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The Second Bounded Cohomology of a Group Acting on a Gromov-Hyperbolic Space

by

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Dedicated to Professor David B.A. Epstein on his 60th birthday

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1 Introduction

In [9], Epstein and the author showed that a non-elementary word-hyperbolic group has infinite dimensional second bounded cohomology group — see Corollary 1.5 in this paper. For example, the fundamental group of a closed negatively curved Riemannian manifold is a non-elementary word-hyperbolic group. In this paper, we prove a more general result — see Theorem 1.1.

To define the bounded cohomology group of a discrete group G, let

$$C_b^k(G;A) = \{f : G^k \to A \mid f \text{ has bounded range}\},$$

where $A = \mathbb{Z}$ or $\mathbb{R}$. The boundary $\delta : C_b^k(G;A) \to C_b^{k+1}(G;A)$ is given by

$$\begin{aligned}\delta f(g_0, \ldots, g_k) &= f(g_1, \ldots, g_k) + \sum_{i=1}^{k} (-1)^i f(g_0, \ldots, g_{i-1}g_i, \ldots, g_k) \\ &\quad + (-1)^{k+1} f(g_0, \ldots, g_{k-1}).\end{aligned}$$

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The cohomology of the complex $\{C_b^k(G;A), \delta\}$ is the *bounded cohomology group* of G, denoted by $H_b^*(G;A)$. See [12], [14] as general references for the theory of bounded cohomology. It is well known that $H_b^1(G;A)$ is trivial for any group G, and that $H_b^n(G;\mathbb{R})$ is trivial for all $n \geq 1$ if G is amenable.

In order to state our results, we recall that l^1 denotes the Banach space of summable sequences of real numbers with the norm $||(x_i)|| = \sum_{i=1}^{\infty} |x_i|$. It is well known that the $\mathbb{R}$-vector space l^1 has dimension equal to the cardinal of the continuum. We denote by e_i the sequence which is zero except at the i-th place where it is equal to one. Other definitions in the following statements are given in the next section.

Theorem 1.1 *Let G be a group and X a Gromov-hyperbolic space. Suppose G acts on X by isometries. Assume the action is properly discontinuous and the limit set of the action has at least three points. Then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G;\mathbb{R})$ such that, for each $i (1 \leq i < \infty)$, $\omega(e_i)$ is the image of a class in $H_b^2(G;\mathbb{Z})$. In particular, the dimension of $H_b^2(G;\mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

Remark Suppose in the above theorem that X is locally compact. Then the limit set is empty if and only if G is finite, and the limit set has two points if and only if G is virtually infinite cyclic. In both cases G is amenable, thus $H_b^n(G;\mathbb{R})$ is trivial for $n \geq 1$.

Theorem 1.1 has applications in Riemannian geometry.

Corollary 1.2 *Let M be a complete Riemannian manifold such that the sectional curvature K satisfies $-a \leq K \leq -b < 0$ for some $a, b > 0$. Suppose $G = \pi_1(M)$ is not almost nilpotent. Then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G;\mathbb{R})$ such that, for each $i (1 \leq i < \infty)$, $\omega(e_i)$ is the image of a class in $H_b^2(G;\mathbb{Z})$. In particular, the dimension of $H_b^2(G;\mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

Remark If $\pi_1(M)$ is almost nilpotent, then it is amenable and $H_b^n(\pi_1(M);\mathbb{R})$ is trivial for all $n \geq 1$.

The next result is a special case of Corollary 1.2.

Corollary 1.3 *Let M be a complete Riemannian manifold such that $-a \leq K \leq -b < 0$ for some $a, b > 0$. Suppose the volume of M is finite. Let $G = \pi_1(M)$. Then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G;\mathbb{R})$ such* *that, for each $i(1 \leq i < \infty)$, $\omega(e_i)$ is the image of a class in $H_b^2(G;\mathbb{Z})$. In particular, the dimension of $H_b^2(G;\mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

Theorem 1.1 applies to word-hyperbolic groups. Note that if a group is finite or a finite extension of a finitely generated abelian group, then the group is called *elementary*. If a subgroup H of a word-hyperbolic group is elementary, then H is either finite or a finite extension of $\mathbb{Z}$.

Corollary 1.4 *Let G be a word-hyperbolic group and H a subgroup. Suppose H is non-elementary. Then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(H;\mathbb{R})$ such that, for each $i(1 \leq i < \infty)$, $\omega(e_i)$ is the image of a class in $H_b^2(H;\mathbb{Z})$. In particular, the dimension of $H_b^2(H;\mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

Remark (1) If H is elementary, then it is amenable and the bounded cohomology group is trivial.
(2) In the above corollary H is not word-hyperbolic in general. There exists a word-hyperbolic group containing a finitely presented subgroup which is not word-hyperbolic [5].
Corollary 1.4 obviously implies the following.

Corollary 1.5 ([9]) *Let G be a non-elementary word-hyperbolic group. Then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G;\mathbb{R})$ such that, for each $i(1 \leq i < \infty)$, $\omega(e_i)$ is the image of a class in $H_b^2(G;\mathbb{Z})$. In particular, the dimension of $H_b^2(G;\mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

If G is a knot group of hyperbolic type, then, by Corollary 1.3, $H_b^2(G;\mathbb{R})$ is infinite dimensional. The following theorem shows that in fact the conclusion holds for all non-trivial knot groups.

Theorem 1.6 *Suppose G is a knot group such that $G \not\cong \mathbb{Z}$. Then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G;\mathbb{R})$ such that, for each $i(1 \leq i < \infty)$, $\omega(e_i)$ is the image of a class in $H_b^2(G;\mathbb{Z})$. In particular, the dimension of $H_b^2(G;\mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

We briefly outline the proof of Theorem 1.1. The main idea of the proof is originally due to Brooks, who showed that the second bounded cohomology of free groups are non trivial, [6]. Later this result was generalized to all non-elementary word-hyperbolic groups, [9], whose techniques are somehow improved in this paper. To show our result, we construct infinitely many, linearly independent second bounded cohomology classes $\{[\phi_i]\}_i$ for G. We will obtain second bounded cocycles ϕ_i as the coboundaries of some first cochains h_i. First cochains are nothing but maps from G to $\mathbb{R}$. We choose cochains h_i so that their boundaries are bounded. Then, if h_i is not bounded, its boundary δh_i might give a non-trivial element as a second bounded cohomology class. Therefore our strategy is to construct unbounded maps $h_i : G \to \mathbb{R}$ whose coboundaries are bounded.

To obtain such h_i we use the space X the group is acting on. For the case of word-hyperbolic groups, we may take X to be the Cayley graph of G, which is a geometric realization of the group (more precisely, X and G are quasi-isometric to each other). The construction of h_i are combinatorial, which is done by some counting argument (see Figure 1). That h_i are unbounded is shown in Lemma 5.2 and Prop 5.8. We use the hyperbolicity of the space X to conclude the boundaries δh_i are bounded (Lemma 3.8). Since we assume the limit set of the action of G on X has more than two points, there is abundant choice for h_i. We choose h_i carefully so that the coboundary class of each h_i is non-trivial in $H_b^2(G)$, and moreover, they are linearly independent. One might say $[\delta h_i] \in H_b^2(G)$ is non-trivial essentially because h_i is unbonded, but this requires an argument(see section 5).

One of the improvement made in this paper after [9] is that one does not assume the action of G on X is cocompact (thus G and X are not quasi-isometric in general). Therefore, we are able to conclude the second bounded cohomology group of all hyperbolic manifold is non-trivial and in fact infinite dimensional as long as its fundamental group is not almost nilpotent (Cor 1.2).

One nice corollary is that the second bounded cohomology of all non-trivial knot groups are infinite dimensional, in particular non-trivial (Th 1.6). In fact this is a part of the following result. If a three manifold M is geometric in the sense of Thurston, then its second bounded cohomology group is either trivial or infinite dimensional. Furthermore, one can determine when it is trivial in terms of the geometry of M. The proof will be given in another work later.

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2 Some basics of hyperbolic spaces and groups

In this section we collect some definitions and results on hyperbolic spaces and groups acting on them. General reference are [4], [13], [17]. Let (X,d) be a metric space. Suppose a group G acts on X by isometries sending x to $g\cdot x$. Note that $(gh)\cdot x = g\cdot(h\cdot x)$. For $x \in X$ and $r > 0$, we set $B_r(x) = \{y \in X | d(x,y) < r\}$. If the subset of G defined by $\{g \in G | B_r(x) \cap g \cdot B_r(x) \neq \emptyset\}$ is finite for any $r > 0$ and any $x \in X$, then we say that the action is *properly discontinuous*.

If any two points in X are joined by a geodesic, then we say X is *geodesic*. For $\delta \geq 0$, if any side of a triangle is contained in the δ-neighborhood of the union of the other two sides, then the triangle is called *δ-thin*. For $\delta \geq 0$, if every geodesic triangle in X is δ-thin, then we say that X is *δ-hyperbolic*. A *Gromov-hyperbolic* space is a space which is δ-hyperbolic for some $\delta \geq 0$, [13]. The following fact is standard — see for example [17].

Lemma 2.1 *Let X be a δ-hyperbolic space and $A, B, C \in X$. Then there exist A', B', C' on $[B,C], [A,C], [A,B]$ respectively such that*

$$d(A',B'),\ d(B',C'),\ d(C',A') \leq 4\delta.$$

Definition Let α be a path in a geodesic space X. If we have for some K and ε

$$\frac{|t-s|}{K} - \varepsilon \leq d(\alpha(t), \alpha(s))$$

for all t and s, then α is called *(K,ε)-quasi-geodesic*.

Let S be a subset in X and $L \geq 0$. We denote the L-neighborhood of S by $N_L(S)$. We recall one important result (see for example Proposition 4.9 of [4]).

