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GROUPS OF QUANTUM VOLUME PRESERVING DIFFEOMORPHISMS AND THEIR BEREZIN REPRESENTATION

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Introduction

In [O.2], the first author gives the notion of *generalized Lie groups.* An idea of defining this notion is intuitively to view one parameter subgroups in a topological group as a sort of straight lines. Since one can regard infinitesimal components of *differentiable curves* as infinitesimal segments of one parameter subgroups, it is possible to make a notion of *derivatives* in terms of group operations. Therefore, infinite products of such infinitesimal components are defined as product integrals.

A generalized Lie group is in a sense a minimal notion in which *derivatives* and *product integrals* are well-defined, but the manifold structure is not requested. Though one may loose a manifold structure, this notion has a categorical advantage:

(1) *every closed subgroup of a generalized Lie group is a generalized Lie group*
(2) *every factor group by a closed normal subgroup is a generalized Lie group.*

Since we are dealing only with first derivatives and first order differential equations, it is not so strongly requested that the total space is a locally Euclidean space. Every projective limit of a system of Banach Lie groups is a generalized Lie group, and hence every locally compact group is a generalized Lie group.

However, a manifold structure is the most fundamental structure to consider higher-order calculus. A *regular Fréchet Lie group* defined in [OMYK1] or in [Mil] is a combined notion of generalized Lie groups and C^∞ Fréchet manifolds where the group operations are C^∞. In other words, a regular Fréchet Lie group is a Fréchet Lie group defined in [Les] on which product integrals converge. *Any covering group of a regular Fréchet Lie group is a regular Fréchet Lie group.*

A *strong ILH-Lie group* or a *strong ILB-Lie group* defined in [O.1] is a combined notion of Fréchet Lie groups and manifold structures given by inverse limits of separable Hilbert or Banach manifolds. Every Hilbert (resp. Banach) Lie group is a strong ILH-Lie (resp. ILB-Lie) group, and an automorphism group of a primitive structure on a closed manifold is a typical example of a strong ILH-Lie group (cf. [O.2]).

However, strong ILH-Lie groups provide a very solid notion. Such groups appear only in a concrete examples.

In contrast, regular Fréchet Lie groups are amenable objects to treat. The reason is that the following two theorems holds which has been developed in [OMYK1]:

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Theorem I. *Let G be a connected, simply connected regular Fréchet Lie group, and let H be a regular Fréchet Lie group. Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of G, H respectively. If there is a continuous homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$, then there is a C^∞ homomorphism $\Phi : G \to H$ such that $d\Phi = \varphi$.*

Thus, the local structure of a regular Fréchet Lie group is determined by its Lie algebra.

Theorem II. (Extension theorem) *In the exact sequence of Fréchet Lie groups*

$$1 \to N \xrightarrow{i} G \xrightarrow{\pi} \tilde{G} \to 1,$$

suppose that N and $\tilde{G}$ are regular Fréchet Lie groups, i and π are C^∞ and there exists a C^∞ local section $\gamma : \tilde{U} \to G$ where $\tilde{U}$ is an open neighborhood of $\tilde{e}$ of $\tilde{G}$ satisfying $\pi\gamma = id.$. Then, G is a regular Fréchet Lie group.

In this paper, we apply Theorems I, II to make an example where these three kind of infinite dimensional Lie groups appear on the same stage.

Statement of theorems and comments

Let $\mathcal{D}_\Omega(S^2)$ be the group of all volume preserving C^∞ diffeomorphisms on S^2. This is a connected strong ILH-Lie group with the Lie algebra of all divergence free vector fields $\Gamma_\Omega(T_{S^2})$ (cf. [O.2]). $\Gamma_\Omega(T_{S^2})$ is isomorphic to $(C^\infty_R(S^2)/\mathbf{R}, \{\,,\,\})$; the space of all C^∞ potential functions with the canonical Poisson bracket $\{\,,\,\}$.

$\Gamma_\Omega(T_{S^2})$ and $\mathcal{D}_\Omega(S^2)$ coincide with the space of all derivations of the Poisson algebra $(C^\infty_R(S^2), \cdot, \{\,,\,\})$ and the group of all automorphisms of $(C^\infty_R(S^2), \cdot, \{\,,\,\})$ respectively.

In this paper, we present the following items:

(C.1) a complete topological associative algebra $(\mathcal{V}, *)$ over $\mathbf{C}$ with a special central element μ as a quantum version of the Poisson algebra $(C^\infty_C(S^2), \cdot, \{\,,\,\})$.

(C.2) a matrix representation of the algebra $(\mathcal{V}, *)$ such that μ is represented as a diagonal matrix $\hat{\mu} = -\mathrm{diag}\{I_1, 2^{-1}I_2, \cdots, k^{-1}I_k, \cdots\}$ where I_k is the identity matrix of rank k.

Moreover, we show

(C.3) Matrices obtained by (C.2) coincides with the Berezin representation on the Riemann sphere by using Kähler polarization (cf.[CGR]).

Minding that $\hat{\mu}$ is invertible, we consider an algebra $\mathcal{V}[\mu^{-1}]$ by joining the inverse μ^{-1} to the algebra $\mathcal{V}$. Since μ is represented as above and every $a \in \mathcal{V}$ commute with μ by (C.1), $a \in \mathcal{V}$ is represented as a blockwise diagonal matrix such that each block is finite rank. Next, we show:

(C.4) $\mu^{-1} * \mathcal{V}$ forms a Lie algebra under the commutator bracket, and there exists a matrix representation of the Lie algebra $(\mu^{-1} * \mathcal{V}, [\,,\,])$ as a projective limit of finite dimensional Lie algebras of matrices.

By (C.4), we can consider the group $\mathcal{G}_\mathcal{V}$ generated by $\exp(\mu^{-1} * \mathcal{V})$ as a projective limit of finite dimensional Lie groups, hence $\mathcal{G}_\mathcal{V}$ is a generalized Lie group. We denote these by

$$\mathcal{G}_\mathcal{V} = \varprojlim G_n, \quad \mu^{-1} * \mathcal{V} = \varprojlim \mathfrak{g}_n.$$

Remark that the projective limit topology on $\mu^{-1} * \mathcal{V}$ is much weaker than the original topology.

On the other hand, we see that $\mathcal{V}$ consists of functions which is written asymptotically in the form

$$f \sim f_0 + \mu * f_1 + \cdots + \mu^k * f_k + \cdots , \quad f_i \in C^\infty_C(S^2).$$

Let $\mu^{-1} * \mathcal{V}_R$ be the Lie subalgebra of all $\mu^{-1} * f$ such that $f_0 \in \sqrt{-1}C^\infty_R(S^2)$ with the relative topology from $\mu^{-1} * \mathcal{V}$. Such Lie algebra is called a *top term pure imaginary Lie subalgebra*. The following is the list of claims for the Lie algebra $\mu^{-1} * \mathcal{V}_R$:

(C.5) There exists a simply connected regular Fréchet Lie group $\tilde{G}_\mathcal{V}$ with the Lie algebra $\mu^{-1} * \mathcal{V}_R$.

(C.6) There exists a continuous homomorphism π from $\tilde{G}_\mathcal{V}$ onto $\mathcal{D}_\Omega(S^2)$, whose kernel is the group generated by the Lie subalgebra with constant top terms $\{f_0 = const.\}$.

(C.7) There is a homomorphism Φ from $\tilde{G}_\mathcal{V}$ into $\mathcal{G}_\mathcal{V}$ such that the derivative $(d\Phi)_e$ at the identity is the natural inclusion of the Lie algebra $\mu^{-1} * \mathcal{V}_R$ into $\mu^{-1} * \mathcal{V}$.

We remark that the existence of Φ in (C.7) is a direct conclusion by Theorem I. Let p_n be the natural projection $\mu^{-1} * \mathcal{V} \to \mathfrak{g}_n$, which is viewed as a Lie algebra homomorphism from $\mu^{-1} * \mathcal{V}$ with the original topology. Since $\tilde{G}_\mathcal{V}$ and G_n are regular Fréchet Lie groups and $\tilde{G}_\mathcal{V}$ is simply connected, there exists a C^∞ homomorphism $\Phi_n : \tilde{G}_\mathcal{V} \to G_n$, and $\Phi = \varprojlim \Phi_n$ is the desired homomorphism.

It is known that there is no strong ILB-Lie group with the complexified Lie algebra $\Gamma_\Omega(T_{S^2}) \otimes \mathbf{C}$ as the Lie algebra (cf. [O.2] §VII, Corollary 4.4 and a remark after Definition 4.5). There may be no regular Fréchet Lie group with the Lie algebra $\mu^{-1} * \mathcal{V}$. The claim (C.4) shows that there is a generalized Lie group with the Lie algebra $\mu^{-1} * \mathcal{V}$ if the projective limit topology from finite dimensional Lie algebras is given to this algebra.

To obtain the regular Fréchet Lie group $\tilde{G}_\mathcal{V}$, we first construct the regular Fréchet Lie group $G\mathcal{V}_0$ whose Lie algebra is the Lie algebra $(\mathcal{V}, [\ ,\ ]_*)$ as the group of all invertible elements of the algebra $(\mathcal{V}, *)$. In fact, $G\mathcal{V}_0$ is an open subset of $\mathcal{V}$.

Since $\mu^{-1} * \mathcal{V}_R/\mathcal{V} \cong C^\infty_R(S^2)$, we next construct a regular Fréchet Lie group $G_{(-1)}$ with the Lie algebra $(C^\infty_R(S^2), \{\ ,\ \})$. Remarking that this Lie algebra is isomorphic to the direct product $\mathbf{R} \times \Gamma_\Omega(T_{S^2})$ (cf. [O.2] §VIII, Theorem 3.2.), we see $G_{(-1)} = S^1 \times \mathcal{D}_\Omega(S^2)$. This is in fact a strong ILH-Lie group.

