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By

Makoto Maejima and Ken-iti Sato

Makoto Maejima
Department of Mathematics
Keio University

Ken-iti Sato

Department of Mathematics
Keio University

Hiyoshi 3-14-1, Kohoku-ku, Yokohama, 223 Japan

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Semi-selfsimilar processes

Makoto Maejima[1] and Ken-iti Sato[2]

Abstract. A notion of semi-selfsimilarity of $\mathbf{R}^d$-valued stochastic processes is introduced as a natural extension of the selfsimilarity. Several topics on semi-selfsimilar processes are studied: 1. The existence of the exponent for semi-selfsimilar processes. 2. Characterization of semi-selfsimilar processes as scaling limits. 3. Relationship between semi-selfsimilar processes with independent increments and semi-selfdecomposable distributions, and examples. 4. Construction of semi-selfsimilar processes with stationary increments. 5. Extension of the Lamperti transformation. Semi-stable processes where all joint distributions are multivariate semi-stable are also discussed in connection with semi-selfsimilar processes. A wide-sense semi-selfsimilarity is defined and shown to be reduced to semi-selfsimilarity.

KEY WORDS: Selfsimilar process; semi-selfsimilar process; semi-stable process; Lévy process.

1. INTRODUCTION

An $\mathbf{R}^d$-valued Lévy process $\{X(t), t \geq 0\}$ with α-semi-stable marginal distribution at each t is, in general, not selfsimilar, but it has the following property: For some $a \in (0,1) \cup (1,\infty)$ and $c \in \mathbf{R}^d$,

$$\{X(at), t \geq 0\} \stackrel{d}{=} \{a^{1/\alpha} X(t) + tc, t \geq 0\}, \tag{1.1}$$

[1]Department of Mathematics, Keio University, Hiyoshi, Yokohama 223, Japan.

[2]Hachimanyama 1101-5-103, Tenpaku-ku, Nagoya 468, Japan.

where $\stackrel{d}{=}$ denotes the equality in all finite-dimensional distributions. Here, by a Lévy process, we mean a stochastically continuous process starting at the origin with independent and stationary increments.

Being motivated by the property (1.1), we introduce a new notion of *semi-selfsimilarity* as follows.

Definition 1.1. An $\mathbf{R}^d$-valued stochastic process $\{X(t), t \geq 0\}$ is said to be *semi-selfsimilar* if there exist $a \in (0,1) \cup (1,\infty)$ and $b > 0$ such that

$$\{X(at), t \geq 0\} \stackrel{d}{=} \{bX(t), t \geq 0\}. \tag{1.2}$$

If

$$\{X(at), t \geq 0\} \stackrel{d}{=} \{bX(t) + c(t), t \geq 0\} \tag{1.3}$$

for some $a \in (0,1) \cup (1,\infty)$, $b > 0$, and a nonrandom function $c : [0,\infty) \to \mathbf{R}^d$, then $\{X(t)\}$ is said to be *wide-sense semi-selfsimilar*.

Recall that $\{X(t)\}$ is said to be *selfsimilar* if, for every $a > 0$, there is $b > 0$ satisfying (1.2) and that it is said to be *wide-sense selfsimilar* if, for every $a > 0$, there are $b > 0$ and $c : [0,\infty) \to \mathbf{R}^d$ satisfying (1.3). Thus the notion of semi-selfsimilarity extends that of selfsimilarity, and semi-selfsimilar processes will be characterized as limiting processes of some subsequences of normalized processes (Theorem 2.1 below), whereas limiting processes of (full sequences of) normalized processes are selfsimilar. Besides semi-stable Lévy processes, processes with property (1.2) are found in the literature about diffusions on Sierpinski gaskets (Example 3.1 below).

A lot of works on selfsimilar processes have been done for last two decades. (See, e.g., Taqqu(17) , Maejima(9), and Samorodnitsky and Taqqu(12).) However, as far as the authors know, the notion of semi-selfsimilarity in the sense of (1.2) or (1.3) has not been well recognized.