Lemma 2.2 *Let X be a δ-hyperbolic space. Given $K \geq 1$ and $\varepsilon \geq 0$, there exists $L(K, \varepsilon, \delta) \geq 0$ with the following property. Let α and β be (K, ε)-quasi-geodesics. Suppose the starting points of α and β coincide and the finishing points coincide. Then $\alpha \subset N_L(\beta)$ and $\beta \subset N_L(\alpha)$. The same applies if the endpoints are at infinity.*

Now we recall the definition of the boundary of a hyperbolic space. Let X be a Gromov-hyperbolic space. Take a base point $x_0 \in X$. Given $x, y \in X$, we define

$$(x \cdot y) = \frac{1}{2}(d(x_0, x) + d(x_0, y) - d(x, y)).$$

A sequence $(x_i)_i$ of points in X is called *convergent at infinity* if

$$\lim_{i,j \to \infty} (x_i \cdot x_j) = \infty.$$

We define a relation denoted by " $\sim$ " on the set of sequences which are convergent at infinity by

$$(x_i) \sim (y_i) \Leftrightarrow \lim_{i,j \to \infty} (x_i \cdot y_j) = \infty.$$

This relation is an equivalence relation if X is Gromov-hyperbolic. The *boundary* ∂X(1.8 in [13]) is the set of the equivalence classes of sequences in X which are convergent at infinity. If $(x_i)_i$ is in a class $a \in \partial X$, we write $\lim_i x_i = a$. Let α be a quasi-geodesic. Then the sequences $(\alpha(i))_i$ and $(\alpha(-i))_i$ are convergent at infinity. We write

$$\alpha(\pm\infty) = \lim_{i \to \pm\infty} \alpha(i).$$

Suppose a group G acts on X. Then G acts on ∂X by $g \cdot (x_i) = (g \cdot x_i)$. The *limit set* $L(G) \subset \partial X$ of the action is defined by

$$L(G) = \{(x_i)_i \in \partial X |\, x_i = g_i \cdot x_0,\, g_i \in G,\, 1 \leq i < \infty\}.$$

It is well known that the number of the points in $L(G)$ is 0, 1, 2, or ∞.

Definition Suppose G acts on X and $x_0 \in X$. Let $g \in G$. If $\{g^i \cdot x_0\}_{i \in \mathbb{Z}}$ is quasi-isometric to $\mathbb{Z}$ with its standard word metric, then g is called a *hyperbolic* isometry(8.1 in [13]).

Definition Suppose $g \in G$ is a hyperbolic isometry of a Gromov-hyperbolic space X. Let $x_0 \in X$. We define $g^{\pm\infty} \in \partial X$ by

$$g^{\pm\infty} = \lim_{i \to \pm\infty} g^i \cdot x_0.$$

Note that the result does not depend on choice of x_0.

The following result is well known — see Lemma 8.1.A in [13].

Lemma 2.3 *Let X be a Gromov-hyperbolic space. Suppose G acts on X. If $|L(G)| \geq 2$, then G has a hyperbolic isometry.*

Lemma 2.4 *Let X be a Gromov-hyperbolic space. Suppose G acts on X. If $|L(G)| \geq 3$, then there exist hyperbolic isometries $g_1, g_2 \in G$ such that*

$$g_2^{+\infty} \neq g_1^{\pm\infty}.$$

Proof. Take a hyperbolic isometry $g_1 \in G$ by Lemma 2.3. Let α be a quasi-geodesic such that $\alpha(0) = x_0$ and $\alpha(\pm\infty) = g_1^{\pm\infty}$. We first show that there exists $k \in G$ such that

$$k \cdot g_1^{\pm\infty} \neq g_1^{\pm\infty}.$$

To show this by contradiction, suppose $k \cdot g_1^{\pm\infty} = g_1^{\pm\infty}$ for all $k \in G$. Then $k \cdot \alpha$ joins $g_1^{+\infty}$ and $g_1^{-\infty}$ for all $k \in G$. By Lemma 2.2, we find $L \geq 0$ such that $k \cdot \alpha \subset N_L(\alpha)$ for all $k \in G$. This implies $G \cdot x_0 \subset N_L(\alpha)$ and hence $L(G) = \{g_1^{\pm\infty}\}$. This is a contradiction. Therefore, there exists $k \in G$ such that $k \cdot g_1^{\pm\infty} \neq g_1^{\pm\infty}$. Suppose $k \cdot g_1^{+\infty}$(or $k \cdot g_1^{-\infty}$) $\neq g_1^{\pm\infty}$. Set $g_2 = kg_1k^{-1}$(or $kg_1^{-1}k^{-1}$, respectively). Then g_2 is hyperbolic and $g_2^{+\infty} = k \cdot g_1^{+\infty}$(or $k \cdot g_1^{-\infty}$, respectively), hence $g_2^{+\infty} \neq g_1^{\pm\infty}$.

The conclusion of the following lemma implies G contains a subgroup F that is isomorphic to the rank-2 free group(Prop4.3).

Lemma 2.5 *Let X be a Gromov-hyperbolic space. Suppose G acts on X properly discontinuously. If $|L(G)| \geq 3$, then there exist hyperbolic isometries $g_1, g_2 \in G$ such that*

$$g_2^{+\infty} \neq g_1^{\pm\infty} \text{ and } g_2^{-\infty} \neq g_1^{\pm\infty}.$$

Proof. By Lemma 2.4, there exist hyperbolic isometries $g_1, g_2 \in G$ such that $g_2^{+\infty} \neq g_1^{\pm\infty}$. We will show $g_2^{-\infty} \neq g_1^{\pm\infty}$ as well. First, to show $g_2^{-\infty} \neq g_1^{-\infty}$ by contradiction, suppose $g_2^{-\infty} = g_1^{-\infty}$. Let α, β be infinite paths defined by

$$\alpha = \bigcup_{n=0}^{\infty} g_1^{-n} \cdot [x_0, g_1^{-1} \cdot x_0],\ \beta = \bigcup_{n=0}^{\infty} g_2^{-n} \cdot [x_0, g_2^{-1} \cdot x_0].$$

Then α and β are quasi-geodesics from x_0 to $g_2^{-\infty} = g_1^{-\infty}$. By Lemma 2.2, there exists $L \geq 0$ such that $\beta \subset N_L(\alpha)$. Thus for each $n \geq 0$, there exists $m(n) \geq 0$ such that

$$d(g_1^{-n} \cdot x_0, g_2^{-m(n)} \cdot x_0) \leq L + l,$$

where $l = |[x_0, g_2^{-1} \cdot x_0]|$. We have $d(x_0, g_1^n g_2^{-m(n)} \cdot x_0) \leq L + l$ for all $n \geq 0$. Since G acts on X properly discontinuously, there exist finitely many elements $k \in G$ such that $d(x_0, k \cdot x_0) \leq L+l$. Therefore there exist $n_2 > n_1 \geq 0$ such that $g_1^{n_1} g_2^{-m(n_1)} = g_1^{n_2} g_2^{-m(n_2)}$. We find $g_1^{n_2-n_1} = g_2^{m(n_2)-m(n_1)}$. This implies $g_2^{+\infty} = g_1^{+\infty}$ or $g_1^{-\infty}$, which is a contradiction. We get $g_2^{-\infty} \neq g_1^{-\infty}$. A similar argument shows $g_2^{-\infty} \neq g_1^{+\infty}$. We obtain the lemma.

We review the notion of local-quasi-geodesics and show that local-quasi-geodesics are in fact quasi-geodesics. We first state an obvious lemma.

Lemma 2.6 *Let α be a (K, ε)-quasi-geodesic and γ a geodesic in a δ-hyperbolic space. For $C \geq 0$, there exists $l_1(K, \varepsilon, C, \delta) > 0$ with the following property. Suppose $\alpha \subset N_C(\gamma)$. Let $t_1 < t_2 < t_3$ such that $t_2 - t_1 \geq l_1$, $t_3 - t_2 \geq l_1$ and $d(\alpha(t_i), \gamma(s_i)) \leq C$ for $1 \leq i \leq 3$. Then we have either*

$$s_1 < s_2 < s_3, \ \textit{or}\ s_1 > s_2 > s_3.$$

To define local-quasi-geodesics in a hyperbolic space, we introduce several constants. Note that those constants depends only on $\varepsilon, \delta \geq 0$ and $K \geq 1$. Using the constant $L(K, \varepsilon, \delta)$ in Lemma 2.2, we first define
$l_2(K, \varepsilon, \delta) = 2K(10\delta + L(K, \varepsilon, \delta) + 1) + \varepsilon$, and $L_1(K, \varepsilon, \delta) = 11 l_2 + 100\delta$.
Let $l_1(K, \varepsilon, L_1, \delta)$ be the constant in Lemma 2.6. We put
$l_3(K, \varepsilon, \delta) = l_1(K, \varepsilon, L_1, \delta) + 2\varepsilon + 4KL_1$, and then $c = l_2 + 2l_3$.

Using this constant $c = c(\delta, K, \varepsilon)$ we define local-quasi-geodesics.

Definition Let α be a path in a δ-hyperbolic space X. If $\alpha|_{[t,s]}$ is a (K, ε)-quasi-geodesic for all $t < s$ with $|t - s| \leq c$, then α is called a *local-(K, ε)-quasi-geodesic.*

Lemma 2.2. says that (K,ε)-quasi-geodesics with common ends stay close to each other. An argument similar to the proof of this fact(see Prop 4.9 [3]) shows that the property holds for local-quasi-geodesics as follows.

Lemma 2.7 *Let X be a δ-hyperbolic space. Suppose $K \geq 1$ and $\varepsilon \geq 0$. Let α be a local-(K,ε)-quasi-geodesic and γ a geodesic. Assume the starting points of α and γ coincide and the finishing points coincide. Then $\alpha \subset N_{L_1}(\gamma)$.*

Using the previous lemma, it is easy to deduce the following significant fact, [17].

Proposition 2.8 *A local-(K,ε)-quasi-geodesic is a $(2K,\varepsilon)$-quasi-geodesic.*

3 Counting copies of a path along paths

Let (X,d) be a geodesic metric space. Suppose a group G acts on X properly discontinuously by isometries. In this section, we define a function on G associated to the action, which was formerly defined in [9] for a word-hyperbolic group and its natural action on the Cayley graph.

Let α be a finite path in X. We denote the length of α by $|\alpha|$, the starting point by $i(\alpha)$, and the finishing point by $t(\alpha)$. We use the action of $g \in G$ on X to define a path $g \cdot \alpha$ starting at the point $g \cdot i(\alpha)$ and finishing at the point $g \cdot t(\alpha)$. We say that $g \cdot \alpha$ is a *copy* of α. Obviously $|g \cdot \alpha| = |\alpha|$.

