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A Confluent Hypergeometric System Associated with Φ_3 and a Confluent Jordan-Pochhammer Equation

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Abstract. We treat a confluent hypergeometric system associated with Φ_3. Near one of the singular loci of irregular type, asymptotic expansions and Stokes multipliers are obtained. Applying our results, we also clarify the asymptotic behaviour of linearly independent solutions of a confluent Jordan-Pochhammer equation.

1. Introduction

The series

$$\Phi_3(\beta, \gamma, x, y) = \sum_{m,n \geq 0} \frac{(\beta)_m}{(\gamma)_{m+n} m! n!} x^m y^n \qquad (|x| < \infty,\ |y| < \infty) \tag{1.1}$$

with $(\beta)_m = \Gamma(\beta + m)/\Gamma(\beta)$ $(m \in \mathbf{Z})$ is one of the confluent hypergeometric functions derived from Appell's hypergeometric function $F_1(\alpha, \beta, \beta', \gamma, x, y)$ ([5], [6], [9]). It satisfies a system of partial differential equations

$$\begin{aligned} &x z_{xx} + y z_{xy} + (\gamma - x) z_x - \beta z = 0, \\ &y z_{yy} + x z_{xy} + \gamma z_y - z = 0 \end{aligned} \tag{1.2}$$

([2; §§5.7, 5.9]) for $(x, y) \in P^1(\mathbf{C}) \times P^1(\mathbf{C})$. Since $x z_{xy} - z_x + \beta z_y = 0$, this system is equivalent to a completely integrable Pfaffian system with respect to the unknown vector function $(z, x z_x, y z_y)$, which possesses the singular loci $x = 0$, $x = \infty$, $y = \infty$ of irregular type, and $y = 0$ of regular type. The solutions of (1.2), which are analytic in $\mathcal{R}^2$, constitute a three-dimensional vector space over $\mathbf{C}$, where $\mathcal{R}$ denotes the universal covering of $\mathbf{C} - \{0\}$. In [11], we studied the asymptotic behaviour of linearly independent solutions of (1.2) near the singular loci $x = \infty$ and $x = 0$. Eliminating the derivatives with respect to x from (1.2), and putting $x = \kappa \in \mathbf{C} - \{0\}$, we obtain an ordinary differential equation of the form

$$y \frac{d^3 z}{dy^3} - \left(\frac{y}{\kappa} + (\beta - \gamma - 1) \right) \frac{d^2 z}{dy^2} - \left(1 + \frac{\gamma}{\kappa} \right) \frac{dz}{dy} + \frac{z}{\kappa} = 0, \tag{1.3}$$

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which has the singular points $y = 0$ of regular type and $y = \infty$ of irregular type. It is easy to see that, for each fixed $x = \kappa$ ($\in \mathbf{C} - \{0\}$), every solution of (1.2) satisfies equation (1.3). This is also derived from the Jordan-Pochhammer equation

$$\begin{aligned} &y(1-y)(y-\kappa)\frac{d^3z}{dy^3} \\ &\quad + \Big[\big((\beta'+1)y+\beta\kappa\big)(1-y)+\big((\gamma+1)-(\alpha+\beta'+3)y\big)(y-\kappa)\Big]\frac{d^2z}{dy^2} \\ &\quad + (\beta'+1)\big(\gamma-(\alpha+\beta'+1)y-\beta\kappa-(\alpha+1)(y-\kappa)\big)\frac{dz}{dy}-\alpha\beta'(\beta'+1)z=0 \end{aligned}$$

(see [4; §3.4]) by a process of making a confluence of singular points. In fact, replacing $(\alpha, \beta', \kappa, y)$ by $(1/\varepsilon, 1/\varepsilon, \varepsilon\kappa, \varepsilon^2 y)$ and letting $\varepsilon \to 0$, we arrive at equation (1.3), which is one of the confluent Jordan-Pochhammer equations.

The present paper gives asymptotic expansions and Stokes multipliers of linearly independent solutions of (1.2) near the irregular singular locus $y = \infty$, and clarify the global behaviour of the solutions of equation (1.3). As in [11] we assume that none of the complex numbers $\beta, \gamma, \beta - \gamma$ is an integer, and use the notation

$$e^{(\lambda)} = \exp(2\pi i \lambda) \qquad (\lambda \in \mathbf{C}).$$

Recall solutions of (1.2) expressible in the form

$$\begin{aligned} z_{-1} &= z_{-1}(x,y) = (1-e^{(-\beta)})^{-1}\int_{C_{-1}} f(\beta,\gamma;x,y,t)dt, \\ z_a &= z_a(x,y) = \int_{C_a} f(\beta,\gamma;x,y,t)dt \qquad (a=\pm 2), \end{aligned}$$

where

$$f(\beta,\gamma;x,y,t) = t^{\beta-\gamma}(t-x)^{-\beta}\exp\left(t+\frac{y}{t}\right) \tag{1.4}$$

([11; §2]). This integrand is obtained from one corresponding to $\Phi_2(\beta, 1/\varepsilon, \gamma, x, \varepsilon y)$ ([1]) by the limiting procedure $\varepsilon \to 0$. (For confluences of the cycles of integral representations, see [3].) In each integral, the path and the branch of the integrand are taken under the condition

$$0 \le \arg y < \pi/2 < \arg x \le \pi \tag{1.5}$$

so that they have the following properties:

(1) The path C_{-1} starts from $t=0$, encircles $t=x$ in the positive sense, and returns to $t=0$. Then, along C_{-1}, $(\arg t, \arg(t-x))$ varies from $(\pi+\arg y, -\pi+\arg x)$ to $(\pi+\arg y, \pi+\arg x)$.

(2) The path C_2 (or C_{-2}) starts from $t=0$ and terminates in $t=\infty$. Then, along C_2 (or C_{-2}), $(\arg t, \arg(t-x))$ varies from $(\pi+\arg y, -\pi+\arg x)$ to $(\pi, -\pi)$ (or $(-\pi, -\pi)$).

