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On the Regularity of a Chemical Reaction Interface

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1. Introduction

This paper is concerned with the regularity of the interface which arises in the so-called fast chemical reaction problem. In its simplest form, we may describe the problem as follows: Let Ω be a smooth, bounded domain in $\mathbf{R}^n$, and let α_1 and α_2 be positive constants. Given an initial condition $h : \Omega \to \mathbf{R}$, find a function $u : \Omega \times [0, \infty) \to \mathbf{R}$ satisfying

$$(1.1) \qquad \begin{cases} \alpha_1 u_t = \Delta u & \text{on } \{u > 0\}, \\ \alpha_2 u_t = \Delta u & \text{on } \{u < 0\}, \\ \frac{\partial u}{\partial \nu^+} + \frac{\partial u}{\partial \nu^-} = 0 & \text{on } \{u = 0\}, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = h(x) & \text{on } \Omega. \end{cases}$$

Here, $\nu^{+(-)}$ is the outward-pointing unit normal vector to the subdomain $\{u > 0\} \cap \Omega \times \{t\}$ ($\{u < 0\} \cap \Omega \times \{t\}$) for any t. The set $\{u = 0\}$ is presumably a hyper-surface, which represents an interface separating two diffusing chemical substances.

J. R. Cannon and C. D. Hill in [CH] established the existence, uniqueness and stability of the weak solution to (1.1) (see Section 2 for the definition).

Using the Hölder estimates of [LSU], Cannon and Hill verified that the weak solution u is Hölder continuous in the space and time variables. Hence, the interface $\{u = 0\}$ is a closed set in $\Omega \times (0, \infty)$, while it may or may not contain interior points. The solution u is smooth away from $\{u = 0\}$ since u satisfies the usual heat equation there.

In this paper, we show that, away from degenerate points, the set $\{u = 0\}$ is indeed an n-dimensional hyper-surface, except for a set of small measure. To be more precise, we recall the definition of the n-dimensional parabolic measure in $\Omega \times (0, \infty)$.

Definition 1.1. Let $S \subset \Omega \times (0, \infty)$ be a subset. Define the n-dimensional parabolic measure of S as

$$\mathcal{P}^n(S) \equiv \lim_{\delta \to 0} \inf \left\{ \sum_{j=1}^{\infty} r_j^n \mid S \subset \cup_{j=1}^{\infty} P_{r_j}(x_j, t_j),\ 2r_j < \delta \text{ for all } j \right\}.$$

Here, $P_r(x, t)$ is the parabolic cylinder

$$P_{r_j}(x_j, t_j) \equiv \{(x, t) \in \Omega \times (0, \infty) \mid |x - x_j| < r_j,\ \ |t - t_j| < r_j^2\}.$$

Note that the n-dimensional Hausdorff measure $\mathcal{H}^n$ has the property $\mathcal{H}^n(S) \le c(n)\mathcal{P}^n(S)$ for some constant $c(n)$.

Theorem 1.1. Let u be the weak solution to (1.1) (see Definition 2.1). Then, there exist an open set $\mathcal{O} \subset \Omega \times (0, \infty)$ and a closed set $\mathcal{W} \subset \Omega \times (0, \infty) \cap \{u = 0\}$ with the following properties:
(1) $\Omega \times (0, \infty) = \mathcal{O} \cup \mathcal{W}$ and $\mathcal{O} \cap \mathcal{W} = \emptyset$.
(2) On $\mathcal{O}$, u_t and $\nabla u = (\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n})$ are locally Hölder continuous in the space and time variables.
(3) For any open set $\tilde{\mathcal{O}} \subset\subset \mathcal{O}$, $\nabla^2 u \equiv \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{1 \le i,j \le n}$ are Hölder continuous up to the boundary on each domain $\tilde{\mathcal{O}} \cap \{u > 0\}$ and $\tilde{\mathcal{O}} \cap \{u < 0\}$.
(4) On $\mathcal{O} \cap \{u = 0\}$, $|\nabla u| \ne 0$.
(5) $\mathcal{O} \cap \{u = 0\}$ is an embedded n-dimensional hyper-surface, and locally $C^{2,\alpha}$ in space and $C^{1,\alpha/2}$ in time for some $0 < \alpha < 1$.
(6) $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ with $\mathcal{P}^n(\mathcal{W}_1) = 0$ and

$$\lim_{r \to 0} \frac{1}{r^{n+2}} \int_{P_r(x,t)} |\nabla u|^2 dx dt = 0$$

for all points $(x, t) \in \mathcal{W}_2$.

Property (2) shows that the functions u_t and ∇u are Hölder continuous across $\{u = 0\}$ on $\mathcal{O}$, while (3) shows that $\nabla^2 u$ is Hölder continuous up to

$\{u = 0\}$ on $\mathcal{O}$. In particular, we may define, for each point on the interface $(x,t) \in \mathcal{O} \cap \{u = 0\}$,

$$(\nabla^2 u)^{+(-)}(x,t) = \lim_{(y,s)\to(x,t),\ (y,s)\in\{u>0\ (<0)\}} (\nabla^2 u)(y,s).$$

These two functions $(\nabla^2 u)^+$ and $(\nabla^2 u)^-$ may not coincide on $\{u = 0\}$, resulting in a discontinuity of $\nabla^2 u$ across $\{u = 0\}$. Property (6) shows that the "bad set" $\mathcal{W}$ is decomposed into two sets, $\mathcal{W}_1$ being a set of n-dimensional measure 0 and $\mathcal{W}_2$ being a set of degenerate points ($|\nabla u| = 0$) in the measure-theoretic sense. We conjecture that $\mathcal{P}^n(\mathcal{W}_2) = 0$ when the initial condition is not trivially equal to 0, but we were not able to prove such a statement. Indeed, we were not able to exclude the possibility that $\mathcal{W}_2$ may have non-trivial interior points.

We also derive an interesting formula for the speed of motion of the interface. Let $(x,t) \in \mathcal{O} \cap \{u = 0\}$ be a point on the interface, and let ν be the unit normal n-vector to the $n-1$-dimensional surface $\mathcal{O} \cap \{u = 0\} \cap (\Omega \times \{t\})$ pointing toward the positive domain $\{u > 0\} \cap (\Omega \times \{t\})$.

Theorem 1.2. The speed of motion of the interface $\{u = 0\} \cap (\Omega \times \{t\})$ in the normal direction ν at (x,t) is given by

$$-\frac{u^+_{\nu\nu} - u^-_{\nu\nu}}{(\alpha_1 - \alpha_2)u_\nu}.$$

Here,

$$u^{+(-)}_{\nu\nu} = \lim_{(y,s)\to(x,t),\ (y,s)\in\{u>0\ (<0)\}} \frac{\partial^2 u}{\partial \nu^2}(y,s),$$

and $u_\nu = \frac{\partial u}{\partial \nu}(x,t)$.

These quantities are well-defined at each point $(x,t) \in \mathcal{O} \cap \{u = 0\}$ by Theorem 1.1 (3), and $u_\nu \neq 0$ by (4). The formula is interesting in that we would not see it a priori (even heuristically), unless we establish enough regularity of the solution u.

In Sections 2-4, we establish the Hölder continuity of the space gradient ∇u. The idea of the proof is to show that ∇u is Hölder continuous whenever u is close in a weak sense to some non-degenerate affine plane. Namely, we define

Definition 1.2. Given an H^1 function u and a vector $p \in \mathbf{R}^n$, let

$$E(r,(x,t),p) \equiv \frac{1}{|P_r|}\int_{P_r(x,t)} |\nabla u - p|^2 dxdt.$$

Here, $|P_r| = L^{n+1}(P_r(x,t))$. If $|p| \neq 0$ and $E(r,(x,t),p)$ is small enough, we show in Theorem 3.1 that ∇u is Hölder continuous by using the so-called blow-up argument.

In Sections 5-7, we establish the $C^{2,\alpha}$ regularity up to the interface as well as the Hölder continuity of u_t. The idea of the proof is rather simple: since the graph of u is proved to be a $C^{1,\alpha}$-manifold in space, and the tangent plane, with respect to the parabolic scaling, depends only on the space variables, we view the graph of u as a graph over a vertical plane in $\Omega \times (0,\infty) \times \mathbf{R}$. The function $v : (x_1, \cdots, x_{n-1}, y, t) \in \mathbf{R}^{n+1} \to \mathbf{R}$ which represents this graph satisfies the equation

$$(1.2) \qquad \begin{cases} \alpha_1 v_t = \Delta_x v - \frac{\partial}{\partial y}\left(\frac{1+|\nabla_x v|^2}{v_y}\right) & \text{on } \{y>0\}, \\ \alpha_2 v_t = \Delta_x v - \frac{\partial}{\partial y}\left(\frac{1+|\nabla_x v|^2}{v_y}\right) & \text{on } \{y<0\}. \end{cases}$$

We prove Theorems 1.1 and 1.2 by establishing the regularity for v instead of u.

We note that there are numerous works related to the problem in this paper, see [CH,CF,CD,E,CaY,ChY], for example. In [CH,CF,CD], Cannon *et al.* studied initial boundary-value problems of this type, with various boundary conditions in n space dimensions, and showed the existence, uniqueness and certain stability results. In [E], Evans showed the existence of the classical global solution of the one space dimensional problem. In [CaY], Cannon and Yin studied periodic problems with one space dimension, and showed the existence of the classical global solution. Also related is the so-called Stefan problem (see [M] for the references), in which the speed of the interface motion is given by the gap of the first derivatives across the interface.

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2. Weak formulation and some preliminary estimates

2.1. Weak solutions.

In [CH], the existence and uniqueness of a weak solution to problem (1.1) with the homogeneous Neumann boundary condition are established. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary, and fix some $T > 0$.

