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Prime Geodesic Theorem for Arithmetic Compact Surfaces

by

Shin-ya Koyama

Shin-ya Koyama
Department of Mathematics
Keio University

Department of Mathematics
Faculty of Science and Technology
Keio University

3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522 Japan

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PRIME GEODESIC THEOREM FOR ARITHMETIC COMPACT SURFACES

SHIN-YA KOYAMA

0. Introduction. For a discrete subgroup Γ of $PSL(2,\mathbf{R})$ with finite quotient $\Gamma\backslash PSL(2,\mathbf{R})$, the Selberg zeta function is defined by

$$Z_\Gamma(s) = \prod_P \prod_{k=0}^{\infty} (1 - N(P)^{-s-k}) \tag{0.1}$$

for $\mathrm{Re}(s) > 1$ where $\{P\}$ runs over the set of all primitive hyperbolic conjugacy classes in Γ and the norm $N(P)$ is defined by $N(P) = \alpha^2$ where $\alpha > 1$ is the bigger eigenvalue of P. By regarding $N(P)$ as an analog of prime numbers and by using the Selberg zeta function instead of the Riemann zeta function, we have the same asymptotic distribution as that of the rational primes, namely

$$\pi_\Gamma(x) = \#\{\{P\} \ : \ N(P) \le x\} \sim \mathrm{li}x. \tag{0.2}$$

This is called the Prime Geodesic Theorem, since a primitive hyperbolic class $\{P\}$ corresponds to a prime closed geodesic p of the Riemann surface $\Gamma\backslash\mathbf{H}$, and the norm is expressed as $N(P) = e^{\mathrm{length}(p)}$.

For the classical prime number theorem, finding a good error term is equivalent to obtaining a zero-free region of the Riemann zeta function in the critical strip. The error term has a form $O(x^{\rho+\epsilon})$, where ρ is the supremum of the real part of the non-trivial zeros. Therefore the Riemann Hypothesis $\rho = \frac{1}{2}$ gives the ultimate estimate $O(x^{\frac{1}{2}+\epsilon})$ of the error term.

This situation is quite different in the geometric version (0.2). Even when one assumes the Riemann Hypothesis for the Selberg zeta function, which is equivalent to the fact that the first Laplace eigenvalue λ_1 is larger than $1/4$, the resulting error term is $O(x^{\frac{3}{4}+\epsilon})$, which is far from the conjectured $O(x^{\frac{1}{2}+\epsilon})$. The reason for this phenomenon is, as is explained in [I, §1], abundance of zeros of the zeta functions. Obtaining the error term $O(x^{\frac{3}{4}+\epsilon})$ is equivalent to proving $\lambda_1 > 3/16$, which is weaker than the Riemann Hypothesis and was proved when Γ is congruence type. Therefore the analysis of the Selberg zeta function is not sufficient to improve the error term. We need more elaborated machinery and deeper ideas for this problem.

Iwaniec[I] obtained the error term $O(x^{\frac{35}{48}+\epsilon})$ for $\Gamma = PSL(2, \mathbf{Z})$, which was the first improvement. He proved some estimates towards the Lindelöf hypothesis for automorphic L-functions for cusp forms and used the Kuznetsov formula which provides us with a relation between a sum over eigenvalues of the Laplacian and a certain sum of Kloosterman sums. Luo-Sarnak[LS] broke his record by completing the proof of the mean-Lindelöf hypothesis in the theory of arithmetic quantum chaos. They obtained $O(x^{\frac{7}{10}+\epsilon})$ and this is the current best error term for prime geodesic theorem. Although they proved for $\Gamma = PSL(2, \mathbf{Z})$, the results are valid for congruence subgroups by appealing to the newer results on non-existence of certain small eigenvalues proved by Luo-Rudnick-Sarnak [LRS]. But a generalization to any cocompact Γ seemed difficult, since their method heavily depended on the analysis of Eisenstein series and Poincare series. The best known error term of (0.2) for any compact cases was no better than $O(x^{\frac{3}{4}+\epsilon})$, and improving it has been an open problem ([LS, §1]).

The aim of this paper is to prove the following theorem:

Theorem. *Let Γ_D be the cocompact discrete subgroup of $PSL(2, \mathbf{R})$ coming from a quaternion algebra $D = \left(\frac{a,b}{\mathbf{Q}}\right)$ with $(a, b) = 1$ and the prime 2 being unramified. For any $\epsilon > 0$,*

$$\pi_{\Gamma_D}(x) = \mathrm{li}(x) + O(x^{\frac{7}{10}+\epsilon}).$$

Note that any arithmetic cocompact subgroup of $PSL(2, \mathbf{R})$ is commensurable with the unit group of a quaternion algebra over a number field ([T]). The above theorem suggests that the estimate would be true for any arithmetic cocompact Γ.