The contents of this paper are the following. In the case of selfsimilar processes, the existence of the exponent of the selfsimilarity is known (Lamperti(8)). In the next Section 2, we prove that the unique exponent exists also for (wide-sense) semi-selfsimilar processes, although we are making less assumptions on selfsimilarity. Selfsimilar processes are characterized as limits of the normalized processes $\{b_n^{-1}Y(nt)\}$. In Section 3, we show that semi-selfsimilar processes are given as limits of some kind of subsequences of those processes. Semi-stable Lévy processes are the simplest and the first examples of semi-selfsimilar processes as we mentioned at the beginning of this section. They are characterized, in Section 4, as semi-selfsimilar processes with stationary and independent increments. In Section 5, reduction of general wide-sense semi-selfsimilar processes to semi-selfsimilar processes is shown to be possible. Then, in Section 6, we discuss the class of general semi-selfsimilar processes with independent, but not necessarily stationary, increments. We show that the marginal distributions of such processes are semi-selfdecomposable in the sense introduced by Maejima and Naito[(10)]. We also discuss how to construct such processes, given a set of semi-selfdecomposable distributions as marginal distributions. Section 7 is devoted to examples of semi-selfsimilar processes with independent increments. In Section 8, we consider the so-called Lamperti transformation for semi-selfsimilar processes and discuss the relationship between semi-selfsimilar processes and periodically stationary processes. As a byproduct, we are given the semi-stable Ornstein-Uhlenbeck process. In Section 9, we use stochastic integrals with respect to the random measure induced by a semi-stable Lévy process and give some examples of semi-selfsimilar processes with stationary but not independent increments.

We use the following notation. $\mathcal{P}(\mathbf{R}^d)$ and $I(\mathbf{R}^d)$ are the class of all probability distributions on $\mathbf{R}^d$ and the class of all infinitely divisible distributions on $\mathbf{R}^d$, respectively. $\mathcal{L}(X)$ is the distribution of a random vector X. $\mathcal{B}(\mathbf{R}^d)$ is the class of Borel sets in $\mathbf{R}^d$. $X \overset{d}{\sim} Y$ means that $\mathcal{L}(X) = \mathcal{L}(Y)$. For stochastic processes, $\overset{d}{\Rightarrow}$ denotes the convergence of all finite-dimensional distributions. The characteristic function of a distribution μ is denoted by $\widehat{\mu}(z)$. The distribution concentrated at a point c is denoted by δ_c. The Euclidean inner product of $x, y \in \mathbf{R}^d$ is denoted by $\langle x, y\rangle$ and $|x|$ denotes $\sqrt{\langle x, x\rangle}$. The unit sphere is denoted by $S = \{x \in \mathbf{R}^d \colon |x| = 1\}$.

2. EXISTENCE OF THE EXPONENT

In case $\{X(t)\}$ is (wide-sense) selfsimilar, we know that there exists an H, the exponent of the selfsimilarity, so that b in (1.2) or (1.3) has the form $b = a^H$. (See Lamperti[8].) In this section, we shall prove the same conclusion is also true for wide-sense semi-selfsimilar processes.

Definition 2.1. An $\mathbf{R}^d$-valued random variable X is called *degenerate* if it is a constant a. s. A probability distribution is called degenerate if it is concentrated at a point. An $\mathbf{R}^d$-valued process $\{X(t),\, t \geq 0\}$ is called *trivial* if $X(t)$ is degenerate for every t.

Theorem 2.1. Let $\{X(t),\, t \geq 0\}$ be an $\mathbf{R}^d$-valued, nontrivial, stochastically continuous, wide-sense semi-selfsimilar process. Then the following statements are true.

(i) There exists a unique $H \geq 0$ such that, if $a > 0$, $b > 0$, and a function $c(t)$ satisfy (1.3), then $b = a^H$.

(ii) Let Γ be the set of $a > 0$ such that there are $b > 0$ and a function $c(t)$ satisfying (1.3). Then $\Gamma \cap (1, \infty)$ is nonempty. Denote the infimum of $\Gamma \cap (1, \infty)$ by a_0. If $a_0 > 1$, then $\Gamma = \{{a_0}^n \colon n \in \mathbf{Z}\}$. If $a_0 = 1$, then $\Gamma = (0, \infty)$.

(iii) $X(0) =$ const. a. s. if and only if $H > 0$. There is $h\colon [0, \infty) \to \mathbf{R}^d$ such that $X(t) = X(0) + h(t)$ a. s. if and only if $H = 0$.