Definition 2.1. A weak solution to problem (1.1) with $h \in H^1(\Omega)$ is a bounded, measurable function u defined on $\Omega \times [0,T]$ such that

$$(2.1) \qquad \int_0^T \int_\Omega a(u)\frac{\partial \phi}{\partial t} + u\Delta\phi dxdt + \int_{\Omega\times\{0\}} a(h)\phi dx = 0$$

for all test functions ϕ satisfying

$$\phi \equiv 0$$

on $\Omega \times \{T\}$ and

$$\frac{\partial \phi}{\partial n} \equiv 0$$

on $\partial\Omega \times [0,T]$. Here, $a(u)$ is a Lipschitz function defined by

$$a(u) = \begin{cases} \alpha_1 u & \text{if} \quad u \geq 0 \\ \alpha_2 u & \text{if} \quad u < 0 \end{cases}$$

where α_1 and α_2 are positive constants. One may verify, with some modifications of our proof, that the same regularity results hold if we have

$$a(u) = \begin{cases} \alpha_1(u) & \text{if} \quad u \geq 0 \\ \alpha_2(u) & \text{if} \quad u < 0 \end{cases}$$

with $\alpha_1, \alpha_2 \in C^\infty(\mathbf{R})$, $\alpha_1(0) = \alpha_2(0) = 0$, and $0 < \gamma_1 \leq \alpha_1', \alpha_2' \leq \gamma_2 < \infty$ for some constants γ_1 and γ_2. For simplicity, we only deal with the former case in this paper.

As was shown in [CH], u belongs to $H^1(\Omega \times [0,T])$, i.e.,

$$(2.2) \qquad \int_0^T \int_\Omega (|u|^2 + |\nabla u|^2 + |u_t|^2) dxdt < \infty.$$

Here, ∇ is the gradient with respect to the space variables. Next, for L^1 a.e. t, $a'(u)u_t = \Delta u$ is satisfied in the strong sense on Ω. Also, we have

$$(2.3) \qquad \int_0^T \int_{\tilde\Omega} |\nabla^2 u|^2 dxdt < \infty$$

for all $\tilde\Omega \subset\subset \Omega$. To see these claims, first fix $\psi \in C_c^\infty(\Omega)$ (compactly supported C^∞ function defined on Ω) and $\zeta \in C_c^\infty((0,T))$. We use the function $\phi(x,t) = \psi(x)\zeta(t) \in C_c^\infty(\Omega\times(0,T))$ as a test function in (2.1). Note that the conditions $\phi = 0$ on $\Omega\times\{T\}$ and $\frac{\partial\phi}{\partial n} = 0$ on $\partial\Omega\times[0,T]$ are satisfied for this ϕ. Additionally, $\phi = 0$ on $\Omega \times \{0\}$. By integration by parts,

$$\int_0^T \zeta(t) \int_\Omega a'(u)u_t(x,t)\psi(x) + \nabla u(x,t)\cdot\nabla\psi(x) dxdt = 0.$$

Since ζ is arbitrary, this shows that

$$\int_\Omega a'(u)u_t(x,t)\psi(x) + \nabla u(x,t)\cdot\nabla\psi(x)dx = 0 \tag{2.4}$$

for a.e. $t \in [0,T]$ for this fixed $\psi(x)$. Next, since $H_0^1(\Omega)$ is separable, we may choose a countable set of functions $\{\psi_i(x)\}_{i=1}^\infty$ in $C_c^\infty(\Omega)$ which is dense in $H_0^1(\Omega)$. By the previous argument, (2.4) holds for a.e. $t \in [0,T]$ for each ψ_i, $i = 1, 2, \cdots$. Since the collection $\{\psi_i\}$ is dense, (2.4) holds for any $\psi \in H_0^1(\Omega)$ for a.e. $t \in [0,T]$. Since $u_t(\cdot,t)$ is in $L^2(\Omega)$ for a.e. $t \in [0,T]$ and a' is bounded, the standard elliptic regularity theory shows that $u(\cdot,t) \in H^2_{loc}(\Omega)$ for a.e. $t \in [0,T]$ and the equation is satisfied pointwise for a.e. $x \in \Omega$ and a.e. $t \in [0,T]$.

2.2. A time derivative estimate.

Throughout the rest of the paper, all constants depend only on n, α_1 and α_2 unless stated otherwise.

Proposition 2.1. Let u be a weak solution satisfying (2.1) and let $P_{2r} \equiv P_{2r}(x,t) \subset\subset \Omega \times [0,T]$ be a parabolic cylinder centered at (x,t). Then there exists a constant c_1 such that

$$r^2 \int_{P_r} |u_t|^2 dxdt \le c_1 \int_{P_{2r}} |\nabla u - p|^2 dxdt, \tag{2.5}$$

where $p \in \mathbf{R}^n$ is an arbitrary n-vector.

Proof. By [CH], we may approximate the problem (2.1) by

$$\begin{cases} \frac{\partial}{\partial t} a_m(u_m) = \Delta u_m & \text{on} \quad \Omega \times [0,T] \\ u_m = h_m & \text{on} \quad \Omega \times \{0\} \\ \frac{\partial u_m}{\partial n} = 0 & \text{on} \quad \partial\Omega \times [0,T], \end{cases}$$

where a_m is a smooth approximation of a. Let $\phi(y) = \phi(|y-x|)$ be a smooth function such that

$$\phi = \begin{cases} 1 & \text{on} \quad B_r(x) \\ 0 & \text{on} \quad B_{2r}^c(x), \end{cases}$$

and $|\nabla\phi| \le 2/r$. By Fubini's Theorem, we may choose $\tilde{t}_1 \in (-(2r)^2, -r^2)$ and $\tilde{t}_2 \in (r^2, (2r)^2)$ such that

$$\int_{B_{2r}(x)\times\{\tilde{t}_i\}} |\nabla u_m - p|^2\phi^2 dx \le \frac{2}{3r^2} \int_{P_{2r}} |\nabla u_m - p|^2 \phi^2 dxdt$$

for $i = 1, 2$. Multiply the equation by $(u_m)_t\phi^2$ and integrate over $B_{2r} \times (\tilde{t}_1, \tilde{t}_2)$. Then by integration by parts, it follows that there exists a constant c_1 such that

$$\int_{B_{2r}\times(\tilde{t}_1,\tilde{t}_2)} |(u_m)_t|^2\phi^2 dxdt \le \frac{c_1}{r^2}\int_{P_{2r}} |\nabla(u_m) - p|^2 dxdt$$

for any $p \in \mathbf{R}^n$. Hence, we have the inequality (2.5) for u_m. To verify that the limit u satisfies the same inequality, let $\phi \in C_c^\infty(\Omega \times [0, T])$ be a non-negative function such that $\phi = 1$ on P_{2r}. By subtracting the equations, we have

$$\frac{\partial}{\partial t}(a_m(u_m) - a(u)) = \Delta(u_m - u).$$

Multiply the equation by $\phi(u_m - u)$ and integrate. Then, we have

$$\int_0^T\int_\Omega \left(\frac{\partial}{\partial t}(a_m(u_m) - a(u))\right)\phi(u_m - u)dxdt = \int_0^T\int_\Omega \phi(u_m - u)\Delta(u_m - u)dxdt.$$

We integrate by parts, which yields

$$\int_0^T\int_\Omega \left(\frac{\partial}{\partial t}(a_m(u_m) - a(u))\right)\phi(u_m - u)dxdt$$

$$= -\int_0^T\int_\Omega |\nabla u_m - \nabla u|^2\phi dxdt - \int_0^T\int_\Omega (u_m - u)\nabla\phi\cdot\nabla(u_m - u)dxdt.$$

By using Hölder's inequality, we obtain

$$\int_0^T\int_\Omega |\nabla u_m - \nabla u|^2\phi dxdt$$

$$\le \left(\int_0^T\int_\Omega \left(\frac{\partial}{\partial t}(a_m(u_m) - a(u))\right)^2\phi dxdt\right)^{1/2}\left(\int_0^T\int_\Omega (u_m - u)^2\phi dxdt\right)^{1/2}$$

$$+\left(\int_0^T\int_\Omega |\nabla(u_m - u)|^2|\nabla\phi|^2 dxdt\right)^{1/2}\left(\int_0^T\int_\Omega |u_m - u|^2 dxdt\right)^{1/2}.$$

By the strong $L^2(\Omega \times [0, T])$ convergence and the uniform $H^1(\Omega \times [0, T])$ bound for $\{u_m\}$ and u, we see that

$$\int_{P_{2r}} |\nabla(u_m - u)|^2 dxdt \to 0$$

as $m \to \infty$. The sequence of functions $\{(u_m)_t\}$ may not converge strongly in $L^2(\Omega \times [0, T])$, but the lower semicontinuity under the weak convergence is enough to conclude (2.5) from the inequalities satisfied by the approximate solutions. □

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3. Decay estimate

In this section, we first show

Proposition 3.1. Let u be a weak solution of (2.1) on P_2 and suppose that $M > 0$ is given. Then there exist constants $\varepsilon = \varepsilon(M, n, \alpha_1, \alpha_2) > 0$ and $\kappa = \kappa(M, n, \alpha_1, \alpha_2) > 0$ such that the following property holds: Whenever

$$E(2) \equiv \frac{1}{|P_2|}\int_{P_2} |\nabla u - p|^2 dxdt < \varepsilon$$

holds for some n-vector p with $|p| \geq M^{-1}$, we have

$$E(\kappa) \equiv \frac{1}{|P_\kappa|}\int_{P_\kappa} |\nabla u - (\nabla u)_\kappa|^2 dxdt \leq \frac{1}{2}E(2).$$

Here,

$$(\nabla u)_\kappa = \frac{1}{|P_\kappa|}\int_{P_\kappa} \nabla u \;\; dxdt.$$

Also, there exists a constant $c_{19} = c_{19}(M, n, \alpha_1, \alpha_2)$ such that

$$|(\nabla u)_\kappa - p| \leq c_{19} E(2)^{1/2}.$$

Proposition 3.1 shows the following: If $E(2)$ is small enough and $|p|$ is away from 0, then E in a smaller scale with a different vector can be made smaller by a definite factor. This is the key step to show that ∇u is Hölder continuous. We note that the idea of decay estimates originates from the regularity theory of minimal surfaces and has been used successfully in various areas such as harmonic mappings and free boundary problems in recent years.

