)^\ell (\Delta^\ell k)(n) \geq 0 \quad \text{for } \ell \geq 0, \quad n \in \mathbf{Z}.$$

The following gives us an integral representation of completely monotone sequences.

LEMMA 4.3. *If $\{k(n), n \in \mathbf{Z}\}$ is completely monotone, then there exists a unique measure ρ on $(0,1]$ such that*

$$k(n) = \int_{(0,1]} x^n \rho(dx), \quad n \in \mathbf{Z}. \tag{4.5}$$

Conversely, $\{k(n), n \in \mathbf{Z}\}$ having the representation (4.5) *is completely monotone.*

PROOF. Suppose $\{k(n)\}$ is completely monotone and not identically zero. Then $k(n) > 0$ for all $n \in \mathbf{Z}$. In fact, if $k(n) = 0$ for some n, then, choosing $n_0 \in \mathbf{Z}$ that satisfies $k(n_0) > 0$ and $k(n_0 + p) = 0$ for any $p \geq 1$, we have

$$0 \leq (-1)^\ell \Delta^\ell k(n_0 - 1) = \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j k(n_0 + j - 1) = k(n_0 - 1) - \ell k(n_0)$$

and thus

$$0 < k(n_0) \le \frac{1}{\ell} k(n_0 - 1) \to 0 \quad \text{as } \ell \to \infty,$$

which is a contradiction. For $p \in \mathbf{Z}_+$, we apply Theorem 1 on the Hausdorff moment problem in p.225 of Feller (1971) to $k(n-p)/k(-p)$, $n \in \mathbf{Z}_+$, where $\mathbf{Z}_+$ is the set of all nonnegative integers. Then there exists a unique measure ρ_p on $[0,1]$ such that

$$k(n-p) = \int_{[0,1]} x^n \rho_p(dx), \qquad n \in \mathbf{Z}_+. \tag{4.6}$$

In particular,

$$k(n) = \int_{[0,1]} x^n \rho_0(dx), \qquad n \in \mathbf{Z}_+. \tag{4.7}$$

On the other hand, since

$$k(n) = k(n+p-p) = \int_{[0,1]} x^{n+p} \rho_p(dx),$$

we have by the uniqueness of ρ_p that

$$\rho_0(dx) = x^p \rho_p(dx), \tag{4.8}$$

implying $\rho_0(\{0\}) = 0$. This together with (4.7) implies (4.5) for $n \ge 0$, if we take ρ_0 as ρ. Furthermore, since

$$k(n-p) = k(n+1-p-1) = \int_{[0,1]} x^{n+1} \rho_{p+1}(dx), \qquad n \in \mathbf{Z}_+,$$

we have again by the uniqueness of ρ_p that

$$\rho_p(dx) = x \rho_{p+1}(dx),$$

implying $\rho_p(\{0\}) = 0$. Thus from (4.6) with $n = 0$ and (4.8),

$$k(-p) = \int_{[0,1]} \rho_p(dx) = \int_{(0,1]} x^{-p} \rho_0(dx) = \int_{(0,1]} x^{-p} \rho(dx),$$

which is (4.5) for $n < 0$. The uniqueness of ρ is trivial. Conversely, $k(n)$ in (4.5) satisfies

$$\begin{aligned}(-1)^\ell(\Delta^\ell k)(n) &= \sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^j k(n+j)\\ &= \sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^j\int_{(0,1]} x^{n+j}\rho(dx)\\ &= \int_{(0,1]} x^n(1-x)^\ell\rho(dx) \geq 0,\end{aligned}$$

and hence $\{k(n), n\in\mathbf{Z}\}$ is completely monotone. □

COROLLARY 4.1. *Let* $0 < b < 1$. *If* $\{k(n), n \in \mathbf{Z}\}$ *is completely monotone, then there exists a unique measure* Γ *on* $[0,\infty)$ *such that*

$$k(n) = \int_{[0,\infty)} b^{n\alpha}\Gamma(d\alpha).$$

PROOF. In (4.5), change the variable $x \in (0,1]$ to $\alpha \in [0,\infty)$ by $\alpha = \log x/\log b$, and define for $E \in \mathcal{B}([0,\infty))$,

$$\Gamma(E) = \rho\left(\left\{x : \frac{\log x}{\log b} \in E\right\}\right).$$

Then from (4.5)

$$k(n) = \int_{(0,1]} x^n\rho(dx) = \int_{[0,\infty)} b^{n\alpha}\Gamma(d\alpha).$$

This completes the proof. □

We extend the notion of complete monotonicity for functions on $\mathbf{Z}$ to that for functions on $\mathbf{Z}^k$, $k \geq 1$. Let F be a function on $\mathbf{Z}^k$. For $1 \leq j \leq k$, define

$$(\Delta_j F)(n_1,\ldots,n_k) = F(n_1,\ldots,n_j+1,\ldots,n_k) - F(n_1,\ldots,n_j,\ldots,n_k),$$

and denote the ℓ times iteration of Δ_j by Δ_j^ℓ.

DEFINITION 4.1. A function F on $\mathbf{Z}^k$ is said to be completely monotone if for any $\ell_j \in \mathbf{Z}_+$ $(1 \leq j \leq k)$ and $(n_1,\ldots,n_k) \in \mathbf{Z}^k$,

$$(-1)^{\ell_1+\cdots+\ell_k}\Delta_1^{\ell_1}\cdots\Delta_k^{\ell_k}F(n_1,\ldots,n_k) \geq 0.$$

LEMMA 4.4. *If F on $\mathbf{Z}^k$ is completely monotone, then there exists a unique finite measure ρ on $(0,1]^k$ such that*

$$F(n_1,\dots,n_k)=\int_{(0,1]^k} x_1^{n_1}\cdots x_k^{n_k}\rho(dx), \tag{4.9}$$

where $x=(x_1,\dots,x_k)$. Conversely, if (4.9) holds for F on $\mathbf{Z}^k$, then F is completely monotone.