Let w be a finite path. We define

$|\alpha|_w = \{$the number of times you see copies of w in α without overlapping$\}$.

Suppose $A, B \in X$. We use $[A,B]$ to denote some choice of a geodesic from A to B. Let W be a number with $0 < W < |w|$. We define

$$c_{w,W}([A,B]) = d(A,B) - \inf_{\alpha}(|\alpha| - W|\alpha|_w),$$

where α ranges over all the paths from A to B — see Figure 1. Obviously $c_{w,W}([A,B])$ does not depend on the choice of a geodesic $[A,B]$. Thus $c_{w,W}$ is in fact a (non-symmetric) function on $X \times X$. The following result is clear.

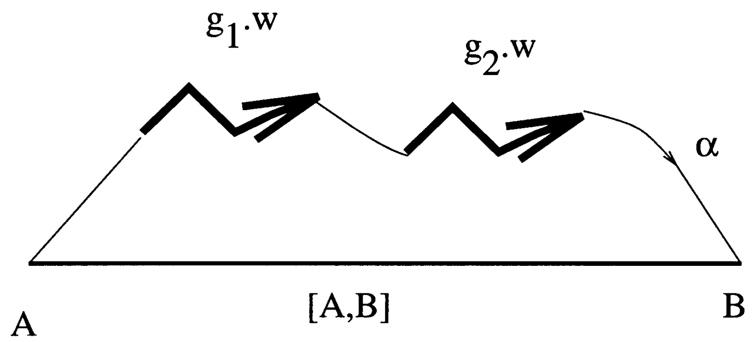

Figure 1: Definition of $|\alpha|_w$. We see $|\alpha|_w = 2$ in this case.

Lemma 3.1

$$c_{w,W}([A,B]) = c_{w^{-1},W}([B,A]).$$

If α attains the infimum in the definition of $c_{w,W}([A,B])$, we say that α *realizes* $c_{w,W}$ at $[A,B]$. It is easy to see that since the action of G on X is properly discontinuous, for all w, W, A, B, there is a path which realizes $c_{w,W}$ at $[A,B]$.

Lemma 3.2 *If α is a geodesic, then*

$$\frac{W}{|w|}|\alpha| \geq c_{w,W}(\alpha) \geq W|\alpha|_w.$$

Proof. Let α' be a path which realizes $c_{w,W}$ at α. Then, since $|\alpha'| - W|\alpha'|_w \leq |\alpha| - W|\alpha|_w$, we find

$$c_{w,W}(\alpha) = |\alpha| - (|\alpha'| - W|\alpha'|_w) \geq W|\alpha|_w.$$

To show the other inequality, note that $|\alpha'|_w \leq \frac{|\alpha'|}{|w|}$. This implies

$$|\alpha'| - W|\alpha'|_w \geq |\alpha'| - \frac{|\alpha'|}{|w|}W = \left(1 - \frac{W}{|w|}\right)|\alpha'| \geq \left(1 - \frac{W}{|w|}\right)|\alpha|.$$

Thus,

$$c_{w,W}(\alpha) = |\alpha| - (|\alpha'| - W|\alpha'|_w) \leq \frac{|\alpha|}{|w|}W.$$

Lemma 3.3 *Suppose β realizes $c_{w,W}$ at some geodesic. Then β is $\left(\frac{|w|}{|w|-W}, \frac{2W|w|}{|w|-W}\right)$-quasi-geodesic.*

Proof. Let $t < s$ and set $\beta' = \beta|_{[t,s]}$. Note that $|\beta'| = |t - s|$. Let γ be a geodesic from $\beta(t)$ to $\beta(s)$. Then since β is a realizing path,

$$|\beta'| - W(|\beta'|_w + 2) \leq |\gamma| - W|\gamma|_w.$$

The constant 2 arises from the fact that copies of w might overlap each of the two ends of γ. Clearly $|\beta'|_w \leq |\beta'|/|w|$. Therefore

$$\begin{aligned} d(\beta(t), \beta(s)) = |\gamma| &\geq |\gamma| - W|\gamma|_w \geq |\beta'| - W|\beta'|_w - 2W \\ &\geq |\beta'| - \frac{W}{|w|}|\beta'| - 2W = \frac{|w| - W}{|w|}|\beta'| - 2W. \end{aligned}$$

The proof is completed.

Lemma 3.4 *Let A, B, C be three points in X. Then*

$$|c_{w,W}([A, B]) - c_{w,W}([A, C])| \leq 2d(B, C).$$

Proof. Let $\sigma = [C, B]$. Let α and β be paths which realize $c_{w,W}$ at $[A, B]$ and $[A, C]$, respectively. Then

$$\begin{aligned} |\alpha| - W|\alpha|_w &\leq |\beta\sigma| - W|\beta\sigma|_w \leq |\beta| + |\sigma| - W(|\beta|_w + |\sigma|_w) \\ &\leq |\beta| - W|\beta|_w + |\sigma|. \end{aligned}$$

Therefore

$$|c_{w,W}([A, B]) - c_{w,W}([A, C])| \leq |d(A, B) - d(A, C)| + |\sigma| \leq 2d(B, C).$$

The following result is an immediate consequence of Lemmas 2.2 and 3.3.

Lemma 3.5 *Let X be a δ-hyperbolic space. Suppose α is a geodesic. If β realizes $c_{w,W}$ at α, then $\beta \subset N_{L_0}(\alpha)$, where*

$$L_0 = L\left(\frac{|w|}{|w| - W}, \frac{2W|w|}{|w| - W}, \delta\right).$$

The next lemma is obvious.

Lemma 3.6 *Let α be a geodesic. Suppose there is no $g \in G$ such that $g \cdot w \subset N_{L_0}(\alpha)$. Then $c_{w,W}(\alpha) = 0$ and $c_{w^{-1},W}(\alpha) = 0$.*

Lemma 3.7 *Let X be a δ-hyperbolic space. Let A, B, C be three points in X with $C \in [A, B]$ and let $\alpha = [A, C], \beta = [C, B], \gamma = [A, B]$. Then*

$$c_{w,W}(\gamma) \leq c_{w,W}(\alpha) + c_{w,W}(\beta) \leq c_{w,W}(\gamma) + 2L_0 + W.$$

Proof. Let α', β', γ' be paths which realize $c_{w,W}$ at α, β and γ, respectively. Then

$$|\gamma'| - W|\gamma'|_w \leq |\alpha'\beta'| - W|\alpha'\beta'|_w \leq |\alpha'| + |\beta'| - W(|\alpha'|_w + |\beta'|_w).$$

Therefore $c_{w,W}(\gamma) \leq c_{w,W}(\alpha) + c_{w,W}(\beta)$. To prove the other inequality, note that we can take $C' \in \gamma'$ such that $d(C, C') \leq L_0$, since $\gamma' \subset N_{L_0}(\gamma)$. Let $\sigma = [C', C]$ and divide γ' into γ_1 and γ_2 at C'. Then

$$|\alpha'| - W|\alpha'|_w \leq |\gamma_1\sigma| - W|\gamma_1\sigma|_w \leq |\gamma_1| + L_0 - W|\gamma_1|_w,$$

since $|\sigma| = L_0$ and $|\gamma_1\sigma|_w \geq |\gamma_1|_w$.

Also,

$$|\beta'| - W|\beta'|_w \leq |\gamma_2| + L_0 - W|\gamma_2|_w.$$

Therefore $c_{w,W}(\alpha) + c_{w,W}(\beta) \leq c_{w,W}(\gamma) + 2L_0 + W$, since $|\gamma'| = |\gamma_1| + |\gamma_2|$ and $|\gamma'|_w \leq |\gamma_1|_w + |\gamma_2|_w + 1$. We obtain the lemma.

Let α be a geodesic. We define

$$h_{w,W}(\alpha) = c_{w,W}(\alpha) - c_{w^{-1},W}(\alpha).$$

Lemma 3.8 *Suppose X is a δ-hyperbolic space. Let w be a path and W a number such that $0 < W < |w|$. Suppose A, B, C are three points in X. Then*

$$|h_{w,W}([A, B]) + h_{w,W}([B, C]) - h_{w,W}([A, C])| \leq 12L_0 + 6W + 48\delta.$$

Proof. By Lemma 2.1, there exist A', B', C' on $[B, C], [A, C]$ and $[A, B]$, respectively, such that $d(A', B')$, $d(B', C')$, $d(C', A') \leq 4\delta$. Let $\alpha = [B, C]$, $\beta = [A, C]$, $\gamma = [A, B]$. We divide α, β, γ into two geodesics at A', B', C', respectively, such that $\alpha = \alpha_1\alpha_2$, $\beta = \beta_1\beta_2$, $\gamma = \gamma_1\gamma_2$. By Lemma 3.7,

$$\begin{aligned}
|c_{w^{\pm1},W}(\alpha) - c_{w^{\pm1},W}(\alpha_1) - c_{w^{\pm1},W}(\alpha_2)| &\leq 2L_0 + W, \\
|c_{w^{\pm1},W}(\beta) - c_{w^{\pm1},W}(\beta_1) - c_{w^{\pm1},W}(\beta_2)| &\leq 2L_0 + W, \\
|c_{w^{\pm1},W}(\gamma) - c_{w^{\pm1},W}(\gamma_1) - c_{w^{\pm1},W}(\gamma_2)| &\leq 2L_0 + W.
\end{aligned}$$

By Lemma 3.4,

$$\begin{aligned}
|c_{w^{\pm1},W}(\beta_1) - c_{w^{\pm1},W}(\gamma_1)| &\leq 2d(C', B') \leq 8\delta \\
|c_{w^{\pm1},W}(\gamma_2^{-1}) - c_{w^{\pm1},W}(\alpha_1)| &\leq 2d(C', A') \leq 8\delta \\
|c_{w^{\pm1},W}(\beta_2^{-1}) - c_{w^{\pm1},W}(\alpha_2^{-1})| &\leq 2d(A', B') \leq 8\delta.
\end{aligned}$$

Collecting these inequalities together, by Lemma 3.1, we obtain the desired inequality $|h_{w,W}(\alpha) + h_{w,W}(\beta) - h_{w,W}(\gamma)| \leq 12L_0 + 6W + 48\delta$.