Each integral is continued analytically to the whole domain $\mathcal{R}^2$, if we modify the path continuously preserving conditions imposed on $(\arg t, \arg(t-x))$ at both ends of it. We consider the triplet of linearly independent solutions z_{-1}, z_2, z_{-2} of (1.2) near the sigular locus $y=\infty$. (The linearly independence follows from [11; Proposition 2.1 and Theorem 3.1].) The main results concerning asymptotic expansions and Stokes multipliers of these solutions are stated in Section 2. The proofs of them are given in Section 3 and Section 4. In the calculation of asymptotic expansions, the saddle point method is employed, and in the derivation of Stokes multipliers, the monodromy matrices obtained in [11] are used. It may be interesting to treat these Stokes multipliers from a group-theoretic point of view ([8]). In the final section, we apply our results to equation (1.3), and clarify the global behaviour of its solutions, namely asymptotic expansions, Stokes multipliers (near $y=\infty$), and convergent series expansions in $0<|y|<\infty$. They are described explicitly by well-known special functions. These solutions of (1.3) are expected to be applicable to a global study of a third or higher order linear differential equation with one or more irregular singularities (cf. [7], [10]).

2. Main results

In what follows, δ denotes an arbitrary small positive constant, R an arbitrary one satisfying $R \geq 2.44$, and δ_R an arbitrary one satisfying

$$\sin^{-1}(2R(R^2-1)^{-1}) + \sin^{-1}(R^{-2}) < \delta_R < \pi/2. \tag{2.1}$$

For example, we can take $\delta_R = \pi/100$ (if $R \geq 65$), $\delta_R = \pi/5$ (if $R \geq 4$), and $\delta_R = \pi/2 - \pi/821$ (if $R \geq 2.44$).

2.1. Asymptotic expansions

Let $P_m^{(a,b)}(s)$ be the Jacobi polynomial

$$P_m^{(a,b)}(s) = \sum_{j=0}^{m} \binom{a+m}{j} \binom{b+m}{m-j} \left(\frac{s+1}{2}\right)^j \left(\frac{s-1}{2}\right)^{m-j}$$

(see [2; §10.8,(12),(16)]).

THEOREM 2.1. *The solution* z_{-1} *admits an asymptotic expansion of the form*

$$\begin{aligned} z_{-1} \sim & U_{-1}(x,y) = -e^{(\beta)}\Gamma(1-\beta)x^{-\beta-\gamma+2}y^{\beta-1}e^{y/x+x} \\ & \times \sum_{m\ge 0}(1-\beta)_m(1-x^2y^{-1})^{\beta-1-2m}P_m^{(1-\gamma,\beta-1-2m)}(1-2x^2y^{-1})(y/x)^{-m} \end{aligned}$$

uniformly for $|xy^{-1/2}| < 1/R$ *as* y *tends to* ∞ *through the sector* $|\arg(y/x)+\pi| < 3\pi/2 - \delta_R$.

THEOREM 2.2. (i) *The solution* z_2 *admits an asymptotic expansion of the form*

$$\begin{aligned} z_2 \sim & U_2(x,y) = -\sqrt{\pi}e^{(2\beta-\gamma)\pi i}y^{-\gamma/2+1/4}\exp(-2y^{1/2}) \\ & \times \sum_{m\ge 0}(m+1)_m 4^{-m}(1+xy^{-1/2})^{-\beta-2m}P_{2m}^{(-3/2+\gamma-m,-\beta-2m)}(1+2xy^{-1/2})y^{-m/2} \end{aligned}$$

uniformly for $|xy^{-1/2}| < 1/R$ *as* y *tends to* ∞ *through the sector* $|\arg y| < 3\pi - \delta,\ |\arg(y/x)| < 3\pi/2 - \delta_R$.

(ii) *The solution* z_{-2} *admits an asymptotic expansion of the form*

$$z_{-2} \sim U_{-2}(x,y) = e^{(\gamma-\beta)}U_2(x, e^{2\pi i}y)$$

uniformly for $|xy^{-1/2}| < 1/R$ *as* y *tends to* ∞ *through the sector* $|\arg y + 2\pi| < 3\pi - \delta,\ |\arg(y/x)+2\pi| < 3\pi/2 - \delta_R$.

2.2. Stokes multipliers

Let $S = S(\theta_1,\theta_2)$ denote a sector defined by

$$S(\theta_1,\theta_2) = \big\{(x,y) \in \mathcal{R}^2 \ \big|\ |\arg y - \theta_1| < 2\pi - \delta,\ |\arg(y/x) - \theta_2| < \pi - \delta_R\big\}.$$

We call a matrix $T(S)$ ($\in GL(3,\mathbf{C})$) a Stokes multiplier corresponding to the sector S with respect to (z_{-1}, z_2, z_{-2}), if linearly independent solutions $z_{-1}^S,\ z_2^S,\ z_{-2}^S$ such that

$${}^t(z_{-1}, z_2, z_{-2}) = T(S)\,{}^t(z_{-1}^S, z_2^S, z_{-2}^S)$$

satisfy

$$z_{-1}^S \sim U_{-1}(x,y), \quad z_2^S \sim U_2(x,y), \quad z_{-2}^S \sim U_{-2}(x,y)$$

uniformly for $|xy^{-1/2}| < 1/R$ as y tends to ∞ through the sector S.

