Remarks on Semi-Selfsimilar Processes

by

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REMARKS ON SEMI-SELFSIMILAR PROCESSES

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1. Introduction

In the paper [MSat97], two of the authors have introduced the notion of semi-selfsimilar processes extending that of selfsimilar processes. In this paper, we give some supplementary results on semi-selfsimilar processes.

An $\mathbf{R}^d$-valued stochastic process $\{X(t), t \geq 0\}$ is called semi-selfsimilar if there exist $a, b \in (0, 1) \cup (1, \infty)$ such that

$$\{X(at), t \geq 0\} \overset{d}{=} \{bX(t), t \geq 0\},$$

where $\overset{d}{=}$ denotes the equality in all finite-dimensional distributions. If

$$(1.1) \qquad \{X(at), t \geq 0\} \overset{d}{=} \{bX(t) + c(t), t \geq 0\}$$

for some $a, b \in (0, 1) \cup (1, \infty)$, and a nonrandom function $c : [0, \infty) \to \mathbf{R}^d$, then $\{X(t)\}$ is called wide-sense semi-selfsimilar.

In Section 2, we improve a result on the existence of exponents of semi-selfsimilar processes in [MSat97] by relaxing the condition on their stochastic continuity, and give some examples. In Section 3, we discuss the joint distributions of selfsimilar and semi-selfsimilar processes with independent increments. In Section 4, we give some examples of infinitely divisible semi-selfsimilar processes.

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2. Existence of the exponent

A stochastic process $\{X(t), t \geq 0\}$ is called trivial if, for each t, $P\{X(t) = \text{const.}\} = 1$. Otherwise it is called nontrivial. We prove the following.

Theorem 2.1. Let $\{X(t), t \geq 0\}$ be an $\mathbf{R}^d$-valued, nontrivial, wide-sense semi-selfsimilar process. Suppose that it is stochastically continuous at $t = 0$. Then there exists a unique $H > 0$ such that, if $a, b \in (0,1) \cup (1, \infty)$ and $c(t)$ satisfy (1.1), then $b = a^H$.

The constant H is called the exponent of $\{X(t)\}$. In [MSat97] the same conclusion has been proved under the condition that $\{X(t)\}$ is stochastically continuous in $t \geq 0$. Our theorem above shows that it is enough to assume the stochastic continuity only at $t = 0$. By this, the class of semi-selfsimilar processes having unique exponents is actually enlarged, as will be seen in Example 2.1 below. In Example 2.2 we shall further show that a nontrivial semi-selfsimilar process does not necessarily have an exponent, unless it is stochastically continuous at $t = 0$.

Proof of Theorem 2.1. Let Γ be the set of $a > 0$ such that there are $b > 0$ and a function $c(t)$ satisfying (1.1). By Lemma 2.2 of [MSat97], the nontriviality of $\{X(t)\}$ implies that b and $c(t)$ are uniquely determined by a. Thus we write $b = b(a)$ and $c(t) = c(t, a)$ for $a \in \Gamma$. We know that $b(a) > 1$ if $a \in \Gamma \cap (1, \infty)$, since the proof of this fact in the proof of Theorem 2.1 of [MSat97] uses only the stochastic continuity at $t = 0$. Denote by $\log \Gamma$ the set of $\log a$ with $a \in \Gamma$. As in [MSat97], $\log \Gamma$ is an additive subgroup of $\mathbf{R}$ and $(\log \Gamma) \cap (0, \infty) \neq \varnothing$. Let r_0 be the infimum of $(\log \Gamma) \cap (0, \infty)$.

Suppose that $r_0 > 0$. Then $r_0 \in \log \Gamma$. In fact, if $r_0 \notin \log \Gamma$, then there are $s_n, n = 1, 2, \cdots$, in $\log \Gamma$ strictly decreasing to r_0 and we have $r_0 > s_n - s_{n+1} \in (\log \Gamma) \cap (0, \infty)$ for sufficiently large n, contrary to the definition of r_0. As in [MSat97], $r_0 \in \log \Gamma$ implies $\log \Gamma = \{nr_0 : n \in \mathbf{Z}\}$, and hence there is a unique

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exponent $H > 0$.

In the rest of the proof, assume that $r_0 = 0$. In this case the proof for that $\log \Gamma = \mathbf{R}$ in [MSat97] does not work, since the argument to show the closedness of $\log \Gamma$ uses the stochastic continuity of $\{X(t)\}$ in $t \geq 0$. (Actually $\log \Gamma$ is not necessarily closed under the condition that $\{X(t)\}$ is stochastically continuous only at $t = 0$ as will be seen in Remark 2.1 below.) So we have to use another idea to show the existence of an exponent. Suppose that $H > 0$ with the desired property does not exist. Then there exist $0 < H_1 < H_2$ such that, for $i = 1, 2$, the set Γ_i defined by $\Gamma_i = \{a \in \Gamma : b(a) = a^{H_i}\}$ contains some $a_i \neq 1$. If $a \in \Gamma_i$, then $a^{-1} \in \Gamma_i$, since $b(a^{-1}) = b(a)^{-1} = a^{-H_i}$. Hence, for each i, there is $a_i \in \Gamma_i \cap (1, \infty)$. For any sufficiently large integer m, there exists a positive integer n such that

$$\left| n - \frac{H_2 \log a_2}{H_1 \log a_1} m \right| \leq 1.$$

Therefore we can find two sequences $\{m_k\}, \{n_k\}$ such that $m_k, n_k \to \infty$ and

$$-n_k H_1 \log a_1 + m_k H_2 \log a_2 \to b \quad \text{as } k \to \infty$$

for some $b \in (-\infty, \infty)$. Let $s_k = a_1^{-n_k} a_2^{m_k}$. Since

$$\frac{n_k}{m_k} \to \frac{H_2 \log a_2}{H_1 \log a_1} \quad \text{as } k \to \infty,$$