Definition 2.2. The real number H is called the *exponent* of the (wide-sense) semi-selfsimilar process. In order to signify it, we call $\{X(t)\}$ (wide-sense) H-semi-selfsimilar. Any $a \in \Gamma \cap (1, \infty)$ is called an *epoch* of $\{X(t)\}$.

It follows from Theorem 2.1 that if $a_0 = 1$, then the (wide-sense) semi-selfsimilar process is (wide-sense) H-selfsimilar.

Let us give a proof of Theorem 2.1. The following lemma is well-known[3,14].

Lemma 2.1. Let X be a nondegenerate random variable on $\mathbf{R}^d$. If $b_1 X + c_1 \overset{d}{\sim} b_2 X + c_2$ with b_1, $b_2 > 0$ and c_1, $c_2 \in \mathbf{R}^d$, then $b_1 = b_2$ and $c_1 = c_2$.

Lemma 2.2. Let $\{X(t),\, t \geq 0\}$ be a nontrivial $\mathbf{R}^d$-valued process. If $a > 0$ satisfies (1.3) with some $b > 0$ and $c(t)$, then b and $c(t)$ are uniquely determined by a.

Proof. Suppose that $\{X(at)\} \stackrel{d}{=} \{b_1 X(t) + c_1(t)\} \stackrel{d}{=} \{b_2 X(t) + c_2(t)\}$. If $X(t)$ is nondegenerate, then we have $b_1 = b_2$ and $c_1(t) = c_2(t)$ for this t by Lemma 2.1. By the nontriviality, such a t exists. Hence $b_1 = b_2$. Now $c_1(t) = c_2(t)$ follows even if $X(t)$ is degenerate at t, because $b_1 X(t) + c_1(t) \stackrel{d}{\sim} b_1 X(t) + c_2(t)$. □

Proof of Theorem 2.1. By Lemma 2.2, b and $c(t)$ in (1.3) are uniquely determined by a. We denote $b = b(a)$ and $c(t) = c(t, a)$. Let us examine the properties of the set Γ. By the definition, Γ contains an element of $(0,1) \cup (1,\infty)$. Obviously $1 \in \Gamma$ and $b(1) = 1$. If $a \in \Gamma$, then $a^{-1} \in \Gamma$ and $b(a^{-1}) = b(a)^{-1}$, because (1.3) is equivalent to

$$\{X(a^{-1}t), t \geq 0\} \stackrel{d}{=} \{b^{-1}X(t) - b^{-1}c(a^{-1}t), t \geq 0\}. \tag{2.1}$$

Hence $\Gamma \cap (1,\infty)$ is nonempty. If a and a' are in Γ, then $aa' \in \Gamma$ and $b(aa') = b(a)b(a')$, because

$$\{X(aa't)\} \stackrel{d}{=} \{b(a)X(a't) + c(a't, a)\} \stackrel{d}{=} \{b(a)b(a')X(t) + c(a't, a) + b(a)c(t, a')\}.$$

Suppose that $a_n \in \Gamma$ $(n = 1, 2, \dots)$ and $a_n \to a$ with $0 < a < \infty$. Let us show that $a \in \Gamma$ and $b(a_n) \to b(a)$. Denote $b_n = b(a_n)$ and $c_n(t) = c(t, a_n)$. Let b_∞ be a limit point of $\{b_n\}$ in $[0,\infty]$. For simplicity, a subsequence of $\{b_n\}$ approaching b_∞ is identified with $\{b_n\}$. Denote $\mu_t = \mathcal{L}(X(t))$. We have

$$\widehat{\mu}_{a_n t}(z) = \widehat{\mu}_t(b_n z) e^{i\langle z, c_n(t)\rangle}, \quad \forall z \in \mathbf{R}^d. \tag{2.2}$$