Take $x_0 \in X$ as a base point. Let $g \in G$. We define functions $c_{w,W}$ and $h_{w,W} : G \to \mathbb{R}$ by

$$c_{w,W}(g) = c_{w,W}([x_0, g \cdot x_0]),$$

$$h_{w,W}(g) = h_{w,W}([x_0, g \cdot x_0]).$$

The following lemma is clear from Lemma 3.2.

Lemma 3.9 *For all $g \in G$*

$$\frac{W}{|w|}|[x_0, g \cdot x_0]| \geq c_{w,W}(g) \geq W|[x_0, g \cdot x_0]|_w.$$

Definition Let $f : G \to \mathbb{R}$ be a function. If there exists $D > 0$ such that $|f(g) + f(h) - f(gh)| \leq D$ for all $g, h \in G$, then we say that f is a *quasi-homomorphism* with *defect D*.

Proposition 3.10 *Let X is a δ-hyperbolic space. Suppose w is a path and W a number with $0 < W < |w|$. Then $h_{w,W} : G \to \mathbb{R}$ is a quasi-homomorphism with defect $Q = 12L_0 + 6W + 48\delta$, where $L_0 = L\left(\frac{|w|}{|w|-W}, \frac{2W|w|}{|w|-W}, \delta\right)$.*

Proof. Let $g_1, g_2 \in G$. Apply Lemma 3.8 to $A = x_0$, $B = g_1 \cdot x_0$, and $C = g_1 g_2 \cdot x_0$, then we obtain $|h_{w,W}(g_1) + h_{w,W}(g_2) - h_{w,W}(g_1 g_2)| \leq Q$.

4 Quasi-geodesic rank-2 free subgroups

Suppose G acts on a hyperbolic space X with a base point x_0 such that the limit set of the action has more than two points. It is easy to show G contains a subgroup F isomorphic to the rank-2 free group. In this section we will show that one may further assume that F is quasi-convex and quasi-geodesic in G with respect to the word metric, the orbit $F \cdot x_0$ is embedded in X in a quasi-convex and quasi-geodesic way, and F and $F \cdot x_0$ are quasi-isometric to each other. One may think that $F \cdot x_0$ is a nice geometric realization of F in X.

We first state the following obvious lemma.

Lemma 4.1 *Let $\alpha = \alpha_1\alpha_2$ be a path in a δ-hyperbolic space with $\delta > 0$. Suppose α_1, α_2 are geodesics and there exists $D \geq 100\delta$ such that $|\alpha_1|, |\alpha_2| \geq D$ and*

$$d(\alpha_1^{-1}(D), \alpha_2(D)) \geq 10\delta.$$

Then α is a $(1, 10D)$-quasi-geodesic.

The next lemma is useful.

Lemma 4.2 *Let X be a δ-hyperbolic space with $\delta > 0$. Let $\alpha = \alpha_1 \cdots \alpha_n$ be a path such that α_i is a geodesic for all i. Suppose there is $D \geq 100\delta$ such that $|\alpha_i| \geq D$ for all i and*

$$d(\alpha_i^{-1}(D), \alpha_{i+1}(D)) \geq 10\delta$$

for all $1 \leq i \leq n-1$. Set $C_0 = L(2, 10D, \delta)$. Let $K > 1$ be given and set

$$l(K, D, \varepsilon) = \max\left\{\frac{4C_0K}{K-1}, l_1(2, 10D, C_0, \delta), c(1, 10D, \delta)\right\}.$$

If $|\alpha_i| \geq l$ for all i, then α is an $\left(\frac{l}{l-2C_0}, \frac{2lC_0}{l-2C_0}\right)$-quasi-geodesic. In particular α is a $(K, 4C_0)$-quasi-geodesic.

Proof. Note that $l \geq c(1, 10D, \delta)$. Since $|\alpha_i| \geq l \geq c(1, 10D, \delta)$ for all i, α is a local-$(1, 10D)$-quasi-geodesic by Lemma 4.1. By Proposition 2.8, α is a $(2, 10D)$-quasi-geodesic. We denote the finishing point of α_i by P_i for $1 \leq i \leq n$. Let $x = \alpha(t)$ and $y = \alpha(s)$, $t \leq s$. Suppose $x \in \alpha_I, y \in \alpha_J$ for $I \leq J$. Since $|\alpha_i| \geq l$, we have

$$\frac{t-s}{l} \geq J - I - 1.$$

Let γ be a geodesic from x to y. Since α is a $(2, 10D)$-quasi-geodesic, there are points $Q_I, Q_{I+1}, \cdots, Q_{J-1}$ on γ such that

$$d(P_i, Q_i) \leq C_0, \text{ for all } I \leq i \leq J-1.$$

Since $l_1(2, 10D, C_0, \delta) \leq l$, by Lemma 2.6, the points $Q_I, \cdots, Q_{J-1}$ appear in this order when we travel along γ from x to y. This is because α is a $(2, 10D)$-quasi-geodesic and γ is a geodesic which are C_0-close to each other, the points P_i are separated from each other at least by $l \geq l_1$ on α, and $d(P_i, Q_i) \leq C_0$ for all i. Having this property we obtain

$$\begin{aligned} d(\alpha(t), \alpha(s)) &= d(\alpha(t), Q_I) + \sum_{i=I}^{J-2} d(Q_i, Q_{i+1}) + d(Q_{J-1}, \alpha(s)) \\ &\geq d(\alpha(t), P_I) - C_0 + \sum_{i=I}^{J-2} (d(P_i, P_{i+1}) - 2C_0) + d(P_{J-1}, \alpha(s)) - C_0 \\ &= |t-s| - 2(J-I)C_0 \geq \left(1 - \frac{2C_0}{l}\right) |t-s| - 2C_0. \end{aligned}$$

This shows that α is an $\left(\frac{l}{l-2C_0}, \frac{2lC_0}{l-2C_0}\right)$-quasi-geodesic. Now, since $l \geq \frac{4C_0K}{K-1}$, we have

$$\frac{l}{l-2C_0} \leq K, \quad \frac{2lC_0}{l-2C_0} \leq 4C_0.$$

Therefore it follows that α is a $(K, 4C_0)$-quasi-geodesic. This completes the proof.

Suppose G acts on X. Let $g, h \in G$ and let α_g, α_h be paths from x_0 to $g \cdot x_0, h \cdot x_0$ respectively. Then $\alpha_g \cup g \cdot \alpha_h$ is a path from x_0 to $gh \cdot x_0$ through $g \cdot x_0$. We simply denote this path $\alpha_g \alpha_h$.

$$\alpha_g \alpha_h = \alpha_g \cup g \cdot \alpha_h.$$

Inductively we define α_g^n for $n \geq 1$.

Let $a, b \in G$ and let $F =< a, b >$ in G. We choose geodesics γ_a, γ_b from x_0 to $a \cdot x_0$, $b \cdot x_0$, respectively. Let $\gamma_{a^{-1}}$, $\gamma_{b^{-1}}$ be the geodesics from x_0 to $a^{-1} \cdot x_0$ and $b^{-1} \cdot x_0$, respectively, defined by

$$\gamma_{a^{-1}} = a^{-1} \cdot \gamma_a^{-1}, \; \gamma_{b^{-1}} = b^{-1} \cdot \gamma_b^{-1}.$$

Let $g \in F$ and suppose

$$g = a^{n_1} b^{m_1} \cdots a^{n_i} b^{m_i}$$

is a reduced word in a and b, where a^{n_1} and b^{m_i} may be empty. We define a path w_g from x_0 to $g \cdot x_0$ by

$$w_g = \gamma_a^{n_1}\gamma_b^{m_1}\cdots\gamma_a^{n_i}\gamma_b^{m_i},$$

where if $n_i < 0$, then $\gamma_a^{n_i}$ means $\gamma_{a^{-1}}^{-n_i}$, and if $m_i < 0$, then $\gamma_b^{m_i}$ means $\gamma_{b^{-1}}^{-m_i}$.

Proposition 4.3 *Let X be a Gromov-hyperbolic space and G a group acting on X. Suppose $g_1, g_2 \in G$ are hyperbolic isometries of X such that $g_2^{+\infty} \neq g_1^{\pm\infty}$ and $g_2^{-\infty} \neq g_1^{\pm\infty}$. Then there exists $C > 0$, which depends on g_1, g_2, with the following property; for $K > 1$ given, there exists an integer $N \geq 1$ such that (1), (2), (3) and (4) are satisfied for all $n, m \geq N$.*
(1) Set $a = g_1^n$ and $b = g_2^m$. Then the subgroup $F =< a, b >$ is free of rank 2.
(2) For all $g \in F$, w_g is a (K, C)-quasi-geodesic.
(3) For all $1 \neq g \in F$, we have

$$d(x_0, g\cdot x_0) \geq 3C.$$

(4) For all cyclically reduced $g \in F$ and all $1 \leq n < \infty$, we have

$$d(x_0, g^n\cdot x_0) \geq n(d(x_0, g\cdot x_0) - C).$$

Proof. For $g \in G$, let γ_g be some choice of a geodesic from x_0 to $g \cdot x_0$. Without loss of generality, we may assume

$$\gamma_{g^{-1}} = g^{-1}\cdot\gamma_g^{-1}$$

for all $g \in G$. Since $g_2^{+\infty} \neq g_1^{\pm\infty}$ and $g_2^{-\infty} \neq g_1^{\pm\infty}$, there exist $D > 100\delta$ and $N' > 0$ such that if $n_1, n_2 \geq N'$, then

$$|\gamma_{g_1^{\pm n_1}}|, |\gamma_{g_2^{\pm n_2}}| \geq D$$

and

$$d(\gamma_{g_1^{\pm n_1}}(D), \gamma_{g_2^{\pm n_2}}(D)) \geq 10\delta,\ d(\gamma_{g_1^{n_1}}(D), \gamma_{g_1^{-n_1}}(D)) \geq 10\delta,$$
$$d(\gamma_{g_2^{n_2}}(D), \gamma_{g_2^{-n_2}}(D)) \geq 10\delta.$$

For $K > 1$ and $D > 100\delta$, let $l > 0$ be the constant in Lemma 4.2. Set

$$C_1 = L(K, 4C_0, \delta),\ C_0 = L(2, 10D, \delta),\ C = 1 + \max\{2C_1, 4C_0\} > 0,$$

and $l' = \max\{l, l_1(K, 4C_0, C_1, \delta), 3CK + 4C_0\}$.