we have as $k \to \infty$

$$\log s_k = m_k \left(-\frac{n_k}{m_k} \log a_1 + \log a_2 \right) \to -\infty,$$

namely $s_k \to 0$. Take $t_0 > 0$ so that $X(t_0) \neq \text{const.}$ a.s. Then

$$(2.1) \qquad X(s_k t_0) = X(a_1^{-n_k} a_2^{m_k} t_0) \overset{d}{\sim} a_1^{-H_1 n_k} a_2^{H_2 m_k} X(t_0) + c_k$$

for some $c_k \in \mathbf{R}^d$ expressible by the function $c(t, a)$. Here $\overset{d}{\sim}$ means the equality in distribution. Denote the distribution of $X(t)$ by μ_t, and its characteristic function by $\hat{\mu}_t$. Then by (2.1)

$$\left| \hat{\mu}_{s_k t_0}(z) \right| = \left| \hat{\mu}_{t_0}(a_1^{-H_1 n_k} a_2^{H_2 m_k} z) \right|.$$

Let k tend to ∞ here. Use the stochastic continuity of $\{X(t)\}$ at $t = 0$ and the fact that $X(0) = $ const. proved in Remark 1.1 of [MSat97]. Then we have $|\hat{\mu}_{t_0}(e^b z)| = 1$, which contradicts that $X(t_0) \neq $ const. a.s. Therefore the exponent H uniquely exists. This completes the proof of Theorem 2.1. $\square$

We give below three examples. Throughout those examples, Let $\{Y(t), t \geq 0\}$ be a stochastically continuous, H-selfsimilar process in the sense that $\{Y(at), t \geq 0\} \overset{d}{=} \{a^H Y(t)\}$ for any $a > 0$, such that $Y(t)$ is nonconstant for every $t > 0$. Note that $Y(0) = 0$ a.s., since $Y(0) \overset{d}{\sim} a^H Y(0)$ for any $a > 0$.

The first example is a nontrivial H-semi-selfsimilar process which is stochastically continuous at $t = 0$ but not at any other $t > 0$.

Example 2.1. Define $\{X(t), t \geq 0\}$ by
$$X(t) = \begin{cases} 0, & \text{if } t = 0 \text{ or } \log t \notin \mathbf{Q}, \\ Y(t), & \text{if } \log t \in \mathbf{Q}, \end{cases}$$
where $\mathbf{Q}$ is the set of all rational numbers. Obviously $\{X(t)\}$ is stochastically continuous at $t = 0$ but not at any other $t > 0$. We have

(2.2) $$\{X(at)\} \overset{d}{=} \{a^H X(t)\}, \quad \text{if } \log a \in \mathbf{Q},$$

because $\log at \in \mathbf{Q}$ if and only if $\log t \in \mathbf{Q}$. If $a > 0$ and $\log a \notin \mathbf{Q}$, then there are no $b > 0$ and $c(t)$ satisfying (1.1), as is seen by choosing $t = a^{-1}$. Thus we have $\log \Gamma = \mathbf{Q}$, and it follows from (2.2) that $\{X(t)\}$ is H-semi-selfsimilar.

Remark 2.1. This example shows that, under the condition in Theorem 2.1., $\log \Gamma$ is not necessarily closed.

The second example is a nontrivial semi-selfsimilar process which does not have an exponent and therefore, by Theorem 2.1., is not stochastically continuous at $t = 0$. By this example, we see that we cannot entirely remove the assumption of the stochastic continuity at $t = 0$ to prove the existence of a unique exponent.

Example 2.2. Define $\{X(t), t \geq 0\}$ by
$$X(t) = \begin{cases} 0, & \text{if } t = 0 \text{ or } \log t \notin \mathbf{Q} + \sqrt{2}\mathbf{Q}, \\ Y(e^{r+s}), & \text{if } \log t = r + s\sqrt{2} \in \mathbf{Q} + \sqrt{2}\mathbf{Q}. \end{cases}$$

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It is easily seen that $\{X(t)\}$ is not stochastically continuous at any $t \geq 0$. Let $\alpha \in \mathbf{Q}$. Then

$$X(e^\alpha t) = X(e^{\alpha\sqrt{2}}t) = \begin{cases} 0, & \text{if } t = 0 \text{ or } \log t \notin \mathbf{Q} + \sqrt{2}\mathbf{Q}, \\ Y(e^{\alpha+r+s}), & \text{if } \log t = r + s\sqrt{2} \in \mathbf{Q} + \sqrt{2}\mathbf{Q}. \end{cases}$$

It follows that $\{X(e^\alpha t)\} \overset{d}{=} \{X(e^{\alpha\sqrt{2}}t)\} \overset{d}{=} \{e^{\alpha H}X(t)\}$. Thus $\{X(t)\}$ is semi-selfsimilar but does not have an exponent, and $\log \Gamma = \mathbf{Q} + \sqrt{2}\mathbf{Q}$.

If $\{X(t)\}$ is H-selfsimilar $(H > 0)$, then $\{X(t)\}$ is always stochastically continuous at $t = 0$. For, $X(0) = 0$ a.s. and

$$P\{|X(t)| > \varepsilon\} = P\{t^H|X(1)| > \varepsilon\} \to 0$$

when $t \to 0$. This is not true for semi-selfsimilar processes. The third example below is a nontrivial H-semi-selfsimilar process which is not stochastically continuous at $t = 0$.