If $b_\infty = 0$, then, taking the limit of the absolute values, we get $|\widehat{\mu}_{at}(z)| = |\widehat{\mu}_t(0)| = 1$, which shows that $X(at)$ is degenerate for every t, contradicting the assumption of the nontriviality. Hence $b_\infty > 0$. It also follows that $b_\infty < \infty$. In fact, if $b_\infty = \infty$, then $b(a_n^{-1}) = b_n^{-1} \to 0$ with $a_n^{-1} \to a^{-1} > 0$, which contradicts the fact just shown. To each fixed t, there is $\varepsilon > 0$ such that $\widehat{\mu}_t(b_\infty z) \neq 0$ for $|z| \leq \varepsilon$. It follows from (2.2) that $e^{i\langle z, c_n(t)\rangle} \to \widehat{\mu}_{at}(z)/\widehat{\mu}_t(b_\infty z)$ uniformly in z with $|z| \leq \varepsilon$. Hence

$$i\langle z, c_n(t)\rangle = \log(e^{i\langle z, c_n(t)\rangle}) \to \log \frac{\widehat{\mu}_{at}(z)}{\widehat{\mu}_t(b_\infty z)}$$

for $|z| \leq \varepsilon$, where the branches of the logarithms here are taken continuous in z and equal to 0 at $z = 0$. Hence, $c_n(t)$ tends to some $c_\infty(t) \in \mathbf{R}^d$ for each t. Now we have

$\{X(at)\} \stackrel{d}{=} \{b_\infty X(t) + c_\infty(t)\}$. Therefore $a \in \Gamma$ and $b_\infty = b(a)$. This shows that the original sequence $\{b_n\}$ tends to $b(a)$.

We denote the set of $\log a$ with $a \in \Gamma$ by $\log \Gamma$. Then, by the properties that we have proved, $\log \Gamma$ is a closed additive subgroup of $\mathbf{R}$ and $(\log \Gamma) \cap (0, \infty) \neq \emptyset$. Denote the infimum of $(\log \Gamma) \cap (0, \infty)$ by r_0. Then we have:

(1) If $r_0 > 0$, then $\log \Gamma = r_0 \mathbf{Z} = \{r_0 n : n \in \mathbf{Z}\}$.

(2) If $r_0 = 0$, then $\log \Gamma = \mathbf{R}$.

To see (1), let $r_0 > 0$. Then, obviously, $r_0 \mathbf{Z} \subset \log \Gamma$. If there is $r \in (\log \Gamma) \setminus (r_0 \mathbf{Z})$, then $nr_0 < r < (n+1) r_0$ with some $n \in \mathbf{Z}$, and hence $r - nr_0 \in \log \Gamma$ and $0 < r - nr_0 < r_0$, which is a contradiction. To see (2), suppose that $r_0 = 0$ and that there is r in $\mathbf{R} \setminus (\log \Gamma)$. As $\log \Gamma$ is closed, we have that $(r - \varepsilon, r + \varepsilon) \subset \mathbf{R} \setminus (\log \Gamma)$ with some $\varepsilon > 0$. Choose $s \in \log \Gamma$ satisfying $0 < s < 2\varepsilon$. Then $r - \varepsilon < ns < r + \varepsilon$ with some $n \in \mathbf{Z}$, which is absurd. This shows (2). Letting $a_0 = e^{r_0}$, we see that the assertion (ii) of the theorem is proved.

We claim the following.

(3) If $X(0) = \text{const. a.s.}$, then $b(a) > 1$ for any $a \in \Gamma \cap (1, \infty)$.

(4) If $b(a) \neq 1$ for some $a \in \Gamma \cap (1, \infty)$, then $X(0) = \text{const. a.s.}$

(5) If $b(a) = 1$ for some $a \in \Gamma \cap (1, \infty)$, then there is $h(t)$ such that $X(t) = X(0) + h(t)$ a.s.

By induction we see that

$$c(t, a^n) = \sum_{j=0}^{n-1} b(a)^j c(a^{n-1-j} t, a), \qquad n = 1, 2, \ldots. \tag{2.3}$$

Likewise, we see from (2.1) that

$$c(t, a^{-n}) = -\sum_{j=0}^{n-1} b(a)^{-j-1} c(a^{j-n} t, a), \qquad n = 1, 2, \ldots. \tag{2.4}$$