PROOF. The latter part is obvious, because

$$\begin{aligned}(-1)^{\ell_1+\cdots+\ell_k}&\Delta_1^{\ell_1}\cdots\Delta_k^{\ell_k}F(n_1,\dots,n_k)\\ &=\int_{(0,1]^k} x_1^{n_1}(1-x_1)^{\ell_1}\cdots x_k^{n_k}(1-x_k)^{\ell_k}\rho(dx)\ge 0.\end{aligned}$$

We show the first part by induction with respect to k. The case $k=1$ is Lemma 4.3. Suppose the assertion is true for k. Let $F(n_1,\dots,n_{k+1})$ be completely monotone on $\mathbf{Z}^{k+1}$. For a fixed $n\in\mathbf{Z}$, $F(n_1,\dots,n_k,n)$, $(n_1,\dots,n_k)\in\mathbf{Z}^k$, is completely monotone. Thus for any $n\in\mathbf{Z}$, there exists a unique finite measure ρ_n on $(0,1]^k$ such that

$$F(n_1,\dots,n_k,n)=\int_{(0,1]^k} x_1^{n_1}\cdots x_k^{n_k}\rho_n(dx). \tag{4.10}$$

Using the Hahn decomposition, we can verify the uniqueness even among the class of finite signed measures. Since $F(n_1,\dots,n_k,n)$ is completely monotone on $\mathbf{Z}^{k+1}$,

$$\begin{aligned}\widetilde{F}_{(\ell,n)}(n_1,\dots,n_k):=&(-1)^\ell\Delta_{k+1}^\ell F(n_1,\dots,n_k,n)\\ =&\int_{(0,1]^k} x_1^{n_1}\cdots x_k^{n_k}(-1)^\ell\Delta^\ell\rho_n(dx)\end{aligned}$$

is completely monotone on $\mathbf{Z}^k$. Here we are using the notation $\Delta\rho_n=\rho_{n+1}-\rho_n$. Thus by the induction hypothesis, $(-1)^\ell\Delta^\ell\rho_n$ should be the unique measure in the representation of $\widetilde{F}_{(\ell,n)}(n_1,\dots,n_k)$, and so

$$(-1)^\ell\Delta^\ell\rho_n\ge 0,\qquad \forall\ell\ge 0,\quad \forall n\in\mathbf{Z}. \tag{4.11}$$

From this, we observe that ρ_n is absolutely continuous with respect to ρ_0 for any $n \in \mathbf{Z}$. Let E satisfy $\rho_0(E) = 0$. Then (4.11) implies that $\rho_n(E) = 0$ for any $n \in \mathbf{Z}$ by the argument at the beginning of the proof of Lemma 4.3. Thus we get the absolute continuity and there exists F_x such that

$$\rho_n(dx) = F_x(n)\rho_0(dx). \tag{4.12}$$

It follows from (4.11) and (4.12) that

$$(-1)^\ell \Delta^\ell F_x(n) \geq 0, \qquad \forall \ell \geq 1,$$

for ρ_0-a.e. x. Use Lemma 4.3. Then, for ρ_0-a.e. x, there exists a unique finite measure ρ_x on $(0,1]$ such that

$$F_x(n) = \int_{(0,1]} y^n \rho_x(dy) \tag{4.13}$$

and $\rho_x(E)$ is measurable in x for any $E \in \mathcal{B}((0,1])$. Combining (4.10), (4.12), and (4.13), we have

$$F(n_1, \ldots, n_k, n_{k+1}) = \int_{(0,1]^{k+1}} x_1^{n_1} \cdots x_{k+1}^{n_{k+1}} \rho(d(x, x_{k+1})),$$

where

$$\rho(E) = \int_{(0,1]^k} \rho_0(dx) \int_{(0,1]} \rho_x(dy) 1_E(x,y), \quad \forall E \in \mathcal{B}((0,1]^{k+1}),$$

and conclude that the representation (4.9) is true for $k+1$ in place of k. The uniqueness of ρ in the representation (4.9) is evident by the standard argument, as the class of functions $x_1^{n_1} \cdots x_k^{n_k}$ with $(n_1, \ldots, n_k) \in \mathbf{Z}_+^k$ generates all continuous functions on $[0,1]^k$. □

We are now ready to prove Theorem 4.4.

Proof of Theorem 4.4.

Step 1. Fix d and let I_G and I_N be the class of Gaussian distributions on $\mathbf{R}^d$ and that of purely non-Gaussian infinitely divisible distributions on $\mathbf{R}^d$, respectively. We let point distributions belong to both I_G and I_N. Any $\mu \in I(\mathbf{R}^d)$ has the decomposition $\mu = \mu_G * \mu_N$ with $\mu_G \in I_G$ and $\mu_N \in I_N$, uniquely up to factors of point distributions. Theorem 3.1 says that $\mu \in L_\infty(b, Q)$ if and only if $\mu \in I(\mathbf{R}^d)$ and both μ_G and μ_N are in $L_\infty(b, Q)$. Also, by Lemma 4.2, we have that $\mu \in L_\infty(C, Q)$ if and only if $\mu \in I(\mathbf{R}^d)$ and both μ_G and μ_N belong to $L_\infty(C, Q)$. Hence, in order to prove the theorem, we can handle $L_\infty(C, Q) \cap I_G$ and $L_\infty(C, Q) \cap I_N$ separately.

Step 2. Let us prove that

(i)' $\quad L_\infty(C, Q) \cap I_N = L_\infty(b_0, Q) \cap I_N$, when $\Xi(C) \neq \emptyset$,

(ii)' $\quad L_\infty(C, Q) \cap I_N = L_\infty([0, 1], Q) \cap I_N$, when $\Xi(C) = \emptyset$.