Then there exists $N > N'$ such that if $n \geq N$, then

$$|\gamma_{g_1^n}|, |\gamma_{g_2^n}| \geq l'.$$

Set

$$a = g_1^n,\ b = g_2^n.$$

Let

$$g = a^{n_1} b^{m_1} \cdots a^{n_i} b^{m_i}$$

be a reduced word in a, b. Note that $\gamma_a^{\pm 1}, \gamma_b^{\pm 1}$ are geodesics with $|\gamma_a^{\pm 1}|, |\gamma_b^{\pm 1}| \geq l' \geq l$. By Lemma 4.2,

$$w_g = \gamma_a^{n_1} \gamma_b^{m_1} \cdots \gamma_a^{n_i} \gamma_b^{m_i}$$

is a $(K, 4C_0)$-quasi-geodesic. Since $C \geq 4C_0$, we obtain (2).

Suppose g is cyclically reduced, then g^n is reduced and $w_g^n = w_{g^n}$. Since w_g^n is a $(K, 4C_0)$-quasi-geodesic, we see that

$$w_g^n \subset N_{C_1}(\gamma_{g^n}),$$

for $C_1 = L(K, 4C_0, \delta)$. For each $i, 1 \leq i \leq n-1$, since $g^i \cdot x_0$ is on w_g^n, we can find points P_i on γ_{g^n} such that

$$d(P_i, g^i \cdot x_0) \leq C_1.$$

Set $P_0 = x_0$ and $P_n = g^n \cdot x_0$. We have

$$d(x_0, g \cdot x_0) - 2C_1 = d(g^i \cdot x_0, g^{i+1} \cdot x_0) - 2C_1 \leq d(P_i, P_{i+1})$$

for all $0 \leq i \leq n-1$. Since $l' \geq l_1(K, 4C_0, C_1, \delta)$, the points $P_0, P_1, \cdots, P_n$ appear on γ_{g^n} in this order. To see this, note that w_g^n is a $(K, 4C_0)$-quasi-geodesic, $d(g^i \cdot x_0, g^{i+1} \cdot x_0) \geq l' \geq l_1$ and $d(P_i, g^i \cdot x_0) \leq C_1$. Having this property on $P_0, P_1, \cdots, P_n$, we obtain

$$d(x_0, g^n \cdot x_0) = \sum_{i=0}^{n-1} d(P_i, P_{i+1}) \geq \sum_{i=0}^{n-1} (d(g^i \cdot x_0, g^{i+1} \cdot x_0) - 2C_1) = n(d(x_0, g \cdot x_0) - 2C_1).$$

Since $C \geq 2C_1$, we obtain (4).

Note that

$$\frac{l' - 4C_0}{K} \geq 3C.$$

If the reduced word g is not empty, then $|w_g| \geq l'$. Since w_g is a $(K, 4C_0)$-quasi-geodesic,

$$d(x_0, g \cdot x_0) \geq \frac{l' - 4C_0}{K} \geq 3C.$$

We obtain (3).

The previous argument implies that any reduced non-empty word in a, b is non-trivial as a group element in G. Therefore $F = < a, b >$ is free of rank 2. This proves (1). We obtain the proposition.

5 Proof of Theorem 1.1

In this section, we assume that G acts on a Gromov-hyperbolic space X and that there are two hyperbolic isometries $k_1, k_2 \in G$ with $k_2^{+\infty} \neq k_1^{\pm\infty}$ and $k_2^{-\infty} \neq k_1^{\pm\infty}$. Then Proposition 4.3 gives us a number $C > 0$ for k_1, k_2.

We state a proposition which is essential for the proof of Theorem 1.1. Note that γ_g denotes a geodesic from x_0 to $g \cdot x_0$ for $g \in G$.

Proposition 5.1 *There exist elements $g_i \in G$, $1 \leq i < \infty$ such that $|\gamma_{g_i}| \geq 3C$ for all $i \geq 1$. It follows that $c_{\gamma_{g_i}, 2C}$ is well-defined as an element of $C^1(G; \mathbb{R})$. Further, the elements g_i satisfy the following properties.*
(1) For all $1 \leq n < \infty$ and all $1 \leq i < \infty$, we have

$$c_{\gamma_{g_i}, 2C}(g_i^n) \geq nC.$$

(2) For all $1 \leq n < \infty$ and all $1 \leq i < \infty$, we have

$$c_{\gamma_{g_i}^{-1}, 2C}(g_i^n) = 0.$$

(3) For all $1 \leq n < \infty$ and all $1 \leq i < j < \infty$, we have

$$c_{\gamma_{g_j}^{\pm 1}, 2C}(g_i^n) = 0.$$

(4) For all $1 \leq i < \infty$,

$$g_i \in [G, G].$$

(5) $\lim_{i \to \infty} |\gamma_{g_i}| = \infty$.

Let $K > 1$ be some constant. Then Proposition 4.3(1) gives us $a, b \in G$. Let $F =< a, b >$. From now we fix F in G throughout this section. For all $1 \neq g \in F$, we have $|\gamma_g| \geq 3C$ by Proposition 4.3(3). It follows that $c_{\gamma_g,2C}$ is well-defined. We will prove Proposition 5.1 later by choosing elements $g_i \neq 1$ from F.

Lemma 5.2 *For all cyclically reduced* $1 \neq g \in F$ *and all* $n \geq 1$*, we have*

$$c_{\gamma_g,2C}(g^n) \geq nC.$$

Proof. By Proposition 4.3(4), for all $1 \leq n < \infty$,

$$d(x_0, g^n \cdot x_0) \geq n(d(x_0, g \cdot x_0) - C).$$

Note that γ_g^n is a path from x_0 to $g^n \cdot x_0$ and $|\gamma_g^n| = n|\gamma_g|$, $|\gamma_g^n|_{\gamma_g} = n$. Thus

$$\begin{aligned} c_{\gamma_g,2C}(g^n) &\geq d(x_0, g^n \cdot x_0) - (|\gamma_g^n| - 2C|\gamma_g^n|_{\gamma_g}) \\ &\geq n(|\gamma_g| - C) - (n|\gamma_g| - 2nC) = nC. \end{aligned}$$

Lemma 5.3 *Let* $1 \neq g \in F$ *and* $n \geq 1$*. Suppose* α *realizes* $c_{\gamma_g,2C}$ *at* γ_{g^n}*. Then* α *is a* $(3, 12C)$*-quasi-geodesic and* $\alpha \subset N_{L_1}(\gamma_{g^n})$*, where* $L_1 = L(3, 12C, \delta)$*.*

Proof. Since α is a realizing path, by Lemma 3.3, α is a $\left(\frac{|\gamma_g|}{|\gamma_g|-2C}, \frac{4C|\gamma_g|}{|\gamma_g|-2C}\right)$-quasi-geodesic. Since $|\gamma_g| \geq 3C$ by Proposition 4.3(3), α is a $(3, 12C)$-quasi-geodesic. Therefore $\alpha \subset N_{L_1}(\gamma_{g^n})$ by Lemma 2.2. The proof is completed.

We denote the word length of $g \in F$ in a, b by $||g||$. For $n \geq 1$, we define

$$\psi(n) = \sharp\{g \in F | \, ||g|| \leq n, ^\exists k \in G \text{ s.t. } k \cdot w_g \subset N_L(\gamma_a^\infty)\}.$$

Lemma 5.4 *There exist* $A, B > 0$ *such that*

$$\psi(n) \leq An + B$$

for all $1 \leq n < \infty$*.*

Proof. Set $S = \{g \in G \,|\, d(x_0, g \cdot x_0) \leq L + |\gamma_a|\}$. Then $|S| < \infty$. Let $S = \{s_1, \cdots, s_I\}$. Suppose $||g|| \leq n$ and $k \cdot w_g \subset N_L(\gamma_a^\infty)$ for some $k \in G$. Then we find $m_1, m_2 \geq 0$ such that $d(a^{m_1} \cdot x_0, k \cdot x_0) \leq L + |\gamma_a|$ and $d(a^{m_2} \cdot x_0, kg \cdot x_0) \leq L + |\gamma_a|$. Thus

$$d(a^{m_1} \cdot x_0, a^{m_2} \cdot x_0) \leq |w_g| + 2L + 2|\gamma_a|.$$

Also, there exist $s_1, s_2 \in S$ such that

$$k^{-1}a^{m_1} = s_1,\ (kg)^{-1}a^{m_2} = s_2.$$

We have $g = s_1 a^{m_1 - m_2} s_2^{-1}$. We now obtain an upper bound for $|m_1 - m_2|$ as follows. Note that there exists $Q > 0$ such that $||g||/Q \leq |w_g| \leq Q||g||$ for all $g \in G$. Since $\gamma_a^{m_1-m_2}$ is a (K, C)-quasi-geodesic by Proposition 4.3(2), we have

$$\begin{aligned} |m_1 - m_2||\gamma_a| &= |\gamma_a^{m_1-m_2}| \leq Kd(a^{m_1} \cdot x_0, a^{m_2} \cdot x_0) + C \\ &\leq K(|w_g| + 2L + 2|\gamma_a|) + C \leq KQ||g|| + 2LK + 2K|\gamma_a| + C \\ &\leq KQn + (2LK + 2K|\gamma_a| + C). \end{aligned}$$

Therefore

$$\psi(n) \leq I^2 \sup\{|m_1 - m_2|\} \leq \frac{I^2}{|\gamma_a|}(KQn + 2LK + 2K|\gamma_a| + C).$$

This is the desired inequality. We obtain the lemma.

Let

$$\phi(n) = \sharp\{g \in F \,|\, ||g|| = n,\ g \text{ starts and ends with } b\}.$$

The following lemma is clear.