To see (3), suppose that $b(a) \leq 1$ for some $a \in \Gamma \cap (1, \infty)$ and that $X(0) = \text{const.}$ a.s. Fix t. Then $\widehat{\mu}_{a^n t}(z) = \widehat{\mu}_t(b(a)^n z) e^{i\langle z, c(t, a^n)\rangle}$, and hence $|\widehat{\mu}_{a^n t}(b(a)^{-n} z)| = |\widehat{\mu}_t(z)|$ for every $n \in \mathbf{Z}$ and $z \in \mathbf{R}^d$. Since $X(0) = \text{const.}$ a.s., we have $|\widehat{\mu}_{a^n t}(w)| \to 1$ uniformly in w in any compact set as $n \to -\infty$. Hence $|\widehat{\mu}_{a^n t}(b(a)^{-n} z)| \to 1$ as $n \to -\infty$. It follows that $|\widehat{\mu}_t(z)| = 1$, that is, $X(t)$ is degenerate. Since t is arbitrary, this contradicts the nontriviality. This proves (3). To see (4), let $b(a) < 1$ for some a and note that $X(0) \overset{d}{\sim} b(a)^n X(0) + c(0, a^n)$ and $c(0, a^n) = c(0, a) \sum_{j=0}^{n-1} b(a)^j$ by (2.3), which implies that $X(0) = c(0, a)(1 - b(a))^{-1}$ a. s. Similarly, if $b(a) > 1$ for some a, then $X(0) = \text{const.}$ a. s. by (2.4). To prove (5), note that, since $\{X(a^{-n}t)\} \overset{d}{=} \{X(t) + c(t, a^{-n})\}$ by $b(a) = 1$, we have

$$P\{|X(t) + c(t, a^{-n}) - X(0) - c(0, a^{-n})| > \varepsilon\} = P\{|X(a^{-n}t) - X(0)| > \varepsilon\} \to 0$$

as $n \to \infty$. Hence $\lim_{n\to\infty}(c(t, a^{-n}) - c(0, a^{-n}))$ exists $(= -h(t)$, say) and $X(t) = X(0) + h(t)$ a. s.

Now we prove the assertion (i). It follows from (3) and (4) that $b(a) \geq 1$ for $a \in \Gamma \cap (1, \infty)$. Suppose $a_0 > 1$. Let $H = (\log b(a_0))/(\log a_0)$. Then $H \geq 0$. Any a in Γ is written as $a = a_0^n$ with $n \in \mathbf{Z}$. Hence $b(a) = b(a_0)^n$. It follows that $\log b(a) = n \log b(a_0) = nH \log a_0 = H \log a$, that is, $b(a) = a^H$. In case $a_0 = 1$, we have $\Gamma = (0, \infty)$ and there exists $H \geq 0$ satisfying $b(a) = a^H$, since $b(a)$ is continuous and satisfies $b(aa') = b(a)b(a')$.

The assertion (iii) is a consequence of (3), (4), and (5). Proof of Theorem 2.1 is now complete. $\square$

3. CONNECTION TO SCALING LIMITS

As is well known, selfsimilar processes are closely related to scaling limits of stochastic processes. We shall show that semi-selfsimilar processes are characterized as limiting processes of some subsequences of the usually normalized processes.

Theorem 3.1. (i) Suppose that an $\mathbf{R}^d$-valued process $\{X(t), t \geq 0\}$ is stochastically continuous at $t = 0$. Suppose that there exist another $\mathbf{R}^d$-valued process $\{Y(t), t \geq 0\}$, $0 < b_n \uparrow \infty$, $0 < a_n \uparrow \infty$, and $c_n \colon [0, \infty) \to \mathbf{R}^d$ such that, for some $a > 1$,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a, \tag{3.1}$$

$$(3.2)\qquad \frac{1}{b_n}\{Y(a_{n+1}t) - Y(a\cdot a_n t)\} \to 0 \quad \text{in probability},$$

$$(3.3)\qquad \left\{\frac{1}{b_n}Y(a_n t) + c_n(t), t \geq 0\right\} \stackrel{d}{\Rightarrow} \{X(t), t \geq 0\}.$$

Suppose further that there exists $t_0 > 0$ such that $X(t_0)$ and $X(at_0)$ are nondegenerate. Then $\{X(t)\}$ is wide-sense H-semi-selfsimilar with some $H > 0$ and a is an epoch. In case $c_n(t) = 0$ for every n and t, $\{X(t)\}$ is H-semi-selfsimilar.