We first show (i)'. Suppose $\mu \in L_\infty(C, Q) \cap I_N$ with Lévy measure ν. Let $D \in \mathcal{B}_0(\mathbf{R}^d)$. For any $k \geq 1$, any $b_j \in C \setminus \{0, 1\}$, $1 \leq j \leq k$, and $(n_1, \ldots, n_k) \in \mathbf{Z}^k$, define with $B_j = b_j^Q$,

$$F_D(n_1, \ldots, n_k) = \nu(B_1^{-n_1} \cdots B_k^{-n_k} D). \tag{4.14}$$

Notice that for any $\ell_j \geq 1$, $1 \leq j \leq k$,

$$\begin{aligned}(-1)^{\ell_1 + \cdots + \ell_k} &\Delta_1^{\ell_1} \cdots \Delta_k^{\ell_k} F_D(n_1, \ldots, n_k) \\ &= (I - T_{B_1})^{\ell_1} \cdots (I - T_{B_k})^{\ell_k} \nu(B_1^{-n_1} \cdots B_k^{-n_k} D).\end{aligned}$$

In Lemma 4.2, we can choose the identical b_j as many times as we want. Hence

$$(I - T_{B_1})^{\ell_1} \cdots (I - T_{B_k})^{\ell_k} \nu \geq 0 \quad \text{on } \mathcal{B}_0(\mathbf{R}^d).$$

Thus, by Lemma 4.4, there exists a finite measure ρ_D on $(0, 1]^k$ such that

$$F_D(n_1, \ldots, n_k) = \int_{(0,1]^k} x_1^{n_1} \cdots x_k^{n_k} \rho_D(dx).$$

As in Corollary 4.1, by the change of variable $x = (x_1, \ldots, x_k)$ to $\alpha = (\alpha_1, \ldots, \alpha_k)$ by $\alpha_j = \log x_j / \log b_j$ and by defining $\Gamma_D(d\alpha) = \rho_D(dx)$, we have

$$F_D(n_1, \ldots, n_k) = \int_{[0,\infty)^k} b_1^{n_1 \alpha_1} \cdots b_k^{n_k \alpha_k} \Gamma_D(d\alpha). \tag{4.15}$$

Since $b_0 \in \Xi(C)$, there exist $(p_1, \cdots, p_k) \in \mathbf{Z}^k$ and $b_j \in C \setminus \{0,1\}$, $1 \le j \le k$, such that

$$b_0 = \prod_{j=1}^{k} b_j^{p_j}. \tag{4.16}$$

To show this, we start with that any $b \in C \setminus \{0,1\}$ can be expressed as $b = b_0^{\ell(b)}$ for some $\ell(b) \in \mathbf{Z}_+$. Let

$$C_n = \{b \in C \setminus \{0,1\} : \ell(b) \le n\},$$

then $C_n \uparrow C \setminus \{0,1\}$, and thus $C_n \neq \emptyset$ for sufficiently large n. If we let $g(n) = g.c.d.\{\ell(b) : b \in C_n\}$, then $\Xi(C_n) = b_0^{g(n)}$. Since $g(n)(\ge 1)$ is nonincreasing as $n \uparrow \infty$, there exists n_0 such that $g(n) = g(n_0)$ for $n \ge n_0$. Thus

$$C_n \subset \left\{ (b_0^{g(n_0)})^k \right\}_{k=1}^{\infty}.$$

Since $C_n \uparrow C \setminus \{0,1\}$, we have $b_0^{g(n_0)} \in \Xi(C)$. Since $b_0 \in \Xi(C)$ is the minimum of $\Xi(C)$, it must be that $g(n_0) = 1$. Thus

$$g(n_0) = g.c.d.\{\ell(b) : b \in C_{n_0}\} = 1,$$

and there exist $k \ge 1, b_j \in C_{n_0}$ and $p_j \in \mathbf{Z}$ $(j = 1, 2, \cdots, k)$ such that $\sum_{j=1}^{k} \ell(b_j) p_j = 1$. Thus

$$b_0 = b_0^{\sum_{j=1}^{k} \ell(b_j) p_j} = \prod_{j=1}^{k} b_j^{p_j},$$

showing (4.16). Furthermore, for any $1 \le i < j \le k$, there exist $q_i, q_j \in \mathbf{Z}_+$ such that $b_i^{q_i} = b_j^{q_j}$. Hence

$$F_D(n_1, \cdots, n_i + q_i, \cdots, n_j, \ldots, n_k) = F_D(n_1, \cdots, n_i, \cdots, n_j + q_j, \ldots, n_k),$$

which we denote by $\widetilde{F}_D(n_1, \cdots, n_k)$. By (4.15),

$$\begin{aligned} \widetilde{F}_D(n_1, \cdots, n_k) &= \int_{[0,\infty)^k} b_1^{n_1\alpha_1} \cdots b_k^{n_k\alpha_k} b_i^{q_i\alpha_i} \Gamma_D(d\alpha) \\ &= \int_{[0,\infty)^k} b_1^{n_1\alpha_1} \cdots b_k^{n_k\alpha_k} b_j^{q_j\alpha_j} \Gamma_D(d\alpha). \end{aligned}$$

Since $\widetilde{F}_D$ is completely monotone in $\mathbf{Z}^k$, by the uniqueness of ρ in the representation (4.9), we have

$$b_i^{q_i\alpha_i} = b_j^{q_j\alpha_j} \quad \text{for } \Gamma_D\text{-a.e. } \alpha$$

implying $\alpha_i = \alpha_j$. Thus

$$\Gamma_D([0,\infty)^k \setminus \{\alpha_1 = \alpha_2 = \cdots = \alpha_k\}) = 0,$$

and by (4.14) and (4.15),

$$\begin{aligned}\nu(b_0^{-nQ}D) &= F_D(np_1, \ldots, np_k)\\ &= \int_{[0,\infty)} b_1^{np_1\beta}\cdots b_k^{np_k\beta}\bar{\Gamma}_D(d\beta).\\ &= \int_{[0,\infty)} b_0^{n\beta}\bar{\Gamma}_D(d\beta),\end{aligned}$$

where we define $\bar{\Gamma}_D(E) = \Gamma_D(\{(\beta,\ldots,\beta) : \beta \in E\})$ for $E \in \mathcal{B}([0,\infty))$. Hence $\nu(b_0^{-nQ}D)$ is completely monotone with respect to n, and thus for any $\ell \geq 1$,

$$(I - T_{b_0^Q})^\ell \nu(D) \geq 0.$$

Thus by Theorem 3.1 ($m = \infty$), we conclude that $\mu \in L_\infty(b_0, Q) \cap I_N$.