Suppose now there is an abstract group G without topology, such that the following exact sequence holds:

$$1 \to G\mathcal{V}_0 \xrightarrow{i} G \xrightarrow{\pi} G_{(-1)} \to 1 \tag{0.1}$$

To apply extension theorem (Theorem II), we first have to make G a Fréchet Lie group such that i, and π are C^∞, and there is a C^∞ local section γ. Such structure is given, if there is an open neighborhood $\tilde{U}$ of the identity $\tilde{e}$ in $G_{(-1)}$ and a mapping $\gamma : \tilde{U} \to G$ such that $\pi\gamma = \text{id.}$ and with the following properties:

(Ext.1) The mapping $(\tilde{g}, n) \to \gamma(\tilde{g})n$ gives a one-to-one correspondence of $\tilde{U} \times G\mathcal{V}_0$ onto $\pi^{-1}(\tilde{U})$.

(Ext.2) The mapping $r_\gamma : \tilde{V} \times \tilde{V} \to G\mathcal{V}_0$ defined by $r_\gamma(\tilde{g}, \tilde{h}) = \gamma(\tilde{g}\tilde{h})^{-1}\gamma(\tilde{g})\gamma(\tilde{h})$, is C^∞, where $\tilde{V}$ is a neighborhood of $\tilde{e}$ such that $\tilde{V}^2 \subset \tilde{U}$ and $\tilde{V}^{-1} = \tilde{V}$.

(Ext.3) The mapping $\alpha_\gamma : \tilde{V} \times G\mathcal{V}_0 \to G\mathcal{V}_0$ defined by $\alpha_\gamma(\tilde{g}, m) = \gamma(\tilde{g})^{-1}m\gamma(\tilde{g})$ is C^∞.

The group G without topology in (0.1) is constructed at first in the group GF^0 of invertible Fourier integral operators on $\mathbf{R}^4$ (cf. Theorem 1.1), and the properties (Ext.1-3) are shown also in this group. Hence, we make G a regular Fréchet Lie group. $\tilde{G}_{\mathcal{V}}$ is the universal covering group of G.

We use this routine at several stages to construct regular Fréchet Lie groups, though we do not give details.

1. Wick algebra and its extension

We recall the Wick algebra W: a noncommutative associative algebra W over $\mathbf{C}$ generated by $\{\hbar, \zeta_1, \bar{\zeta}_1, \zeta_2, \bar{\zeta}_2\}$ with the relations:

$$[\zeta_i, \bar{\zeta}_j] = -2\hbar\delta_{ij},\ [\zeta_i, \zeta_j] = [\bar{\zeta}_i, \bar{\zeta}_j] = 0, \tag{1.1}$$

where $\hbar$ is a positive real parameter. The parameter $\hbar$ commutes with every element of W. W has a canonical involutive anti-automorphism $a \to \bar{a}$. We emphasize the product in W by $*$. Set ρ as

$$\rho = \bar{\zeta}_1 * \zeta_1 + \zeta_2 * \bar{\zeta}_2 = \zeta_1 * \bar{\zeta}_1 + \bar{\zeta}_2 * \zeta_2. \tag{1.2}$$

Our first task is to make a topological completion (in a sense) of W. Let $\zeta_1, \bar{\zeta}_1, \zeta_2, \bar{\zeta}_2$ be complex coordinates on $\mathbf{C}^2 \times \mathbf{R}_+$. We consider $\mathbf{C}[\zeta, \bar{\zeta}, \hbar]$ as the set of polynomials on $\mathbf{C}^2 \times \mathbf{R}_+$. The Wick algebra W is linearly isomorphic to $\mathbf{C}[\zeta, \bar{\zeta}, \hbar]$ and the product $*$ is given by the *Moyal product formula*:

$$a * b = a \exp \hbar\{\overleftarrow{\partial_{\bar{\zeta}}} \cdot \overrightarrow{\partial_{\zeta}} - \overleftarrow{\partial_{\zeta}} \cdot \overrightarrow{\partial_{\bar{\zeta}}}\} b. \tag{1.3}$$

where

$$f(\overleftarrow{\partial_{\bar{\zeta}}} \cdot \overrightarrow{\partial_{\zeta}} - \overleftarrow{\partial_{\zeta}} \cdot \overrightarrow{\partial_{\bar{\zeta}}})g = \sum_i (\partial_{\bar{\zeta}_i} f \cdot \partial_{\zeta_i} g - \partial_{\zeta_i} f \cdot \partial_{\bar{\zeta}_i} g)$$

(cf. [MO] for these notations and several properties). Thus, we have

$$\rho = \bar{\zeta}_1 \cdot \zeta_1 + \zeta_2 \cdot \bar{\zeta}_2. \tag{1.4}$$

Now, we define a class $\Sigma^m(\mathbf{C}^2 \times \mathbf{R}_+)$ ($m \in \mathbf{Z}$) of smooth functions on $\mathbf{C}^2 \times \mathbf{R}_+$ as follows: Take a cut-off function $\phi(t)$ such that $\phi(t) = 0$ around $t = 0$, and $\phi(t) = 1$ for $t > 1/2$. Then, f is an element of $\Sigma^m(\mathbf{C}^2 \times \mathbf{R}_+)$ if for every integer $k \le m$, there exists a set of functions $f_m, f_{m-1}, \cdots, f_k$ on $S^3 \times \mathbf{R}_+$ such that

$$\phi(\rho)(f - f_m\rho^{\frac{m}{2}} - f_{m-1} \cdot \rho^{\frac{m-1}{2}} - \cdots - f_k \cdot \rho^{\frac{k}{2}}) \cdot \rho^{-\frac{k+1}{2}}$$

is smooth and bounded on $\mathbf{C}^2 \times \mathbf{R}_+$, where $\rho^{-\alpha}$ is computed by the ordinary product $\cdot$. We write

$$f \sim f_m\rho^{\frac{m}{2}} + f_{m-1} \cdot \rho^{\frac{m-1}{2}} + \cdots + f_k \cdot \rho^{\frac{k}{2}} + \cdots, \quad f_k = f_k(p, \hbar) \in C^\infty(S^3 \times \mathbf{R}_+). \tag{1.5}$$

and we call the right hand side an *asymptotic expansion* of f.

We set

$$\Sigma^{-\infty}(\mathbf{C}^2 \times \mathbf{R}_+) = \bigcap_m \Sigma^{-m}(\mathbf{C}^2 \times \mathbf{R}_+), \quad \Sigma^{\infty}(\mathbf{C}^2 \times \mathbf{R}_+) = \bigcup_m \Sigma^{m}(\mathbf{C}^2 \times \mathbf{R}_+).$$

The topology for $\Sigma^0(\mathbf{C}^2 \times \mathbf{R}_+)$ is given as follows: For every $m \in \mathbf{Z}_+$, we set

$$\Sigma^0(\mathbf{C}^2 \times \mathbf{R}_+) = \bigoplus_{k=0}^{m} C^{\infty}(S^3 \times \mathbf{R}_+)\phi(\rho)\rho^{-\frac{k}{2}} \oplus \Sigma^{-(m+1)}(\mathbf{C}^2 \times \mathbf{R}_+). \tag{1.6}$$

Giving the product topology of the uniform C^∞ topology, which is denoted by T_m, we give the projective limit topology $\varprojlim T_m$ to $\Sigma^0(\mathbf{C}^2 \times \mathbf{R}_+)$. $\Sigma^0(\mathbf{C}^2 \times \mathbf{R}_+)$ is then a complete topological vector space containing $\Sigma^{-\infty}(\mathbf{C}^2 \times \mathbf{R}_+)$ as a closed subspace.

We make $\Sigma^m(\mathbf{C}^2 \times \mathbf{R}_+)$ $(m > 0)$ a topological space by identifying this space with $\Sigma^0(\mathbf{C}^2 \times \mathbf{R}_+)(\phi(\rho)\rho)^m$, and give the inductive limit topology for $\Sigma^\infty(\mathbf{C}^2 \times \mathbf{R}_+)$.

Identifying $\mathbf{C}^2$ with $\mathbf{R}^4$, $*$-product extends continuously to the space $\Sigma^\infty(\mathbf{C}^2 \times \mathbf{R}_+)$ by the following oscillatory integral formula:

$$f * g = os\text{-}\iint f(x + X, y + \hbar Y)g(x + X', y + \hbar Y')e^{\frac{2}{i}(XY'-YX')}\, dX\, dY\, dX'\, dY',$$

where $x = (x_1, x_2)$, $XY' - YX' = \sum(X_iY_i' - Y_iX_i')$ and $dX = dX_1 dX_2$. By taking the Taylor expansions of f and g with respect to X, Y, X', Y' up to an appropriate order, $*$-product $f * g$ is actually obtained from the above oscillatory integral.