To construct such a process, let $g : \mathbf{R} \to \mathbf{R}$ be a function satisfying

$$(2.3) \qquad\qquad g(u + v) = g(u) + g(v), \quad \forall u, v \in \mathbf{R},$$

$$(2.4) \qquad\qquad g(1) > 0,$$

$$(2.5) \qquad\qquad \limsup_{u \to -\infty} g(u) = +\infty, \quad \liminf_{u \to -\infty} g(u) = -\infty.$$

The existence of such a function is shown in [H05]. It follows easily from (2.3) that

$$(2.6) \qquad\qquad g(ru) = rg(u), \quad \forall r \in \mathbf{Q}, \forall u \in \mathbf{R}.$$

Therefore

$$(2.7) \qquad\qquad g(r + u) = rg(1) + g(u), \quad \forall r \in \mathbf{Q}, \forall u \in \mathbf{R}.$$

Let $f(t) = e^{g(\log t)}, t > 0$, and $H = g(1)$. We see from (2.7) that

$$f(at) = a^H f(t), \quad \text{if } \log a \in \mathbf{Q}.$$

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Example 2.3. Define $\{X(t), t \geq 0\}$ by

$$X(t) = \begin{cases} 0, & \text{if } t = 0, \\ f(t), & \text{if } t > 0 \text{ and } \log t \notin \mathbf{Q}, \\ Y(t), & \text{if } \log t \in \mathbf{Q}, \end{cases}$$

By the same reasoning as in Example 2.1, we have $\{X(at)\} \stackrel{d}{=} \{a^H X(t)\}$, if $\log a \in \mathbf{Q}$. Also if $a > 0$ and $\log a \notin \mathbf{Q}$, then there are no $b > 0$ and $c(t)$ satisfying (1.1). Thus we have $\log \Gamma = \mathbf{Q}$, and $\{X(t)\}$ is semi-selfsimilar with a unique exponent H. On the other hand, since

$$\lim_{\substack{u \to -\infty \\ u \in \mathbf{Q}}} g(u) = -\infty$$

by (2.4) and (2.6), we have

$$\limsup_{\substack{u \to -\infty \\ u \notin \mathbf{Q}}} g(u) = +\infty$$

by (2.5). Hence,

$$\limsup_{\substack{t \downarrow 0 \\ \log t \notin \mathbf{Q}}} f(t) = +\infty.$$

Namely, $\{X(t)\}$ is not stochastically continuous at $t = 0$. (Actually, this $\{X(t)\}$ is not stochastically continuous at any $t \geq 0$.)

3. Joint distributions of selfsimilar and
semi-selfsimilar processes with independent increments

If $\{X(t), t \geq 0\}$ is a stochastically continuous selfsimilar process with independent increments on $\mathbf{R}^d$, then, for each t, $X(t)$ has a selfdecomposable distribution. Conversely, any selfdecomposable distribution induces such a process uniquely in some sense ([Sat91]). But the joint distribution of $(X(t_1), \cdots, X(t_n))$ for the process $\{X(t)\}$ is not always selfdecomposable (Proposition 4.2 of

[Sat91]). We shall give conditions for the joint distributions to be selfdecomposable, and further, conditions for them to belong to the subclasses L_m. The relations between selfsimilar processes with independent increments and self-decomposable distributions are generalized in [MSat97] to those between semi-selfsimilar processes with independent increments and semi-selfdecomposable distributions. We shall also discuss joint distributions of those processes.

Let us recall some definitions. An $\mathbf{R}^d$-valued stochastic process $\{X(t)\}$ is called selfsimilar if, for any $a > 0$, there exists $b > 0$ such that $\{X(at), t \geq 0\} \overset{d}{=} \{bX(t), t \geq 0\}$. It is called wide-sense selfsimilar if, for any $a > 0$, there exist $b > 0$ and $c(t)$ such that $\{X(at), t \geq 0\} \overset{d}{=} \{bX(t) + c(t), t \geq 0\}$. The class of selfdecomposable distributions on $\mathbf{R}^d$ is called the class L and denoted by L_0 or $L_0(\mathbf{R}^d)$. A sequence of its subclasses $L_m = L_m(\mathbf{R}^d), m = 0, 1, \cdots, \infty$, is studied in [U72], [Sat80], and others. A description of the classes is as follows. Let $\mathcal{P}(\mathbf{R}^d)$ and $I(\mathbf{R}^d)$ be the collections of all probability measures on $\mathbf{R}^d$ and all infinitely divisible distributions on $\mathbf{R}^d$, respectively. A distribution $\mu \in \mathcal{P}(\mathbf{R}^d)$ belongs to $L_0(\mathbf{R}^d)$ if and only if, for any $b \in (0, 1)$, there is $\rho_b \in \mathcal{P}(\mathbf{R}^d)$ such that

$$(3.1) \qquad \hat{\mu}(z) = \hat{\mu}(bz)\hat{\rho}_b(z), \quad \forall z \in \mathbf{R}^d.$$

If $\mu \in L_0(\mathbf{R}^d)$, then $\mu \in I(\mathbf{R}^d)$, ρ_b is uniquely determined by μ and b, and $\rho_b \in I(\mathbf{R}^d)$. Let m be a positive integer. A distribution $\mu \in \mathcal{P}(\mathbf{R}^d)$ belongs to $L_m(\mathbf{R}^d)$ if and only if $\mu \in L_0(\mathbf{R}^d)$ and, for every $b \in (0, 1), \rho_b$ in (3.1) belongs to $L_{m-1}(\mathbf{R}^d)$. The class $L_\infty(\mathbf{R}^d)$ is the intersection of the classes $L_m(\mathbf{R}^d), m = 0, 1, \cdots$. Thus we have

$$I(\mathbf{R}^d) \supset L_0(\mathbf{R}^d) \supset L_1(\mathbf{R}^d) \supset \cdots \supset L_\infty(\mathbf{R}^d) \supset S(\mathbf{R}^d),$$

where $S(\mathbf{R}^d)$ is the class of all stable distributions on $\mathbf{R}^d$.