Conversely suppose $\mu \in L_\infty(b_0, Q) \cap I_N$. Let $b_1, \ldots, b_n \in C \setminus \{0,1\}$. Note that there exists a positive integer $m(j)$ such that $b_j = b_0^{m(j)}$. Hence

$$\prod_{j=1}^n (I - T_{b_j^Q})\nu = \left(\prod_{j=1}^n (I + T_{b_0^Q} + T_{b_0^{2Q}} + \cdots + T_{b_0^{(m(j)-1)Q}})\right)(I - T_{b_0^Q})^n \nu \geq 0$$

by Theorem 3.1 ($m = \infty$). Thus by Lemma 4.2, $\mu \in L_\infty(C, Q) \cap I_N$. This proves (i)'.

We next show (ii)'. Since $L_\infty(C,Q) \cap I_N \supset L_\infty([0,1],Q) \cap I_N$ by Proposition 1.4, it is enough to show that $L_\infty(C,Q) \cap I_N \subset L_\infty([0,1],Q) \cap I_N$. We consider two cases. Let $\mathbf{Q}$ be the set of all rational numbers.

Case 1. There exist $b_1, b_2 \in C \setminus \{0,1\}$ such that

$$\frac{\log b_2}{\log b_1} \notin \mathbf{Q}. \tag{4.17}$$

Case 2. For any $b_1, b_2 \in C \setminus \{0,1\}$,

$$\frac{\log b_2}{\log b_1} \in \mathbf{Q}. \tag{4.18}$$

We first treat Case 1. Suppose $\mu \in L_\infty(C,Q) \cap I_N$. Let $D \in \mathcal{B}_0(\mathbf{R}^d)$ such that $\nu(\partial D) = 0$, where ∂D means the boundary of D. Choose $b_1, b_2 \in C \setminus \{0,1\}$ satisfying (4.17). Let

$$F_D(n_1,n_2) = \nu(b_1^{-n_1 Q} b_2^{-n_2 Q} D), \qquad (n_1,n_2) \in \mathbf{Z}^2.$$

We have proved (4.15) without using that $\Xi(C) \neq \emptyset$. Hence

$$F_D(n_1,n_2) = \int_{[0,\infty)^2} b_1^{n_1\alpha_1} b_2^{n_2\alpha_2} \Gamma_D(d\alpha),$$

where $\alpha = (\alpha_1,\alpha_2)$. We can choose, by (4.17), $m(k), n(k) \in \mathbf{Z}_+$ such that $m(k) \to \infty, n(k) \to \infty$ and $a_k := b_1^{m(k)} b_2^{-n(k)} \to 1$ as $k \to \infty$. Thus

$$\begin{aligned}\nu(b_1^{-m(k)Q} b_2^{n(k)Q} D) &= F_D(m(k), -n(k)) \\ &= \int_{[0,\infty)^2} b_1^{m(k)(\alpha_1-\alpha_2)} a_k^{\alpha_2} \Gamma_D(d\alpha).\end{aligned}$$

If $\Gamma_D(\{\alpha_1 < \alpha_2\}) > 0$, then we have a contradiction from the above equality, by letting $k \to \infty$ and using Fatou's lemma. Thus, $\Gamma_D(\{\alpha_1 < \alpha_2\}) = 0$. Similarly we can show that $\Gamma_D(\{\alpha_1 > \alpha_2\}) = 0$ and conclude that $\Gamma_D(\{\alpha_1 \neq \alpha_2\}) = 0$. Hence

$$F_D(n_1,n_2) = \int_{[0,\infty)} (b_1^{n_1} b_2^{n_2})^\beta \bar{\Gamma}_D(d\beta),$$

where $\bar{\Gamma}_D$ is defined as before. By (4.17), for any $a > 0$, we can choose $m(k), n(k) \in \mathbf{Z}$ such that $b_1^{-m(k)} b_2^{-n(k)}$ decreases to a as $k \to \infty$. Thus

$$\nu(a^Q D) = \lim_{k\to\infty} F_D(m(k), n(k)) = \int_{[0,\infty)} a^{-\beta} \bar{\Gamma}_D(d\beta).$$

This means that, for any $b \in (0,1)$ and $n \geq 1$,

$$\nu(b^{-nQ}D) = \int_{[0,\infty)} b^{n\beta}\bar{\Gamma}_D(d\beta)$$

and thus, for any $\ell \geq 1$,

$$(I - T_{b^Q})^{\ell}\nu(D) \geq 0. \tag{4.19}$$

By approximation, we observe that (4.19) is true for any $D \in \mathcal{B}_0(\mathbf{R}^d)$. Thus by Theorem 3.1, we get that $\mu \in L_\infty(b,Q) \cap I_N$. Hence, by Theorem 2.2, $\mu \in L_\infty([0,1],Q) \cap I_N$.

We next consider Case 2. In Section 1 we have defined $\Xi(C)$ for $C \in \mathfrak{C}$. But, here we will use $\Xi(C)$ for any $C \subset [0,1]$ in the same definition. Let $\mu \in L_\infty(C,Q) \cap I_N$. By the assumption (4.18), we can find $\{C_n, n = 1,2,\dots\}$, a sequence of finite subsets of C, such that $b_n := \inf \Xi(C_n)$, $n \geq 1$, satisfy

$$\frac{\log b_n}{\log b_{n+1}} \in \mathbf{Z}_+ \tag{4.20}$$

and $b_n \uparrow 1$ as $n \to \infty$. To show this, we start with choosing an $a_1 \in C \setminus \{0,1\}$ and defining $C_1 = \{a_1\}$. Trivially, $b_1 = \inf \Xi(C_1) = a_1$. Next suppose that we are given $C_n \subset C \setminus \{0,1\}$ consisting of n elements. By (4.18), $\Xi(C_n) \neq \emptyset$. Then as in (4.16), there exist $k \geq 1, e_j \in C_n$ and $p_j \in \mathbf{Z}$ $(j = 1,2,\cdots,k)$ such that