Proof of Proposition 3.1. For the purpose of eventually obtaining a contradiction, consider a sequence of solutions $\{u^i\}_{i=1}^\infty$ satisfying (2.1) on the parabolic cylinder $P_2 \equiv \{(x,t) \mid |x| < 2, \; |t| < 2^2\}$, which are getting closer to some non-degenerate affine functions. Namely, let $p_1, p_2, \cdots \in \mathbf{R}^n$ be vectors with

$$0 < M^{-1} \leq |p_i| \tag{3.1}$$

and let

$$E_i = \int_{P_2} |\nabla u^i - p_i|^2 dxdt.$$

We assume that $E_i \to 0$ as $i \to \infty$. Denote

$$\bar{u}^i = \frac{1}{|P_1|}\int_{P_1} u^i \; dxdt$$

for $i = 1, 2, \cdots$. By Proposition 2.1, we have

$$\int_{P_{7/4}} |u_t^i|^2 + |\nabla^2 u^i|^2 dxdt \le c_2 E_i. \tag{3.2}$$

With (3.2), one can show (see [EG 6.6.2] and [HT, section 4]) that there exists a Lipschitz function g^i defined on $P_{3/2}$ such that

$$\sup_{P_{3/2}} (|\nabla g^i - p_i| + |g_t^i|) \le E_i^{1/8}, \tag{3.3}$$

$$L^{n+1}(P_{3/2} \cap \{u^i \ne g^i\}) \le c_3 E_i^{3/4}, \tag{3.4}$$

$$\int_{P_{3/2}} |\nabla g^i - p_i|^2 + |g_t^i|^2 dxdt \le c_3 E_i, \tag{3.5}$$

where c_3 depends only on n. We then estimate the distance between g^i and $\bar{u}^i + p_i \cdot x$. Since $|g^i - \bar{u}^i - p_i \cdot x|$ is a continuous function on P_1, there exists a point $(x^i, t^i) \in P_1$ such that

$$\begin{aligned} |g^i(x^i, t^i) - \bar{u}^i - p_i \cdot x| &\le \frac{1}{|P_1|} \int_{P_1} |g^i - \bar{u}^i - p_i \cdot x| dxdt \\ &\le \frac{1}{|P_1|} \int_{P_1} (|g^i - u^i| + |u^i - \bar{u}^i - p_i \cdot x|) dxdt \\ &\le c_4 \int_{P_1} (|\nabla g^i - \nabla u^i| + |g_t^i - u_t^i| + |\nabla u^i - p_i| + |u_t^i|) \\ &\le c_5 E_i^{1/2}. \end{aligned}$$

We used Poincaré's inequality, and the last inequality holds by (3.2) and (3.5). With (3.3), we obtain

$$\sup_{(x,t) \in P_1} |g^i(x,t) - \bar{u}^i - p_i \cdot x| \le 4 E_i^{1/8} + c_5 E_i^{1/2}. \tag{3.6}$$

Now let

$$Q_+^i \equiv P_1 \cap \{\bar{u}^i + p_i \cdot x \ge 0\}.$$

Since the sequence of sets $\{Q_+^i\}_{i=1}^\infty$ is precompact in the Hausdorff metric, we can choose a subsequence of $\{Q_+^i\}_{i=1}^\infty$ (and we again call it $\{Q_+^i\}_{i=1}^\infty$) so that $Q_+^i \to Q_+$ in the Hausdorff metric for some closed set $Q_+ \subset P_1$. Note that Q_+

may be empty. Since it is a limit of the sequence of half spaces in the space variables, Q_+ is of the form

$$Q_+ = P_1 \cap \{b + q \cdot x \geq 0\}$$

for some $b \in \mathbf{R}$ and $q \in \mathbf{R}^n$. Note that we are not excluding the possibility that $\bar{u}^i$ or $|p_i|$ may go off to infinity. We also define $Q_- = P_1 \setminus Q_+$. By using the non-degeneracy condition (3.1), and the estimates (3.4) and (3.6), one sees that

$$L^{n+1}(Q_+ \cap \{u^i \leq 0\}) \to 0, \tag{3.7}$$

$$L^{n+1}(Q_- \cap \{u^i \geq 0\}) \to 0 \tag{3.8}$$

as $i \to \infty$. Note here that the non-degeneracy condition (3.1) is essential. Since $|p_i| \geq M^{-1}$ and g^i deviates from $\bar{u}^i + p_i \cdot x$ only slightly due to (3.6), we conclude that g^i is positive on most of Q^i_+. This allows us to conclude that u^i is mostly positive on Q^i_+ since g^i and u^i coincide on most of Q^i_+ by (3.4). (See Remark 3.3 and 3.4 for more discussions.)

We define a sequence of functions $\{w^i\}$ by

$$w^i \equiv (u^i - \bar{u}^i - p_i \cdot x)E_i^{-1/2}.$$

By (3.2) and the Poincaré inequality, we have

$$\int_{P_1} |w^i|^2 + |w^i_t|^2 + |\nabla^2 w^i|^2 \leq c_6, \tag{3.9}$$

$$\int_{P_2} |\nabla w^i|^2 = 1. \tag{3.10}$$

Because of these estimates, there exists a subsequence of $\{w^i\}_{i=1}^\infty$ (again called $\{w^i\}_{i=1}^\infty$) and $w^\infty \in H^1(P_1)$ such that the w^i converges to w^∞ strongly in $L^2(P_1)$ and weakly in $H^1(P_1)$, and w^∞ satisfies

$$\int_{P_1} (|w^\infty|^2 + |w^\infty_t|^2 + |\nabla w^\infty|^2) \leq c_7. \tag{3.11}$$

We claim that w^∞ satisfies the equation

$$\int_{P_1} w^\infty_t(\alpha_1 \chi_{Q_+} + \alpha_2 \chi_{Q_-})\phi + \nabla w^\infty \nabla \phi = 0 \tag{3.12}$$

for all test functions $\phi \in C_0^\infty(P_1)$. Here, χ_{Q_+} and χ_{Q_-} are the characteristic functions for the sets Q_+ and Q_-, respectively.

Proof of claim. By dividing (2.4) by $E_i^{1/2}$, note that w^i satisfies

$$\int_{P_1} a'(u^i)w^i_t\phi + \nabla w^i \nabla \phi \, dxdt = 0.$$

The second term converges to $\int_{P_1} \nabla w^\infty \nabla \phi$ by the weak convergence in H^1. Next, by a telescopic argument,

$$\left| \int_{P_1} a'(u^i)w^i_t\phi - \int_{P_1} w^\infty_t(\alpha_1\chi_{Q+} + \alpha_2\chi_{Q-})\phi \right|$$

$$\le \left| \int_{P_1} (\alpha_1\chi_{Q+} + \alpha_2\chi_{Q-})(w^i_t - w^\infty_t)\phi \right|$$

$$+ \left| \int_{P_1} \{a'(u^i)w^i_t - (\alpha_1\chi_{Q+} + \alpha_2\chi_{Q-})w^i_t\}\phi \right|$$

$$\le \left| \int_{P_1} (\alpha_1\chi_{Q+} + \alpha_2\chi_{Q-})(w^i_t - w^\infty_t)\phi \right|$$

$$+ \left(\int_{P_1} \phi |a'(u^i) - (\alpha_1\chi_{Q+} + \alpha_2\chi_{Q-})|^2 \right)^{1/2} \left(\int_{P_1} |w^i_t|^2 \phi \right)^{1/2}.$$

The first term goes to 0 by the weak convergence in H^1, and the second term also goes to 0 by (3.7), (3.8) and (3.9). Thus, (3.12) holds. □

Remark 3.1. By subtracting equations, we obtain

$$\int_{P_1} \{w^\infty_t(\alpha_1\chi_{Q+} + \alpha_2\chi_{Q-}) - a'(u^i)w^i_t\}\phi + \nabla(w^\infty - w^i)\nabla\phi = 0$$

for $\phi \in C_0^\infty(P_1)$. Let $\zeta \in C_c^\infty(P_1)$ be a function such that $|\zeta| \le 1$ and $\zeta = 1$ on $P_{7/8}$. By using $\phi = (w^\infty - w^i)\zeta$ as the test function in the above expression and using the strong $L^2(P_1)$ convergence, we conclude that

$$\int_{P_{7/8}} |\nabla(w^\infty - w^i)|^2 \to 0 \tag{3.13}$$

as $i \to \infty$. □

We next analyze the solution w^∞ to the equation (3.12). We show that w^∞ is $C^{1,\alpha}(P_1)$ function for any exponent $\alpha < 1$. We denote w^∞ by w in the following, for notational simplicity.