For any fixed $b \in (0, 1)$ a sequence of the subclasses $L_m(b) = L_m(b, \mathbf{R}^d)$, $m = 0, 1, \cdots, \infty$, is recently introduced by [MN97]. A distribution $\mu \in \mathcal{P}(\mathbf{R}^d)$

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is in $L_0(b, \mathbf{R}^d)$ if and only if (3.1) holds with some $\rho_b \in I(\mathbf{R}^d)$. If $\mu \in L_0(b, \mathbf{R}^d)$ with some $b \in (0,1)$, it is called semi-selfdecomposable. Again, if $\mu \in L_0(b, \mathbf{R}^d)$, then $\mu \in I(\mathbf{R}^d)$ and ρ_b is uniquely determined. A distribution $\mu \in \mathcal{P}(\mathbf{R}^d)$ belongs to $L_m(b, \mathbf{R}^d)$ with a positive integer m if and only if $\mu \in L_0(b, \mathbf{R}^d)$ and $\rho_b \in L_{m-1}(b, \mathbf{R}^d)$. The class $L_\infty(b, \mathbf{R}^d)$ is the intersection of all $L_m(b, \mathbf{R}^d), m = 0, 1, \cdots$. We have

$$I(\mathbf{R}^d) \supset L_0(b, \mathbf{R}^d) \supset L_1(b, \mathbf{R}^d) \supset \cdots \supset L_\infty(b, \mathbf{R}^d).$$

We see that

$$L_m(\mathbf{R}^d) = \bigcap_{0<b<1} L_m(b, \mathbf{R}^d), \quad m = 0, 1, \cdots, \infty.$$

This follows from the characterization of Lévy measures of distributions in $L_m(\mathbf{R}^d)$ and $L_m(b, \mathbf{R}^d)$, actually from the combination of Theorem 3.2 of [Sat80] and Theorem 4.2 of [MN97].

Now let us prove the following result. The distribution of a random vector X is denoted by $\mathcal{L}(X)$.

Theorem 3.1. Let $\{X(t), t \geq 0\}$ be a stochastically continuous, wide-sense selfsimilar process with independent increments. Let m be a positive integer or ∞. Then the following four conditions are equivalent. We understand $m-1 = \infty$ if $m = \infty$.

(i) $\mathcal{L}(X(t)) \in L_m(\mathbf{R}^d), \quad \forall t \geq 0.$

(ii) $\mathcal{L}((X(t_1), \cdots, X(t_n))) \in L_{m-1}(\mathbf{R}^{nd}), \quad \forall n, \forall t_1, \cdots, t_n \geq 0.$

(iii) $\mathcal{L}\left(\sum_{k=1}^n c_k X(t_k)\right) \in L_{m-1}(\mathbf{R}^d), \quad \forall n, \forall t_1, \cdots, t_n \geq 0, \forall c_1, \cdots, c_n \in \mathbf{R}.$

(iv) $\mathcal{L}(X(t) - X(s)) \in L_{m-1}(\mathbf{R}^d), \quad \forall s, t \geq 0.$

Lemma 3.1. Let $m \in \{0, 1, \cdots, \infty\}$. Let $d_1, \cdots, d_n$ be positive integers. If $\mu \in L_m(\mathbf{R}^{d_1})$ and if T is a linear transformation from $\mathbf{R}^{d_1}$ to $\mathbf{R}^{d_2}$, then $\mu T^{-1} \in$

$L_m(\mathbf{R}^{d_2})$, where $(\mu T^{-1})(B) = \mu(T^{-1}(B))$. If $\mu_k \in L_m(\mathbf{R}^{d_k})$ for $k = 1, \cdots, n$, then $\mu_1 \times \cdots \times \mu_n \in L_m(\mathbf{R}^d)$ with $d = d_1 + \cdots + d_n$.

This lemma is essentially found in Theorem 2.4 of [Sat80]. The proof is based on the decomposition (3.1).

Proof of Theorem 3.1. Let $0 \leq t_1 \leq \cdots \leq t_n$. Let $Y_1 = X(t_1)$ and $Y_k = X(t_k) - X(t_{k-1})$ for $k = 2, \cdots, n$. Then $X(t_k) = Y_1 + \cdots + Y_k$. By Lemma 3.1 we see that $\mathcal{L}((X(t_1), \cdots, X(t_n))) \in L_{m-1}(\mathbf{R}^{nd})$ if and only if $\mathcal{L}((Y_1, \cdots .Y_n)) \in L_{m-1}(\mathbf{R}^{nd})$. Since $Y_1, \cdots, Y_n$ are independent, Lemma 3.1 shows that $\mathcal{L}((Y_1, \cdots, Y_n)) \in L_{m-1}(\mathbf{R}^{nd})$ if and only if $\mathcal{L}(Y_k) \in L_{m-1}(\mathbf{R}^d)$ for $k = 1, \cdots, n$. Hence we see that (ii) and (iv) are equivalent. By Lemma 3.1, (ii) implies (iii). Obviously (iii) implies (iv). Hence (iii) is equivalent to (ii) and (iv).

Let us prove the equivalence of (i) and (iv). The process $\{X(t)\}$ has an exponent $H > 0$, that is

$$(3.2) \qquad \{X(at), t \geq 0\} \overset{d}{=} \{a^H X(t) + c(t, a), t \geq 0\}, \quad \forall a > 0.$$

Let $0 \leq s \leq t$. Denote $\mu_t = \mathcal{L}(X(t))$ and $\mu_{s,t} = \mathcal{L}(X(t) - X(s))$. Then

$$(3.3) \qquad \hat{\mu}_t(z) = \hat{\mu}_s(z)\hat{\mu}_{s,t}(z) = \hat{\mu}_t\left(\left(\frac{s}{t}\right)^H z\right) e^{i<c(t,s/t),z>}\hat{\mu}_{s,t}(z),$$

where we have used the independent increments property and (3.2) with $a = s/t$. On the othe hand, as we have mentioned at the beginning of this section, Sato [Sat91] proved that if $\{X(t)\}$ is the process assumed in Theorem 3.1, then $\mu_t \in L_0(\mathbf{R}^d)$. Thus for any $b \in (0, 1)$, there exists $\rho_{t,b} \in I(\mathbf{R}^d)$ such that

$$(3.4) \qquad \hat{\mu}_t(z) = \hat{\mu}_t(bz)\hat{\rho}_{t,b}(z), \quad \forall z \in \mathbf{R}^d.$$

Since $\hat{\mu}_t(z) \neq 0$, it follows from (3.3) and (3.4) that

$$\rho_{t,(s/t)^H} = \mu_{s,t} * \delta_{c(t,s/t)}.$$

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Hence $\rho_{t,(s/t)^H} \in L_{m-1}(\mathbf{R}^d)$ if and only if $\mu_{s,t} \in L_{m-1}(\mathbf{R}^d)$, comcluding that (i) and (iv) are equivalent. $\square$

Note that if $\mathcal{L}(X(1)) \in L_m(\mathbf{R}^d)$, then (i) is true. This is because $X(t) \overset{d}{\sim} t^H X(1) + c(1, t)$ by (3.2).