The proof uses the Jacquet-Langlands correspondence. The key lemma (Lemma 1) reveals that its image is equal to the set of new forms for a certain level, which is equal to the product of all ramified primes. It implies that the spectra for such arithmetic compact surfaces are completely expressed via those for a certain congruence surface. It gives a new prospect for the study of such compact surfaces. For instance we see arithmetic compact surfaces are isospectral if they have the same sets of ramified primes.

Corollary. *The first Laplace eigenvalue* λ_1 *for* $\Gamma_D\backslash H$ *satisfies*

$$\lambda_1 \geq \frac{21}{100}.$$

The previously best record was $\lambda_1 \geq \frac{3}{16}$ [H, p.186]. (cf. [SX, Corollary 2])

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1. Jacquet-Langrands correspondence. Let $D = \left(\frac{a,b}{\mathbf{Q}}\right)$ be the quaternion algebra over $\mathbf{Q}$ linearly generated by $1, \omega, \Omega, \omega\Omega$, where $\omega^2 = a$, $\Omega^2 = b$, $\omega\Omega + \Omega\omega = 0$, a, b are square-free positive integers. Let S be the set of primes p such that $D_p = D \otimes_{\mathbf{Q}} \mathbf{Q}_p$ is a division algebra. Elements of S are called ramified. Throughout this paper we assume $(a, b) = 1$ and $2 \notin S$ Let R be a maximal order of D. For $m \geq 1$, put $R(m) = \{x \in R \mid N(x) = m\}$. Denote by θ the map $D \ni x \mapsto \begin{pmatrix} \xi & \eta \\ b\bar{\eta} & \bar{\xi} \end{pmatrix} \in M_2(\mathbf{Q}(\sqrt{a}))$, where $x = x_0 + x_1\omega + x_2\Omega + x_3\omega\Omega = \xi + \eta\Omega$. Let $\theta(R(1)) = \Gamma_D \subset SL(2, \mathbf{R})$. It is well-known that the surface $\Gamma_D\backslash H$ is compact [GGP, p.115].

Lemma 1. *The set of Laplace eigenvalues for* $\Gamma_D\backslash H$ *is equal to the set of those of new forms for* $\Gamma_0(N)\backslash H$*, where* N *is an integer depending only on* D*.*

Proof. According to Hejhal [H], who explicitly writes the image of the Jacquet-Langlands correspondence, the Laplace eigenvalue of a Maass cusp form for Γ_D is equal to that of its image via the correspondence. All we need prove is that the image is a new form for a congruence subgroup which depends only on D. We will prove it is $\Gamma_0(N)$ with $N = \prod_{p \in S} p$.

For a cusp form $\pi' = \otimes \pi'_p$ for Γ_D, we first prove the conductor $c(\pi)$ of its image $\pi = \otimes \pi_p$ by the Jacquet-Langlands correspondence is equal to N.

If $p \notin S$, π'_p is equivalent to π_p by [G, Theorem 10.5]. So π_p is class 1 and its conductor is 1 by [G, Remark 4.25].

If $p \in S$, π_p is given by the irreducible component of the Weil representation [G, Theorem 10.5]. In case $p \in S$, π_p is defined via the quaternion algebra D_p. By [G, Theorem 7.6], π_p is either supercuspidal or special.

We claim it is special. Assume π_p is supercuspidal. Then by [G, (4.20)] the conductor is $c(\pi_p) = p^e$ for some $e \geq 2$. So the global conductor $c(\pi)$ contains an odd square. On the other hand we have $c(\pi) | 4ab$ by [H, Theorem 5.2]. It shows $c(\pi)$ does not contain any odd squares, which gives a contradiction.

Since π_p is special for $p \in S$, we have by [G, (4.20)] that the conductor $c(\pi_p)$ is either p or the square of the conductor of a certain character μ. From [G, Remark 4.25] the latter case happens only when μ is nontrivial on $\mathbf{Z}_p^*$, which means the conductor again contains an odd square and leads to the contradiction. Hence $c(\pi_p) = p$ for all $p \in S$.

Considering the non-holomorphic version of [G, Proposition 5.21], we deduce π is a new form of $\Gamma_0(N)$. Taking into account [G, Theorem 5.19], the image agrees to the set of all new forms. $\square$

Remark. According to [V, p.58], $p \in S$ if and only if the Hilbert symbol $(a,b)_p = -1$. We have the following properties:

(1) If $p = \infty$, $(a,b)_p = \begin{cases} 1 & (a > 0 \text{ or } b > 0) \\ -1 & (a, b < 0) \end{cases}$

(2) If $p \neq 2$, $(a,b)_p = 1$ if $p \nmid a$ and $p \nmid b$.

Since we assume $a > 0$, we have $\infty \notin S$ by (1). Any element in S is a divisor of ab by (2). Hence the level N is a divisor of ab.

2. New Forms. In this section we compute the contribution from non-exceptional eigenvalues $\lambda = \frac{1}{4} + r^2$ with $r \in \mathbf{R}$.