It is known that

$$\begin{aligned} &\Sigma^k(\mathbf{C}^2 \times \mathbf{R}_+) * \Sigma^l(\mathbf{C}^2 \times \mathbf{R}_+) \subset \Sigma^{k+l}(\mathbf{C}^2 \times \mathbf{R}_+) \\ &[\Sigma^k(\mathbf{C}^2 \times \mathbf{R}_+), \Sigma^l(\mathbf{C}^2 \times \mathbf{R}_+)] \subset \Sigma^{k+l-2}(\mathbf{C}^2 \times \mathbf{R}_+). \end{aligned} \tag{1.7}$$

It follows that $\Sigma^2(\mathbf{C}^2 \times \mathbf{R}_+)$ forms a Lie algebra under the commutator bracket. Let $\Sigma^2_R(\mathbf{C}^2 \times \mathbf{R}_+)$ be the top term pure imaginary Lie subalgebra of $\Sigma^2(\mathbf{C}^2 \times \mathbf{R}_+)$; i.e.

$$\begin{aligned} &f \in \Sigma^2_R(\mathbf{C}^2 \times \mathbf{R}_+) \iff \\ &f \sim \sqrt{-1}f_{-2}\rho + f_{-1}\rho^{\frac{1}{2}} + f_0 + f_1\rho^{-\frac{1}{2}} + \cdots, \quad f_{-2}(p, \hbar) \in C^\infty_R(S^3 \times \mathbf{R}_+). \end{aligned} \tag{1.8}$$

Let $\mathcal{D}_\omega(S^3)$ be the group of all contact transformations on S^3. This is a strong ILH-Lie group (cf.[O.1]) with the Lie algebra $\Gamma_\omega(S^3)$ of all infinitesimal contact transformations. $\Gamma_\omega(S^3)$ is the space $C^\infty_R(S^3)$ with the Liouville bracket $\{\ ,\ \}_c$. It is easy to see that the factor space $\Sigma^2_R(\mathbf{C}^2 \times \mathbf{R}_+)/\Sigma^1(\mathbf{C}^2 \times \mathbf{R}_+)$ is the Lie algebra of all smooth maps from $\mathbf{R}_+$ into $\Gamma_\omega(S^3)$. The factor space $\Sigma^2_R(\mathbf{C}^2 \times \mathbf{R}_+)/\Sigma^0(\mathbf{C}^2 \times \mathbf{R}_+)$ is a semi-direct product of the Lie algebra $\Sigma^2_R(\mathbf{C}^2 \times \mathbf{R}_+)/\Sigma^1(\mathbf{C}^2 \times \mathbf{R}_+)$ and the commutative Lie algebra $\Sigma^1_R(\mathbf{C}^2 \times \mathbf{R}_+)/\Sigma^0(\mathbf{C}^2 \times \mathbf{R}_+)$, and we see that

$$C^\infty(S^3 \times \mathbf{R}_+)[[\rho^{-\frac{1}{2}}]] \cong \Sigma^0(\mathbf{C}^2 \times \mathbf{R}_+)/\Sigma^{-\infty}(\mathbf{C}^2 \times \mathbf{R}_+). \tag{1.9}$$

Thus, the algebra $(\Sigma^0(\mathbf{C}^2 \times \mathbf{R}_+), *)$ defines an associative product $*$ on the space $C^\infty(S^3 \times \mathbf{R}_+)[[\rho^{-\frac{1}{2}}]]$.

Using these facts combined with Theorem II in the introduction, we can see the following:

Theorem 1.1. *There is a regular Fréchet Lie group GF^0 with the top term pure imaginary Lie algebra $\Sigma^2_R(\mathbf{C}^2 \times \mathbf{R}_+)$ as its Lie algebra.*

The above theorem is proved by a similar manner as in [OMYK], and indeed this has been proved by Miyazaki [M] by a careful calculation of the product formulas of the invertible Fourier integral operators involving the parameter $\hbar$ on $T^*\mathbf{R}^2$. Here we remark for the later use that Fourier integral operators used in [M] is written in the form:

$$Fu(x) = os\text{-}\iint a(\tilde{x}, y, x) e^{\frac{i}{\hbar}(H(\tilde{x},y,x) - y\cdot\tilde{x})} u(\tilde{x})\, d\tilde{x}\, dy. \tag{1.10}$$

2. Existence of r_*, r_*^{-1} in $(\Sigma^\infty(\mathbf{C}^2 \times \mathbf{R}_+), *)$.

It is not obvious that the square root $\sqrt[*]{\rho}$ or its inverse $\sqrt[*]{\rho}^{-1}$ of ρ are defined in the algebra $(\Sigma^\infty(\mathbf{C}^2 \times \mathbf{R}_+), *)$. To ensure the existence of such elements, we compute first the $*$-exponential $e_*^{-t\frac{1}{2}\rho}$.

Suppose for a while that $F_t(\rho) = e_*^{-t\frac{1}{2}\rho}$, and $F_t(s)$ is a function of one variable. Differentiating and using the product formula (1.3), we have

$$\frac{\partial}{\partial t}F_t(\rho) = -\frac{1}{2}\rho * F_t(\rho) = -\frac{1}{2}\rho\cdot F_t(\rho) + \hbar^2 F_t'(\rho) + \hbar^2\frac{1}{2}\rho\cdot F_t''(\rho). \tag{2.1}$$

Hence, F_t must satisfy the differential equation

$$\frac{\partial}{\partial t}F_t(s) = -\frac{1}{2}sF_t(s) + \hbar^2(F_t'(s) + \frac{1}{2}sF_t''(s)) \tag{2.2}$$

with the initial condition $F_0(s) = 1$. Solving (2.2), we have

Lemma 2.1. *The $*$-exponential $e_*^{-\frac{t}{2}\rho}$ is given by*

$$e_*^{-\frac{t}{2}\rho} = \frac{4e^{\hbar t}}{(e^{\hbar t}+1)^2}\exp\{-\frac{\rho}{\hbar}\tanh\frac{\hbar t}{2}\} \tag{2.3}$$

In particular, for every $t > 0$, $e_*^{-\frac{t}{2}\rho} \in \Sigma^{-\infty}(\mathbf{C}^2 \times \mathbf{R}_+)$.

$$\lim_{t\to\infty} e_*^{-\frac{t}{2}\rho} = 0, \quad \textit{but} \quad \lim_{t\to\infty} e_*^{-\frac{t}{2}(\zeta_1 * \bar{\zeta}_1 + \zeta_2 * \bar{\zeta}_2)} = \lim_{t\to\infty} e_*^{-\frac{t}{2}\rho}e^{\hbar t} = 4e^{-\frac{\rho}{\hbar}}. \tag{2.4}$$

The second equality of (2.4) plays an important role. In the following, we denote the limit $\lim_{t\to\infty} e_*^{-\frac{t}{2}\rho}e^{\hbar t} = 4e^{-\frac{\rho}{\hbar}}$ by $\varpi \in \Sigma^{-\infty}(\mathbf{C}^2 \times \mathbf{R}_+)$.

Corollary 2.2. *$\frac{1}{2}\rho - z$ is invertible in $(\Sigma^0(\mathbf{C}^2 \times \mathbf{R}_+), *)$ for $Re\, z < \hbar$. In particular, ρ is invertible.*

Proof. By Lemma 2.1, we see $\int_0^\infty e_*^{-t(\frac{1}{2}\rho - z)}dt$ exists for $\mathrm{Re}\, z < \hbar$. Since

$$(\frac{1}{2}\rho - z) * \int_0^\infty e_*^{-t(\frac{1}{2}\rho-z)}dt = -\int_0^\infty \frac{d}{dt}e_*^{-t(\frac{1}{2}\rho-z)}dt = 1 - \lim_{t\to\infty} e_*^{-t(\frac{1}{2}\rho-z)} = 1,$$

$\frac{1}{2}\rho - z$ has the inverse $\int_0^\infty e_*^{-t(\frac{1}{2}\rho - z)}dt$. ■

We denote the inverse by ρ_*^{-1}. Using the formula of Laplace transform, we define

$$\sqrt[*]{\rho_*^{-1}} = \frac{1}{\sqrt{\pi}}\int_0^\infty \sqrt{t}^{-1} e_*^{-t\rho} dt, \tag{2.5}$$

and denote $r_*^{-1} = \sqrt[*]{\rho_*^{-1}}$. Then, we see $(r_*^{-1})^2 = \rho_*^{-1}$, $r_*^{-1} \in \Sigma^{-1}(\mathbf{C}^2 \times \mathbf{R}_+)$. We define r_* by $r_* = r_*^{-1} * \rho$.

By the product formula (1.3), we see that

$$\bar{\zeta}_i * \varpi = 0 = \varpi * \zeta_i, \quad i = 1, 2. \tag{2.6}$$

This shows also that

$$\rho * \varpi = 2\hbar\varpi \tag{2.7}$$

By (2.6) we view ϖ as the vacuum. Since the right multiplication $*\varpi$ by ϖ kills out every $\bar{\zeta}_i$ by (2.6), the polynomial approximation theorem gives that $f(\zeta, \bar{\zeta}, \hbar) * \varpi$ is a holomorphic function in $\Sigma^{-\infty}(\mathbf{C}^2 \times \mathbf{R}_+)$. Hence the space $\Sigma^\infty(\mathbf{C}^2 \times \mathbf{R}_+) * \varpi$ is reduced to the space $C^\infty(\mathbf{R}_+)\hat{\otimes}\mathcal{H}_\varpi[\zeta_1, \zeta_2] * \varpi$, where $\mathcal{H}_\varpi[\zeta_1, \zeta_2]$ is the space of all holomorphic functions $h(\zeta_1, \zeta_2)$ such that $h * \varpi \in \Sigma^{-\infty}(\mathbf{C}^2 \times \mathbf{R}_+)$. We use this space as the regular representation space. That is, for every $f \in \Sigma^\infty(\mathbf{C}^2 \times \mathbf{R}_+)$ we define the operator

$$\hat{f} : C^\infty(\mathbf{R}_+)\hat{\otimes}\mathcal{H}_\varpi[\zeta_1, \zeta_2] * \varpi \to C^\infty(\mathbf{R}_+)\hat{\otimes}\mathcal{H}_\varpi[\zeta_1, \zeta_2] * \varpi \tag{2.8}$$

by $\hat{f}(a * \varpi) = f * a * \varpi$.