Example 3.1. Let $d = 1$. Consider the case where

$$\hat{\mu}(z) = \exp\left(\int_0^\infty (e^{izx} - 1)\frac{k(x)}{x}dx\right).$$

Let $h(x) = k(e^{-x})$. We have that $\mu \in L_0(\mathbf{R})$ if and only if $k(x)$ is nonicreasing. The necessary and sufficient condition for that $\mu \in L_1(\mathbf{R})$ is the convexity of $h(x)$, (see Theorem 3.2 of [Sat80]).

(i) Let $k(x) = ce^{-ax}$ with $a, c > 0$. Then μ is a gamma distribution. It is in $L_0(\mathbf{R})$ but not in $L_1(\mathbf{R})$, since $h(x)$ is not convex. Thus some joint distributions (in fact, all joint distributions with dimension greater than or equal to 2) of the corresponding process $\{X(t)\}$ are outside the class L_0. More properties of this example are studied in [Sat91] and [W96].

(ii) Let $k(x) = cx^{-\alpha}e^{-ax}$ with $a, c > 0$ and $0 < \alpha < 2$. It is easy to find the condition for the convexity of $h(x)$. Thus $\mu \in L_1(\mathbf{R})$ if and only if $\alpha \geq 1/4$. Hence inverse Gaussian distributions (that is, $\alpha = 1/2$) are in $L_1(\mathbf{R})$. (As to inverse Gaussian distributions, see, e.g. [Se93].)

Let us consider a generalization of Theorem 3.1 to semi-selfsimilar case. Let $\{X(t), t \geq 0\}$ be a nontrivial, stochastically continuous, wide-sense semi-selfsimilar process on $\mathbf{R}^d$ with independent increments. Let H be its exponent. We assume that it is not wide-sense selfsimilar. As in Section 2, let Γ be the set of $a > 0$ such that there are $b > 0$ and $c(t)$ satisfying (1.1). Then there is $a_0 > 1$ such that $\Gamma = \{a_0^n : n \in \mathbf{Z}\}$, (see the proof of Theorem 2.1 of [MSat97]). We have $\mathcal{L}(X(t)) \in L_0(a_0^{-H}, \mathbf{R}^d)$ for every t. The distributions $\{\mathcal{L}(X(t)) : t \in [1, a_0)\}$ determine all distributions of $X(t), t \geq 0$, modulo translations. Theorem 3.1 has the following counterpart.

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Theorem 3.2. Let $\{X(t), t \geq 0\}$, H, and a_0 be as above. Let m be a positive integer or ∞. Then the following four conditions are equivalent.

(i) $\mathcal{L}(X(t)) \in L_m(a_0^{-H}, \mathbf{R}^d), \quad \forall t \geq 0.$

(ii) $\mathcal{L}((X(u_1 t), \cdots, X(u_n t))) \in L_{m-1}(a_0^{-H}, \mathbf{R}^{nd}), \quad \forall n, \forall t \geq 0, \forall u_1, \cdots, u_n \in \Gamma.$

(iii) $\mathcal{L}\left(\sum_{k=1}^{n} c_k X(u_k t)\right) \in L_{m-1}(a_0^{-H}, \mathbf{R}^d),$

$$\forall n, \forall t \geq 0, \forall u_1, \cdots, u_n \in \Gamma, \forall c_1, \cdots, c_n \in \mathbf{R}.$$

(iv) $\mathcal{L}(X(u_2 t) - X(u_1 t)) \in L_{m-1}(a_0^{-H}, \mathbf{R}^d), \quad \forall t \geq 0, \forall u_1, u_2 \in \Gamma.$

Lemma 3.2. Let $0 < b < 1$. The statement of Lemma 3.1 remains true if we replace $L_m(\mathbf{R}^{d_k})$ and $L_m(\mathbf{R}^d)$ by $L_m(b, \mathbf{R}^{d_k})$ and $L_m(b, \mathbf{R}^d)$, respectively.

This is proved similarly to Lemma 3.1.

Proof of Theorem 3.2. Using Lemma 3.2 in place of Lemma 3.1, we can prove the equivalence of (ii), (iii), and (iv) in the same way as in the proof of Theorem 3.1. To show the equivalence of (i) and (iv), let $u, v \in \Gamma$ with $0 < u < v$ and let $t \geq 0$. Then, since $u/v \in \Gamma$,

$$\hat{\mu}_{vt}(z) = \hat{\mu}_{ut}(z)\hat{\mu}_{ut,vt}(z)$$
$$= \hat{\mu}_{vt}\left(\left(\frac{u}{v}\right)^H z\right) e^{i<c,z>}\hat{\mu}_{ut,vt}(z)$$
$$= \hat{\mu}_{vt}(a_0^{-nH} z)e^{i<c,z>}\hat{\mu}_{ut,vt}(z)$$

with some $c \in \mathbf{R}^d$ and some positive integer n. To see that (iv) implies (i), it is enough to choose $v = 1$ and $u = a_0^{-1}$ in the identity above. Conversely, suppose that (i) is satisfied. Then

$$\hat{\mu}_{vt}(z) = \hat{\mu}_{vt}(a_0^{-H} z)\hat{\rho}(z)$$

with $\rho \in L_{m-1}(a_0^{-H}, \mathbf{R}^d)$. Hence

$$\hat{\mu}_{vt}(z) = \hat{\mu}_{vt}(a_0^{-2H} z)\hat{\rho}(a_0^{-H} z)\hat{\rho}(z)$$
$$= \hat{\mu}_{vt}(a_0^{-nH} z)\hat{\rho}(a_0^{-(n-1)H} z) \cdots \hat{\rho}(z).$$