Lemma 2. *Put* $\lambda_j = \frac{1}{4} + r_j^2$ $(j = M+1, M+2, \ldots)$ *be non-exceptional Laplace eigenvalues on* $L^2(\Gamma_0(N)\backslash H)$, *then*

$$\sum_{j:\ \text{new form for } \Gamma_0(N)} X^{ir_j} \exp(-r_j/T) \ll T^{\frac{5}{4}+\epsilon} X^{\frac{1}{8}}. \tag{1}$$

Proof. Let l be the number of prime divisors of N. We prove by induction on l. When $l = 1$, letting $N = p$ be a prime, the space of old forms for $\Gamma_0(p)$ is spanned by

$$S_1 = \{f(z),\ f(pz) \mid f \text{ is a cusp form for } \Gamma(1)\}.$$

The set of eigenvalues coming from old forms for $\Gamma_0(p)$ is the same as that for $\Gamma(1)$. So we have

$$\sum_{j:\ \text{old form for } \Gamma_0(N)} X^{ir_j} \exp(-r_j/T) \ll \sum_{j:\ \text{for } \Gamma(1)} X^{ir_j^{(1)}} \exp(-r_j^{(1)}/T),$$

where $\lambda_j^{(1)} = \frac{1}{4} + {r_j^{(1)}}^2$ is an eigenvalue for $\Gamma(1)$. The last sum is estimated by $T^{\frac{5}{4}+\epsilon} X^{\frac{1}{8}}$ by [LS]. On the other hand, the sum over all cusp forms for $\Gamma_0(N)$ should satisfy the same bound, since the Kuznetsov formula [K] is generalized to congruence subgroups by Proskurin [P] and Deshouillers-Iwaniec [DI]. For applying the Kuznetsov formula, we have to take orthogonal system consisting of cusp forms. As the Laplacian is self-adjoint, all eigenspaces are orthogonal. For functions which belong to the same eigenspace, we can modify them by Schmidt's orthogonalization to get orthogonal basis. By applying the resulting basis to the Kuznetsov formula, we have the desired result for $l = 1$.

Next assume we have proved for all cases when the number of prime divisors of the level is smaller than l. It suffices to estimate the sum over old forms for $\Gamma_0(N)$.

The space of old forms for $\Gamma_0(N)$ is spanned by the set

$$S_N = \{f(mz) \mid f \text{ is a new form of } \Gamma_0(d),\ m|(N/d),\ d|N,\ d \neq N\}.$$

Let $\lambda_j^{(d)} = \frac{1}{4} + r_j^{(d)^2}$ be non-exceptional eigenvalues for $\Gamma_0(d)$. The set of eigenvalues concerning S_N agrees with $\{\lambda_j^{(d)} \mid m|\frac{N}{d},\ d|N,\ d \neq N\}$. Therefore

$$\begin{aligned} \sum_{j:\ \text{old form for } \Gamma_0(N)} & X^{ir_j} \exp(-r_j/T) \\ & \ll \sum_{\substack{d|N \\ d \neq N}} \sum_{m|\frac{N}{d}} \sum_{j:\ \text{new form for } \Gamma_0(d)} \left(X^{ir_j^{(d)}} \exp(-r_j^{(d)}/T) \right) \\ & \ll T^{\frac{5}{4}+\epsilon} X^{\frac{1}{8}}, \end{aligned}$$

by the assumption. □

3. Proofs. In this section we complete the proof of Theorem. For hyperbolic conjugacy class P of $\Gamma = \Gamma_D$, we put $\Lambda(P) = \log N(P_0)$ if P is a power of the prmitive hyperbolic class P_0. Put $\Psi_\Gamma(x) = \sum_{N(P) \leq X} \Lambda(P)$. We have the following explicit formula (cf. [I, Lemma 1]): for $1 \leq T \leq \sqrt{X}(\log X)^{-2}$,

$$\Psi_\Gamma(X) = X + \sum_{j=1}^{M} \frac{X^{s_j}}{s_j} + X^{\frac{1}{2}} \sum_{j>M,\ |r_j| \leq T} \frac{X^{ir_j}}{s_j} + O\left(\frac{X}{T} \log^2 X\right) \tag{3.1}$$

where s_j $(j = 1, 2, 3, ..., M)$ are the real zeros of $Z_\Gamma(s)$ with $0 < \mathrm{Re}(s_j) < 1$, $\mathrm{Re}(s_j) \neq \frac{1}{2}$, and s_j $(j = M+1, M+2, ...)$ are those with $\mathrm{Re}(s_j) = \frac{1}{2}$ and $r_j = \mathrm{Im}(s_j)$. Lemma 1 implies that for all j the contribution of a zero s_j is equal to that of a zero for some fixed congruence subgroup. By [LRS, Thoerem 1.1], for any congruence subgroup, small eigenvalues $\lambda = s(1-s)$ with $s > \frac{7}{10}$ do not exist. Therefore the contribution from real zeros satisfies $O(X^{\frac{7}{10}+\epsilon})$. The sum over non-exceptional zeros can be estimated by Lemma 2. By following the method of [I] we have Theorem in §0. □

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