Lemma 5.5 *There exists $E > 0$ such that*

$$\phi(n) \geq 2^n - E$$

for all $1 \leq n < \infty$.

Lemma 5.6 *Let $L > 0$ and $l > 0$. There exists an element $h \in F =< a, b >$ with the following properties.*
(1) h starts and ends with the letter b.
(2) There is no $g \in G$ such that $g \cdot w_h \subset N_L(\gamma_a^\infty)$.
(3) $|w_h| \geq l$.

Proof. By Lemmas 5.4 and 5.5, we find n_0 such that $\psi(n) < \phi(n)$ for all $n \geq n_0$. Set $n_1 = \max\{n_0, Ql\}$. Then since $\psi(n_1) < \phi(n_1)$, there exists an element $h \in F$ satisfying (1), (2) and $||h|| = n_1$, hence $l \leq n_1/Q \leq |w_h|$. We obtain the lemma.

Set

$$L_1 = L(3, 12C + 3, \delta),\ L_2 = L(K, C, \delta),\ L_3 = L_1 + 2L_2.$$

We choose a number $l_0 > 0$ as follows. Let $I, J \geq 1$ be integers.

Let $h \in F$ be a reduced word which starts and ends with the letter b. We define $g_i \in F$, $1 \leq i < \infty$ by

$$g_i = a^{IJ^i}ha^{I^2J^i}ha^{I^3J^i}ha^{-IJ^i}h^{-1}a^{-I^2J^i}h^{-1}a^{-I^3J^i}h^{-1}. \tag{1}$$

By definition,

$$w_{g_i} = \gamma_a^{IJ^i}w_h\gamma_a^{I^2J^i}w_h\gamma_a^{I^3J^i}w_h\gamma_a^{-IJ^i}w_h^{-1}\gamma_a^{-I^2J^i}w_h^{-1}\gamma_a^{-I^3J^i}w_h^{-1}.$$

Since h starts and ends with b, g_i is reduced and cyclically reduced for all $i \geq 1$. Note that w_{g_i} is a (K, C)-quasi-geodesic by Proposition 4.3. Therefore there exists $l_0 > 0$ such that if $|w_h| \geq l_0$, then

$$N_{L_3}(\gamma_a^{IJ^i}), N_{L_3}(\gamma_a^{I^2J^i}), \cdots, N_{L_3}(\gamma_a^{-I^3J^i})$$

are disjoint from each other for all $i, I, J \geq 1$.

Letting $l = l_0, L = L_3$ in Lemma 5.6, we obtain an element $h \in F$. For this h we obtain elements $g_i \in F$, $1 \leq i < \infty$ by (1).

Lemma 5.7 $\lim_{i\to\infty} |[x_0, g_i \cdot x_0]| = \infty$.

Proof. First note that $\lim_i |w_{g_i}| = \infty$. Since w_{g_i} is a (K, C)-quasi-geodesic from x_0 to $g_i \cdot x_0$, we obtain $\lim_i |[x_0, g_i \cdot x_0]| = \infty$.

Proposition 5.8 *There exists I_0 such that if $I > I_0$, then for all $1 \leq i < \infty$, all $1 \leq n < \infty$ and all $J \geq 1$, we have*

$$c_{\gamma_{g_i}^{-1}, 2C}(g_i^n) = 0.$$

Proof. We argue by contradiction. Suppose there exists arbitrary large $I \geq 1$ such that for some $J \geq 1$, $i \geq 1$ and $n \geq 1$, we have

$$c_{\gamma_{g_i}^{-1},2C}(g_i^n) > 0.$$

Note that by Lemma 5.3, there exists a $(3, 12C)$-quasi-geodesic α from $g_i^n \cdot x_0$ to x_0 such that

$$|\alpha|_{\gamma_{g_i}} > 0.$$

This means there is a $g \in G$ such that

$$g \cdot \gamma_{g_i} \subset \alpha \subset N_{L_1}(\gamma_{g_i^n}).$$

Since w_{g_i} is a (K, C)-quasi-geodesic by Proposition 4.3, we have

$$g \cdot w_{g_i} \subset N_{L_2}(g \cdot \gamma_{g_i}),\ \gamma_{g_i^n} \subset N_{L_2}(w_{g_i}^n).$$

This implies

$$g \cdot w_{g_i} \subset N_{L_3}(w_{g_i}^n),$$

since $L_1 + 2L_2 = L_3$.

We label $g \cdot w_{g_i}$ as

$$g \cdot w_{g_i} = A_1H_1A_2H_2A_3H_3A_4H_4A_5H_5A_6H_6$$

in the obvious way such that $A_m, 1 \leq m \leq 6$ are copies of powers of $\gamma_a^{\pm 1}$ and $H_m, 1 \leq m \leq 6$ are copies of $w_h^{\pm 1}$.

Let P be the finishing point of A_6, Q the starting point of A_6 and R the finishing point of A_5. Let n_0 be an integer with $0 \leq n_0 \leq n-1$ such that

$$P \in N_{L_3}(g_i^{n_0} \cdot w_{g_i}).$$

We also label $g_i^{n_0} \cdot w_{g_i}$ as

$$g_i^{n_0} \cdot w_{g_i} = A_1'H_1'A_2'H_2'A_3'H_3'A_4'H_4'A_5'H_5'A_6'H_6'.$$

Set $L_4 = L_3 + |w_h|$, $L_5 = K(2L_3 + |w_h|) + C + L_3$. We state two lemmas.

Lemma 5.9 *$P \in N_{L_4}(H_m')$ for some $1 \leq m \leq 6$.*

Proof. Suppose $P \notin N_{L_4}(H'_m)$ for all m. Then $H_6 \cap N_{L_3}(H'_m) = \emptyset$ for all m since $|H_6| = |w_h|$. Thus $H_6 \subset N_{L_3}(\cup_{m=1}^{6} A'_m)$. Since we chose h so that $N_{L_3}(A'_m), 1 \le m \le 6$ are disjoint from each other, we find $H_6 \subset N_{L_3}(A'_m)$ for some m. This implies $g \cdot w_h \subset N_{L_3}(\gamma_a^\infty)$ for some $g \in G$, which contradicts the choice of h. We obtain the lemma.

Lemma 5.10 *There exists I_0 such that if $P \in N_{L_4}(H'_m)$ for some $1 \le m \le 6$, some i and some J, then $I \le I_0$.*

Proof. To argue by contradiction, suppose for some $m, 1 \le m \le 6$, there exists arbitrary large I such that $P \in N_{L_4}(H'_m)$ for some i and J. Note that for all $m, 1 \le m \le 6$, we have

$$\lim_{I \to \infty} |A_m| = \infty$$

uniformly for i and J. Also if $A_m, 1 \le m \le 6$ are sufficiently long, then they travel in the negative direction along $w_{g_i}^n$ in the L_3-neighborhood of $w_{g_i}^n$ since α is a quasi-geodesic from $g_i^n \cdot x_0$ to x_0. Therefore, in the following argument, we may assume
(1) I is very big compared to K and C.
(2) $A_1, \cdots, A_6$ are very long compared to $|w_h|, L_4$ and L_5, for all i, J.
(3) $A_1, \cdots, A_6$ travel in the negative direction along $w_{g_i}^n$ in $N_{L_3}(w_{g_i}^n)$ for all i, J.

Suppose $m = 1$. Then since A_6 and A'_2 are very long compared to $|w_h|$ and L_4, the finishing point of A_6 and the starting point of A'_2 are very close compared to the length of A_6 and A'_2. Note that $|A_6| = I|A'_2|$, and both paths are (K, C)-quasi-geodesics. Since I is very big compared to K, C, we have $H'_2 \subset N_{L_3}(A_6)$ — see Figure 2. But $H'_2 \subset N_{L_3}(A_6)$ is impossible. Thus $m \ne 1$.

Suppose $m = 2$. Since A_6 is very long and travels in the negative direction along $w_{g_i}^n$, we have

$$Q \in N_{L_3}(A'_3 H'_3 A'_4 H'_4 \cdots).$$

We divide A'_4 into two at the middle point and write

$$A'_4 = A'_{41} A'_{42}.$$

We have five cases according to the position of Q as follows. Note that they are not mutually exclusive.

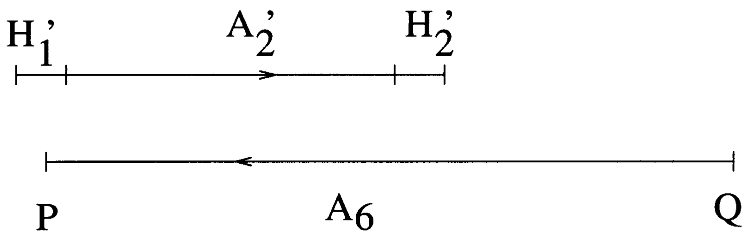

Figure 2: We see $H_2' \subset N_{L_3}(A_6)$.

(i) $Q \in N_{L_3}(A_3')$,
(ii) $Q \in N_{L_3}(H_3')$,
(iii) $Q \in N_{L_3}(A_{41}')$,
(iv) $Q \in N_{L_3}(A_{42}')$,
(v) $Q \in N_{L_3}(H_4' A_5' \cdots)$.

Suppose (i) holds. Choose a point $Q' \in A_3'$ such that $d(Q, Q') \leq L_3$. Since $H_5 \not\subset N_{L_3}(A_3')$, there exists a point $S \in H_5$ such that

$$S \notin N_{L_3}(A_3').$$

Choose $S' \in w_{g_i}^n$ with $d(S, S') \leq L_3$. Then since A_3' and A_6 are very long compared to the distance between the finishing point of A_3' and the starting point of A_6, we have

$$S' \in A_3' H_3' A_4' H_4' \cdots.$$

But since $S' \notin A_3'$, otherwise we would have $S \in N_{L_3}(A_3')$, we find

$$S' \in H_3' A_4' H_4' \cdots.$$

Also,

$$d(Q', S') \leq 2L_3 + d(Q, S) \leq 2L_3 + |w_h|.$$

Note that Q' and S' are on the path $w_{g_i}^n$, which is a (K, C)-quasi-geodesic. Let $Q' = w_{g_i}^n(q)$, $S' = w_{g_i}^n(s)$. We have

$$|q - s| \leq K d(Q', S') + C \leq K(2L_3 + |w_h|) + C.$$

Let T' be the finishing point of A_3'. Then since $S' \in H_3' A_4' H_4' \cdots$, we see

$$d(T', Q) \leq |q - s| + d(Q, Q') \leq K(2L_3 + |w_h|) + C + L_3 = L_5.$$

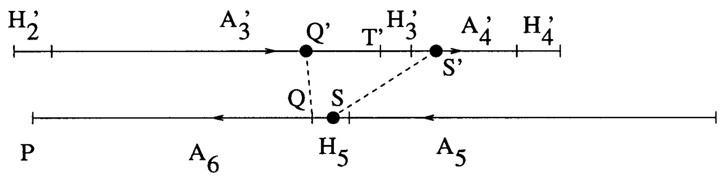

Figure 3: We see $H'_4 \subset N_{L_3}(A_5)$.