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Since $\hat{\mu}_{vt}(z) \neq 0$, it follows that

$$\hat{\mu}_{ut,vt}(z) = \hat{\rho}(a_0^{-(n-1)H}z)\cdots\hat{\rho}(z)e^{-i<c,z>},$$

where $u = va_0^{-n}$. Since $L_{m-1}(a_0^{-H}, \mathbf{R}^d)$ is closed under convolution (Theorem 3.3 of [MN97]), we see that $\mu_{ut,vt} \in L_{m-1}(a_0^{-H}, \mathbf{R}^d)$. That is, we get the condition (iv). $\square$

We note that (i) is true if $\mathcal{L}(X(t)) \in L_{m-1}(a_0^{-H}, \mathbf{R}^d)$ for $1 \leq t < a_0$.

Comparing Theorems 3.1 and 3.2, one might ask whether in Theorem 3.2 the condition (i) implies that all $n \times d$-dimensional joint distributions of $\{X(t)\}$ are in $L_{m-1}(a_0^{-H}, \mathbf{R}^{nd})$. We show, by an example, that the answer is negative.

Example 3.2. Let $d = 1$ and $0 < b < 1$. Consider an infinitely divisible distribution with Lévy measure

$$\nu = \sum_{n\in\mathbf{Z}} k_n \delta_{b^{-n}},$$

where $k_n \geq 0$ and $\sum_{n\geq 0} k_n + \sum_{n<0} b^{-2n}k_n < \infty$. Then, $\mu \in L_0(b, \mathbf{R})$ if and only if

$$(3.5) \qquad\qquad k_n - k_{n+1} \geq 0, \quad \forall n \in \mathbf{Z},$$

and $\mu \in L_1(b, \mathbf{R})$ if and only if, in addition to (3.5),

$$(3.6) \qquad\qquad (k_n - k_{n+1}) - (k_{n+1} - k_{n+2}) \geq 0, \quad \forall n \in \mathbf{Z},$$

(see [MN97]). Suppose that we are given $\mu \in L_0(b, \mathbf{R})$ with Lévy measure of this form. Choose $a > 1$ and $H > 0$ such that $b = a^{-H}$. If $g_n(t), n \in \mathbf{Z}$, are chosen to be nondecreasing continuous functions on $[1, a)$ satisfying $g_n(1) = k_n$ and $\lim_{t\uparrow a} g_n(t) = k_{n-1}$, then we can construct a stochastically continuous, H-semi-selfsimilar process $\{X(t), t \geq 0\}$ with independent increments such that $\mathcal{L}(X(1)) = \mu$ and $\mathcal{L}(X(t))$ has Lévy measure

$$(3.7) \qquad\qquad \nu_t = \sum_{n\in\mathbf{Z}} g_n(t)\delta_{b^{-n}}$$

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for $1 \leq t < a$ (Theorem 6.2 of [MSat97]). Now assume that the given $k_n, n \in \mathbf{Z}$, satisfy (3.5) and (3.6) with strict inequalities (e.g. $k_n = b^{n/H}$ with $H > 1/2$). Let

$$h_n(t) = k_n \frac{a - t}{a - 1} + k_{n-1} \frac{t - 1}{a - 1}, \quad 1 \leq t < a.$$

Then, for any fixed t, $h_n(t), n \in \mathbf{Z}$, satisfy the inequalities corresponding to (3.5) and (3.6). Choose $\varepsilon > 0$ so small that

$$k_j - 2h_{j+1}(1 + 2\varepsilon) + k_{j+2} > 0 \quad \text{for } j = 0, -1, -2.$$

This is possible, since $k_j - 2h_{j+1}(t) + k_{j+2} \to k_j - 2k_{j+1} + k_{j+2} > 0$ as $t \downarrow 1$. Next choose $g_n(t), n \in \mathbf{Z}$, as follows:

$$g_n(t) = h_n(t), \quad 1 \leq \forall t < a, \forall n \neq 0,$$
$$g_0(t) = h_0(t), \quad 1 + 2\varepsilon \leq \forall t < a,$$
$$g_0(t) = k_0, \quad 1 \leq \forall t \leq 1 + \varepsilon,$$

and $g_0(t)$ is continuous and nondecreasing for $1 + \varepsilon \leq t \leq 1 + 2\varepsilon$. We claim that

$$(3.8) \qquad g_n(t) - g_{n+1}(t) \geq 0, \quad 1 \leq \forall t < a, \forall n \in \mathbf{Z},$$

$$(3.9) \qquad (g_n(t) - g_{n+1}(t)) - (g_{n+1}(t) - g_{n+2}(t)) \geq 0, \quad 1 \leq \forall t < a, \forall n \in \mathbf{Z},$$

$$(3.10) \qquad g_0(t) - k_0 < g_1(t) - k_1, \quad 1 < \forall t \leq 1 + \varepsilon.$$

In fact, (3.8) follows from that $g_n(t) \geq k_n \geq g_{n+1}(t)$. If $n \neq 0, -1, -2$ and $t \in [1, a)$ or if $n \in \{0, -1, -2\}$ and $t \in [1 + 2\varepsilon, a)$, then (3.9) is identical with the corresponding inequality for $h_n(t), n \in \mathbf{Z}$. If $j \in \{0, -1, -2\}$ and $t \in [1, 1 + 2\varepsilon)$, then

$$(g_j(t) - g_{j+1}(t)) - (g_{j+1}(t) - g_{j+2}(t)) \geq k_j - 2h_{j+1}(1 + 2\varepsilon) + k_{j+2} > 0.$$