Since the representation space $C^\infty(\mathbf{R}_+)\hat{\otimes}\mathcal{H}_\varpi[\zeta_1, \zeta_2] * \varpi$ is the topological completion of space:

$$C^\infty(\mathbf{R}_+) \otimes \{\text{polynomials of } \zeta_1, \zeta_2 \text{ degree up to } n\}$$

in the projective limit topology of $\{T_m\}$, every element $w \in \Sigma^\infty(\mathbf{C}^2 \times \mathbf{R}_+)$ is represented by the following blockwise matrix:

$$\hat{w} = \begin{bmatrix} B_{1,1} & B_{1,2} & \cdot & \cdots & \cdots & \cdots \\ B_{2,1} & B_{2,2} & B_{2,3} & \cdot & \cdots & \cdots \\ \cdot & B_{3,2} & B_{3,3} & B_{3,4} & \cdot & \cdots \\ \cdots & \cdot & B_{4,3} & B_{4,4} & B_{4,5} & \cdot \\ \cdots & \cdots & \cdot & B_{5,4} & B_{5,5} & B_{5,6} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{2.9}$$

where $B_{i,j}$ is $i \times j$-matrix with a suitable growth condition for $|i - j| \gg 1$. In fact, for the generators of W matrix representation satisfy $B_{i,j} = 0$ for $|i - j| \geq 2$ and have the following form:

$$B_{s+1,s}(\zeta_1) = \sqrt{2\hbar}\begin{bmatrix} \sqrt{s} & \cdot & \cdot & \cdot \\ \vdots & \vdots & \ddots & \\ \cdot & \cdot & \sqrt{2} & \\ & & 0 & \sqrt{1} \\ 0, & \cdots & \cdots & 0 \end{bmatrix}, \qquad \text{other blockes are } 0.$$

$$B_{s+1,s}(\zeta_2) = \sqrt{2\hbar}\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \sqrt{1} & \cdot & \cdot & \cdot \\ \cdot & \sqrt{2} & \cdot & \cdot \\ \vdots & \vdots & \ddots & \\ & & & \sqrt{s} \end{bmatrix}, \qquad \text{other blockes are 0.}$$

We also have $\hat{\bar{\zeta}}_i = {}^t\hat{\zeta}_i$. The matrix expressions of $\varpi = 2e^{-\frac{\rho}{\hbar}}$, and r_* are given by

$$\begin{aligned} \hat{\varpi} &= \mathrm{diag}\{1,0,0,\cdots,0,\cdots\} \\ \hat{r} &= \sqrt{2\hbar}\,\mathrm{diag}\{I_1, \sqrt{2}I_2, \cdots \sqrt{k}I_k \cdots\}. \end{aligned} \tag{2.10}$$

We construct matrix representations of $\mathcal{V}$ etc. as subalgebras of the above matrix algebra. A matrix is called *blockwise diagonal*, if it has the blockwise form such that $B_{i,j} = 0$ for $i \neq j$.

By (2.10), every element which commutes with r_* is represented by blockwise diagonal matrices with finite rank blocks.

Remark. In [OMMY1], the parameter $\hbar$ and the element μ after section §2 was treated as formal parameters so that the element ϖ did not appear. Thus, we had to consider asymptotic behaviors of represented operators to describe the relation to the Berezin representation. Such unnatural description is removed in this paper by using the element ϖ.

3. Energy surface

In classical mechanics, we take here a Hamiltonian $H = r^2/2$ and consider the energy surface $H = c$. However, it is difficult to define the notion of submanifold in terms of noncommutative algebra in general. Thus, instead of energy submanifold, we consider a subalgebra $\mathcal{C}$ of $(\Sigma^0(\mathbf{C}^2 \times \mathbf{R}_+), *)$ defined as follows: Set

$$R(e^t)\zeta_i = e^t\zeta_i, \ R(e^t)\bar{\zeta}_i = e^t\bar{\zeta}_i, \ R(e^t)\hbar = e^{2t}\hbar \tag{3.1}$$

on generators. We see easily that $R(e^t)$ extends to a one parameter group of automorphisms of $(\Sigma^0(\mathbf{C}^2 \times \mathbf{R}_+), *)$. We denote by $\mathcal{C}$ the closed subalgebra of all $R(e^t)$-invariant elements.

By identifying $\mathbf{C}^2$ with $T^*\mathbf{R}^2 = \mathbf{R}^4$ with coordinate functions x_1, x_2, y_1, y_2, $R(e^t)$ acts $R(e^t)x_i = e^t x_i$, $R(e^t)y_i = e^t y_i$, $R(e^t)\hbar = e^{2t}\hbar$. Using the form (1.10), we see that $R(e^t)$ also acts on the group GF^0. $G_{\mathcal{C}}$ denotes the subgroup of GF^0 consisting of all $R(e^t)$-invariant elements.

Knowing the existence of r_*^{-1} in $\Sigma^0(\mathbf{C}^2 \times \mathbf{R}_+)$, we see that $\mathcal{C}$ is the minimal closed subalgebra containing the elements

$$\mu = -2\hbar r_*^{-2}, \quad \xi_1 = r_*^{-1} * \zeta_1, \quad \xi_2 = r_*^{-1} * \zeta_2, \quad \bar{\xi}_1 = \bar{\zeta}_1 * r_*^{-1}, \quad \bar{\xi}_2 = \bar{\zeta}_2 * r_*^{-1}.$$

Since $\bar{\xi}_i * \varpi = 0$ and $\mu^{-1} * \varpi = -\varpi$ by (2.6) and (2.7), we have

$$\mathcal{C} * \varpi = \mathcal{H}_{\varpi}[\xi_1, \xi_2] * \varpi. \tag{3.2}$$

This can be used as the regular representation space of the algebra $\mathcal{C}$.

We see that

$$-\hat{\mu}^{-1} = \mathrm{diag}\{I_1, \cdots, kI_k, \cdots\} \tag{3.3}$$

where I_k is the $k \times k$ identity matrix, and $\hat{\xi}_i$, $\hat{\bar{\xi}}_i$ are easily obtained via the representations of r_* and ζ_i, $\bar{\zeta}_i$.

By using the product formula (1.3) carefully we have (cf.[OMMY1])

Lemma 3.1. *$\mu, \xi_i, \bar{\xi}_i$ are in $\Sigma^0(\mathbf{C}^2 \times \mathbf{R}_+)$ with the following relations:*

$$\bar{\xi}_1 * \xi_1 + \bar{\xi}_2 * \xi_2 = 1, \quad \xi_1 * \bar{\xi}_1 + \xi_2 * \bar{\xi}_2 = 1 + \mu$$
$$[\mu^{-1}, \xi_i] = -\xi_i, \qquad [\mu^{-1}, \bar{\xi}_i] = \bar{\xi}_i,$$
$$[\xi_1, \xi_2] = [\bar{\xi}_1, \bar{\xi}_2] = 0, \quad [\xi_i, \bar{\xi}_j] = \mu * (\delta_{ij} - \bar{\xi}_j * \xi_i) \quad \textit{for} \quad i, j = 1, 2.$$

Equalities of first line of Lemma 3.1 give a constraint in classical level, and the equalities in the second line show that every polynomial of ξ_i, $\bar{\xi}_i$ is an eigenfunction of $\mathrm{ad}(\mu^{-1})$.

Let $\mathcal{C}^{-\infty} = \mathcal{C} \bigcap \Sigma^{-\infty}(\mathbf{C}^2 \times \mathbf{R}_+)$. Then, we see that $\mathcal{C}/\mathcal{C}^{-\infty}$ is linearly isomorphic to the space $C^\infty(S^3)[[\mu]]$. Thus, $(\mathcal{C}, *)$ defines an associative product $*$ on the space $C^\infty(S^3)[[\mu]]$.

$(\mathcal{C}, *)$ can be viewed as a deformation quantization of the contact algebra algebra $C^\infty(S^3)$, and $(C^\infty(S^3)[[\mu]], *)$ can be viewed as its formal deformation quantization. Actually, we give the following property:

Theorem 3.2. *Set $B = C^\infty(S^3)$.*

(A.1) $[\mu, \mathcal{C}] \subset \mu{*}\mathcal{C}{*}\mu$.
(A.2) $[\mathcal{C}, \mathcal{C}] \subset \mu{*}\mathcal{C}$.
(A.3) $\mathcal{C} = B \oplus \mu{*}\mathcal{C}$ *(topological direct sum).*
(A.4) *Mappings $\mu* : \mathcal{C} \to \mu{*}\mathcal{C}$, $*\mu : \mathcal{C} \to \mathcal{C}{*}\mu$ defined by $a \to \mu * a$, $a \to a * \mu$ respectively are linear isomorphisms.*
(A.5) *$a \to \bar{a}$ is an involutive anti-automorphism such that $\bar{\mu} = \mu$.*
(A.6) $\bigcap_k \mu^k{*}\mathcal{C} = \mathcal{C}^{-\infty}$.

By the property (A.3), $\mathcal{C}$ is decomposed as

$$\mathcal{C} = B \oplus \mu{*}B \oplus \cdots \oplus \mu^{N-1}{*}B \oplus \mu^N{*}\mathcal{C} \tag{3.4}$$

for every positive integer N.

In [OMMY1], the algebra $\mathcal{C}$ is called the *non-commutative contact algebra* on S^3. The notion is motivated as follows: Set for every $a, b \in B$

$$a * b = \textstyle\sum_{k \geq 0} \mu^k * \pi_k(a, b), \quad \pi_k(a, b) \in B. \tag{3.5}$$

We see $\pi_0(a, b) = a \cdot b$ (usual commutative product), and the skew part π_1^- of π_1 gives a biderivation of $B \times B$ into B. By setting $[\mu^{-1}, a] = -\mu^{-1} * [\mu, a] * \mu^{-1}$, $\mathrm{ad}(\mu^{-1})$ gives a derivation of $\mathcal{C}$, and it is decomposed as

$$\mathrm{ad}(\mu^{-1})(a) = \xi_0(a) + \cdots + \mu^k * \xi_k(a) + \cdots, \tag{3.6}$$

where ξ_0 is a derivation of $(B, \cdot)$. Hence $\sqrt{-1}\xi_0$ is a vector field on S^3, which is called the *characteristic vector field*, It gives an S^1-free action on S^3. $(B, \cdot, \sqrt{-1}\xi_0, \pi_1^-)$ defines the ordinary contact structure on S^3.