THEOREM 2.3. *We write* $S_1 = S(-\pi, -3\pi/2)$, $S_2 = S(-\pi, -\pi/2)$, $S_3 = S(\pi, -3\pi/2)$, $S_4 = S(\pi, -\pi/2)$. *Then the Stokes multipliers* $T_j = T(S_j)$ $(j = 1, 2, 3, 4)$ *corresponding to these sectors with respect to* (z_{-1}, z_2, z_{-2}) *are given by*

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 - e^{(\beta)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 - e^{(-\beta)} & 0 & 1 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 - e^{(\beta)} & 1 & 0 \\ 0 & e^{(-\beta)} + e^{(\gamma-\beta)} & 1 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 - e^{(-\beta)} & e^{(-\beta)} + e^{(\gamma-\beta)} & 1 \end{pmatrix}.$$

3. Proofs of Theorems 2.1 and 2.2

3.1. Preliminaries

Consider the functions

$$\tau = g(t) = t + y/t,$$
$$\tau = h(t) = g(t) + (\beta - \gamma)\log t - \beta \log(t - x),$$

where $\operatorname{Im}\log s = \arg s$. Integrand (1.4) is written in the form $f(\beta, \gamma; x, y, t) = \exp h(t)$. In the proof of Theorem 2.2, we use the saddle points of $h(t)$ and $g(t)$, namely the roots of $h'(t) = 0$ and $g'(t) = 0$. In the following three lemmas, we assume that $|xy^{-1/2}| < 1/2$, and that $|y|$ is sufficiently large.

LEMMA 3.1. *The saddle points of* $g(t)$ *are* $t_\pm = \pm y^{1/2}$, *and those of* $h(t)$ *are* $t_{\pm1}$ *and* t_2, *where* $t_{\pm 1} = \pm y^{1/2} + O(1)$, $t_2 = x - \beta x^2 y^{-1}(1 - x^2 y^{-1})^{-1}(1 + O(y^{-1/2}))$.

LEMMA 3.2. *Let* μ *be an arbitrary positive constant.*

(i) *For* $|t - t_-| \le |y|^{1/2-\mu}$,

$$g(t) - g(t_-) = -y^{-1/2}(t - t_-)^2(1 + O(|y|^{-\mu})). \tag{3.1}$$

(ii) *For* $|t - t_{\pm1}| \le |y|^{1/2-\mu}$,

$$h(t) - h(t_{\pm1}) = \pm y^{-1/2}(t - t_{\pm1})^2(1 + O(|y|^{-\mu})), \tag{3.2}$$

$$h'(t) = \pm 2y^{-1/2}(t - t_{\pm1})(1 + O(|y|^{-\mu})). \tag{3.3}$$

(iii) *For* $|t - t_2| \le |y|^{-\mu}|x^2 y^{-1}|$ $(0 < \mu \le 1/2)$,

$$h(t) - h(t_2) = (2\beta)^{-1}x^{-4}y^2(1 - x^2y^{-1})^2(t - t_2)^2(1 + O(|y|^{-\mu})), \tag{3.4}$$

$$h'(t) = \beta^{-1}x^{-4}y^2(1 - x^2y^{-1})^2(t - t_2)(1 + O(|y|^{-\mu})). \tag{3.5}$$

LEMMA 3.3. *We have*

$$h(t_{\pm 1}) = \pm 2y^{1/2} - (\gamma/2)\log y + O(1), \tag{3.6}$$

$$h(t_2) = x^{-1}y(1 + x^2y^{-1} + O(y^{-1/3})). \tag{3.7}$$

Let α be an arbitrary complex constant. For every non-negative integer k and for a fixed positive integer N, we write

$$R_{N+1}(\alpha - k, \sigma) = (1-\sigma)^{\alpha-k} - \sum_{n \le N} \frac{(k-\alpha)_n}{n!}\sigma^n,$$

where the branch of $(1-\sigma)^{\alpha-k}$ is taken such that $\arg(1-\sigma) = 0$ for $\sigma < 1$. The following lemma is a special case of [11; Lemma 5.1].

LEMMA 3.4. *If $N \ge \operatorname{Re}\alpha$, then $|R_{N+1}(\alpha - k, \sigma)| \le 2^k(k+1)^{N+1}K_N|\sigma|^{N+1}$ in the domain $|\sigma| < 1/2$, where K_N is a positive constant independent of k.*

LEMMA 3.5 ([11; Lemma 5.2]). *For any complex numbers a, b and for any non-negative integer m,*

$$\sum_{k \ge 0} \frac{(b)_k(a+k)_m}{k!}\xi^k = m!(1-\xi)^{-b-m}P_m^{(a-1,-b-m)}(1-2\xi) \tag{3.8}$$

in the domain $|\xi| < 1$.

In order to calculate the asymptotic expansion of z_2 as y tends to ∞, we need to modify the path of integration C_2, for each (x, y) satisfying $|xy^{-1/2}| < 1/R$, in such a way that C_2 possesses the following properties.

(a) C_2 consists of three curves Γ_-, Γ_0, Γ_+ such that

(a.1) Γ_0 is an arc passing through $t = t_-$ and lying inside the circle K_0 defined by $|t - t_-| = |y|^{1/3}$;

(a.2) both ends a_+, a_- of Γ_0 are located on K_0;

(a.3) Γ_- (or Γ_+) is a curve starting from a_- (or a_+), tending to ∞ (or 0), and lying outside the circle K_0.

(b) C_2 lies outside the circles $|t - t_1| = |y|^{1/4}$, $|t - t_2| = |\beta||y|^{-1/4}|x^2y^{-1}|$.

(c) $g(t) - g(t_-) \le 0$ for $t \in \Gamma_0$.

(d) $(d/d\rho)\operatorname{Re}h(t) \le -c$, for $t \in \Gamma_-$ (or $t \in \Gamma_+$), in which c is a positive constant and $\rho = \rho(t)$ denote the length of a part of $h(\Gamma_-)$ (or $h(\Gamma_+)$) from $h(a_-)$ (or $h(a_+)$) to $h(t)$.

LEMMA 3.6. *If (x,y) $(\in \mathcal{R}^2)$ satisfies $|xy^{-1/2}| < 1/R$ $(R \geq 2.44)$, $|y| > R_\infty$,*

$$|\arg y| < 3\pi - \delta, \quad |\arg(y/x)| < 3\pi/2 - \delta_R, \tag{3.9}$$

then we can modify the path C_2 continuously with respect to (x,y) preserving the properties above, where δ and δ_R are positive constants given in Section 2 and R_∞ is a sufficiently large positive constant.