(ii) Conversely, if $\{X(t)\}$ is nontrivial, wide-sense H-semi-selfsimilar with $H > 0$, and stochastically continuous at $t = 0$, then $\{X(t)\}$ is such a limit.

We remark that if $a_n = a^n$ with $a > 1$, then (3.1) and (3.2) are automatically satisfied. Lamperti(8) gets a conclusion similar to Theorem 3.1 for selfsimilar processes, characterizing them as the limiting processes of $\{\frac{1}{b(\xi)}Y(\xi t)\}$ as $\xi \to \infty$ with $b(\xi) \uparrow \infty$. Thus, in case $c_n(t) = 0$, our sequence $\{\frac{1}{b_n}Y(a_n t)\}$ can be considered as a special kind of subsequences of Lamperti's normalization. Note that he assumes that $\{X(t)\}$ is proper in the sense that $X(t)$ is nondegenerate for every $t > 0$.

Example 3.1. The diffusions on Sierpinski gaskets $\{X(t)\}$ on $\mathbf{R}^d$ are constructed by

$$\left\{\frac{1}{2^n}Y((d+3)^n t)\right\} \stackrel{d}{\Rightarrow} \{X(t)\}$$

for some $\{Y(t)\}$ and $X(t)$ is shown to be nondegenerate for $t > 0$. (See Kusuoka(7), Goldstein(4) and Barlow and Perkins(1).) Hence $\{X(t)\}$ is semi-selfsimilar. Indeed, $b_{n+1}/b_n = 2$ in this case, and $\{X(t)\}$ satisfies (1.2) with $a = d+3$ and $b = 2$, as will be seen in the proof of Theorem 3.1 (i). Theorem 2.1 says that the exponent H is uniquely determined by $b = a^H$. Therefore the diffusions in this example have the exponent $H = \log 2/\log(d+3)$.

To prove Theorem 3.1, we need a lemma.

Lemma 3.1 (Lamperti(8)). Suppose that $\alpha_n Y_n + \beta_n \stackrel{d}{\to} X_1$ and $\gamma_n Y_n + \delta_n \stackrel{d}{\to} X_2$ for some α_n, $\beta_n > 0$, γ_n, $\delta_n \in \mathbf{R}^d$, and for nondegenerate $\mathbf{R}^d$-valued random vectors X_1 and X_2. Then $0 < \lim_{n\to\infty} \alpha_n/\gamma_n < \infty$.

Lamperti[8] proved this lemma for $d = 1$, but the same proof works for $d \geq 2$.

Proof of Theorem 3.1. We first show the assertion (i). Let

$$\begin{aligned} I_n(t) :=& \frac{1}{b_n} Y(a_{n+1}t) + c_n(at) \\ =& \left(\frac{1}{b_n} Y(a \cdot a_n t) + c_n(at) \right) + \frac{1}{b_n}(Y(a_{n+1}t) - Y(a \cdot a_n t)) \\ =:& \, I_{n1}(t) + I_{n2}(t). \end{aligned}$$

We have from (3.3) that $\{I_{n1}(t)\} \overset{d}{\Rightarrow} \{X(at)\}$ and from (3.2) that $I_{n2}(t) \to 0$ in probability. Therefore

$$\{I_n(t)\} \overset{d}{\Rightarrow} \{X(at)\}. \tag{3.4}$$

For $t = t_0$ the limit $X(at_0)$ is nondegenerate. On the other hand, we have $\frac{1}{b_{n+1}} Y(a_{n+1}t_0) + c_{n+1}(t_0) \overset{d}{\to} X(t_0)$ by (3.3) and $X(t_0)$ is nondegenerate. Lemma 3.1 assures that

$$b := \lim_{n\to\infty} \frac{b_{n+1}}{b_n} \in (0, \infty) \tag{3.5}$$

exists. Since $\{b_n\}$ is an increasing sequence, we have that $b \geq 1$.