$$b_n = \prod_{j=1}^{k} e_j^{p_j}. \tag{4.21}$$

Since $\Xi(C) = \emptyset$, there exists $a_{n+1} \in C \setminus \{0,1\}$ such that

$$\frac{\log a_{n+1}}{\log b_n} \notin \mathbf{Z},$$

and define C_{n+1} by $C_{n+1} = C_n \cup \{a_{n+1}\}$. By (4.18) again, $\Xi(C_{n+1}) \neq \emptyset$, and since $\Xi(C_{n+1}) \subset \Xi(C_n)$, we have that $b_{n+1} \geq b_n$. Since $e_j \in C_n \subset C_{n+1}, j = 1,2,\cdots,k$, we have

$$e_j = b_{n+1}^{\ell_{n+1}(e_j)} \quad \text{for some } \ell_{n+1}(e_j) \in \mathbf{Z}_+.$$

Thus by (4.21),

$$b_n = \prod_{j=1}^{k} b_{n+1}^{p_j \ell_{n+1}(e_j)} = b_{n+1}^{\sum_{j=1}^{k} p_j \ell_{n+1}(e_j)}.$$

Hence

$$\frac{\log b_n}{\log b_{n+1}} = \sum_{j=1}^{k} p_j \ell_{n+1}(e_j) \in \mathbf{Z}_+,$$

which is (4.20). Since $b_{n+1} > b_n$, $\log b_n / \log b_{n+1} \geq 2$. Thus $b_n \uparrow 1$ as $n \to \infty$. By (4.20), we see that $b_n = b_{n+1}^m$ for some $m \geq 2$. Thus, by the repeated use of Proposition 2.2,

$$L_\infty(b_{n+1}, Q) \subset L_\infty(b_n, Q).$$

If we let $\widetilde{C}_n$ ($\subset C$) be the smallest closed multiplicative subsemigroup including $C_n \cup \{0, 1\}$, then the assertion (i)' of this theorem gives us

$$L_\infty(\widetilde{C}_n, Q) \cap I_N = L_\infty(b_n, Q) \cap I_N,$$

and thus

$$L_\infty(C, Q) \cap I_N \subset \bigcap_{n \geq 1} L_\infty(b_n, Q) \cap I_N = \bigcap_{m<\infty} \bigcap_{n \geq 1} L_m(b_n, Q) \cap I_N.$$

Since $b_n \uparrow 1$, for any $b \in (0,1)$, we can find $n(k), m(k) \geq 1$ such that $n(k), m(k) \to \infty$ and $b_{n(k)}^{m(k)} \to b$ as $k \to \infty$. Hence we can show, as in the proof of Theorem 2.2, that $\bigcap_{n \geq 1} L_m(b_n, Q) \subset L_m(b, Q)$. Thus by Theorem 2.2, $L_\infty(C, Q) \cap I_N \subset L_\infty([0,1], Q) \cap I_N$ and we conclude the assertion (ii)'.

Step 3. Next we consider $L_\infty(C, Q) \cap I_G$. Let $\mu \in I_G$ with Gaussian covariance matrix A. Define, for any $k \geq 1$ and $b_j \in C \setminus \{0, 1\}, 1 \leq j \leq k$,

$$F_z(n_1, \cdots, n_k) = \left\langle A \prod_{j=1}^{k} b_j^{n_j Q'} z, \prod_{j=1}^{k} b_j^{n_j Q'} z \right\rangle. \tag{4.22}$$

We note the following two facts about $\mu \in I_G$. By Theorem 3.1,

(i) $\mu \in L_\infty(b, Q)$ if and only if for any $z \in \mathbf{R}^d$,

$$k_z(n) := \langle A b^{nQ'} z, b^{nQ'} z \rangle$$

is completely monotone in $\mathbf{Z}$, and by Lemma 4.2,

(ii) $\mu \in L_\infty(C,Q)$ if and only if for any $k \geq 1, b_j \in C \setminus \{0,1\}, 1 \leq j \leq k$, and any $z \in \mathbf{R}^d$, F_z in (4.22) is completely monotone in $\mathbf{Z}^k$.
Then we can apply the argument in Step 2 to show the statement of Theorem 4.4 for $\mu \in L_\infty(C,Q) \cap I_G$ by using F_z above in place of F_D in Step 2. □

REMARK 4.2. As we promised in Remark 3.1, we apply Theorem 4.4 to show that

$$L_m(C,Q) \supsetneqq L_{m+1}(C,Q) \tag{4.23}$$

for any $C \in \mathfrak{C}$. To show (4.23), we first note that Theorem 6.2 of Jurek (1983) implies that (4.23) is true for $C = [0,1]$, namely

$$L_m([0,1],Q) \supsetneqq L_{m+1}([0,1],Q). \tag{4.24}$$

Now suppose $L_m(C,Q) = L_{m+1}(C,Q)$ for some $C \in \mathfrak{C}$ and $m < \infty$. Then

$$L_{m+2}(C,Q) = K(L_{m+1}(C,Q),C,Q) = K(L_m(C,Q),C,Q) = L_{m+1}(C,Q),$$

and thus

$$L_m(C,Q) = L_\infty(C,Q).$$

By Theorem 4.4, $L_\infty(C,Q)$ equals either $L_\infty(b_0,Q)$ or $L_\infty([0,1],Q)$. If $L_\infty(C,Q) = L_\infty(b_0,Q)$, then $C \subset \{b_0^n\}_{n=0}^\infty \cup \{0\}$, and thus $L_m(b_0,Q) \subset L_m(C,Q)$ by Proposition 1.4 and by the repeated use of Proposition 2.3, which implies

$$L_m(b_0,Q) \subset L_\infty(C,Q) = L_\infty(b_0,Q),$$

and contradicts that $L_m(b_0,Q) \supsetneqq L_{m+1}(b_0,Q)$ as shown in Remark 3.1. If $L_\infty(C,Q) = L_\infty([0,1],Q)$, then

$$L_m([0,1],Q) \subset L_m(C,Q) = L_\infty(C,Q) = L_\infty([0,1],Q),$$

which contradicts (4.24). We thus conclude (4.23).