PROOF. First consider the special case where $\arg x = \arg y = 0$, $|xy^{-1/2}| < 1/R$, and β, $\beta-\gamma \in \mathbf{R} - \mathbf{Z}$. Take the path C_2 to be the negative real axis passing through $t = t_- = -y^{1/2}$. It is expressed as $C_2 = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$ with $\Gamma_- : t \leq a_-^0$, $\Gamma_0 : a_-^0 \leq t \leq a_+^0$, $\Gamma_+ : a_+^0 \leq t < 0$, where $a_-^0 = t_- - y^{1/3}$, $a_+^0 = t_- + y^{1/3}$. Then the images $S_0 = g(\Gamma_0)$, $T_-^0 = h(\Gamma_-)$, $T_+^0 = h(\Gamma_+)$ are included in the negative real axis and expressed as $S_0 : g(a_+^0) \leq \tau \leq g(t_-) = -2y^{1/2}$, $T_-^0 : \tau \leq h(a_-^0)$, $T_+^0 : \tau \leq h(a_+^0)$, respectively. Observing that $t_1 - (-t_-) = O(1)$, we can verify that C_2 has the properties above.

Next we consider the case where $\arg x = \arg y = 0$ is not necessarily satisfied and β, $\beta - \gamma \in \mathbf{C} - \mathbf{Z}$. Take the segment $S : \tau = g(t_-) - \sigma$ $(-2|y|^{1/6} \leq \sigma \leq 0)$ in the τ-plane. By (3.1) the inverse image $g^{-1}(S)$ passes through t_- and intersects the circle $|t - t_-| = |y|^{1/3}$ at a_-, a_+, which are continuous in y and, in case $\arg x = \arg y = 0$, coincide with a_-^0, a_+^0, respectively. We wish to choose curves T_- and T_+ in the τ-plane with the following properties.

(i) T_- (or T_+) is a curve starting from $h(a_-)$ (or $h(a_+)$) and tending to ∞, and lies outside the circles $|\tau - h(t_1)| = 2$, $|\tau - h(t_2)| = |\beta|(1+R^2)^2$.

(ii) $(d/d\rho)\operatorname{Re}\tau \leq -c$, for $\tau \in T_-$ (or $\tau \in T_+$), where ρ denotes the length of a part of T_- (or T_+) from a_- (or a_+) to τ.

(iii) T_- (or T_+) is a continuous modification of T_-^0 (or T_+^0).

Let δ_R' be a sufficiently small positive constant such that

$$\delta_R > \sin^{-1}(2R(R^2-1)^{-1}) + \sin^{-1}(R^{-2}) + \delta_R' \tag{3.10}$$

(cf. (2.1)). Note that $g(\pm t_-) = \mp 2y^{1/2}$, and that $(a_\pm - t_{-1})^2/(a_\pm - t_-)^2 = 1 + O(|y|^{-1/2})$. We have $H_\pm = (h(a_\pm) - h(t_{-1}))/(g(a_\pm) - g(t_-)) = 1 + o(1)$ (cf. (3.1), (3.2)). Hence, by (3.1) and by the definition of S given above,

$$h(a_\pm) = h(t_{-1}) + H_\pm(g(a_\pm) - g(t_-)) = h(t_{-1}) - |y|^{1/6}(1 + o(1)) \tag{3.11.}$$

Furthermore (3.6) implies that

$$h(t_{\mp 1}) = g(\pm t_-) - (\gamma/2)\log y + O(1). \tag{3.12}$$

Since $|xy^{-1/2}| < 1/R$, $R \geq 2.44$, it follows from (3.7) that

$$|g(\pm t_-)|/|h(t_2)| < 2R(R^2-1)^{-1} + o(1) < 0.99 + o(1). \tag{3.13}$$

By these estimates, if $|y|$ is sufficiently large, as long as

$$|\arg g(-t_-)| < 3\pi/2 - \delta/2, \tag{3.14}$$

$$|\arg h(t_2)| < 3\pi/2 - \theta(x,y) - \delta'_R \tag{3.15}$$

with $\theta(x,y) = \sin^{-1}(|g(\pm t_-)|/|h(t_2)|) < \pi/2$, we can draw the curves T_- and T_+ with the properties above (cf. Figures 3.1 and 3.2).

FIGURE 3.1.

FIGURE 3.2.

Once these curves are constructed, we obtain the desired modification $C_2 = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$, where Γ_- (or Γ_+) is one of the connected components of the inverse image $h^{-1}(\Gamma_-)$ (or $h^{-1}(\Gamma_+)$) tending to $t = \infty$ (or $t = 0$), and $\Gamma_0 = \{t \in g^{-1}(S) \,|\, |t - t_-| \leq |y|^{1/3}\}$. Since (3.14) is written as $|\arg y| < 3\pi - \delta$, it remains to verify that (3.15) is valid in sector (3.9). Note that $\arg h(t_2) = \arg(y/x) + \arg(1 + x^2y^{-1} + o(1))$ (cf. (3.7)). For sufficiently large $|y|$, using (3.10), (3.13) and the inequality $|\arg(1 + x^2y^{-1} + o(1))| < \sin^{-1}(R^{-2}) + o(1)$, we derive (3.15) from (3.9). Thus the lemma is proved. □

LEMMA 3.7. *Under the same hypotheses as in Lemma* 3.6, *for the path* $C_2 = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$ *given above, we have*

$$\int_{\Gamma_- \cup \Gamma_+} \exp h(t)\, dt = y^{-\gamma/2+1/4} \exp(-2y^{1/2}) E(x,y)$$

with $E(x,y) = O(\exp(-|y|^{1/6}/2))$.