Lemma 3.1. Suppose that $w \in H^1(P_1)$ satisfies (3.12) for all test functions $\phi \in C_0^\infty(P_1)$, with the estimate (3.11). Then, for any $\alpha < 1$, there exists a constant c_8 such that

$$||w||_{C^{1,\alpha}(P_{1/2})} \le c_8. \tag{3.14}$$

Proof. Let $\eta = \eta(|x|)$ be a smooth function such that

$$\eta(x) = \begin{cases} 1 & \text{on } B_{7/8} \\ 0 & \text{on } B_1^c \end{cases}$$

and $|\nabla \eta| \le 10$. Also let $\zeta = \zeta(t)$ be a smooth function such that

$$\zeta(t) = \begin{cases} 1 & \text{for } |t| \le \left(\frac{7}{8}\right)^2 \\ 0 & \text{for } |t| > 1 \end{cases}$$

and $|\zeta'| \le 100$. In (3.12), we use $\phi = \eta^2\zeta^2 w$. Then,

$$\int_{P_1} w_t(\alpha_1\chi_{Q_+} + \alpha_2\chi_{Q_-})\eta^2\zeta^2 w + |\nabla w|^2\eta^2\zeta^2 + 2\eta\zeta^2 w\nabla w\nabla\eta = 0.$$

By integration by parts and Hölder inequality,

$$\int_{P_1} |\nabla w|^2\zeta^2\eta^2 \le \frac{1}{2}\int_{P_1} |\nabla w|^2\eta^2\zeta^2 + c_9\int_{P_1} |\nabla\eta|^2\zeta^2 w^2$$

$$+\int_{P_1} w^2(\alpha_1\chi_{Q_+} + \alpha_2\chi_{Q_-})2\zeta|\zeta'|\eta^2.$$

This shows, with a suitable choice of c_{10}, that

$$\int_{P_{7/8}} |\nabla w|^2 \le c_{35}\int_{P_1} w^2. \tag{3.15}$$

Next, let η and ζ be as before. Let w^ε be the usual mollification of w as a function in $\mathbf{R}^{n+1}$. Namely, define

$$w_\varepsilon(x,t) \equiv \int_{\Omega\times[0,T]} w(y,s)\rho_\varepsilon(x-y,t-s)dyds.$$

Here, $\rho \in C_0^\infty(B_1^{n+1}(0,0))$, $\int \rho dxdt = 1$, and

$$\rho(x,t) = \varepsilon^{-n-1}\rho(x\varepsilon^{-1}, t\varepsilon^{-1}).$$

We use $(w^\varepsilon)_t\eta^2\zeta^2$ as a test function in (3.12). Then we compute that

$$\int_{P_1} w_t(\alpha_1\chi_{Q_+} + \alpha_2\chi_{Q_-})(w^\varepsilon)_t\eta^2\zeta^2$$

$$+\eta^2\zeta^2\nabla w\nabla(w^\varepsilon)_t + 2\eta(w^\varepsilon)_t\zeta^2\nabla w\nabla\eta = 0.$$

We use the properties of the mollifier that $(w^\varepsilon)_t = (w_t)^\varepsilon$ and $\nabla(w^\varepsilon) = (\nabla w)^\varepsilon$. Note that, as $\varepsilon \to 0$,

$$\left|\int \eta^2\zeta^2\left(\nabla w^\varepsilon \nabla (w^\varepsilon)_t - \nabla w \nabla (w^\varepsilon)_t\right)\right| \le \left|\int (\nabla w^\varepsilon - \nabla w)\nabla (w^\varepsilon)_t \eta^2\zeta^2\right|$$

$$= \left| - \int (\Delta w^\varepsilon - \Delta w)(w^\varepsilon)_t \eta^2\zeta^2 + \int 2\eta\zeta^2 (w^\varepsilon)_t (\nabla w^\varepsilon - \nabla w)\nabla\eta\right| \to 0$$

by using $\Delta w,\ \nabla w \in L^2_{loc}(P_1)$. Since

$$\int \eta^2\zeta^2 \nabla w^\varepsilon \nabla (w^\varepsilon)_t = \frac{1}{2}\int \eta^2\zeta^2 \frac{d}{dt}|\nabla w^\varepsilon|^2 = -\frac{1}{2}\int |\nabla w^\varepsilon|^2 2\zeta\zeta'\eta^2,$$

with a suitable choice of c_{11}, it follows that

$$\int_{P_{7/8}} |w_t|^2 \le c_{11}\int_{P_1} |\nabla w|^2. \tag{3.16}$$

Using (3.15), (3.16) (with slight modifications of η and ζ) and equation (3.12), it follows that

$$\int_{P_{7/8}} |\nabla^2 w|^2 \le c_{12}\int_{P_1} |w|^2. \tag{3.17}$$

Note also that all of the above estimates are valid if we replace w by the difference quotients approximating $\frac{\partial^j w}{\partial t^j}$ for any $j \ge 1$, since (3.12) is satisfied by such quotients. Since the estimates do not depend on the approximation by the quotients, they are also valid for $\frac{\partial^j w}{\partial t^j}$ as well. After a bootstrap argument, we see that there exist constants $\{c_{13}(j)\}_{j=1}^\infty$ such that

$$||D_t^j w||_{H^2(P_{3/4})} \le c_{13}(j) \tag{3.18}$$

for $j = 0, 1, 2, \cdots$. By the Sobolev inequality applied to $P_{3/4} \subset \mathbf{R}^{n+1}$,

$$||D_t^j w||_{L^{\frac{2(n+1)}{(n+1)-4}}(P_{5/8})} \le c_{14}(j) \tag{3.19}$$

for $n > 3$, while

$$||D_t^j w||_{L^q(P_{5/8})} \le c_{14}(j, q) \tag{3.20}$$

with any $q < \infty$ for $n \le 3$. By the L^p estimate ([GT]),

$$||D_t^j w||_{W^{2,s}(P_{1/2})} \le c_{15}(j)\left(||D_t^j w||_{L^s(P_{5/8})} + ||\Delta D_t^j w + D_t^{j+2} w||_{L^s(P_{5/8})}\right)$$

for $j = 0, 1, 2, \cdots$ and $s = \frac{2(n+1)}{(n+1)-4}$. Note that $\Delta D_t^j w = D_t^j \Delta w = (\alpha_1 \chi_{Q_+} + \alpha_2 \chi_{Q_-}) D_t^{j+1} w$, hence, the right-hand side is bounded by (3.19) (or (3.20) for $n \le 3$ with s replaced by some large $q < \infty$). We may repeat such estimates to obtain

$$||w||_{W^{2,q}(P_{1/2})} \le c_{16}(q)$$

for any $q < \infty$. For any given $\alpha < 1$, we may choose $q = q(n, \alpha)$ such that

$$||w||_{C^{1,\alpha}(P_{1/2})} \le c_{17} ||w||_{W^{2,q}(P_{5/8})},$$

by Sobolev's inequality. Hence, we obtain (3.14). □

Remark 3.2. The estimate (3.14) shows, by choosing $\alpha = 1/2$, that there exists a constant c_{18} such that

$$\frac{1}{|P_\kappa|} \int_{P_\kappa} |\nabla w - (\nabla w)_\kappa|^2 \le c_{18} \kappa \tag{3.21}$$

for all $0 < \kappa < 1/2$. Here,

$$(\nabla w)_\kappa \equiv \frac{1}{|P_\kappa|} \int_{P_\kappa} \nabla w \, dxdt. \quad \square$$

We are now ready to prove the decay estimate. We choose $\kappa > 0$ as $\kappa = \min\{\frac{1}{4} c_{18}^{-1}, 1/2\}$. Assume the contrary to the statement of the proposition. Then, there exists a sequence of solutions $\{u^i\}_{i=1}^\infty \subset H^1(P_2)$ and $\{p_i\}_{i=1}^\infty \subset \mathbf{R}^n$ with $|p_i| \ge M^{-1}$,

$$E_i(2) \equiv \frac{1}{|P_2|} \int_{P_2} |\nabla u^i - p_i|^2 dxdt \to 0,$$

$$\frac{1}{2} E_i(2) < E_i(\kappa) \equiv \frac{1}{|P_\kappa|} \int_{P_\kappa} |\nabla u^i - (\nabla u^i)_\kappa|^2 dxdt.$$

After choosing a subsequence, let $w^1, w^2, \cdots, w^\infty \in H^1(P_2)$ be the corresponding sequence of functions and its limit, as was discussed previously. By the strong convergence (3.13), note that

$$\frac{1}{2} E_i(2) < \frac{1}{|P_\kappa|} \int_{P_\kappa} |\nabla u^i - (\nabla u^i)_\kappa|^2 dxdt$$

$$\le \frac{1}{|P_\kappa|} \int_{P_\kappa} |\nabla u^i - p_i - (\nabla w^\infty)_\kappa E_i(2)^{1/2}|^2 dxdt$$

$$= E_i(2) \cdot \frac{1}{|P_\kappa|} \int_{P_\kappa} |\nabla w^i - (\nabla w^\infty)_\kappa|^2$$

$$= E_i(2)\left(\frac{1}{|P_\kappa|}\int_{P_\kappa}|\nabla w^\infty - (\nabla w^\infty)_\kappa|^2 + o(1)\right) \le E_i(2)\left(\frac{1}{4} + o(1)\right),$$

which is a contradiction. Thus the first part of the theorem is proved. The second part follows from

$$|(\nabla u)_\kappa - p| \le \frac{1}{|P_\kappa|}\int_{P_\kappa}|\nabla u - p|dxdt$$

$$\le c_{19}\left(\frac{1}{|P_2|}\int_{P_2}|\nabla u - p|^2 dxdt\right)^{1/2} \le c_{19}E(2)^{1/2}. \quad \square$$

Remark 3.3. We note that we cannot use the Hölder continuity estimates of [LSU] in showing the conclusions (3.7) and (3.8). The reason is the following: To obtain the Hölder continuity estimates, one would need to convert the problem so that the equation in question is in divergence form. Namely, one defines $v = a(u)$, where a is the function defined in section 2.1, and notes that v satisfies

$$v_t = div((a^{-1})'(v)\nabla v)$$

in an appropriate weak sense. Here, a^{-1} is the inverse function of a. Since $(a^{-1})'(v)$ is a bounded positive measurable function, one may apply the result from [LSU] to obtain the Hölder continuity estimates. However, we may not conclude, for example, that v is close to any affine function or piece-wise affine function, since we would lose the control of the H^1 norm of v after such a conversion. We have no control over the upper bound of $|p_i|$ in the proof, so that such an attempt would destroy the control of the H^1 norm inevitably. The Lipschitz approximation thus fills the gap nicely to conclude (3.7) and (3.8). $\square$

Remark 3.4. The statement of Proposition 3.1 can be improved slightly, even though it is unnecessary to do so to prove Theorem 3.1. Namely, given $0 < \delta < \frac{1}{2}$, we can replace the non-degenerate condition

$$|p| \ge M^{-1} \quad \text{by} \quad |p| \ge E(2)^{\frac{1}{2}-\delta}$$

in the statement of Proposition 3.1, with ε, κ and c_{19} depending only on δ, n, α_1 and α_2. The proof is exactly the same, except that we replace the exponent $1/8$ in (3.3) by $\frac{1}{2} - \delta$ and the exponent $3/4$ in (3.4) by 2δ (see [HT]).