It is easy to see that

$$[\mu^{-1}*\mathcal{C}, \mathcal{C}] \subset \mathcal{C}, \quad [\mu^{-1}*\mathcal{C}, \mu^{-1}*\mathcal{C}] \subset \mu^{-1}*\mathcal{C}.$$

Since $\mathcal{C}$ is characterized as the subalgebra of all $R(e^t)$-invariant elements in $\Sigma^0(\mathbf{C}^2 \times \mathbf{R}_+)$, we see that the group $G_{\mathcal{C}}^{(0)}$ of all invertible elements in $\mathcal{C}$ is an open subset of $\mathcal{C}$ and hence a regular Fréchet Lie group.

Let $\mathcal{C}_R$ be the top term pure imaginary subspace of $\mathcal{C}$. Then, we see that $\mu^{-1} * \mathcal{C}_R$ forms a Lie algebra. The Lie algebra $\mu^{-1} * \mathcal{C}_R/\mathcal{C}$ is isomorphic to the Lie algebra $\Gamma_\omega(T^*S^3)$ of all infinitesimal contact transformations of S^3. Using this fact and the extension theorem, we have the following theorem by the similar proof that Theorem A in [OMYK], and Theorem A in [M]:

Theorem 3.3. *Let $G_{\mathcal{C}}$ be the subgroup of GF_0^0 consisting of all $R(e^t)$-invariant elements. Then, $G_{\mathcal{C}}$ is a regular Fréchet Lie group with the Lie algebra $\mu^{-1} * \mathcal{C}_R$.*

Remark that elements of $G_{\mathcal{C}}$ in general may not be regularly represented, but as it is seen in §3, its Lie algebra is regularly represented as blockwise matrices and this representation is faithful.

4. Reduction

We now consider the subalgebra $\mathcal{V} = \{f \in \mathcal{C}; [\mu, f] = 0\}$ of $\Sigma^0(\mathbf{C}^2 \times \mathbf{R}_+)$. Viewing μ as a Hamiltonian $\frac{1}{2}r^2$, we think $\mathcal{V}$ as the algebra obtained by the ordinary reduction procedure.

By Lemma 3.1, we have $[\mu^{-1}, \xi_i] = -\xi_i$, $\quad [\mu^{-1}, \bar{\xi}_i] = \bar{\xi}_i$. It follows that $\mathcal{V}$ is generated (topologically) by the elements

$$\mu, \ \xi_1*\bar{\xi}_1, \ \xi_1*\bar{\xi}_2, \ \xi_2*\bar{\xi}_1, \ \xi_1^k*\xi_2^l*\varpi*\bar{\xi}_1^m*\bar{\xi}_2^n; \ (k+l=m+n). \tag{4.1}$$

Remark that $\xi_2*\bar{\xi}_1 = (\xi_1*\bar{\xi}_2)^-$ and that $\xi_1^k*\xi_2^l*\varpi*\bar{\xi}_1^m*\bar{\xi}_2^n$ is the $(k \otimes l) \times (m \otimes n)$ matrix element. (See (2.6), (2.7).)

Remark that $\varpi * \mathcal{V} = \mathcal{V} * \varpi = \mathbf{C}\varpi$. Thus, we continue to use $\mathcal{C} * \varpi$ as the regular representation space of $\mathcal{V}$. Since every element of $\mathcal{V}$ commute with μ, every element of $\mathcal{V}$ is faithfully represented as blockwise diagonal matrices. Setting $\mathcal{V}^{-\infty} = \mathcal{V} \bigcap \Sigma^{-\infty}(\mathbf{C}^2 \times \mathbf{R}_+)$, we see

$$\mathcal{V}/\mathcal{V}^{-\infty} \cong C^\infty(S^2)[[\mu]]. \tag{4.2}$$

Through the identification (4.2), $(\mathcal{V}, *)$ defines a noncommutative associative product on the space $C^\infty(S^2)[[\mu]]$. This associative product gives a deformation quantization of the Poisson algebra $(C^\infty(S^2), \cdot, \{\ ,\ \})$. By the uniqueness of the deformation quantization on S^2 (cf. [G], [OMY3]), $(C^\infty(S^2)[[\mu]], *)$ is the algebra obtained by the standard deformation quantization of the Poisson algebra $(C^\infty(S^2), \cdot, \{\ ,\ \})$. Note that the Lie algebra $(C^\infty(S^2), \{\ ,\ \})$ is the central extension of the Lie algebra of all volume preserving diffeomorphisms on S^2. In fact, this is the direct product $\mathbf{R} \times \Gamma(T^*_{S^2})$.

The Lie algebra $(\mu^{-1}*\mathcal{V}, [\ ,\ ])$ can be viewed as a quantization of the Poisson Lie algebra of functions $(C^\infty(S^2), \{\ ,\ \})$. If we set

$$H = \xi_1 * \bar{\xi}_1 - \tfrac{1+\mu}{2}, \qquad Z = \xi_1 * \bar{\xi}_2, \qquad Z^* = \xi_2 * \bar{\xi}_1, \tag{4.3}$$

Lemma 3.1 shows that the algebra generated by (4.3) is an enveloping algebra of the Lie algebra $sl_\mu(2;\mathbf{C})$:

$$[H, Z] = -\mu * Z,\ [H, Z^*] = \mu * Z^*,\ [Z, Z^*] = -2\mu * H \tag{4.4}$$

constrained by

$$(H + \frac{\mu}{2})^2 + Z * Z^* = \frac{1}{4} \tag{4.5}$$

via the constraint relations in Lemma 3.1. Note that $(H + \frac{\mu}{2})^2 + Z*Z^*$ is in the center of the enveloping algebra.

The matrix representations for H and Z are given as follows:

$$\begin{aligned} H &= \mathrm{diag}\{B_{1,1}, B_{2,2}, \cdots, B_{k,k}, \cdots\}, \\ Z &= \mathrm{diag}\{B'_{1,1}, B'_{2,2}, \cdots, B'_{k,k}, \cdots\}, \end{aligned} \tag{4.6}$$

with $Z^* = {}^tZ$. Here we set

$$B_{k,k} = \frac{1}{2k}\mathrm{diag}\{k-1, k-3, \cdots, -(k-3), -(k-1)\},$$

$$B'_{k,k} = k^{-1}\begin{bmatrix} 0, & \sqrt{(k-1)1}, & & & \\ & 0, & \sqrt{(k-2)2}, & & \\ & & \ddots & \ddots & \\ & & & 0, & \sqrt{1(k-1)} \\ & & & & 0 \end{bmatrix}$$

Let $\mathcal{V}_R$ be the top term pure imaginary subspace of $\mathcal{V}$. Then, we see that $\mu^{-1} * \mathcal{V}_R$ forms a Lie algebra. The Lie algebra $\mu^{-1} * \mathcal{V}_R/\mathcal{V}$ is isomorphic to the Lie algebra of all infinitesimal volume preserving transformations of S^2. By a similar proof in Theorem A in [OMYK] or Theorem A in [M], we have:

Theorem 4.1. *Let $G_\mathcal{V} = \{g \in G_C; \mu * g = g * \mu\}$. Then, $G_\mathcal{V}$ is a regular Fréchet Lie group with the Lie algebra $\mu^{-1} * \mathcal{V}_R$.*

The universal covering group $\tilde{G}_\mathcal{V}$ of $G_\mathcal{V}$ is also a regular Fréchet Lie group.

Since μ is represented as a blockwise scalar times identity matrix (cf (3.3)), every element of $\mathcal{V}$ is represented as a blockwise diagonal matrix. The claims (C.1), (C.2), (C.4), (C.5) in §I are easily obtained. The claim (C.7) has been already proved in §1. In particular it follows that $\tilde{G}_\mathcal{V}$ has a series of finite codimensional normal subgroups N_k such that $N_k \supset N_{k+1}$, $\bigcap N_k = \{D\}$; a discrete subgroup.

To prove the claim (C.6) in §1, we remark that $\tilde{G}_\mathcal{V}$ and $\mathcal{D}_\Omega(S^2)$ are regular Fréchet Lie groups and that the Lie algebra of $\mathcal{D}_\Omega(S^2)$ is isomorphic to the factor algebra $\mu^{-1} * \mathcal{V}/\mathcal{V}$. Since $\mathcal{D}_\Omega(S^2)$ is connected and $\tilde{G}_\mathcal{V}$ is 1-connected, the natural projection of Lie algebras is lifted to a C^∞ homomorphism of $\tilde{G}_\mathcal{V}$ onto $\mathcal{D}_\Omega(S^2)$.

Thus, the claim (C.3) in §1 only remains to be proved.

5. LOCAL GENERATORS

We show that the representation given in §4 coincides with the Berezin representation (cf.[CGR]) via the Kähler polarization on S^2.

For $i = 1, 2$, we define the algebra $\Sigma^0_{[i]}(\mathbf{C}\times\mathbf{R}_+)$ by a similar manner as in §1 by using only $\hbar, \zeta_i, \bar{\zeta}_i$ together with the commutation relation $[\zeta_i, \bar{\zeta}_i] = -2\hbar$. The space $\Sigma^m_{[i]}(\mathbf{C}\times\mathbf{R}_+)$ is defined by the similar manner.

By similar computations as in ρ, we see that for $i = 1, 2$,

$$e_*^{-\frac{t}{2}\zeta_i\bar{\zeta}_i} = \frac{2e^{\hbar t/2}}{e^{\hbar t}+1}\exp\{-\frac{\zeta_i\bar{\zeta}_i}{\hbar}\tanh\frac{\hbar t}{2}\} \tag{5.1}$$

and $e_*^{-\frac{t}{2}\zeta_i\bar{\zeta}_i} \in \Sigma^{-\infty}_{[i]}(\mathbf{C}\times\mathbf{R}_+)$. Remark here that $e_*^{-\frac{t}{2}\bar{\zeta}_i*\zeta_i} = e_*^{-\frac{t}{2}\zeta_i\bar{\zeta}_i}e^{-\frac{t}{2}\hbar}$.