PROOF. Since $1/h'(t)$ is analytic at $t \neq t_{\pm 1}$, t_2, from (b), (3.3), (3.5) combined with the maximum modulus principle, it follows that $|dt| = |1/h'(t)||dh/d\rho| d\rho = O(|y|^{1/4}) d\rho$ for $t \in \Gamma_-$. The property (d) yields $\mathrm{Re}(h(t) - h(a_-)) \leq -c\rho$ for $t \in \Gamma_-$. Using (3.6), (3.11) and this inequality, we obtain

$$\begin{aligned} |\exp h(t)| \leq e^{-c\rho} |\exp h(a_-)| &= e^{-c\rho} \Big| \exp\big(h(t_{-1}) - |y|^{1/6}(1 + o(1))\big) \Big| \\ &\leq e^{-c\rho} |y^{-\gamma/2} \exp(-2y^{1/2})| \exp(-|y|^{1/6}/2) \end{aligned}$$

for $t \in \Gamma_-$. From this estimate and a similar one for $t \in \Gamma_+$, the lemma immediately follows. □

3.2. Proof of Theorem 2.2

It is sufficient to show the asymptotic representation of z_2, from which we can derive that of z_{-2} by using the relation

$$z_{-2}(x,y) = e^{(\gamma-\beta)} z_2(x, e^{2\pi i} y) \tag{3.16}$$

(see [11; Theorem 3.2]). Assume that (x,y) satisfies the hypotheses of Lemma 3.6, and that the path C_2 has the properties (a),...,(d). Consider an integral of the form

$$I = \int_{\Gamma_0} t^{\beta-\gamma} (t-x)^{-\beta} \exp(t + y/t)\, dt. \tag{3.17}$$

We put $t = y^{1/2}(\sigma - 1)$, in which σ moves along a curve Γ_0^* inside the circle $|\sigma| = |y|^{-1/6}$. Taking $\arg t$ and $\arg(t-x)$ into consideration, we can write $t = e^{\pi i} y^{1/2}(1-\sigma)$, $t - x = e^{-\pi i} y^{1/2} (1-\sigma)(1 + xy^{-1/2}(1-\sigma)^{-1})$ along Γ_0, where $\arg(1-\sigma) \to 0$, $\arg(1 + xy^{-1/2}(1-\sigma)^{-1}) \to 0$ as $\sigma \to 0$, $xy^{-1/2} \to 0$. Observe that $g(t) = t + y/t = -2y^{1/2} - y^{1/2}\sigma^2 - y^{1/2}\sigma^3(1-\sigma)^{-1}$. We wish to calculate an asymptotic expansion of the integral

$$J = e^{(\gamma - 2\beta)\pi i} y^{(\gamma-1)/2} \exp(2y^{1/2})\, I = \int_{\Gamma_0^*} w(x,y,\sigma) \exp(-y^{1/2}\sigma^2)\, d\sigma, \tag{3.18}$$

where

$$
\begin{aligned}
(3.19)\qquad w(x,y,\sigma) &= (1-\sigma)^{-\gamma}\left(1+\frac{xy^{-1/2}}{1-\sigma}\right)^{-\beta}\exp\left(-\frac{y^{1/2}\sigma^3}{1-\sigma}\right)\\
&= \sum_{k\ge 0,p\ge 0}\frac{(\beta)_k}{k!p!}(-xy^{-1/2})^k(-y^{1/2})^p\sigma^{3p}(1-\sigma)^{-\gamma-k-p}
\end{aligned}
$$

for $|xy^{-1/2}| < 1/R$, $|\sigma| \le |y|^{-1/6}$. Let N be an arbitrary large fixed positive integer. By Lemma 3.4,

$$
(1-\sigma)^{-\gamma-k-p} = \sum_{n=0}^{N}\frac{(\gamma+k+p)_n}{n!}\sigma^n + O\left(2^{k+p}(k+p+1)^{N+1}\sigma^{N+1}\right).
$$

Hence series (3.19) is written in the form

$$
(3.20)\qquad \sum_{k\ge 0}\sum_{p=0}^{N}\sum_{n=0}^{N}\frac{(\beta)_k(\gamma+k+p)_n}{k!p!n!}(-xy^{-1/2})^k(-y^{1/2})^p\sigma^{3p+n} + E(x,y,\sigma).
$$

Here, for $|xy^{-1/2}| < 1/R$, $|\sigma| \le |y|^{-1/6}$,

$$
E(x,y,\sigma) = O\left(\sum_{k\ge 0}\frac{|(\beta)_k|}{k!}G_k(N,y,\sigma)R^{-k}\right)
$$

with

$$
\begin{aligned}
G_k(N,y,\sigma) = &\sum_{p\ge N+1}\sum_{n=0}^{N}\frac{|(\gamma+k+p)_n||y^{1/2}\sigma^3|^p}{p!n!}\\
&+ 2^k\sum_{p\ge 0}\frac{(k+p+1)^{N+1}}{p!}|\sigma|^{N+1}|2y^{1/2}\sigma^3|^p.
\end{aligned}
$$

Observing that $\sum_{n=0}^{N}(1/n!)|(\gamma+k+p)_n| = O(p^N(k+|\gamma|+N+1)^N)$ uniformly for $p \ge N+1$, $k \ge 0$, and that $(k+p+1)^{N+1} \le (k+1)^{N+1}(p+1)^{N+1}$ uniformly for $p \ge 0$, $k \ge 0$, we have

$$
(3.21)\qquad E(x,y,\sigma) = O(|y^{1/2}\sigma^3|^{N+1} + |\sigma|^{N+1}).
$$

From (c) and the fact that, in case $\arg y = 0$, the path Γ_0^* coincides with the segment from $t = t_- + y^{-1/6}$ to $t = t_- - y^{-1/6}$, it follows that

$$
\begin{aligned}
(3.22)\qquad &\int_{\Gamma_0^*}\sigma^q\exp(-y^{1/2}\sigma^2)\,d\sigma\\
&=\begin{cases} -\Gamma((q+1)/2)y^{-(q+1)/4} + O(\exp(-|y|^{1/6})) & (q : \text{even}),\\ O(\exp(-|y|^{1/6})) & (q : \text{odd}).\end{cases}
\end{aligned}
$$