We also note that, if we could replace the non-degenerate condition

$$|p| \ge E(2)^{\frac{1}{2}-\delta} \quad \text{by} \quad |p| \ge E(2)^{\frac{1}{2}+\delta}$$

for some positive $\delta > 0$, then we would be able to obtain a Hölder continuous estimate of ∇u at degenerate points as well. At present, we do not see how to prove that such estimates exist. $\Box$

By using Proposition 3.1 and by iteration argument, we show

Theorem 3.1. Let $u : \Omega \times [0, \infty) \to \mathbf{R}$ be the weak solution of the problem (1.1) and suppose $P_{2r}(x_0, t_0) \subset\subset \Omega \times (0, \infty)$. For any $M > 0$, there exist $\varepsilon_0 > 0$, $1 > \alpha > 0$ and c_{20} depending only on M, n, α_1 and α_2, with the following property: Whenever

$$\frac{1}{|P_{2r}|} \int_{P_{2r}(x_0,t_0)} |\nabla u - p|^2 dxdt < \varepsilon_0$$

holds for some $p \in \mathbf{R}^n$ with $|p| \geq M^{-1}$, we have

$$\sup_{(x,t) \in P_r(x_0,t_0)} |\nabla u(x,t) - p| \leq (2M)^{-1},$$

$$\sup_{(x,t),(y,s) \in P_r(x_0,t_0)} r^{\alpha} \frac{|\nabla u(x,t) - \nabla u(y,s)|}{|x-y|^{\alpha} + |t-s|^{\alpha/2}} \leq c_{20},$$

$$\sup_{(x,s),(x,t) \in P_r(x_0,t_0)} r^{\alpha} \frac{|u(x,s) - u(x,t)|}{|s-t|^{(1+\alpha)/2}} \leq c_{20}.$$

Proof. It is enough to prove the case $r = 1$ and $(x_0, t_0) = (0, 0)$, since all the relevant quantities are invariant under the scaling $\frac{1}{r} u(rx + x_0, r^2 t + t_0)$. Let

$$\varepsilon_0 \equiv (2^{n+2})^{-1} \min\{\varepsilon, (\sqrt{2} - 1)^2 / (2\sqrt{2} M c_{19})^2\},$$

where ε and c_{19} are the constants in Proposition 3.1 corresponding to $2M$ (instead of M). Now, assume that

$$E(2) \equiv \frac{1}{|P_2|} \int_{P_2} |\nabla u - p|^2 dxdt < \varepsilon_0$$

with $|p| \geq M^{-1}$. Since $P_1(x, t) \subset P_2$ for any point $(x, t) \in P_1$,

$$E(1, (x,t), p) = \frac{1}{|P_1|} \int_{P_1(x,t)} |\nabla u - p|^2 dxdt \tag{3.22}$$

$$\leq \frac{2^{n+2}}{|P_2|} \int_{P_2} |\nabla u - p|^2 dxdt < 2^{n+2} \varepsilon_0 \leq \varepsilon.$$

By Proposition 3.1, we may conclude that

$$E(\kappa/2, (x,t), (\nabla u)_{\kappa/2,(x,t)}) \leq \frac{1}{2} E(1, (x,t), p)$$

with the inequality

$$|(\nabla u)_{\kappa/2,(x,t)} - p| \le c_{19} E(1,(x,t),p)^{1/2} \le (2M)^{-1}.$$

Here, we denote

$$(\nabla u)_{\kappa/2,(x,t)} \equiv \frac{1}{|P_{\kappa/2}|} \int_{P_{\kappa/2}(x,t)} \nabla u \; dxdt.$$

The last inequality is by (3.22) and the choice of ε_0. Hence,

$$|(\nabla u)_{\kappa/2,(x,t)}| \ge |p| - |(\nabla u)_{\kappa/2,(x,t)} - p| \ge (2M)^{-1}.$$

Therefore, we can apply Proposition 3.1 again with P_2 replaced by $P_{\kappa/2}(x,t)$ and p replaced by $(\nabla u)_{\kappa/2,(x,t)}$ (and with an appropriate scaling). Inductively, assume, for $i = 2, \cdots, l$, that we have

$$E(\kappa^i/2,(x,t),(\nabla u)_{\kappa^i/2,(x,t)}) \le \left(\frac{1}{2}\right)^i E(1,(x,t),p)$$

and

$$|(\nabla u)_{\kappa^i/2,(x,t)} - (\nabla u)_{\kappa^{i-1}/2,(x,t)}| \le c_{19} E(\kappa^{i-1}/2,(x,t),(\nabla u)_{\kappa^{i-1}/2,(x,t)})^{1/2}.$$

To proceed with the induction, we only need to prove that

$$|(\nabla u)_{\kappa^l/2,(x,t)}| \ge (2M)^{-1}.$$

We may compute

$$|(\nabla u)_{\kappa^l/2,(x,t)}| \ge |p| - |p - (\nabla u)_{\kappa/2,(x,t)}| - \sum_{j=2}^{l} |(\nabla u)_{\kappa^j/2,(x,t)} - (\nabla u)_{\kappa^{j-1}/2,(x,t)}|$$

$$\ge M^{-1} - c_{19}\left(E(1,(x,t),p)^{1/2} + \sum_{j=2}^{l} E(\kappa^{j-1}/2,(x,t),(\nabla u)_{\kappa^{j-1}/2,(x,t)})^{1/2}\right)$$

$$\ge M^{-1} - c_{19} E(1,(x,t),p)^{1/2} \sum_{j=1}^{l} \left(\frac{1}{2}\right)^{(j-1)/2}$$

(by the inductive assumptions)

$$\ge M^{-1} - c_{19} \frac{\sqrt{2}}{\sqrt{2}-1} E(1,(x,t),p)^{1/2} \ge (2M)^{-1}$$

by the choice of ε_0. Hence, we may indefinitely continue the iterations. This also shows that

$$\lim_{i\to\infty} |(\nabla u)_{\kappa^i/2,(x,t)} - (\nabla u)_{\kappa^j/2,(x,t)}| \le c_{21} E(\kappa^j/2,(x,t),(\nabla u)_{\kappa^j/2,(x,t)})^{1/2}$$

for all integers $j \ge 1$ for some constant c_{21}. By the Lebesgue differentiation Theorem applied with parabolic cylinders and a simple interpolation argument, this leads to the estimate

$$|\nabla u(x,t) - (\nabla u)_{s,(x,t)}| \le c_{22} s^{\alpha}$$

for $(x,t) \in P_1$, L^{n+1} a.e. for all $0 < s < 1/2$. Here, c_{22} and α may be computed explicitly from c_{21} and κ. The theorem follows from a simple modifications of the argument in [EG 6.6.2], for example. □

4. Covering argument

In this section, we show

Theorem 4.1. There exist an open set $\mathcal{O} \subset \Omega \times (0,\infty)$ and a closed set $\mathcal{W} \subset \Omega \times (0,\infty) \cap \{u = 0\}$ such that
(1) $\Omega \times (0,\infty) = \mathcal{O} \cup \mathcal{W}$ and $\mathcal{O} \cap \mathcal{W} = \emptyset$,
(2) on $\mathcal{O}$, $\nabla u = (\frac{\partial u}{\partial x_1},\cdots,\frac{\partial u}{\partial x_n})$ is locally Hölder continuous in the space and time variables, and $|\nabla u| \ne 0$ on $\mathcal{O} \cap \{u = 0\}$,
(3) $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ with $\mathcal{P}^n(\mathcal{W}_1) = 0$ and

$$\lim_{r\to 0} \frac{1}{r^{n+2}} \int_{P_r(x,t)} |\nabla u|^2 dxdt = 0$$

for all points $(x,t) \in \mathcal{W}_2$.

Proof. We use Theorem 3.1 and the Vitali covering lemma with parabolic cylinders. Let

$$N \equiv \left\{ (x,t) \in \Omega \times (0,T) \;\middle|\; \frac{1}{r^n} \int_{P_r(x,t)} (|u_t|^2 + |\nabla^2 u|^2) dxdt \to 0 \text{ as } r \to 0 \right\}$$
$$\cap \{u = 0\}.$$

The estimates (2.2), (2.3) and the standard Vitali covering lemma (see [EG], for example) shows that

$$\mathcal{P}^n(\Omega \times (0,T) \cap \{u = 0\} \setminus N) = 0.$$

We set this measure 0 set to be $\mathcal{W}_1$. We show that every point in N is a Lebesgue point for ∇u with respect to the shrinking parabolic cylinders.