By (5.1), we have

$$\lim_{t\to\infty} e_*^{-\frac{t}{2}\zeta_i*\bar{\zeta}_i} = \lim_{t\to\infty} e_*^{-\frac{t}{2}\zeta_i\bar{\zeta}_i}e^{\frac{t}{2}\hbar} = 2e^{-\frac{\zeta_i\bar{\zeta}_i}{\hbar}}. \tag{5.2}$$

As in (2.4), we set $\varpi_i = 2e^{-\frac{\zeta_i\bar{\zeta}_i}{\hbar}}$.

Similar to Corollary 2.2 , we see that $\zeta_i\cdot\bar{\zeta}_i$ and $\bar{\zeta}_i*\zeta_i$ are invertible but $\zeta_i*\bar{\zeta}_i$ is not invertible.

For computations, the following general lemma is very powerful, although we do not give the precise statement and the proof here (see [OMMY1] for the proof):

Lemma 5.1. (Bumping Lemma) *For a fairly wide class of functions $f(t)$ involving entire functions on $\mathbf{C}$, the equality*

$$f_*(z*\bar{z})*z = z*f_*(\bar{z}*z) \quad \bar{z}*f_*(z*\bar{z}) = f_*(\bar{z}*z)*\bar{z}$$

holds. Hence we have for instance

$$f_*(\zeta_i\bar{\zeta}_i - \hbar)*\zeta_i = \zeta_i * f_*(\zeta_i\bar{\zeta}_i+\hbar).$$

Obviously, ζ_i has a left inverse $(\bar{\zeta}_i*\zeta_i)^{-1}*\bar{\zeta}_i$, but by expressing $(\bar{\zeta}_i*\zeta_i)^{-1}$ as $\int_0^\infty e_*^{-t\bar{\zeta}_i*\zeta_i}dt$ and using the bumping lemma, we see that

$$\zeta_i*((\bar{\zeta}_i*\zeta_i)^{-1}*\bar{\zeta}_i) = 1-\varpi_i. \tag{5.3}$$

It is not hard to obtain following formulas:

$$\begin{aligned}&\varpi_1*\varpi_2 = \varpi_2*\varpi_1 = \varpi, \quad \varpi_i*\varpi_i = \varpi_i, \quad \bar{\zeta}_i*\varpi_i = 0 = \varpi_i*\zeta_i\\ &\zeta_i*\varpi_j = \varpi_j*\zeta_i \quad \text{for } i\neq j\\ &\varpi_i*\mu = \mu*\varpi_i, \quad \varpi_i*r_* = r_**\varpi_i.\end{aligned} \tag{5.4}$$

Using the formula of Laplace transform, we define

$$\sqrt[*]{\bar{\zeta}_i*\zeta_i}^{\,-1} = \tfrac{1}{\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{t}}\,e_*^{-t\bar{\zeta}_i*\zeta_i}dt. \tag{5.5}$$

Lemma 5.2. $\zeta_i * \sqrt[*]{\bar{\zeta}_i * \zeta_i}^{-1} \in \Sigma^0_{[i]}(\mathbf{C}\times\mathbf{R}_+)$, *but this is not the unitary part of* ζ_i.

Set as follows:

$$(5.6) \qquad \tau_i = e_*^{-\frac{1}{2}\bar{\zeta}_i*\zeta_i}, \quad T_i = \zeta_i * \sqrt[*]{\bar{\zeta}_i * \zeta_i}^{-1}, \quad T_i^* = \sqrt[*]{\bar{\zeta}_i * \zeta_i}^{-1} * \bar{\zeta}_i :$$

Using the bumping lemma, we have

Lemma 5.3. $T_i^* * T_i = 1, \quad T_i * T_i^* = 1 - \varpi_i, \quad \tau_i * T_i = e^{-\hbar} T_i * \tau_i,$
$[T_i, T_j] = 0, \ [T_i, T_j^*] = 0 \ (i \neq j).$

In fact, $\{T_i, T_i^*\}$ generates a Töplitz algebra. We see easily that $T_i^* * \varpi_i = 0 = \varpi_i * T_i$, but $T_i^* * \varpi_j = \varpi_j * T_i^*$, $T_i * \varpi_j = \varpi_j * T_i$.

We denote by $\tilde{\Sigma}^0_{[i]}(\mathbf{C}^2\times\mathbf{R}_+)$ for $i = 1, 2$ the algebra generated by

$$\Sigma^0(\mathbf{C}^2\times\mathbf{R}_+) \quad \text{and} \quad \Sigma^0_{[i]}(\mathbf{C}\times\mathbf{R}_+).$$

Let $\tilde{\mathcal{C}}_{[i]}$ be the subalgebra of all $R(e^t)$-invariant elements, and let $\tilde{\mathcal{V}}_{[i]}$ be the subalgebra of $\tilde{\mathcal{C}}_{[i]}$ consisting of elements which commute with μ.

Now, we fix the notion of *localization* of the algebra $\Sigma^0(\mathbf{C}^2\times\mathbf{R}_+)$, $\mathcal{C}$ or $\mathcal{V}$.

This means simply its embedding into other algebras, or more generally a homomorphism into another algebra. Thus, $\tilde{\Sigma}^0_{[i]}(\mathbf{C}^2\times\mathbf{R}_+)$ is considered as a localization of $\Sigma^0(\mathbf{C}^2\times\mathbf{R}_+)$.

Since $1-\mu$ is invertible and $r_*^2 * \zeta_i = \zeta_i * (1-\mu) * r_*^2$, there exists the inverse $(\bar{\xi}_i * \xi_i)^{-1}$ in the localized algebra $\tilde{\mathcal{C}}_{[i]}$, but $\xi_i * \bar{\xi}_i$ is not invertible. Hence ξ_i has the right inverse $(\bar{\xi}_i * \xi_i)^{-1} * \bar{\xi}_i$, though we have

$$(5.7) \qquad \xi_i * ((\bar{\xi}_i * \xi_i)^{-1} * \bar{\xi}_i) = 1 - \varpi_i.$$

Hence ξ_i is invertible in the factor algebra $\tilde{\mathcal{C}}_{[i]}/\{\varpi_i\}$ where $\{\varpi_i\}$ is the two-sided ideal generated by ϖ_i.

Define $z \in \tilde{\mathcal{V}}_{[1]}$, $w \in \tilde{\mathcal{V}}_{[2]}$ by

$$(5.8) \qquad \begin{aligned} z &= \xi_2 * \{(\bar{\xi}_1 * \xi_1)^{-1} * \bar{\xi}_1\} = \zeta_2 * \{(\bar{\zeta}_1 * \zeta_1)^{-1} * \bar{\zeta}_1\}, \\ w &= \xi_1 * \{(\bar{\xi}_2 * \xi_2)^{-1} * \bar{\xi}_2\} = \zeta_1 * \{(\bar{\zeta}_2 * \zeta_2)^{-1} * \bar{\zeta}_2\} \end{aligned}$$

Though $(\bar{\xi}_i * \xi_i)^{-1} * \bar{\xi}_i$ is not the genuine inverse of ξ_i, z, w play the role of complex coordinate $z = \xi_2 * \xi_1^{-1}$, $w = \xi_1 * \xi_2^{-1}$ of Riemann sphere. Namely, $\mathcal{V}_{[i]}$ is viewed as the algebra of $\mathcal{V}$ localized on the set $\mathbf{C}$ by a stereographic projection of S^2 from north/south pole.

Remark that (5.4) gives

$$(5.9) \qquad z * \varpi_1 = \varpi_2 * z = 0, \quad w * \varpi_2 = \varpi_1 * w = 0, \quad \varpi = \varpi_1 * \varpi_2 = \varpi_2 * \varpi_1.$$

Using these we have

Lemma 5.4. $1+z*\bar{z}$, $1+\bar{z}*z$ *are invertible in* $\tilde{\mathcal{V}}_{[1]}$ *and*

$$(1+z*\bar{z})^{-1}=\xi_1*\bar{\xi}_1-\mu\in\mathcal{V}$$
$$(1+\bar{z}*z)^{-1}=\xi_1*\bar{\xi}_1+\varpi_1\in\tilde{\mathcal{V}}_{[1]}$$

Proof. Since $1+z*\bar{z}=r^2*(\bar{\zeta}_1*\zeta_1)^{-1}$, we see $(1+z*\bar{z})^{-1}$ exists and using $\xi_1*\bar{\xi}_1=r_*^{-1}*\zeta_1*\bar{\zeta}_1*r_*^{-1}=r_*^{-2}*\zeta_1*\bar{\zeta}_1$, we have the inverse is given as $\xi_1*\bar{\xi}_1-\mu$. Using the bumping lemma, we have $1-\bar{z}*(1+z*\bar{z})^{-1}*z$ is the inverse of $1+\bar{z}*z$. For the second equality, remark that $\xi_1*z=\xi_2*(1-\varpi_1)$, $\varpi_1*\xi_1=0$. Using Lemma 3.1, (5.3-4) and (5.7-9), we obtain Lemma 5.4. ∎

We see easily that $z*w=1-\varpi_2\in\tilde{\mathcal{V}}_{[2]}$, $w*z=1-\varpi_1\in\tilde{\mathcal{V}}_{[1]}$. Note that

$$\xi_2=z*\xi_1,\quad \xi_1=w*\xi_2,\quad \varpi_1*\xi_1=0,\quad \varpi_2*\xi_2=0. \tag{5.10}$$

By the first two equalities, ξ_i can be viewed as a section of the canonical line bundle L over S^2. We see also that $\{\mu,z,\bar{z},\xi_1,\bar{\xi}_1\}$ and $\{\mu,w,\bar{w},\xi_2,\bar{\xi}_2\}$ topologically generate $\tilde{\mathcal{C}}_{[1]}$ and $\tilde{\mathcal{C}}_{[2]}$ respectively.