Substitute (3.20) and (3.21) into (3.18), and put $N = 2M$, $n + p = 2m$. Then, by (3.22), the integral J becomes

$$(3.23)\quad -\sqrt{\pi}y^{-1/4}\sum_{m=0}^{M}(1/2)_m y^{-m/2}\sum_{k\geq 0}\frac{(\beta)_k}{k!}K_{k,m}(-xy^{-1/2})^k + O(y^{-(M+1)/2}),$$

where

$$K_{k,m} = \sum_{p=0}^{2m}\frac{(m+1/2)_p(-\gamma-k-2m+1)_{2m-p}}{p!(2m-p)!} = \frac{(\gamma-1/2+k-m)_{2m}}{(2m)!}.$$

Using (3.8), we have an asymptotic expansion of I:

$$I = e^{(2\beta-\gamma)\pi i}y^{-(\gamma-1)/2}\exp(-2y^{1/2})\, J \sim U_2(x,y)$$

uniformly for $|xy^{-1/2}| < 1/R$ as y tends to ∞ through (3.9). Combining this formula with Lemma 3.7, we arrive at the asymptotic representation of z_2.

3.3. Proof of Theorem 2.1

By [11; Proposition 2.2,(2.8)], we have

$$z_{-1}(\beta,\gamma;x,y) = e^{\beta\pi i}e^{(\beta-\gamma)}x^{-\beta}y^{\beta-\gamma+1}z_1(\beta,\beta-\gamma+2;e^{2\pi i}y/x,y),$$

in which $z_1(\beta,\gamma;x,y)$ is a solution of (1.2) admitting an asymptotic expansion of the form

$$\begin{aligned} z_1(\beta,\gamma;x,y) &\sim -e^{-\beta\pi i}\Gamma(1-\beta)x^{\beta-\gamma}e^{x+y/x} \\ &\times\sum_{m\geq 0}(1-\beta)_m(1-x^{-2}y)^{\beta-1-2m}P_m^{(\gamma-\beta-1,\beta-1-2m)}(1-2x^{-2}y)x^{-m} \end{aligned}$$

uniformly for $|x^{-2}y| < 1/R_1$ as x tends to ∞ through the sector $|\arg x - \pi| < 3\pi/2 - \delta_{R_1}$ (cf. [11; Theorem 4.1]). Here R_1 and δ_{R_1} are arbitrary constants satisfying $R_1 > 2$ and $2\sin^{-1}(R_1^{-1/2}) < \delta_{R_1} < \pi/2$. Putting $R_1 = R^2$, $\delta_{R_1} = \delta_R$, from the fact above, we obtain the desired asymptotic representation as y/x tends to ∞.

4. Proof of Theorem 2.3

4.1. Preliminaries

Consider the column vector functions $\mathbf{u}(x,y) = {}^t(z_{-1},z_2,z_{-2})$, $\mathbf{U}(x,y) = {}^t(U_{-1}(x,y),U_2(x,y),U_{-2}(x,y))$ (cf. Section 2). From [11; Proposition 2.1,(2.3) and Theorem 3.2], we derive the following lemma.

LEMMA 4.1. *We have* $\mathbf{u}(xe^{2\pi i}, y) = M_1'\mathbf{u}(x,y)$, $\mathbf{u}(x, ye^{2\pi i}) = M_2'\mathbf{u}(x,y)$, *where*

$$M_1' = \begin{pmatrix} e^{(-\gamma)} & e^{(-\beta)} & -e^{(-\gamma)} \\ e^{(-\gamma)} - e^{(\beta-\gamma)} & e^{(-\beta)} & e^{(\beta-\gamma)} - e^{(-\gamma)} \\ e^{(-\beta)} - 1 & 0 & 1 \end{pmatrix},$$
$$M_2' = \begin{pmatrix} 1 & -1 & e^{(\beta-\gamma)} \\ 0 & 0 & e^{(\beta-\gamma)} \\ 0 & -1 & e^{(\beta-\gamma)} + 1 \end{pmatrix}.$$

The formal monodromy matrices are given by the following lemma.

LEMMA 4.2. *We have* $\mathbf{U}(xe^{2\pi i}, y) = P_1\mathbf{U}(x,y)$, $\mathbf{U}(x, ye^{2\pi i}) = P_2\mathbf{U}(x,y)$, *where*

$$P_1 = \begin{pmatrix} e^{(-\beta-\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} e^{(\beta)} & 0 & 0 \\ 0 & 0 & e^{(\beta-\gamma)} \\ 0 & -e^{(-\beta)} & 0 \end{pmatrix}.$$

4.2. Proof of Theorem 2.3

By Lemma 4.1,

$$z_2(xe^{-2\pi i}, y) = (e^{(\beta)} - 1)z_{-1}(x,y) + z_2(x,y), \tag{4.1}$$
$$z_{-2}(xe^{2\pi i}, y) = (e^{(-\beta)} - 1)z_{-1}(x,y) + z_{-2}(x,y). \tag{4.2}$$