Let (x,t) be a point in N, and assume that r is small enough so that $P_r \equiv P_r(x,t) \subset\subset \Omega \times (0,T)$. Choose $\tilde{r} \in [r, 2r]$ so that

$$\int_{\partial B_{\tilde{r}}(x)\times[t-(2r)^2,t+(2r)^2]} |u_t|^2 dS_x dt \leq \frac{2}{r}\int_{P_{2r}} |u_t|^2 dx dt. \tag{4.1}$$

For $i = 1, \cdots, n$ and $t-(2r)^2 < s_1 < s_2 < t+(2r)^2$,

$$\begin{aligned}
&\left|\int_{B_{\tilde{r}}(x)\times\{s_1\}} u_{x_i} dx - \int_{B_{\tilde{r}}(x)\times\{s_2\}} u_{x_i} dx\right| \\
&= \left|\int_{\partial B_{\tilde{r}}(x)\times\{s_1\}} u\nu_i dS_x - \int_{\partial B_{\tilde{r}}(x)\times\{s_2\}} u\nu_i dS_x\right| \\
&\leq \int_{\partial B_{\tilde{r}}(x)\times[s_1,s_2]} |u_t| dS_x dt \leq c_{21}(r^{n+1})^{1/2}\left(\int_{\partial B_{\tilde{r}}(x)\times[s_1,s_2]} |u_t|^2 dS_x dt\right)^{1/2} \\
&\leq 2c_{21}(r^n)^{1/2}\left(\int_{P_{2r}} |u_t|^2 dx dt\right)^{1/2}
\end{aligned}$$

by (4.1). This shows

$$\left|\frac{1}{|P_{\tilde{r}}|}\int_{P_{\tilde{r}}} \nabla u \, dx dt - \frac{1}{|B_{\tilde{r}}|}\int_{B_{\tilde{r}}(x)\times\{s\}} \nabla u dx\right| \leq c_{23}\left(\frac{1}{r^n}\int_{P_{2r}} |u_t|^2 dx dt\right)^{1/2}$$

for almost all $t-(2r)^2 < s < t+(2r)^2$ in L^1 measure. Using this, we may estimate

$$\begin{aligned}
\frac{1}{|P_{\tilde{r}}|}\int_{P_{\tilde{r}}} |\nabla u - (\nabla u)_{\tilde{r}}|^2 dx ds &\leq \frac{2}{|P_{\tilde{r}}|}\int_{P_{\tilde{r}}} |\nabla u - (\nabla u)_{\tilde{r},s}|^2 dx ds \\
&\quad + c_{24}\left(\frac{1}{r^n}\int_{P_{2r}} |u_t|^2 dx ds\right),
\end{aligned}$$

where

$$(\nabla u)_{\tilde{r},s} \equiv \frac{1}{|B_{\tilde{r}}|}\int_{B_{\tilde{r}}(x)\times\{s\}} \nabla u \, dx$$

and

$$(\nabla u)_{\tilde{r}} \equiv \frac{1}{|P_{\tilde{r}}|}\int_{P_{\tilde{r}}} \nabla u \, dx dt.$$

The first term can be bounded by the Poincaré inequality applied to each time slice, so that

$$\frac{1}{|P_{\tilde{r}}|}\int_{P_{\tilde{r}}} |\nabla u - (\nabla u)_{\tilde{r}}|^2 dx dt \leq \frac{c_{25}}{r^n}\int_{P_{2r}} (|u_t|^2 + |\nabla^2 u|^2) dx dt.$$

Since $(x,t) \in N$, the right-hand side goes to 0 as $r \to 0$. Thus, unless $|(\nabla u)_{\tilde r}| \to 0$ as $\tilde r \to 0$, Theorem 3.1 shows that ∇u is Hölder continuous in some neighborhood of (x,t) and $|\nabla u| \neq 0$. If $|(\nabla u)_{\tilde r}| \to 0$, then one may verify that $\frac{1}{|P_r|}\int_{P_r} |\nabla u|^2 dxdt \to 0$ as $r \to 0$. This completes the proof. □

5. Derivation of equation (1.2)

Even though it is elementary to derive equation (1.2), we give the detail for the reader's convenience. Let $(x_0, s_0) \in \Omega \times (0,T)$ be an arbitrary point in $\mathcal{O} \cap \{u=0\}$ (as in Theorem 4.1). Suppose $P_r(x_0,s_0) \subset\subset \mathcal{O}$, so that u has the regularity stated in Theorem 3.1 on $P_r(x_0,s_0)$. Define

$$u^s(x,t) = \frac{1}{s} u(sx + x_0, s^2 t + s_0)$$

for $0 < s < r$. Then, u^s satisfies the equation $a'(u^s)(u^s)_t = \Delta u^s$ on $P_1 = P_1(0,0)$, and by the estimates in Theorem 3.1, we have

$$\sup_{(x,t)\in P_1} |u^s(x,t) - (\nabla u)(x_0,s_0)\cdot x| \le c_{26} s^\alpha$$

and

$$\sup_{(x,t)\in P_1} |\nabla u^s(x,t) - \nabla u(x_0,s_0)| \le c_{26} s^\alpha$$

for some $0 < \alpha < 1$ and c_{26}. Next, choose a coordinate system on $\mathbf{R}^n$ so that

$$\nabla u(x_0,s_0) = \nabla u^s(0,0) = (0,\cdots,0,|\nabla u^s(0,0)|)$$

after a suitable rotation and reflection in $\mathbf{R}^n$. Since $\nabla u^s(0,0) = \nabla u(x_0,s_0)$ is a non-zero vector, we may choose a small s so that u^s has an everywhere non-zero space gradient on P_1. Consider a map $\Phi : (x_1,\cdots,x_n,t) \in P_1 \to (x_1,\cdots,x_{n-1},y,t) \in \mathbf{R}^{n+1}$ defined by

$$\Phi(x_1,\cdots,x_n,t) = (x_1,\cdots,x_{n-1},u^s(x_1,\cdots,x_n,t),t).$$

For all sufficiently small s, Φ is an injective map on P_1. Fix such an s. Choose a small $\rho > 0$ so that

$$(5.1)\qquad P_\rho(0,0)$$
$$= \left\{(x_1,\cdots,x_{n-1},y,t) \in \mathbf{R}^n \times \mathbf{R} \mid \sum_{i=1}^{n-1}(x_i)^2 + y^2 < \rho^2, |t| < \rho^2\right\} \subset\subset \Phi(P_1),$$

and define a function $v : P_\rho(0,0) \to \mathbf{R}$ so that

$$(5.2)\qquad u^s(x_1,\cdots,x_{n-1},v(x_1,\cdots,x_{n-1},y,t),t) = y$$

is satisfied for all $(x_1, \cdots, x_{n-1}, y, t) \in P_\rho(0,0)$. Intuitively, this v corresponds to a function which is obtained by viewing the graph of u from the "vertical direction". Again, by a suitable scaling, we may assume that $\rho = 1$. We also write P_1 for $P_1(0,0)$ in (5.1) with $\rho = 1$.

In the following, we derive an equation which v satisfies on P_1. Since v is a smooth function on $\{y > 0\} \cap P_1$ and $\{y < 0\} \cap P_1$, the following computations are all valid away from $\{y = 0\}$. Differentiate (5.2) with respect to y, x_i, $i = 1, \cdots, n-1$ and t. Denoting u^s by u for simplicity, we have

$$1 = u_{x_n} v_y, \qquad 0 = u_{x_i} + u_{x_n} v_{x_i}, \quad i = 1, \cdots, n-1$$

$$\text{and} \quad 0 = u_t + u_{x_n} v_t.$$

By differentiating the first two identities with respect to the space variables, we obtain

$$0 = u_{x_n x_n} v_y^2 + u_{x_n} v_{yy},$$

$$0 = u_{x_n x_n} v_{x_i} v_y + u_{x_n x_i} v_y + u_{x_n} v_{y x_i},$$

$$0 = u_{x_i x_i} + 2u_{x_i x_n} v_{x_i} + u_{x_n x_n} v_{x_i}^2 + u_{x_n} v_{x_i x_i}$$

for $i = 1, \cdots, n-1$. We can solve these equations for u_t, u_{x_i} and $u_{x_i x_i}$, $i = 1, \cdots, n$, to obtain

$$u_{x_n} = v_y^{-1}, \qquad u_{x_i} = -v_y^{-1} v_{x_i}, \qquad u_t = -v_t v_y^{-1}, \tag{5.3}$$

$$u_{x_n x_n} = -v_y^{-3} v_{yy}, \qquad u_{x_i x_i} = 2v_{x_i} v_{y x_i} v_y^{-2} - v_{yy} v_{x_i}^2 v_y^{-3} - v_{x_i x_i} v_y^{-1}$$

for $i = i, \cdots, n-1$. On $\Phi^{-1}(\{y > 0\}) (= \{u > 0\})$, $\alpha_1 u_t = \Delta u$ is satisfied, thus, we have

$$\alpha_1 v_t = \Delta_x v + \frac{v_{yy}}{v_y^2}(|\nabla_x v|^2 + 1) - \frac{2\nabla_x v \cdot \nabla_x v_y}{v_y} \tag{5.4}$$

on $P_1 \cap \{y > 0\}$. Here,$\nabla_x v = (\frac{\partial v}{\partial x_1}, \cdots, \frac{\partial v}{\partial x_{n-1}}) \in \mathbf{R}^{n-1}$ and $\Delta_x v = \frac{\partial^2 v}{\partial x_1^2} + \cdots + \frac{\partial^2 v}{\partial x_{n-1}^2}$. We also have the similar equation for v on $P_1 \cap \{y < 0\}$. Let

$$b_{ij} = \begin{cases} \delta_{ij} & \text{if } 1 \le i, j \le n-1, \\ b_{ji} = -\frac{v_{x_i}}{v_y} & \text{if } 1 \le i \le n-1 \text{ and } j = n, \\ \frac{|\nabla_x v|^2 + 1}{v_y^2} & \text{if } i = j = n. \end{cases}$$