Following relations also hold;

$$\begin{aligned} &[z*\bar{z},\bar{z}*z]=0, && [w*\bar{w},\bar{w}*w]=0,\\ &[z,\xi_1]*(1-\varpi_1)=0, && [w,\xi_2]*(1-\varpi_2)=0\\ &[z,\xi_2]=0, && [w,\xi_1]=0. \end{aligned} \tag{5.11}$$

By Lemma 5.4, we get

$$\varpi_1*(1+z*\bar{z})=-\mu^{-1}*\varpi_1,\quad \varpi_1*(1+\bar{z}*z)=\varpi_1. \tag{5.12}$$

(5.11-12) gives

$$\begin{aligned} [z,\bar{z}]&=\mu*(1+z*\bar{z})*(1+\bar{z}*z)+\mu^{-1}*\varpi_1\\ [w,\bar{w}]&=\mu*(1+w*\bar{w})*(1+\bar{w}*w)+\mu^{-1}*\varpi_2 \end{aligned} \tag{5.13}$$

Though first terms of the right hand side of (5.13) have the same shapes as the standard Kähler form on S^2, we need the terms $\mu^{-1}*\varpi_i$ at the quantum level where $\hbar$ and μ are not formal parameters.

Remark that by Lemma 3.1 and (5.10), w in (5.8) is rewritten as

$$w=\xi_1*(1-\bar{\xi}_1*\xi_1)^{-1}*\bar{\xi}_1*\bar{z},$$

and

$$1-\xi_1*\bar{\xi}_1=\xi_2*\bar{\xi}_2-\mu=-(1-\mu)*\bar{\xi}_2*\xi_2$$

Since $1-\mu$ is invertible, there exists the inverse $(1-\xi_1*\bar{\xi}_1)^{-1}$ in the localized algebra $\tilde{\mathcal{C}}_{[2]}$. Hence we can use the bumping lemma for the first equality, and we have $w=$

$(1-\xi_1 * \bar{\xi}_1)^{-1}\xi_1 * \bar{\xi}_1 * \bar{z}$. By Lemma 5.4, $1-\xi_1 * \bar{\xi}_1 = 1-\mu-(1+z*\bar{z})^{-1}$. Thus, w can be written only by using $\mu, z, \bar{z}$, i.e. $w = f(\mu, z, \bar{z})$, where we need the variable $\bar{z}$.

This means some difficulty to define the notion of holomorphic structure for which $\hbar$ is not treated as a formal parameter.

Remark that $\tilde{\mathcal{V}}_{[1]}$ (resp. $\tilde{\mathcal{V}}_{[2]}$) is the algebra topologically generated by $\{\mu, z, \bar{z}\}$ in $\tilde{\mathcal{C}}_{[1]}$, (resp $\{\mu, w, \bar{w}\}$ in $\tilde{\mathcal{C}}_{[2]}$). We see then $\mathcal{V} = \tilde{\mathcal{V}}_{[1]} \cap \tilde{\mathcal{V}}_{[2]}$.

For the variables ξ_1, $\bar{\xi}_1$, we have the following relations:

$$\bar{\xi}_1 * (1+\bar{z}*z) * \xi_1 = 1, \quad \bar{\xi}_1 * (1+\bar{z}*z) = (\bar{\xi}_1 * \xi_1)^{-1} * \bar{\xi}_1, \tag{5.14}$$

(5.14) gives the constraint relation of the variables ξ_1, $\bar{\xi}_1$. To avoid such constraint relations, we set $\tilde{T}_1$, $\tilde{T}_2$ as follows with some similarity to unitary parts:

$$\tilde{T}_1 = \xi_1 * (\bar{\xi}_1 * \xi_1)^{-1/2} = T_1, \quad \tilde{T}_2 = \xi_2 * (\bar{\xi}_2 * \xi_2)^{-1/2} = T_2 \tag{5.15}$$

which are well-defined respectively in $\tilde{\mathcal{C}}_{[1]}$ and $\tilde{\mathcal{C}}_{[2]}$. Sets of elements $\{\mu, z, \bar{z}, \tilde{T}_1, \tilde{T}_1^*\}$, $\{\mu, w, \bar{w}, \tilde{T}_2, \tilde{T}_2^*\}$ define local generator systems without constraint relations, and by denoting the unitary part of the polar decomposition of z by $e_*^{i\theta_+}$, the coordinate transformation between above two generator systems may be understood as

$$\{\mu,\ w,\ \bar{w},\ \tilde{T}_2\} = \{\mu,\ z^{-1},\ \bar{z}^{-1},\ e_*^{i\theta_+} * \tilde{T}_1\} \tag{5.16}$$

The existence of such a unitary part $e_*^{i\theta_+}$ is given only in $\mathcal{C}/\{\varpi_1, \varpi_2\}$. Intuitively, this corresponds to the fact that the coordinate transformation between z and w is defined on the space $S^2 - \{N, S\}$. By using $z = \xi_2 * \{(\bar{\zeta}_1 * \zeta_1)^{-1} * \bar{\zeta}_1 * r_*\}$, it is not hard to see that

$$e_*^{i\theta_+} * \tilde{T}_1 = \tilde{T}_1 * e_*^{i\theta_+} = \tilde{T}_2 \tag{5.17}$$

The coordinate transformation (5.17) may be understood as that of quantum version of Hopf fibration of S^3 over the Riemann sphere $P^1(\mathbf{C}) = S^2$.

Since $[z, \xi_1] * (1-\varpi_1) = 0$, $[z, \xi_2] = 0$ by (5.11), we see that $\xi_2^k * \varpi = (z * \xi_1)^k * \varpi = z^k * \xi_1^k * \varpi$ (resp. $\xi_1^m = w^m * \xi_2^m * \varpi$). (5.9) gives

$$z^k * \xi_1^m * \varpi = 0, \quad \text{for} \quad k > m. \tag{5.18}$$

Thus, we see that $\xi_1^k * \xi_2^l * \varpi$, $\quad (k+l=m)$ can be viewed as a linear basis of the space $\mathcal{H}_m$ of all holomorphic sections of the holomorphic line bundle L^m. We see easily that $\mathcal{H}_m = \{0\}$ for $m < 0$ and $\mathcal{H}_m = \mathcal{P}_m(z) * \xi_i^m * \varpi$ for $m \geq 0$.

On $\tilde{\mathcal{V}}_{[1]}$ (resp. $\tilde{\mathcal{V}}_{[2]}$), we see that

$$\xi_1^k * \xi_2^l * \varpi = z^l * \xi_1^m * \varpi, \ \text{(resp.} \ = w^k * \xi_2^m * \varpi).$$

The coordinate transformation of the line bundle L^m is given by

$$\xi_1^k * \xi_2^l = w^k * \xi_2^m = (w^k * \xi_1^m) * z^m = z^l * \xi_1^m. \tag{5.19}$$

Combining Lemma 5.3 with the bumping lemma, we have also

$$[z, \bar{z} * (1 + z * \bar{z})^{-1}] = \mu + \varpi_1, \tag{5.20}$$

which implies that $\bar{z} * (1 + z * \bar{z})^{-1}$ play the similar role of the canonical conjugate element of z. Indeed, by (5.18) we must set $z^{m+1} * \xi_1^m * \varpi = 0$. Hence, we see

$$[\hat{z},\ \hat{\mu} * \partial_z] = \{\mu + \varpi_1\}\check{}. \tag{5.21}$$

Remark that $\mu * \varpi = -\varpi$ by (2.6-7).

By direct computation using formulas (5.18-21) and Lemma 2.3, we see that the localized algebras $\mathcal{V}_{[i]}$ can be represented as matrices as follows; for $k + l = m$,

$$\hat{\mu}(\xi_1^k * \xi_2^l * \varpi) = -\frac{1}{m+1}\xi_1^k * \xi_2^l * \varpi. \tag{5.21}$$

$$\begin{aligned} \hat{z}(\xi_1^k * \xi_2^l * \varpi) &= \xi_1^{k-1} * \xi_2^{l+1} * \varpi, \quad \xi_1^{-1} = \xi_2^{m+1} = 0 \\ \hat{w}(\xi_1^k * \xi_2^l * \varpi) &= \xi_1^{k+1} * \xi_2^{l-1} * \varpi, \quad \xi_2^{-1} = \xi_1^{m+1} = 0. \\ \hat{\varpi}_1(\xi_1^k * \xi_2^l * \varpi) &= \delta_{k,0}\xi_2^l * \varpi \\ \hat{\varpi}_2(\xi_1^k * \xi_2^l * \varpi) &= \delta_{0,l}\xi_1^k * \varpi \\ \hat{\varpi}(\xi_1^k * \xi_2^l * \varpi) &= \delta_{k,0}\delta_{0,l} * \varpi \end{aligned} \tag{5.23}$$

We also have for $k + l = m$:

$$\frac{1}{\sqrt{2\hbar}^m}\frac{1}{\sqrt{k!l!}}\zeta_1^k * \zeta_2^l = \sqrt{-\mu}^{-m}\prod_{j=0}^{m-1}\sqrt{1+j\mu} * \frac{1}{\sqrt{k!l!}}\xi_1^k * \xi_2^l. \tag{5.24}$$

Since $\hat{\mu}^{-1} = -(m+1)$ on $\Gamma(L^m)$, we obtain on $\tilde{\mathcal{V}}_{[1]}$;

$$\frac{1}{\sqrt{2\hbar}^m}\frac{1}{\sqrt{k!l!}}\zeta_1^k * \zeta_2^l = \sqrt{\frac{(m+1)!}{k!l!}}\xi_1^k * \xi_2^l = \sqrt{\frac{(m+1)!}{k!l!}}z^l * \xi_1^m \tag{5.25}$$

An orthonormal bases of the space $\mathcal{H}_m$ of all holomorphic sections is given for $k + l = m$ by

$$\frac{\sqrt{(m+1)!}}{\sqrt{k!l!}}\xi_1^k * \xi_2^l * \varpi = \begin{cases} \frac{\sqrt{(m+1)!}}{\sqrt{k!l!}}z^l * \xi_1^m * \varpi & (\text{on } \tilde{\mathcal{V}}_{[1]}) \\ \frac{\sqrt{(m+1)!}}{\sqrt{k!l!}}w^k * \xi_2^m * \varpi & (\text{on } \tilde{\mathcal{V}}_{[2]}) \end{cases} \tag{5.26}$$