Assume that $(x,y) \in S_1$. By Theorems 2.1 and 2.2, we have

$$z_{-1} \sim U_{-1}(x,y), \qquad z_{-2} \sim U_{-2}(x,y). \tag{4.3}$$

Since $|\arg(y/(xe^{-2\pi i})) - \pi/2| < \pi - \delta_R$, it follows from Theorem 2.2 and Lemma 4.2 that $z_2(xe^{-2\pi i}, y) \sim U_2(xe^{-2\pi i}, y) = U_2(x,y)$. This relation and (4.3) combined with (4.1) yield the matrix T_1. In the sector S_2, observing that $|\arg(y/(xe^{2\pi i})) + 5\pi/2| < \pi - \delta_R$ and using (4.2), we can derive T_2 in a similar way. If $(x,y) \in S_3$ then $(xe^{-2\pi i}, ye^{-2\pi i}) \in S_1$. Hence, by Lemmas 4.1 and 4.2, we have $\mathbf{u}(x,y) = M_1'M_2'\mathbf{u}(xe^{-2\pi i}, ye^{-2\pi i}) = M_1'M_2'T_1\mathbf{U}(xe^{-2\pi i}, ye^{-2\pi i}) = M_1'M_2'T_1P_1^{-1}P_2^{-1}\mathbf{U}(x,y)$, from which $T_3 = M_1'M_2'T_1P_1^{-1}P_2^{-1}$ follows. Using the fact that $(x,y) \in S_4$ implies $(xe^{-2\pi i}, ye^{-2\pi i}) \in S_2$, we also derive $T_4 = M_1'M_2'T_2P_1^{-1}P_2^{-1}$. Thus the proof is completed.

5. Confluent Jordan-Pochhammer equation (1.3)

In equation (1.3), assume that the constant $\kappa \in \mathbf{C} - \{0\}$ satisfies $-\varepsilon < \arg\kappa < 2\pi + \varepsilon$, where ε is a small positive constant. Consider the triplet of linearly independent solutions $(z^\kappa_{-1}, z^\kappa_2, z^\kappa_{-2})$ of (1.3) near $y = \infty$, in which $z^\kappa_{-1} = z_{-1}(\kappa, y)$, $z^\kappa_2 = z_2(\kappa, y)$, $z^\kappa_{-2} = z_{-2}(\kappa, y)$. By $L^{(\alpha)}_\nu(\xi)$ we denote the Laguerre polynomial

$$L^{(\alpha)}_\nu(\xi) = \frac{1}{\nu!} e^\xi \xi^{-\alpha} \left(\frac{d}{d\xi}\right)^\nu (e^{-\xi}\xi^{\nu+\alpha}) = \sum_{j=0}^{\nu} \binom{\nu+\alpha}{\nu-j} \frac{(-\xi)^j}{j!}.$$

Let δ be an arbitrary small positive constant.

THEOREM 5.1. *The solution z^κ_{-1} admits an asymptotic expansion of the form*

$$z^\kappa_{-1} \sim U^\kappa_{-1}(y) = -e^{(\beta)}\Gamma(1-\beta)e^\kappa \kappa^{-\beta-\gamma+2} y^{\beta-1} e^{y/\kappa} \sum_{n\geq 0} (1-\beta)_n \kappa^n L^{(1-\gamma)}_n(-\kappa) y^{-n}$$

as y tends to ∞ through the sector $|\arg y - \arg\kappa + \pi| < 3\pi/2 - \delta$.

THEOREM 5.2. (i) *The solution z^κ_2 admits an asymptotic expansion of the form*

$$z^\kappa_2 \sim U^\kappa_2(y) = -\sqrt{\pi} e^{(2\beta-\gamma)\pi i} y^{-\gamma/2+1/4} \exp(-2y^{1/2}) \sum_{n\geq 0} Q_n(\beta,\gamma;\kappa) y^{-n/2}$$

with

$$Q_n(\beta,\gamma;\kappa) = \sum_{m=0}^{n} \frac{(\beta)_{n-m}(3/2-\gamma-n)_{2m}}{4^m (n-m)! m!} (-\kappa)^{n-m}$$

as y tends to ∞ through the sector $-3\pi/2 + \arg\kappa + \delta < \arg y < \min\{3\pi/2 + \arg\kappa, 3\pi\} - \delta$.

(ii) *The solution z^κ_{-2} admits an asymptotic expansion of the form*

$$z^\kappa_{-2} \sim U^\kappa_{-2}(y) = e^{(\gamma-\beta)} U^\kappa_2(e^{2\pi i} y)$$

as y tends to ∞ through the sector $-7\pi/2 + \arg\kappa + \delta < \arg y < \min\{-\pi/2 + \arg\kappa, \pi\} - \delta$.

PROOFS OF THEOREMS 5.1 AND 5.2. It is sufficient to show that, after rearranging the terms of the formal series $U_{-1}(\kappa, y)$ (or $U_2(\kappa, y)$) in Theorem 2.1 (or Theorem 2.2), we obtain the asymptotic expression $U^\kappa_{-1}(y)$ (or $U^\kappa_2(y)$). By (3.8),

$$\left[-e^{(\beta)}\Gamma(1-\beta)\kappa^{-\beta-\gamma+2} y^{\beta-1} e^{y/\kappa+\kappa}\right]^{-1} U_{-1}(\kappa, y)$$
$$= \sum_{m\geq 0}\sum_{k\geq 0} (1-\beta)_{m+k} \frac{(2-\gamma+k)_m}{m!k!} \kappa^{m+2k} y^{-m-k} = \sum_{n\geq 0} (1-\beta)_n \kappa^n L^{(1-\gamma)}_n(-\kappa) y^{-n}$$

$(n = m + k)$, which implies Theorem 5.1. Observing that

$$\Big[-\sqrt{\pi}e^{(2\beta-\gamma)\pi i}y^{1/4-\gamma/2}\exp(-2y^{1/2})\Big]^{-1} U_2(\kappa, y)$$
$$= \sum_{m\ge 0}\sum_{k\ge 0} \frac{(\beta)_k(\gamma - 1/2 + k - m)_{2m}}{4^m m!k!}(-\kappa)^k y^{-(m+k)/2},$$

and putting $m + k = n$, we obtain the asymptotic series $U_2^\kappa(y)$. Thus the theorems are verified. □