With $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, \cdots, n-1$ and $\partial_n = \frac{\partial}{\partial y}$, we may write (5.4) as

$$a'(y) v_t = \sum_{1 \le i,j \le n} b_{ij} \partial_i \partial_j v \tag{5.5}$$

on $P_1 \cap \{y \neq 0\}$. A straightforward computation shows that there exist strictly positive constants c_{27} and c_{28} such that

$$c_{27}|\xi|^2 \leq \sum_{1 \leq i,j \leq n} b_{ij}\xi_i\xi_j \leq c_{28}|\xi|^2 \tag{5.6}$$

for all $\xi = (\xi_1, \cdots, \xi_n) \in \mathbf{R}^n$. The constant c_{28} is determined in terms of the sup bounds on $|\nabla v|$. The constant c_{27} may be computed explicitly as

$$c_{27} = \min\left\{\min_{P_1}\left(1 - \frac{|\nabla_x v|^2}{1+|\nabla_x v|^2}\cdot\gamma\right), \min_{P_1}\frac{1+|\nabla_x v|^2}{v_y^2}(1-\gamma^{-1})\right\} > 0,$$

where $\gamma > 1$ is a fixed constant chosen so that $\gamma \cdot \max_{P_1} \frac{|\nabla_x v|^2}{1+|\nabla_x v|^2} < 1$.

In our analysis, it is useful to note that the last two terms in (5.4) may be expressed as

$$-\frac{\partial}{\partial y}\left(\frac{1+|\nabla_x v|^2}{v_y}\right),$$

hence, we may write (5.4) as

$$a'(y)v_t = \Delta_x v - \frac{\partial}{\partial y}\left(\frac{1+|\nabla_x v|^2}{v_y}\right) \tag{5.7}$$

on $P_1 \cap \{y \neq 0\}$.

Lastly, it is immediate that ∇v is Hölder continuous on P_1. Using the change of variables formula for integration, it is not hard to show that $v \in H^1(P_1)$ and $\nabla^2 v \in L^2(P_1)$. To show the last statement, we also use the fact that v is smooth away from $\{y = 0\}$ and that ∇v is continuous on P_1. Using the continuity of ∇v, we also see that v satisfies

$$\int_{P_1} a'(y)v_t\phi = -\int_{P_1}\left(\nabla_x v \cdot \nabla_x \phi - \frac{1+|\nabla_x v|^2}{v_y}\phi_y\right) \tag{5.8}$$

for all test functions $\phi \in C_c^\infty(P_1)$.

6. Analysis of equation (5.7)

In this section, we prove

Theorem 6.1. Suppose that a function v defined on P_1 satisfies (5.8) with

$$||v||_{H^1(P_1)} + ||\nabla v||_{C^\alpha(P_1)} + ||\nabla^2 v||_{L^2(P_1)} \leq c_{29}$$

for some constants c_{29} and $0 < \alpha < 1$. Then, there exist constants c_{30} and $0 < \beta < 1$ which depend only on n, α_1, α_2, α and c_{29}, such that

$$||v_t||_{C^\beta(P_{1/2})} + \sum_{1\le i,j\le n-1} ||v_{x_i x_j}||_{C^\beta(P_{1/2})} \le c_{30}, \tag{6.1}$$

$$\sum_{i=1}^{n-1} ||v_{x_i y}||_{C^\beta(P_{1/2}\cap\{y>0\})} + ||v_{yy}||_{C^\beta(P_{1/2}\cap\{y>0\})} \le c_{30} \tag{6.2}$$

and

$$\sum_{i=1}^{n-1} ||v_{x_i y}||_{C^\beta(P_{1/2}\cap\{y<0\})} + ||v_{yy}||_{C^\beta(P_{1/2}\cap\{y<0\})} \le c_{30}. \tag{6.3}$$

The theorem shows that v_t and $v_{x_i x_j}$, $1 \le i, j \le n-1$, are Hölder continuous across the hyper-plane $\{y = 0\}$, while v_{yy} and $v_{x_i y}$ are Hölder continuous up to the boundary $\{y = 0\}$ on each side. Thus, we may define $\lim_{y\to 0+} v_{yy}(x, y, t)$ and $\lim_{y\to 0-} v_{yy}(x, y, t)$ for each $x \in B_{1/2}(0) \in \mathbf{R}^{n-1}$ and $|t| < 1/4$, even though the resulting v_{yy} may be discontinuous across $\{y = 0\}$.

We first note the following simple lemma which is used in the proof of Theorem 6.1:

Lemma 6.1. Suppose that $v \in H^1(P_1)$ satisfies

$$\int_{P_1} a'(y) v_t \phi = -\sum_{1\le i,j\le n} \int_{P_1} \partial_i v\, a_{ij} \partial_j \phi - \sum_{i=1}^{n} \int_{P_1} f^i \partial_i \phi \tag{6.4}$$

for all test functions $\phi \in C_c^\infty(P_1)$. Here, $\partial_i = \frac{\partial}{\partial x_i}$ for $i = 1, \cdots, n-1$ and $\partial_n = \frac{\partial}{\partial y}$. Also assume that there exist positive constants $q > n \ge 2$, λ and μ, such that the measurable functions $\{a_{ij}, f^i\}$ satisfy

$$\lambda|\xi|^2 \le \sum_{1\le i,j\le n} a_{ij}\xi_i\xi_j \le \lambda^{-1}|\xi|^2 \tag{6.5}$$

for all $\xi = (\xi_1, \cdots, \xi_n) \in \mathbf{R}^n$ and

$$\left(\int_{P_1} v^2\right)^{1/2} + \sup_{-1\le t\le 1} \left(\sum_{i=1}^{n} \int_{B_1\times\{t\}} (f^i)^q\right)^{1/q} \le \mu. \tag{6.6}$$

Then, for each $r < 1$, there exist constants c_{31} and $0 < \beta < 1$ which depend only on r, n, λ, q, μ, α_1 and α_2, such that

$$||v||_{C^\beta(P_r)} \le c_{31}.$$

Proof. Define

$$\tilde{v}(x_1, \cdots, x_{n-1}, y, t) = v(x_1, \cdots, x_{n-1}, a^{-1}(y), t),$$

where a^{-1} is the inverse function of a. With this change of variable, $\tilde{v}$ satisfies

$$\int_{\tilde{P}_1} \tilde{v}_t \phi = -\sum_{1 \le i,j \le n} \int_{\tilde{P}_1} \partial_i \tilde{v}\, \tilde{a}_{ij} \partial_j \phi - \sum_{i=1}^{n} \int_{\tilde{P}_1} \tilde{f}^i \partial_i \phi$$

for all test functions $\phi \in C_c^\infty(\tilde{P}_1)$, where

$$\tilde{P}_1 = \{(x_1, \cdots, x_{n-1}, a(y), t) \mid (x_1, \cdots, x_{n-1}, y, t) \in P_1\},$$

$$\tilde{a}_{ij} = \begin{cases} a_{ij}/a'(y) & \text{if } 1 \le i, j = 1 \le n-1, \\ a_{in} = a_{ni} & \text{if } 1 \le i \le n-1 \text{ and } j = n, \\ a_{nn} a'(y) & \text{if } i = j = n, \end{cases}$$

$$\tilde{f}^i = \begin{cases} f^i/a'(y) & \text{for } i = 1, \cdots, n-1, \\ f^n & \text{for } i = n. \end{cases}$$

Note that the ellipticity condition (6.5) and the norm bound (6.6) on $\tilde{P}_1$ hold for $\{\tilde{a}_{ij}, \tilde{f}^i\}$ with different constants depending only on λ, μ, α_1 and α_2. Thus, the parabolic Hölder estimates (see [LSU]) give the interior Hölder continuity estimate for $\tilde{v}$ in terms of the listed constants, and hence for v as well. □

Proof of Theorem 6.1.
(1) Estimate of $||v_t||_{C^\beta}$
First, we define difference quotients with respect to the time variable t and the space variables x_i, $1 \le i \le n-1$. For a function f defined on P_1, let

$$f^{h,t}(x_1, \cdots, x_{n-1}, y, t) = \frac{1}{h}\{f(x_1, \cdots, x_{n-1}, y, t+h) - f(x_1, \cdots, x_{n-1}, y, t)\}$$

and

$$f^{h,x_i}(x_1, \cdots, x_{n-1}, y, t) = \frac{1}{h}\{f(\cdots, x_i + h, \cdots) - f(\cdots, x_i, \cdots)\}$$

for $(x_1, \cdots, x_{n-1}, y, t) \in P_{7/8}$ and $h \in \mathbf{R}$ with $|h| < 1/8$. Using $\phi^{-h,t}$ as a test function in (5.8), where we assume $\phi \in C_c^\infty(P_{7/8})$, we have

$$\int_{P_1} a'(y)(v^{h,t})_t \phi = -\int_{P_1} \nabla_x v^{h,t} \cdot \nabla_x \phi + \int_{P_1} \left(\frac{1 + |\nabla_x v|^2}{v_y} \right)^{h,t} \phi_y.$$