5. Examples for the relationship between $L_m(C,1)$ and $\bigcap_{b\in C\setminus\{0,1\}} L_m(b,1)$. Let us recall Theorem 2.2, where we have proved that for $0 \le m \le \infty$,

$$L_m([0,1],Q) = \bigcap_{b\in(0,1)} L_m(b,Q)$$

and

$$\widetilde{L}_m([0,1],Q) = \bigcap_{b\in(0,1)} \widetilde{L}_m(b,Q).$$

Then a natural question arises. If we replace $[0,1]$ by a general $C \in \mathfrak{C}$, then do similar relations hold ? The answer is yes for $m = 0$ just by the definition, and in general, as we have seen in (2.6), $L_m(C,Q) \subset \bigcap_{b\in C\setminus\{0,1\}} L_m(b,Q)$. In the following, we explicitly give $C \in \mathfrak{C}$ for which the reverse inclusion does not hold, to answer the above question negatively.

THEOREM 5.1. *Let $d = 1$ and $1 \le m < \infty$. Let p and q be two prime numbers satisfying $2(m+1) < p < q$, and let*

$$C = \{p^{-n_1}q^{-n_2} : n_1, n_2 \in \mathbf{Z}_+\} \cup \{0\}.$$

Then

(i) $L_m(C,1) \subsetneqq \bigcap_{b\in C\setminus\{0,1\}} L_m(b,1)$

and

(ii) $\widetilde{L}_m(C,1) \subsetneqq \bigcap_{b\in C\setminus\{0,1\}} \widetilde{L}_m(b,1)$.

PROOF. We first show (i). Let $\mu \in I(\mathbf{R}^d)$ be a purely non-Gaussian with Lévy measure ν given by

$$\nu = \sum_{(n_1,n_2)\in D} f(n_1,n_2)\delta_{p^{-n_1}q^{-n_2}}, \tag{5.1}$$

where

$$D = \{(n_1,n_2) : n_1 \ge 0, n_2 \ge 3\} \cup \{(n_1,2) : n_1 \ge 1\} \\ \cup \{(n_1,1) : n_1 \ge 3\} \cup \{(n_1,0) : n_1 \ge 4\}$$

and $f(n_1, n_2)$ is determined below. Let F be the boundary of D, that is,

$$F = \{(0, n_2) : n_2 \geq 3\} \cup \{(n_1, 0) : n_1 \geq 4\} \cup \{(1,2), (2,2), (3,1)\}.$$

Let $D_0 = D \setminus (F \cup \{(3,2)\})$. The function $f(n_1, n_2)$ in (5.1) is determined as follows : First define f for $(n_1, n_2) \in F$ by

$$\begin{cases} f(0, n_2) &= 0, \qquad n_2 \geq 3, \\ f(1,2) &= \frac{2}{m+2}, \\ f(2,2) &= \frac{2m+2}{m+2}, \\ f(3,1) &= 1, \\ f(n_1, 0) &= 0, \qquad n_1 \geq 4, \end{cases} \tag{5.2}$$

and

$$f(3,2) = m + 1. \tag{5.3}$$

For $(n_1, n_2) \in D_0$, define

$$f(n_1, n_2) = (m+1)\{f(n_1 - 1, n_2) + f(n_1, n_2 - 1)\}, \tag{5.4}$$

starting from the nearest points to F successively. We define, for convenience,

$$f(n_1, n_2) = 0, \qquad (n_1, n_2) \in \mathbf{Z}^2 \setminus D.$$

The function $f(n_1, n_2)$ is thus nondecreasing both in n_1 and in n_2. Observe that

$$f(n_1, n_2) \leq (2(m+1))^{n_1+n_2}. \tag{5.5}$$

For, first this relation is obvious for $(n_1, n_2) \in F \cup \{(3,2)\}$. For $(n_1, n_2) \in D_0$, if $f(n_1, n_2 - 1)$ and $f(n_1 - 1, n_2)$ satisfy (5.5), so does $f(n_1, n_2)$ by (5.4). Thus (5.5) is true. It follows from (5.5) that

$$\begin{aligned} \int_0^\infty x\nu(dx) &= \sum_{(n_1,n_2)\in D} p^{-n_1} q^{-n_2} f(n_1, n_2) \\ &\leq \sum_{n_1,n_2 \geq 0} \left(\frac{2(m+1)}{p}\right)^{n_1+n_2} \\ &= \left(\frac{p}{p - 2(m+1)}\right)^2 < \infty. \end{aligned} \tag{5.6}$$

Thus ν in (5.1) can be the Lévy measure of some $\mu \in I(\mathbf{R}^d)$. We see from (5.2)–(5.4) that

(5.7) $f(n_1, n_2) \geq (m+1)\{f(n_1-1, n_2) + f(n_1, n_2-1)\}$ for $(n_1, n_2) \neq (3,2)$

and

$$f(3,2) = \frac{m+2}{2} f(2,2). \tag{5.8}$$

By the monotonicity of $f(n_1, n_2)$ and (5.7), for any $(n_1, n_2) \in \mathbf{Z}^2$ (including $(n_1, n_2) = (3,2)$),

$$f(n_1, n_2) \geq (m+1) f(n_1 - k_1, n_2 - k_2) \tag{5.9}$$

if $k_1 \geq 2$ and $k_2 \geq 0$ or if $k_1 \geq 0$ and $k_2 \geq 1$. Now define, for $k_1, k_2 \in \mathbf{Z}_+$ with $(k_1, k_2) \neq (0,0)$,