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Hence (3.9) is true. The inequality (3.10) is obvious. Consider the process $\{X(t)\}$ that corresponds to $g_n(t), n \in \mathbf{Z}$. That is, for $1 \leq t < a$, $\mathcal{L}(X(t))$ has Lévy measure (3.7). Then $\mathcal{L}(X(t)) \in L_1(b, \mathbf{R})$ for $1 \leq t < a$ by (3.8) and (3.9), and also for all other t by the semi-selfsimilarity. However, $\mathcal{L}(X(t) - X(1)) \notin L_0(b)$ for $1 < t \leq 1 + \varepsilon$ by virtue of (3.10), because

$$\nu_{1,t} = \sum_{n \in \mathbf{Z}} (g_n(t) - k_n) \delta_{b^{-n}}$$

for the Lévy measure $\nu_{1,t}$ of $\mathcal{L}((X(t) - X(1)))$. Hence $\mathcal{L}((X(t), X(1)))$ $\notin L_0(b, \mathbf{R}^2)$ for $1 < t \leq 1 + \varepsilon$.

The proof of Theorem 3.1 shows that, for any process $\{X(t)\}$ with independent increments, the conditions (ii) and (iii) in Theorem 3.1 are equivalent. A related problem is for what more general processes (not necessarily having independent increments) those two conditions are equivalent. This will be discussed in another paper.

4. Infinitely divisible semi-selfsimilar processes

In this section, we shall give several examples of infinitely divisible semi-selfsimilar processes. We say that an $\mathbf{R}^d$-valued process $\{X(t)\}$ is infinitely divisible (resp., α-stable) if for any n and any $0 \leq t_1 < \cdots < t_n$, $d \times n$-dimensional random vector $(X(t_1), \cdots, X(t_n))$ is infinitely divisible (resp., α-stable).

In the following, for $B \subset \mathbf{R}^d$ and $x \in \mathbf{R}^d$, denote

$$\frac{B}{x} = \{t \in \mathbf{R} : tx \in B\} \subset \mathbf{R}.$$

Proposition 4.1. Let X be a real-valued nonnegative infinitely divisible random variable with its Lévy measure ν and with 0 as its drift. Let Z_α be an $\mathbf{R}^d$-valued strictly α-stable random vector, independent of X. Then

$$\tilde{X} := X^{1/\alpha} Z_\alpha$$

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is infinitely divisible on $\mathbf{R}^d$ and its Lévy measure $\tilde{\nu}$ is given by

$$\tilde{\nu}(B) = E\left[(T_\alpha \nu)\left(\frac{B}{Z_\alpha}\right)\right], \quad \forall B \in \mathcal{B}(\mathbf{R}^d \setminus \{0\}),$$

where $T_\alpha \nu$ is a measure on $\mathbf{R}$ defined by

$$(T_\alpha \nu)(A) = \int I_A(t^{1/\alpha})\nu(dt), \quad \forall A \in \mathcal{B}(\mathbf{R})$$

and I_A is the indicator function of the set A.

Proof. Let $\{X(t)\}$ and $\{Z_\alpha(t)\}$ be independent Lévy processes on $\mathbf{R}$ and on $\mathbf{R}^d$, respectively, such that $X(1) \overset{d}{\sim} X$ and $Z_\alpha(1) \overset{d}{\sim} Z_\alpha$. Here by a Lévy process we mean a process which has independent and stationary increments, is stochastically continuous, and starts from the origin. Notice that $X(t)$ is a nondecreasing process. Then a subordination $Y(t) := Z_\alpha(X(t))$ is also a Lévy process.

For any $B \in \mathcal{B}(\mathbf{R}^d)$, we have

$$P\{Y(t) \in B | X(t)\} = [P\{Z_\alpha(s) \in B\}]_{s=X(t)} = \left[P\{s^{1/\alpha} Z_\alpha(1) \in B\}\right]_{s=X(t)}$$

and hence

$$P\{Y(t) \in B\} = P\{X(t)^{1/\alpha} Z_\alpha(1) \in B\}.$$

Thus $\tilde{X} \overset{d}{\sim} Y(1)$, which is infinitely divisible.

The Lévy measure of a subordination is given by

$$\tilde{\nu}(B) = \int_{(0,\infty)} P\{Z_\alpha(t) \in B\}\nu(dt), \quad \forall B \in \mathcal{B}(\mathbf{R}^d \setminus \{0\}),$$

(see [Z58]). Therefore, for any $B \in \mathcal{B}(\mathbf{R}^d \setminus \{0\})$,

$$\begin{aligned}
\tilde{\nu}(B) &= \int_{(0,\infty)} P\{t^{1/\alpha} Z_\alpha(1) \in B\}\nu(dt) \\
&= \int_{(0,\infty)} \nu(dt) \int_{\mathbf{R}^d} I_{B/t^{1/\alpha}}(x)\mathcal{L}(Z_\alpha(1))(dx) \\
&= \int_{\mathbf{R}^d} \mathcal{L}(Z_\alpha(1))(dx) \int_{(0,\infty)} I_{B/x}(t^{1/\alpha})\nu(dt) \\
&= \int_{\mathbf{R}^d} (T_\alpha \nu)\left(\frac{B}{x}\right)\mathcal{L}(Z_\alpha(1))(dx) \\
&= E\left[(T_\alpha \nu)\left(\frac{B}{Z_\alpha}\right)\right].
\end{aligned}$$

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This proves our theorem. □

Example 4.1. Let $\{Z_\alpha(t), t \geq 0\}$ be $\mathbf{R}^d$-valued strictly α-stable, H-semi-selfsimilar. Let X be the same as in Proposition 4.1, independent of $\{Z_\alpha(t)\}$. Then

$$\tilde{X}(t) = X^{1/\alpha} Z_\alpha(t)$$

is an $\mathbf{R}^d$-valued, infinitely divisible, H-semi-selfsimilar process. If $\{Z_\alpha(t)\}$ has stationary increments, then so does $\{\tilde{X}(t)\}$.