The last difference quotient may be expressed as

$$\left(\frac{1+|\nabla_x v|^2}{v_y}\right)^{h,t} = \frac{\{(\nabla_x v)(\cdots,t+h)+(\nabla_x v)(\cdots,t)\}}{v_y(\cdots,t+h)}\cdot\nabla_x v^{h,t}$$
$$-\frac{1+|\nabla_x v|^2(\cdots,t)}{v_y(\cdots,t+h)v_y(\cdots,t)}v_y^{h,t}.$$

With the notation

$$b_{ij,h} = \begin{cases} \delta_{ij} & \text{if } 1\le i,j\le n-1, \\ -\frac{v_{x_i}(\cdots,t+h)+v_{x_i}(\cdots,t)}{2v_y(\cdots,t+h)} & \text{if } 1\le i\le n-1 \text{ and } j=n, \\ \frac{1+|\nabla_x v|^2(\cdots,t)}{v_y(\cdots,t+h)v_y(\cdots,t)} & \text{if } i=j=n, \end{cases}$$

$v^{h,t}$ satisfies

$$a'(y)(v^{h,t})_t = \sum_{1\le i,j\le n} \partial_i(b_{ij,h}\partial_j v^{h,t}) \tag{6.7}$$

in the weak sense on $P_{7/8}$. Since $b_{ij,0}=b_{ij}$ in Section 5, the coefficients $\{b_{ij,h}\}$ are uniformly elliptic for all sufficiently small h. Also, by Fubini's Theorem,

$$\int_{P_{7/8}}(v^{h,t})^2 = \int_{P_{7/8}}\left(\int_0^1 v_t(\cdots,t+hs)ds\right)^2 dx_1\cdots dx_{n-1}dydt \tag{6.8}$$
$$\le \int_0^1 ds\int_{P_{7/8}}|v_t(\cdots,t+hs)|^2 dx_1\cdots dx_{n-1}dydt \le \int_{P_1}|v_t|^2 \le c_{29}^2.$$

Hence, (6.7) and (6.8) with Lemma 6.1 show that there exist constants c_{32} and $0<\beta_1<1$ with

$$||v^{h,t}||_{C^{\beta_1}(P_{3/4})} \le c_{32}$$

for all sufficiently small h. Since c_{32} and β_1 do not depend on h, we conclude that

$$||v_t||_{C^{\beta_1}(P_{3/4})} \le c_{32}. \tag{6.9}$$

(2) Estimates of $||v_{x_ix_j}||_{C^\beta}$, $1\le i,j\le n-1$

First note that the estimate (6.9), the equation (5.5) and the standard L^p estimate for the elliptic PDE applied to each time slice $B_{3/4}\times\{t\}$ show that, for each $p<\infty$,

$$||v(\cdot,t)||_{W^{2,p}(B_{5/8})} \le c_{33} \tag{6.10}$$

for a.e. t with $|t| < (3/4)^2$. Here, c_{33} depends on $p < \infty$, α_1, α_2, c_{29} and c_{32}, but not on t.

Multiply the equation (6.7) by $v^{h,t}\phi^2$, where ϕ is a non-negative function with $\phi \equiv 1$ on $P_{3/4}$ and $\phi \in C_c^\infty(P_{7/8})$, and integrate over P_1. By integration by parts, we have

$$-\int_{P_1} a'(y)(v^{h,t})^2\phi_t\phi = -\sum_{1\le i,j\le n}\int_{P_1} \partial_i v^{h,t} b_{ij,h}\partial_j v^{h,t}\phi^2 + \partial_j v^{h,t} b_{ij,h}\partial_i\phi\, 2v^{h,t}\phi.$$

By Cauchy's inequality and the inequality $2cd \le c^2/2 + 2d^2$,

$$\sum_{1\le i,j\le n}\int_{P_1} \partial_i v^{h,t} b_{ij,h}\partial_j v^{h,t}\phi^2 \le \int_{P_1} a'(y)(v^{h,t})^2|\phi_t|\phi$$

$$+\frac{1}{2}\sum_{1\le i,j\le n}\int_{P_1} \partial_i v^{h,t} b_{ij,h}\partial_j v^{h,t}\phi^2 + 2\sum_{1\le i,j\le n}\int_{P_1}\partial_i\phi\, b_{ij,h}\partial_j\phi(v^{h,t})^2.$$

By the uniform ellipticity of $\{b_{ij,h}\}$ and (6.8), we have

$$\int_{P_{3/4}}\sum_{i=1}^{n-1}|v_{x_i}^{h,t}|^2 + |v_y^{h,t}|^2 \le c_{34}.$$

Since c_{34} does not depend on h, we conclude that the weak derivatives v_{yt} and v_{x_it}, $1 \le i \le n-1$ exist and

$$\int_{P_{3/4}}\sum_{i=1}^{n-1}|v_{x_it}|^2 + |v_{yt}|^2 \le c_{34}. \tag{6.11}$$

To derive the estimate of $||v_{x_{i_0}x_{j_0}}||_{C^\beta}$, fix indices $1 \le i_0, j_0 \le n-1$. Using $\phi_{x_{i_0}}^{-h,x_{j_0}}$ with $\phi \in C_c^\infty(P_{3/4})$ in (5.8), integration by parts and by (6.11), we have

$$\int_{P_1} a'(y)v_{x_{i_0}t}^{h,x_{j_0}}\phi = -\sum_{1\le i,j\le n}\int_{P_1}\partial_i v_{x_{i_0}}^{h,x_{j_0}} b_{ij}\partial_j\phi + \partial_i v_{x_{i_0}} b_{ij}^{h,x_{j_0}}\partial_j\phi.$$

Letting $w = v_{x_{i_0}}^{h,x_{j_0}}$ and $f^j = \sum_{i=1}^n \partial_i v_{x_{i_0}} b_{ij}^{h,x_{j_0}}$, we may write the above as

$$\int_{P_1} a'(y)w_t\phi = -\sum_{1\le i,j\le n}\int_{P_1}\partial_i w\, b_{ij}\partial_j\phi - \sum_{j=1}^n\int_{P_1} f^j\partial_j\phi.$$

To apply Lemma 6.1, we need a uniform L^q norm estimate of f^i for some $q > n$ on each time slice. But (6.10) provides such estimate: for any $p < \infty$,

$$\int_{B_{9/16}}|b_{ij}^{h,x_{j_0}}(\cdot,t)|^p dxdy \le c_{35}\int_{B_{5/8}}|\nabla^2 v(\cdot,t)|^p dxdy \le c_{36},$$

where c_{36} does not depend on h or t. Hence, for a fixed $q > n$ (say, $q = n + 1$),

$$\int_{B_{9/16}} |f^j(\cdot, t)|^q dxdy \leq c_{37}$$

for some constant c_{37}. This combined with Lemma 6.1 shows that there exist c_{38} and $0 < \beta_2 < 1$ such that

$$||v_{x_{i_0}}^{h,x_{j_0}}||_{C^{\beta_2}(P_{1/2})} \leq c_{38}.$$

Since c_{28} and β_2 are independent of h, we have

$$||v_{x_{i_0}x_{j_0}}||_{C^{\beta_2}(P_{1/2})} \leq c_{38}.$$

(3) Conclusion

By (1) and (2), we conclude that v restricted to $\{y = 0\}$ is C^{2,β_2} in the space variables and C^{1,β_1} in the time variable. Since v satisfies

$$\alpha_1 v_t = \sum_{1 \leq i,j \leq n} b_{ij} \partial_i \partial_j v$$

on $P_1 \cap \{y > 0\}$, where $\{b_{ij}\}$ are uniformly elliptic and Hölder continuous, the standard linear parabolic theory (see [F]) shows that v_{yy} and $v_{x_i y}$, $1 \leq i \leq n-1$, are Hölder continuous up to the boundary $\{y = 0\}$. This shows (6.2), and similarly (6.3). □

7. Consequences of Theorem 6.1

In this section, we translate the results for v back to those for u. First of all, note that the restriction of v to $\{y = 0\}$ is the function representing the graph of the interface $\{u = 0\}$ over $P_1 \cap \{y = 0\}$. Hence, we may conclude that the hyper-surface $\{u = 0\}$ in $\mathcal{O}$ has the desired local regularity stated in Theorem 1.1. Since Φ in Section 5 is a Hölder continuous map, the regularities of v_t and $\partial_i \partial_j v$ are all carried over to those of u_t and $\partial_i \partial_j u$ as well, via formula (5.3).

Note also that the value of v_t at the origin represents the speed of motion of the interface $\{u = 0\}$ in the normal direction, since $|\nabla_x v| = 0$ at the origin. By approaching from two opposite directions, $y \to 0+$ and $y \to 0-$, and using (5.4) and the continuity of v_t and $\Delta_x v$, we note that

$$\alpha_1 v_t = \Delta_x v + \frac{v_{yy}^+}{v_y^2} \qquad \text{and} \qquad \alpha_2 v_t = \Delta_x v + \frac{v_{yy}^-}{v_y^2}$$

hold at the origin. Here,

$$v_{yy}^+ = \lim_{y\to 0+} v_{yy}(0,\cdots,0,y,0) \quad \text{and} \quad v_{yy}^- = \lim_{y\to 0-} v_{yy}(0,\cdots,0,y,0).$$

In particular, we obtain

$$v_t = \frac{v_{yy}^+ - v_{yy}^-}{v_y^2(\alpha_1 - \alpha_2)}$$

at the origin. The right-hand side is the expression in Theorem 1.2, when expressed in terms of u via (5.3).

What is interesting is that a simple-minded bootstrap argument does not seem to work for higher order regularity of v. We cannot show, for example, a Hölder continuity of v_{x_it} or $v_{x_ix_jx_k}$, $1 \le i,j,k \le n-1$. It is not at all clear whether v is more regular than shown in this paper.

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