$$\Delta_{(k_1,k_2)} f(n_1, n_2) = f(n_1, n_2) - f(n_1 - k_1, n_2 - k_2),$$

and denote the ℓ times iteration of $\Delta_{(k_1,k_2)}$ by $\Delta^{\ell}_{(k_1,k_2)}$. Write Δ_1 for $\Delta_{(1,0)}$ and Δ_2 for $\Delta_{(0,1)}$. If we let, for $j \geq 0$ and $1 \leq \ell \leq m+1$,

$$G_j = f(n_1 - 2jk_1, n_2 - 2jk_2) - \frac{\ell - 2j}{2j+1} f(n_1 - (2j+1)k_1, n_2 - (2j+1)k_2),$$

then

$$\begin{aligned} \Delta^{\ell}_{(k_1,k_2)} f(n_1, n_2) &= \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j f(n_1 - jk_1, n_2 - jk_2) \\ &= \sum_{j=0}^{[\frac{\ell-1}{2}]} \binom{\ell}{2j} G_j + R, \end{aligned} \tag{5.10}$$

where $[x]$ is the greatest integer less than or equal to x, and

$$R = \begin{cases} f(n_1 - \ell k_1, n_2 - \ell k_2), & \text{if } \ell \text{ is even,} \\ 0, & \text{if } \ell \text{ is odd.} \end{cases}$$

We are now going to show that for any $(n_1, n_2) \in \mathbf{Z}^2$ and any $1 \le \ell \le m+1$,

$$\Delta^{\ell}_{(k_1,k_2)} f(n_1, n_2) \ge 0. \tag{5.11}$$

If $k_1 \ge 2$ or if $k_2 \ge 1$, then (5.11) follows from (5.9) and (5.10), since

$$\frac{\ell - 2j}{2j+1} \le \ell \le m+1 \quad \text{for } j \ge 0.$$

It remains to show (5.11) for $k_1 = 1$ and $k_2 = 0$. Namely, it is enough to show that for $1 \le \ell \le m+1$, $(n_1, n_2) \in \mathbf{Z}^2$,

$$\Delta^{\ell}_1 f(n_1, n_2) \ge 0. \tag{5.12}$$

Rewrite G_j with $k_1 = 1$ and $k_2 = 0$ as

$$g_j = f(n_1 - 2j, n_2) - \frac{\ell - 2j}{2j+1} f(n_1 - 2j - 1, n_2), \qquad j \ge 0.$$

When $n_2 \ne 2$, $g_j \ge 0$ for $j \ge 0$ by (5.7) and thus (5.12) holds by (5.10). Let $n_2 = 2$. Notice that

$$\frac{\ell - 2j}{2j+1} \le \frac{m+2}{2} \quad \text{for } j \ge 1.$$

We have $g_j \ge 0$ for $j \ge 1$ by using (5.7) when $n_1 - 2j \ne 3$ and (5.8) when $n_1 - 2j = 3$. Also, if $n_1 \ne 3$, then $g_0 \ge 0$ by (5.7). Thus if $n_1 \ne 3$, then $g_j \ge 0$ for $j \ge 0$, and we have (5.12) for $n_1 \ne 3$, $n_2 = 2$. Finally, we have

$$\begin{aligned}
\Delta^{\ell}_1 f(3,2) &= f(3,2) - \ell f(2,2) + \frac{\ell(\ell-1)}{2} f(1,2) \\
&= (m+1) - \ell \frac{2(m+1)}{m+2} + \frac{\ell(\ell-1)}{2}\frac{2}{m+2} \\
&= \frac{1}{m+2}\{\ell^2 - (2m+3)\ell + (m+1)(m+2)\} \ge 0.
\end{aligned}$$

This concludes (5.12) for all $(n_1, n_2) \in \mathbf{Z}^2$ and thus (5.11).

We next examine $\Delta^m_2 \Delta_1 f(n_1, n_2)$. For $(n_1, n_2) = (3, 2)$, we see

$$\begin{aligned}
\Delta^m_2 \Delta_1 f(3,2) &= \Delta^m_2 f(3,2) - \Delta^m_2 f(2,2) \\
&= f(3,2) - m f(3,1) - f(2,2) = -\frac{m}{m+2} < 0.
\end{aligned} \tag{5.13}$$

We observe

$$\Delta_2^m \Delta_1 f(n_1, n_2) \geq 0, \qquad \forall (n_1, n_2) \neq (3, 2). \tag{5.14}$$

To show it, let

$$H_j = \Delta_1 f(n_1, n_2 - 2j) - \frac{m-2j}{2j+1} \Delta_1 f(n_1, n_2 - 2j - 1), \qquad j \geq 0.$$

Then

$$\begin{aligned} \Delta_2^m \Delta_1 f(n_1, n_2) &= \sum_{j=0}^{m} \binom{m}{j} (-1)^j \Delta_1 f(n_1, n_2 - j) \\ &= \sum_{j=0}^{[\frac{m-1}{2}]} \binom{m}{2j} H_j + R^*, \end{aligned} \tag{5.15}$$

where

$$R^* = \begin{cases} \Delta_1 f(n_1, n_2 - m), & \text{if } m \text{ is even,} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$$

By (5.7), for $(n_1, n_2) \neq (3, 2)$,

$$\begin{aligned} \Delta_1 f(n_1, n_2) &\geq m f(n_1 - 1, n_2) + (m+1) f(n_1, n_2 - 1) \\ &\geq (m+1) f(n_1, n_2 - 1) \\ &\geq (m+1) \Delta_1 f(n_1, n_2 - 1). \end{aligned} \tag{5.16}$$

From this together with (5.15), we have (5.14) for $n_1 \neq 3$. When $n_1 = 3$, we have $H_j \geq 0$ by (5.16) if $n_2 - 2j \neq 2$. When $n_1 = 3$ and $n_2 - 2j = 2$, note that

$$\Delta_1 f(3, 2) = \frac{m(m+1)}{m+2} \Delta_1 f(3, 1)$$

and

$$\frac{m-2j}{2j+1} \leq \frac{m(m+1)}{m+2} \quad \text{for } j \geq 1.$$

Altogether we have shown that $H_j \geq 0$ for $j \geq 0$ when $n_1 = 3$ and $n_2 \neq 2$, and thus we have (5.14). Particularly, notice that

$$\Delta_2^m \Delta_1 f(1, 2) = \Delta_2^m f(1, 2) = f(1, 2) > 0. \tag{5.17}$$

On the other hand,

(5.18) $$\Delta_2^m \Delta_1 f(n_1, n_2) = 0 \quad \text{for } (n_1, n_2) \text{ satisfying } f(n_1, n_2) = 0,$$

and for $(n_1, n_2) \in D$, by (5.4)