Thus for each non-negative integer m, the matrix representations for the above generators are written as

$$\begin{aligned} (\xi_1 * \bar{\xi}_1)\check{}: \sqrt{\frac{(m+1)!}{(m-l)!l!}}z^l * \xi_1^m * \varpi &\rightarrow \frac{m-l}{m+1}\sqrt{\frac{(m+1)!}{(m-l)!l!}}z^l * \xi_1^m * \varpi, \\ (\xi_1 * \bar{\xi}_2)\check{}: \sqrt{\frac{(m+1)!}{k!l!}}z^l * \xi_1^m * \varpi &\rightarrow \frac{\sqrt{(k+1)l}}{m+1}\sqrt{\frac{(m+1)!}{(k+1)!(l-1)!}}z^{l-1} * \xi_1^m * \varpi, \\ (\xi_2 * \bar{\xi}_1)\check{}: \sqrt{\frac{(m+1)!}{k!l!}}z^l * \xi_1^m * \varpi &\rightarrow \frac{\sqrt{k(l+1)}}{m+1}\sqrt{\frac{(m+1)!}{(k-1)!(l+1)!}}z^{l+1} * \xi_1^m * \varpi, \end{aligned} \tag{5.27}$$

where we set $z^{-1} = z^{m+1} = 0$.

Here, note that the set $\{\bar{\xi}_1 * \xi_1,\ \bar{\xi}_2 * \xi_1,\ \bar{\xi}_1 * \xi_2\}$ also forms a system of generators. To obtain the Berezin representation, it is convenient to use $\{\bar{\xi}_1 * \xi_1,\ \bar{\xi}_2 * \xi_1,\ \bar{\xi}_1 * \xi_2\}$, rather than $\{\xi_1 * \bar{\xi}_1, \xi_1 * \bar{\xi}_2, \xi_2 * \bar{\xi}_1\}$. We have easily:

$$\bar{\xi}_1 * \xi_1 = -\frac{\mu}{2\hbar} * \frac{1}{1-\mu} * \bar{\zeta}_1 * \zeta_1, \quad \bar{\xi}_2 * \xi_1 = -\frac{\mu}{2\hbar} * \frac{1}{1-\mu} * \bar{\zeta}_2 * \zeta_1. \tag{5.28}$$

These have also matrix representations as follows:

$$\begin{aligned}
&(\bar{\xi}_1 * \xi_1)^{\vee}: \sqrt{\tfrac{(m+1)!}{(m-l)!l!}}z^l * \xi_1^m * \varpi \to \tfrac{m+1-l}{m+2}\sqrt{\tfrac{(m+1)!}{(m-l)!l!}}z^l * \xi_1^m * \varpi,\\
&(\bar{\xi}_2 * \xi_1)^{\vee}: \sqrt{\tfrac{(m+1)!}{k!l!}}z^l * \xi_1^m * \varpi \to \tfrac{\sqrt{(k+1)l}}{m+2}\sqrt{\tfrac{(m+1)!}{(k+1)!(l-1)!}}z^{l-1} * \xi_1^m * \varpi,\\
&(\bar{\xi}_1 * \xi_2)^{\vee}: \sqrt{\tfrac{(m+1)!}{k!l!}}z^l * \xi_1^m * \varpi \to \tfrac{\sqrt{k(l+1)}}{m+2}\sqrt{\tfrac{(m+1)!}{(k-1)!(l+1)!}}z^{l+1} * \xi_1^m * \varpi,
\end{aligned} \tag{5.29}$$

where we set $z^{-1} = z^{m+1} = 0$.

To obtain the Berezin representation we now define integral operators:

Lemma 5.4. *Let z and v denote the complex variables on $\mathbf{C}$. Then, for every non-negative integer the mapping*

$$I_m(p)(z) = \tfrac{m+1}{\pi}\int_{\mathbf{C}} p(v)\tfrac{(1+z\bar{v})^m}{(1+v\bar{v})^m}\tfrac{1}{(1+v\bar{v})^2}dvd\bar{v} \tag{5.30}$$

is the identity on the space $\mathcal{P}_m$ of all polynomials of degree up to m.

Since $\frac{1}{(1+v\bar{v})^2}dvd\bar{v}$ is the volume form on S^2, the right hand of (5.30) is considered as the integral over S^2. Using Lemma 5.4, we define the projection operator P_m by

$$(P_m f)(z) = \tfrac{m+1}{\pi}\int_{\mathbf{C}} f(v,\bar{v})\tfrac{(1+z\bar{v})^m}{(1+v\bar{v})^m}\tfrac{1}{(1+v\bar{v})^2}dvd\bar{v} \tag{5.31}$$

For any $a \in C^\infty(S^2)$, we define $B(a)f$ for $f = \sum_m f_m * \xi_1^m$, $f_m \in \mathcal{P}_m$ by

$$B(a)f = \textstyle\sum_{m\geq 0} P_m(a \cdot f_m) * \xi_1^m. \tag{5.32}$$

Then $B(a)$ defines a linear operator of $\sum \oplus \mathcal{P}_m * \xi_1^m * \varpi$ into itself.
A direct computation gives:

$$(\bar{\xi}_1 * \xi_1)^{\vee} = B(\tfrac{1}{1+\bar{z}z}), \quad (\bar{\xi}_2 * \xi_1)^{\vee} = B(\tfrac{\bar{z}}{1+\bar{z}z}), \quad (\bar{\xi}_1 * \xi_2)^{\vee} = B(\tfrac{z}{1+\bar{z}z}) \tag{5.33}$$

This constitutes the Berezin representation. Since (5.33) generates $\mathcal{V}$, we see that the regular representation of the algebra $\mathcal{V}$ coincides with the Berezin representation.

It also coincides with the matrix representation given in (2.9). $B(a)f$ can be expressed as an integral operator

$$B(a)f = \textstyle\sum_{m\geq 0}\{\tfrac{m+1}{\pi}\int_{\mathbf{C}} a(v,\bar{v})f_m(v)\tfrac{(1+z\bar{v})^m}{(1+v\bar{v})^m}\tfrac{1}{(1+v\bar{v})^2}dvd\bar{v}\} * \xi_1^m.$$

We remark again that any element of the algebra $\mathcal{C}$ are represented as matrices, although these are not blockwise diagonal matrices. It extends the Berezin representation.

References

[Be] Berezin, *General concept of quantization*, Comm. Math. Phys. **8** (1975), 153-174.

[BG] L. Boutet de Monvel, V. Guillemin, *The spectral theory of Toeplitz operators, Ann. Math. Studies 99*, Princeton Univ. Press, 1981.

[CGR] M. Cahen, S. Gutt and J. Rawnsley, *Quantization of Kähler manifolds, II*, Trans. Amer. Math. Soc. **337** (1993), 73-98.

[CHZ] C.S. Chu, P.M. Ho, B. Zumino, *The quantum 2-sphere as a complex quantum manifold*, to appear.

[G] S. Gutt, *Equivalence of deformation and associated $*$-product*, Lett. Math. Phys. **3** (1979), 297-309.

[K] A.V. Karabegov, *Deformation quantization with separation of variables on a Kähler manifolds*, to appear.

[L] A. Lichnerowicz, *Varietés de Jacobi et leure algebres des associeès*, J. Math. Pures et Appl. **57** (1978), 453-488.

[Les] J. Leslie, *On a differential structure for the group of diffeomorphisms*, Topology **6** (1967), 263-271.

[MO] T. Masuda and H. Omori, *The noncommutative algebra of the quantum group $SU_q(2)$ as a quantized Poisson manifold*, Contemporary Mathematics **179** (1994), 161-172.

[M] N. Miyazaki, *On regular Fréchet Lie group of invertible inhomogeneous Fourier integral operators on R^n*, Tokyo J. Math. **19** (1996), 1-38.

[Mil] J. Milnor, *Remarks on infinite dimensional Lie groups, In: Relativity, Groups and Topology II*,, Proc.Summer School on Quantum Gravity, ed. B.DeWitt, Les Houches, 1983, North-Holland (1984)..

[O.1] H. Omori, *Infinite dimensional Lie transformation groups, Lec. Note Math.*, vol. 427, Springer, 1974.

[O.2] H. Omori, *Infinite dimensional Lie groups, AMS Translation Monograph*, Amer. Math. Soc., 1997.

[OMY1] H. Omori, Y. Maeda and A. Yoshioka, *Weyl manifolds and deformation quantization*, Adv. in Math. **85** (1991), 224-255.

[OMY2] H. Omori, Y. Maeda and A. Yoshioka, *Deformation quantization of Poisson algebras*, Contemporary Mathematics **179** (1994), 213-241.

[OMY3] H. Omori, Y. Maeda and A. Yoshioka, *The uniqueness of star-products on $P_n(\mathbf{C})$*, Differential geometry, ed. C.H.Gu, H.S.Hu, Y.L.Xin, World Scientific, 1993, pp. 170-176.

[OMMY1] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Noncommutative 3-sphere: A model of noncommutative contact algebras*, to appear in J. Math. Soc. Japan.

[OMMY2] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Poincaré-Cartan class and deformation quantization*, to appear in Com. Math. Phys..

[OMYK1] H. Omori, Y. Maeda and A. Yoshioka, O. Kobayashi, *On regular Fréchet Lie groups IV, V*, Tokyo J. Math. **5,6** (1982, 1983), 365-398, 39-64.

[OMYK] Y. Maeda, H. Omori, A. Yoshioka and O. Kobayashi, *On regular Fréchet Lie groups VIII*, Tokyo J. Math. **18, No. 1** (1985), 1-47.

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