Proof. For any $0 \leq t_1 < \cdots < t_n$, consider

$$(\tilde{X}(t_1), \cdots, \tilde{X}(t_n)) = X^{1/\alpha}(Z_\alpha(t_1), \cdots . Z_\alpha(t_n)).$$

Since $(Z_\alpha(t_1), \cdots . Z_\alpha(t_n))$ is $\mathbf{R}^{nd}$-valued symmetric α-stable, independent of X, it follows from Proposition 4.1 that $(\tilde{X}(t_1), \cdots, \tilde{X}(t_n))$ is $\mathbf{R}^{nd}$-valued infinitely divisible. The H-semi-selfsimilarity and the property of stationary increments (if any) of $\{\tilde{X}(t)\}$ follow from those of $\{Z_\alpha(t)\}$. □

Remark 4.1. In Proposition 4.1, it is known that if X is a nonnegative strictly β-stable random variable ($\beta < 1$), then $\tilde{X}$ is strictly $\alpha\beta$-stable. In [MSam97], it is proved that if X is nonnegative strictly β-semi-stable ($\beta < 1$), then $\tilde{X}$ is strictly $\alpha\beta$-semi-stable. Therefore, if we choose X as a nonnegative strictly semi-stable random variable in Example 4.1, then we can construct examples of strictly semi-stable semi-selfsimilar processes.

Lemma 4.1. (Theorem 3.10.1 and Exercise 3.15 of [SamT94].) Suppose $0 < \alpha < 2$. Let $\{\varepsilon_j\}$ be i.i.d. random variables such that ε_1 takes two values ± 1 with probability $1/2$, respectively, $\{W_j\}$ be i.i.d. $\mathbf{R}^d$-valued random variables with $E[|W_1|^\alpha] < \infty$, and let $\{\Gamma_j\}$ be a sequence of Poisson arrival times with unit rate, namely $\Gamma_j = \sum_{i=1}^{j} e_i$, where $\{e_i\}$ are i.i.d. exponentially distributed random variables with $E[e_1] = 1$. Suppose that $\{\varepsilon_j\}, \{W_j\}$ and $\{\Gamma_j\}$ are independent.

Then

$$(4.1) \qquad X := \sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} W_j$$

converges almost surely and X is $\mathbf{R}^d$-valued symmetric α-stable.

Example 4.2. Let $0 < \alpha < 2$ and let $\{W(t), t \geq 0\}$ be an $\mathbf{R}^d$-valued, H-semi-selfsimilar process with $E[|W(t)|^{\alpha}] < \infty$. Let $\{W_j(t)\}, j = 1, 2, ,\cdots$, be independent copies of $\{W(t)\}$. Then

$$X(t) := \sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} W_j(t)$$

is a symmetric α-stable, $\mathbf{R}^d$-valued, H-semi-selfsimilar process. If $\{W(t)\}$ has stationary increments, then so does $\{X(t)\}$.

Proof. Let $0 \leq t_1 < \cdots < t_n$. Denote the components of $\mathbf{R}^d$-valued random vectors $X(t_k), W_j(t_k)$ by $X_\ell(t_k), W_{j,\ell}(t_k), \ell = 1, \cdots, d$. For any $c_{k,\ell} \in \mathbf{R}$ $(k = 1, \cdots .n; \ell = 1, \cdots, d)$, we have

$$X := \sum_{k=1}^{n} \sum_{\ell=1}^{d} c_{k,\ell} X_\ell(t_k) = \sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} \sum_{k=1}^{n} \sum_{\ell=1}^{d} c_{k,\ell} W_{j,\ell}(t_k)$$

and

$$E\left[\left| \sum_{k=1}^{n} \sum_{\ell=1}^{d} c_{k,\ell} W_{j,\ell}(t_k) \right|^{\alpha} \right] < \infty.$$

Hence X is symmetric α-stable on $\mathbf{R}$ by Lemma 4.1, and thus $(X(t_1), \cdots, X(t_n))$ is symmetric α-stable on $\mathbf{R}^{nd}$ by Theorem 2.1.5 of [SamT94]. As to the H-semi-selfsimilarity of $\{X(t)\}$, since $\{W_j(at)\} \stackrel{d}{=} \{a^H W_j(t)\}$, we have

$$\left\{ \sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} W_j(at) \right\} \stackrel{d}{=} \left\{ \sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} a^H W_j(t) \right\},$$

concluding that $\{X(at)\} \stackrel{d}{=} \{a^H X(t)\}$. As to its increments, by the same reasoning as above, if $\{W(t)\}$ has stationary increments, then so does $\{X(t)\}$. $\square$

An example of $\mathbf{R}^d$-valued H-semi-selfsimilar process $\{W(t)\}$ with stationary increments and with the α-th moment, which we need in Example 4.2, is found in Theorem 9.3 of [MSat97]. It is constructed in the following way: Let $0 < H < 1$ and $0 < \beta \leq 2$ with $H \neq 1/\beta$. Define

$$W(t) = \int_{-\infty}^{\infty} (|t - u|^{H-1/\beta} - |u|^{H-1/\beta}) dS_\beta(u), \quad t \geq 0,$$

where $\{S_\beta(t), t \in \mathbf{R}\}$ is a nontrivial $\mathbf{R}^d$-valued symmetric β-semi-stable Lévy process. Then $\{W(t)\}$ is an H-semi-selfsimilar and β-semi-stable process in the sense of Definition 8.2 of [MSat97]. As is well known, β-semi-stable random vectors have the α-th moment if $\alpha < \beta$. Therefore, if $\beta > \alpha$, then $E[|W(t)|^\alpha] < \infty$.

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