(5.19)
$$\begin{aligned}
&|\Delta_2^m \Delta_1 f(n_1, n_2)| \\
&\le \sum_{j=0}^{m} \binom{m}{j} f(n_1, n_2 - j) + \sum_{j=0}^{m} \binom{m}{j} f(n_1 - 1, n_2 - j) \\
&\le 2 \sum_{j=0}^{m} \binom{m}{j} (2(m+1))^{n_1+n_2} \\
&= 2^{m+1} (2(m+1))^{n_1+n_2}.
\end{aligned}$$

Choose any $b = p^{-k_1} q^{-k_2} \in C \setminus \{0, 1\}$. Then by (5.11), we have that, for any $1 \le \ell \le m+1$,

$$(1 - T_b)^\ell \nu = \sum_{(n_1, n_2) \in \mathbf{Z}^2} \Delta^\ell_{(k_1, k_2)} f(n_1, n_2) \delta_{p^{-n_1} q^{-n_2}} \ge 0.$$

Thus by Theorem 3.1,

$$\mu \in \bigcap_{b \in C \setminus \{0,1\}} L_m(b, 1).$$

On the other hand, it follows from (5.13) that

$$(1 - T_{q^{-1}})^m (1 - T_{p^{-1}}) \nu(\{p^{-3} q^{-2}\}) = \Delta_2^m \Delta_1 f(3, 2) < 0.$$

Thus by Lemma 4.1,

$$\mu \notin L_m(C, 1).$$

This completes the proof of (i).

We next show (ii). For $t > 0$, define $\mu_t = \mu^{*t}$, using the same μ. We can similarly show that

$$\mu_t \in \bigcap_{b \in C \setminus \{0,1\}} L_m(b, 1) \subset \bigcap_{b \in C \setminus \{0,1\}} \widetilde{L}_m(b, 1).$$

Choose the drift to be 0. Then

$$\widehat{\mu}_t(z) = \exp\left\{t\int_0^\infty (e^{izx}-1)\nu(dx)\right\}$$

with ν in (5.1). Let us show that, for any sufficiently small $t > 0$

$$\mu_t \notin \widetilde{L}_m(C,1). \tag{5.20}$$

Suppose that this is not true. Then there is a sequence $t_n \downarrow 0$ such that $\mu_{t_n} \in \widetilde{L}_m(C,1)$. Write $t = t_n$ for a while. Note that $q^{-1} \in C$. Then, there exists $\rho_{t,m-1} \in \widetilde{L}_{m-1}(C,1)$ such that

$$\widehat{\mu}_t(z) = \widehat{\mu}_t(q^{-1}z)\widehat{\rho}_{t,m-1}(z). \tag{5.21}$$

Repeating this, we can find $\rho_{t,j-1} \in \widetilde{L}_{j-1}(C,1)$, $1 \le j \le m-1$, such that

$$\widehat{\rho}_{t,j}(z) = \widehat{\rho}_{t,j}(q^{-1}z)\widehat{\rho}_{t,j-1}(z). \tag{5.22}$$

Now, using $p^{-1} \in C$, we can find $\eta_t \in \mathcal{P}(\mathbf{R}^d)$ such that

$$\widehat{\rho}_{t,0}(z) = \widehat{\rho}_{t,0}(p^{-1}z)\widehat{\eta}_t(z). \tag{5.23}$$

It follows from (5.21)–(5.23) that

$$\widehat{\eta}_t(z) = \exp\left\{t\int_0^\infty (e^{izx}-1)\nu_0(dx)\right\}, \tag{5.24}$$

where

$$\nu_0 = \sum_{(n_1,n_2)\in\mathbf{Z}^2} \Delta_2^m\Delta_1 f(n_1,n_2)\delta_{p^{-n_1}q^{-n_2}}.$$

If we put $c = p^{-3}q^{-2}$ and $\varepsilon = \frac{m}{m+2}$, then, from (5.13), (5.14), (5.17), (5.18), and (5.19), there exists a measure ν_1 on $\{p^{-n_1}q^{-n_2} : (n_1,n_2) \in D\}$ such that

$$\nu_0 = -\varepsilon\delta_c + \nu_1, \tag{5.25}$$

the support of ν_1 does not contain c, and $\int_0^\infty x\nu_1(dx) < \infty$. If we denote by ξ_t an infinitely divisible distribution on $[0,\infty)$ with its characteristic function (5.24) with the replacement of ν_0 by ν_1, then by (5.24) and (5.25),

$$\eta_t = \xi_t * \left(\sum_{n=0}^\infty e^{\varepsilon t}\frac{(-\varepsilon t)^n}{n!}\delta_{cn}\right) = e^{\varepsilon t}\sum_{n=0}^\infty \frac{(\varepsilon t)^n}{n!}(-1)^n\xi_t * \delta_{cn}.$$

If we choose $h \in (0, c)$ as small as $\nu_1([c, c+h]) = 0$, we have

$$\eta_t([c, c+h]) = e^{\varepsilon t}\{\xi_t([c, c+h]) - \varepsilon t \xi_t([0, h])\}. \tag{5.26}$$

Now recall that $t = t_n$ and let $n \to \infty$. Then,

$$\frac{1}{t_n}\xi_{t_n}([c, c+h]) \to \nu_1([c, c+h]) = 0 \tag{5.27}$$

and

$$\xi_{t_n}([0, h]) \to 1. \tag{5.28}$$

Combining (5.26)–(5.28), we see that $\eta_{t_n}([c, c+h]) < 0$ for sufficiently large n, which is absurd. This completes the proof of the theorem. □

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