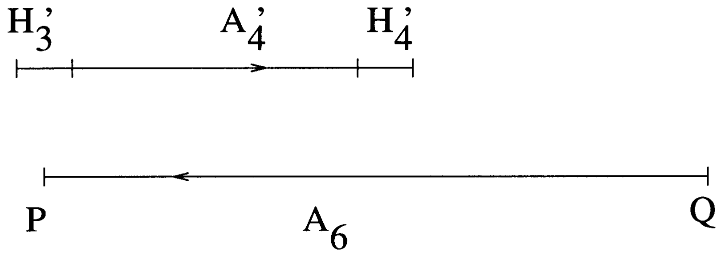

Figure 4: We see $H'_4 \subset N_{L_3}(A_6)$.

Since A'_4 and A_5 are very long compared to L_5, the starting point of A'_4 and the finishing point of A_5 are very close compared to the lengths of A'_4 and A_5. Thus, since $|A_5| = I|A'_4|$, we have $H'_4 \subset N_{L_3}(A_5)$, which is impossible, — see Figure 3.

A similar argument applies to the other cases. Suppose (ii) holds. Then since I is very big, we have $H'_4 \subset N_{L_3}(A_5)$, but this is impossible — see Figure 4.

Suppose (iii) holds. Then $H'_4 \subset N_{L_3}(A_5)$, but this is impossible. Suppose (iv) holds. Then $H'_3 \subset N_{L_3}(A_6)$, but this is impossible. Suppose (v) holds. Then $H'_3 \subset N_{L_3}(A_6)$, but this is impossible.

Therefore, we find contradictions for all cases (i), ..., (v). Thus $m \neq 2$.

Suppose $m = 3$. First note that A'_4 and A_6 are very long compared to H'_3. Since A'_4 and A_6 are (K, C)-quasi-geodesic, and $|A_6| = I^2|A'_4|$, it must be that $H'_4 \subset N_{L_3}(A_6)$ — see Figure 4. This is impossible. Thus $m \neq 3$.

Similarly, we can show $m \neq 4, 5, 6$ as well. We obtain $m \neq 1, 2, 3, 4, 5, 6$, which is a contradiction. We showed Lemma 5.10.

We resume the proof of Proposition 5.8. If $I > I_0$, then Lemmas 5.9 and 5.10 contradict. Thus we conclude that if $I > I_0$, then

$$c_{\gamma_{g_i}^{-1},2C}(g_i^n) = 0$$

for all i, n, J. We obtained Proposition 5.8.

Proposition 5.11 *There exists J_0 such that if $J > J_0$, then*

$$c_{\gamma_{g_j}^{\pm 1},2C}(g_i^n) = 0,$$

for all $I \geq 1$, all $1 \leq i < j < \infty$ and all $1 \leq n < \infty$.

Proof. First, we will show $c_{\gamma_{g_j},2C}(g_i^n) = 0$. To argue by contradiction, suppose there exists arbitrary large $J \geq 1$, such that for some I, some $1 \leq i < j$ and some n, we have

$$c_{\gamma_{g_j},2C}(g_i^n) > 0.$$

Then by Lemma 5.3 and $|\gamma_{g_j}| \geq 3C$, there is a $(3, 12C)$-quasi-geodesic α from x_0 to $g_i^n \cdot x_0$ such that

$$|\alpha|_{\gamma_{g_j}} > 0.$$

Thus we find $g \in G$ such that

$$g \cdot \gamma_{g_j} \subset \alpha \subset N_{L_1}(\gamma_{g_i^n}),\ \gamma_{g_i^n} \subset N_{L_1}(\alpha).$$

We replace the subpath $g \cdot \gamma_{g_j}$ of α by $g \cdot w_{g_j}$ and get a new path β. Clearly β is a (K', C')-quasi-geodesic where K', C' are some constants depending only on C, K, δ since α is a $(3, 12C)$-quasi-geodesic and $g \cdot w_{g_i}$ is a (K, C)-quasi-geodesic.

Since

$$g \cdot w_{g_j} \subset N_{L_2}(g \cdot \gamma_{g_j}),\ g \cdot \gamma_{g_j} \subset N_{L_2}(g \cdot w_{g_j}),$$

we have

$$\alpha \subset N_{L_2}(\beta),\ \beta \subset N_{L_2}(\alpha).$$

Also,

$$\gamma_{g_i^n} \subset N_{L_2}(w_{g_i}^n),\ w_{g_i}^n \subset N_{L_2}(\gamma_{g_i^n}).$$

Therefore we have

$$g \cdot w_{g_j} \subset \beta \subset N_{L_3}(w_{g_i}^n),\ w_{g_i}^n \subset N_{L_3}(\beta).$$

Since

$$|w_{g_j}| = 6|w_h| + 2(I + I^2 + I^3)J^i|\gamma_a|,\ |\gamma_a^{IJ^j}| = IJ^j|\gamma_a|,$$

we get

$$\lim_{J\to\infty} \frac{|\gamma_a^{IJ^j}|}{|w_{g_i}|} = \infty,$$

uniformly for all $i < j$ and $I \geq 1$. Therefore, since β is a (K', C')-quasi-geodesic, there exists J_1 such that if $J > J_1$, then the L_3-neighborhood of each copy of $\gamma_a^{IJ^j}$ in $g \cdot w_{g_j}$ (in β) conatains a copy of w_{g_i} in $w_{g_i}^n$ for all $i < j$.

This implies that there is $g' \in G$ such that

$$g' \cdot w_{g_i} \subset N_{L_3}(\gamma_a^{IJ^j}).$$

Since $g' \cdot w_{g_i}$ contains a copy of w_h, there is $g'' \in G$ such that

$$g'' \cdot w_h \subset N_{L_3}(\gamma_a^{\infty}).$$

This contradicts the choice of h. Now if $J > J_1$, then we have

$$c_{\gamma_{g_j},2C}(g_i^n) = 0$$

for all $1 \leq i < j < \infty$ and all $1 \leq n < \infty$. Similarly, there exists J_1' such that if $J > J_1'$, then we have

$$c_{\gamma_{g_j^{-1}},2C}(g_i^n) = 0$$

for all $1 \leq i < j < \infty$ and all $1 \leq n < \infty$. Setting

$$J_0 = \max\{J_1, J_1'\},$$

we obtain Proposition 5.11.

Proof of Proposition 5.1. Let $K > 1$, then we have a subgroup F by Proposition 4.3. We get a constant l_0 and then $h \in F$ by Lemma 5.6 as in the previous argument. Propositions 5.8 and 5.11 give us I_0 and J_0, respectively. Then we define elements $g_i, 1 \leq i < \infty$ for I_0 and J_0. By Lemma 5.2, we have (1). Proposition 5.8 implies (2). The property (3) follows from Proposition 5.11. We see $g_i \in [G, G]$ for all i, which is (4). Lemma 5.7 means (5).

Proposition 5.12 *There exist elements $g_i \in G$, $1 \leq i < \infty$ such that $|\gamma_{g_i}| \geq 3C$ for all $i \geq 1$, which implies that $h_{\gamma_{g_i},2C}$ is well-defined. Further, they satisfy the following properties.*
(1) For all $1 \leq i < \infty$ and $1 \leq n < \infty$,

$$h_{\gamma_{g_i},2C}(g_i^n) \geq nC.$$

(2) For all $1 \leq i < j < \infty$ and $1 \leq n < \infty$,

$$h_{\gamma_{g_j},2C}(g_i^n) = 0.$$

(3) For all $1 \leq i < \infty$, $1 \leq n < \infty$ and all homomorphisms $\phi : G \to \mathbb{R}$,

$$\phi(g_i^n) = 0.$$

(4) $\lim_{i\to\infty} |\gamma_{g_i}| = \infty$.

Proof. Proposition 5.1 gives us $g_i \in F$, $1 \leq i < \infty$ which satisfies all the properties in Proposition 5.12.
Proof of Theorem 1.1. By Lemma 2.5, we have two hyperbolic isometries k_1 and k_2 such that $k_2^{+\infty} \neq k_1^{\pm\infty}$ and $k_2^{-\infty} \neq k_1^{\pm\infty}$. Then Proposition 5.12 gives us elements $g_i \in G$ for $1 \leq i < \infty$. First note that $|\gamma_{g_i}| \geq 3C$ for all i. Therefore $h_{\gamma_{g_i},2C}$ is well-defined in $C^1(G;\mathbb{R})$. Taking the integer part, we define

$$h_i = [h_{\gamma_{g_i},2C}] \in C^1(G;\mathbb{Z}).$$

For each $g \in G$, there are only finitely many elements h_i such that $h_i(g) \neq 0$. This follows from Lemma 3.9 and Proposition 5.12(4). Therefore, if $(a_i)_i \in l^1$, then $\sum_{i=1}^{\infty} a_i h_i$ is also well-defined as an element of $C^1(G;\mathbb{R})$ since this is in fact a finite sum for each $g \in G$. Also, for the same reason, $\sum_{i=1}^{\infty} a_i \delta h_i$ is a well-defined cocycle, and the following equality holds.

$$\delta\left(\sum_{i=1}^{\infty} a_i h_i\right) = \sum_{i=1}^{\infty} a_i \delta h_i.$$

By Proposition 3.10, for all $1 \leq i < \infty$, we have

$$|\delta h_i| \leq 12L_1 + 12C + 48\delta + 3,$$

where $L_1 = L(3, 12C, \delta)$. Note that 3 occurs in the above inequality since we took the integer part to define h_i. It follows that if $(a_i) \in l^1$, then $\sum_i a_i \delta h_i$ is also a well-defined bounded cocycle. We get a real linear map

$$\omega : l^1 \to H_b^2(G; \mathbb{R}),$$

which sends (a_i) to the cohomology class of $\sum_i a_i \delta h_i$. In order to see that ω is injective, suppose $\omega((a_i)) = 0$. Then

$$\delta(\sum_{i=1}^{\infty} a_i h_i) = \delta b$$

for some $b \in C_b^1(G; \mathbb{R})$. This means

$$\sum_{i=1}^{\infty} a_i h_i - b = \psi$$

for some homomorphism $\psi : G \to \mathbb{R}$. Applying this to $g_1^n \in G$, we find

$$a_1 h_1(g_1^n) - b(g_1^n) = \psi(g_1^n) = 0$$

for all $n \geq 0$. Since b is bounded and $h_1(g_1^n) \geq nC - 1$, we get $a_1 = 0$. By induction on i, a similar argument shows $a_i = 0$ for each i. We obtain the proposition.