For a sector $\Sigma\,(\subset \mathcal{R})$, we call a matrix $T^\kappa(\Sigma)\,(\in GL(3, \mathbf{C}))$ a Stokes multiplier corresponding to Σ with respect to $(z^\kappa_{-1}, z^\kappa_2, z^\kappa_{-2})$, if linearly independent solutions $z^{\kappa,\Sigma}_{-1}$, $z^{\kappa,\Sigma}_2$, $z^{\kappa,\Sigma}_{-2}$ such that

$$^t(z^\kappa_{-1}, z^\kappa_2, z^\kappa_{-2}) = T^\kappa(\Sigma)\ {}^t(z^{\kappa,\Sigma}_{-1}, z^{\kappa,\Sigma}_2, z^{\kappa,\Sigma}_{-2})$$

satisfy

$$z^{\kappa,\Sigma}_{-1} \sim U^\kappa_{-1}(y), \quad z^{\kappa,\Sigma}_2 \sim U^\kappa_2(y), \quad z^{\kappa,\Sigma}_{-2} \sim U^\kappa_{-2}(y)$$

as y tends to ∞ through Σ. Let Σ_- and Σ_+ be sectors defined by

$$\Sigma_- = \big\{y \in \mathcal{R} \ \big|\ -5\pi/2 + \arg\kappa + \delta < \arg y < -\pi/2 + \arg\kappa - \delta\big\},$$
$$\Sigma_+ = \big\{y \in \mathcal{R} \ \big|\ -3\pi/2 + \arg\kappa + \delta < \arg y < \pi/2 + \arg\kappa - \delta\big\}.$$

Then the Stokes multipliers corresponding to these sectors with respect to $(z^\kappa_{-1}, z^\kappa_2, z^\kappa_{-2})$ are given by the following theorem, which immediately follows from Theorem 2.3.

THEOREM 5.3. *We have*

$$T^\kappa(\Sigma_-) = T_1, \quad T^\kappa(\Sigma_+) = T_2, \qquad \textit{if } -\varepsilon < \arg\kappa \le \pi/2,$$
$$T^\kappa(\Sigma_-) = T_1, \quad T^\kappa(\Sigma_+) = T_4, \qquad \textit{if } \pi/2 < \arg\kappa \le 3\pi/2,$$
$$T^\kappa(\Sigma_-) = T_3, \quad T^\kappa(\Sigma_+) = T_4, \qquad \textit{if } 3\pi/2 < \arg\kappa < 2\pi + \varepsilon,$$

where T_j $(j = 1, ..., 4)$ *are matrices given in Theorem* 2.3.

REMARK. When $\arg\kappa = \pi/2$ (or $\arg\kappa = 3\pi/2$), we may also take $T^\kappa(\Sigma_+) = T_4$ (or $T^\kappa(\Sigma_-) = T_3$).

Around the regular singular point $y = 0$, we consider linearly independent solutions z^κ_{-3}, z^κ_0, z^κ_3 expressible by the connection formulas

$$\begin{aligned} z^\kappa_{-3} &= (1 - e^{(-\beta)}) z^\kappa_{-1} + e^{(-\beta)} z^\kappa_2 - z^\kappa_{-2}, \\ z^\kappa_0 &= (e^{(\gamma-\beta)} - 1) z^\kappa_{-1} - e^{(\gamma-\beta)} z^\kappa_2 + z^\kappa_{-2}, \\ z^\kappa_3 &= -e^{(\gamma-\beta)} z^\kappa_2 + z^\kappa_{-2} \end{aligned}$$

(cf. [11; Proposition 2.1 and (3.2)]). From [11; Theorem 3.1], we obtain the convergent series expansions of these solutions.

THEOREM 5.4. *For $y \in \mathcal{R}$, we have*

$$\begin{aligned} z^\kappa_{-3} &= \frac{2\pi i}{\Gamma(\gamma)} \sum_{n \geq 0} \frac{1}{(\gamma)_n n!} F(\beta, \gamma + n, \kappa) y^n, \\ z^\kappa_0 &= \frac{2\pi i e^{\gamma\pi i} \Gamma(1-\beta)}{\Gamma(\gamma-\beta)\Gamma(2-\gamma)} \kappa^{1-\gamma} \\ &\quad \times \sum_{n \geq 0} \frac{(\gamma-1)_n (-\kappa)^{-n}}{(\gamma-\beta)_n n!} F(\beta - \gamma + 1 - n, 2 - \gamma - n, -\kappa) y^n, \\ z^\kappa_3 &= -\frac{2\pi i e^{\beta\pi i}}{\Gamma(\beta - \gamma + 2)} \kappa^{-\beta} y^{\beta-\gamma+1} \\ &\quad \times \sum_{n \geq 0} \frac{1}{(\beta - \gamma + 2)_n n!} \left(\sum_{m=0}^{n} \frac{(\beta)_m (-n)_m (-\kappa)^{-m}}{m!} \right) y^n, \end{aligned}$$

where $F(a, c, x) = {}_1F_1(a, c, x)$ is Kummer's confluent hypergeometric function.

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$g(-t_-) \quad g(-t_-) \quad g(t_-) \quad g(t_-)$

$g(-t_-) \quad g(-t_-) \quad g(t_-) \quad g(t_-)$

$h(t_1) \quad h(t_1) \quad h(t_2) \quad h(t_2) \quad h(a_-) \quad h(a_-)$

$h(t_1) \quad h(t_1) \quad h(t_2) \quad h(t_2) \quad h(a_-) \quad h(a_-)$

$\theta(x,y) \quad \theta(x,y) \quad T_- \quad T_- \quad 0 \quad 0\, \theta(x,y) \quad \theta(x,y) \quad T_- \quad T_- \quad 0 \quad 0$

$\pi/2 < \arg g(-t_-) < 3\pi/2 - \delta/2,$

$-3\pi/2 + \theta(x,y) + \delta'_R < \arg h(t_2) < -\pi/2$

$-\pi/2 \leq \arg g(-t_-) \leq \pi/2,$

$-3\pi/2 + \theta(x,y) + \delta'_R < \arg h(t_2) < -\pi/2$