6 Proofs of the other results

Proposition 6.1 *Let M be a complete Riemannian manifold. Suppose there exist $a > b > 0$ such that $-a \leq K \leq -b < 0$, where K is the sectional curvature. Let $G = \pi_1(M)$. Then we have the following.*
(1) If $|L(G)| = 0$, then $G = \{e\}$.
(2) If $|L(G)| = 1$, then there exists a nilpotent subgroup H of G with finite index.
(3) If $|L(G)| = 2$, then $G \simeq \mathbb{Z}$.

Proof. (1) Note that since M is locally compact, $|L(G)| = 0$ implies G is finite. It is well known that $\pi_1(M)$ is torsion free, [8]. Thus $G = \{e\}$.
(2) Since $|L(G)| = 1$, all elements are parabolic with a common fixed point. Then it is known that G is an almost nilpotent group, [8].

(3) Since $|L(G)| = 2$, all elements of G are hyperbolic. They have a common axis. Since the action is properly discontinuous, we get $G \simeq \mathbb{Z}$.

Proof of Corollary 1.2. By Proposition 6.1, if $|L(G)| \leq 2$, then G is almost nilpotent. Thus if G is not almost nilpotent, then $|L(G)| \geq 3$. Apply Theorem 1.1.

Proof of Corollary 1.3. By Proposition 6.1, it is easy to see that if $|L(G)| \leq 2$, then the volume of M is infinite.

Proof of Corollary 1.4. Let Γ be a Cayley graph of G. Then G acts on Γ properly discontinuously by isometries, and hence so does H. Since G is word-hyperbolic, Γ is a locally compact, Gromov-hyperbolic space. It is well known that there is no parabolic isometries in G, [13]. Thus $|L(H)| = 0, 2,$ or ∞. It is easy to see $|L(H)| = 0$ if and only if $|H| < \infty$ and $|L(H)| = 2$ if and only if H contains $\mathbb{Z}$ with finite index. Therefore, $|L(H)| \leq 2$ implies that H is elementary.

We shall prove Theorem 1.6. Let K be a knot and $G = \pi_1(S^3 \backslash K)$. Suppose $G \not\simeq \mathbb{Z}$, in other words, the knot is non-trivial. By the classification of knots, there are two types of knots, namely, prime knots and non-prime knots. The prime knots are of three types, which are called hyperbolic type, torus type, and satellite type. Group theoretically, we have the following classification. Note that these cases are not mutually exclusive.

Case 1(hyperbolic knot) G is the fundamental group of a hyperbolic manifold M with one cusp. Note that $M = S^3 \backslash K$ and the volume of M is finite.

Case 2(torus knot) G has an epimorphism to $\mathbb{Z}_m * \mathbb{Z}_n$ such that $m, n \geq 2$ and $(m, n) = 1$;

$$G \twoheadrightarrow \mathbb{Z}_m * \mathbb{Z}_n.$$

Case 3(satellite knot) G can be written as $G = A *_{\mathbb{Z}\times\mathbb{Z}} B$ such that $|B/(\mathbb{Z} \times \mathbb{Z})| \geq 2$ and A is either of Case 1 or 2.

Case 4(non-prime knot) G can be written as $G = A *_{\mathbb{Z}} B$ such that $|B/\mathbb{Z}| \geq 2$ and A is either of Case 1, 2, or 3. Note that there are epimorphisms from G to each of A and B.

We need the following results.

Theorem 6.2 (**[3]**) *Let $\phi : G_1 \to G_2$ be a homomorphism and let*

$$\phi_* : H_b^2(G_2; \mathbb{R}) \to H_b^2(G_1; \mathbb{R})$$

be the induced map. If ϕ is surjective, then ϕ_* is injective.

It is known in the previous theorem that if the kernel of ϕ is amenable, then ϕ_* is isomorphic — Section 3.1 in [12], [14].

Theorem 6.3 **([10])** *Let $G = A *_C B$. If $|C\backslash A/C| \geq 3$ and $|B/C| \geq 2$, then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G;\mathbb{R})$ such that, for each $i(1 \leq i < \infty)$, $\omega(e_i)$ is the image of a class in $H_b^2(G;\mathbb{Z})$. In particular, the dimension of $H_b^2(G;\mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

Proposition 6.4 *Let G be a non-elementary word-hyperbolic group and let $\phi : \mathbb{Z} \times \mathbb{Z} \to G$ be a homomorphism. Set $H = \phi(\mathbb{Z} \times \mathbb{Z})$. Then*

$$|H\backslash G/H| = \infty.$$

Proof. Let $\mathbb{Z} \times \mathbb{Z} = \{a, b |\, ab = ba\}$ and $\phi(a) = h_1, \phi(b) = h_2$. If both h_1, h_2 are torsion, then $|H| < \infty$, and clearly, $|H\backslash G/H| = \infty$. Suppose either h_1 or h_2 is not torsion. Without loss of generality, we may assume h_1 is not torsion. Then, h_1 acts on the Cayley graph of G as a hyperbolic isometry. Set

$$x = h_1^{-\infty},\ y = h_1^{\infty} \in \partial G.$$

Since h_1 and h_2 commute, $h_2 \cdot x = x$, and hence $H \cdot x = x$. Note that for all $p \in \partial G$, since only x and y can be the cluster points for the action of H on ∂G, we get

$$\overline{H \cdot p} \subset H \cdot p \cup x \cup y.$$

To argue by contradiction, suppose $|H\backslash G/H| = n < \infty$. Then there exist $g_1, \cdots, g_n \in G$ such that

$$G = \coprod_{i=1}^{n} H g_i H.$$

Then for all i, since $H \cdot x = x$,

$$\overline{H g_i H \cdot x} = \overline{H g_i \cdot x} \subset (H g_i \cdot x) \cup x \cup y.$$

Therefore,

$$\overline{G \cdot x} = \overline{\coprod_{i=1}^{n} H g_i H \cdot x} = \sum_{i=1}^{n} \overline{H g_i H \cdot x} \subset \left(\sum_{i=1}^{n} H g_i \cdot x \right) \cup x \cup y.$$

This shows $\overline{G \cdot x}$ is a countable set. But, it is well-known that $\overline{G \cdot x}$ is an uncountable set for a non-elementary word-hyperbolic group G. We get a contradiction. Therefore $|H\backslash G/H| = \infty$.

Proposition 6.5 *Let M be a three-dimensional, complete hyperbolic manifold with finite volume. Suppose M has one cusp. Let $G = \pi_1(M)$ and $H \simeq \mathbb{Z} \times \mathbb{Z}$ be the cusp group. Then*

$$|H\backslash G/H| = \infty.$$

Proof. Note that there exists $x \in \partial\mathbb{H}^3$ such that H fixes x. To argue by contradiction, suppose $|H\backslash G/H| < \infty$. Then there exist $g_1, \cdots g_n \in G$ such that

$$G = \coprod_{i=1}^{n} Hg_iH.$$

Note that $\overline{Hg_i \cdot x} = (Hg_i \cdot x) \cup x$ for all i. Therefore,

$$\begin{aligned} \overline{G \cdot x} &= \overline{\coprod_{i=1}^{n} Hg_iH \cdot x} = \sum_{i=1}^{n} \overline{Hg_iH \cdot x} \\ &= \sum_{i=1}^{n} \overline{Hg_i \cdot x} = \left(\sum_{i=1}^{n} Hg_i \cdot x\right) \cup x. \end{aligned}$$

This implies $\overline{G \cdot x}$ is a countable set. But, since $\mathbb{H}^3/G$ has finite volume, $\overline{G \cdot x} = S^2$, which is an uncountable set. This is a contradiction. We get $|H\backslash G/H| = \infty$.

Proof of Theorem 1.6. For Case 1, the desired conclusion follows from Corollary 1.3.

For Case 2, note that $\mathbb{Z}_m * \mathbb{Z}_n$ is non-elementary word-hyperbolic since $m, n \geq 2$ and $(m,n) = 1$. Then the conclusion follows from Corollary 1.5 and Theorem 6.2. The same argument was given in section 5 of [11].

For Case 3, if A is of hyperbolic type, then by Proposition 6.5, we see

$$|(\mathbb{Z} \times \mathbb{Z})\backslash A/(\mathbb{Z} \times \mathbb{Z})| = \infty.$$

Since $|B/(\mathbb{Z} \times \mathbb{Z})| \geq 2$, by Theorem 6.3 we have the conclusion. If A is of torus type, a similar argument using Proposition 6.4 gives the conclusion.

For Case 4, since there is an epimorphism from $G = A *_{\mathbb{Z}} B$ to A, by Theorem 6.2, $H_b^2(A;\mathbb{R})$ injects to $H_b^2(G;\mathbb{R})$. We already know that the conclusion holds for $H_b^2(A;\mathbb{R})$ since A is in one of the three previous cases. This implies that the conclusion holds for $H_b^2(G;\mathbb{R})$ as well. We finished the proof of Theorem 1.6.

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