Next we have

$$\begin{aligned} I_n(t) &= \frac{b_{n+1}}{b_n} \left(\frac{1}{b_{n+1}} Y(a_{n+1}t) + c_{n+1}(t) \right) + \left(c_n(at) - \frac{b_{n+1}}{b_n} c_{n+1}(t) \right) \\ &=: J_{n1}(t) + J_{n2}(t). \end{aligned} \tag{3.6}$$

It follows from (3.3) and (3.5) that

$$\{J_{n1}(t)\} \overset{d}{\Rightarrow} \{bX(t)\}. \tag{3.7}$$

Then, by (3.4), (3.6), and (3.7), for each $t \geq 0$

$$\lim_{n\to\infty} J_{n2}(t) (= c(t), \text{ say}) \tag{3.8}$$

must exist. Altogether (3.4), (3.6), (3.7), and (3.8) imply

$$\{X(at)\} \overset{d}{=} \{bX(t) + c(t)\}, \tag{3.9}$$

that is, $\{X(t)\}$ is semi-selfsimilar with epoch a.

It remains to prove that $H > 0$, that is, $b \neq 1$. Suppose $b = 1$. Then, from Theorem 2.1 (iii),

$$X(t) = X(0) + h(t) \quad \text{a.s.} \tag{3.10}$$

for some nonrandom function h. On the other hand, (3.3) implies that $\frac{1}{b_n}Y(0)+c_n(0) \xrightarrow{d} X(0)$. Since $b_n \uparrow \infty$, we have $\lim_{n\to\infty} Y(0)/b_n = 0$ a.s., and hence $\lim_{n\to\infty} c_n(0)(= c$, say) exists. Therefore $X(0) = c$ a.s. This, combined with (3.10), concludes that $X(t) = c + h(t)$ a.s. for any $t \geq 0$, which contradicts that $\{X(t)\}$ is nontrivial. Thus we see that $b > 1$, concluding the proof of the assertion (i).

The assertion (ii) is shown as follows. From the wide-sense semi-selfsimilarity of $\{X(t)\}$ with exponent $H > 0$, we have (1.3) with some $a > 1$, $b > 1$, and $c\colon [0,\infty) \to \mathbf{R}^d$. Then we have $\{X(a^n t)\} \stackrel{d}{=} \{b^n X(t) + c_n(t)\}$ with some $c_n(t)$. Hence

$$\left\{\frac{1}{b^n}X(a^n t) - \frac{1}{b_n}c_n(t)\right\} \stackrel{d}{=} \{X(t)\} \stackrel{d}{\Rightarrow} \{X(t)\}$$

and (3.3) is satisfied with b^n, a^n, and $b^{-n}c_n(t)$ in place of b_n, a_n, and $c_n(t)$. This concludes the assertion (ii). □

4. SEMI-STABLE LÉVY PROCESSES

A probability measure μ on $\mathbf{R}^d$ is called *semi-stable* if it is infinitely divisible and its characteristic function $\widehat{\mu}(z)$ satisfies, for some $a \in (0,1)\cup(1,\infty)$, $b > 0$, and $c \in \mathbf{R}^d$,

$$\widehat{\mu}(z)^a = \widehat{\mu}(bz)e^{i\langle z,c\rangle}, \quad \forall z \in \mathbf{R}^d. \tag{4.1}$$

(In case this a is an integer, the infinite divisibility of μ is automatic; we do not need to assume it.) When (4.1) holds with $c = 0$, it is said to be *strictly semi-stable*. It is known that if μ is semi-stable and nondegenerate, then there exists $\alpha \in (0,2]$ uniquely such that b in (4.1) can be expressed as $b = a^{1/\alpha}$, (see, e.g., Sato[15]). When we want to emphasize this index α, we say that μ is α-semi-stable. This b is called a *span*.

The connection between the semi-selfsimilarity of a Lévy process and the semi-stability of its marginal distributions is given by the following theorem.

Theorem 4.1. Let $\{X(t), t \geq 0\}$ be an $\mathbf{R}^d$-valued Lévy process. Then $\{X(t)\}$ is wide-sense semi-selfsimilar (semi-selfsimilar, resp.) if and only if the distribution of $X(1)$ is semi-stable (strictly semi-stable, resp.).

Remark 4.1. If the wide-sense semi-selfsimilar Lévy process $\{X(t)\}$ in Theorem 4.1 is nontrivial, then the exponent of the semi-selfsimilarity H exists as we have seen in Theorem 2.1. We see that $H = 1/\alpha$, where α is the index of the semi-stability. If $\{X(t)\}$ is a Lévy process with an α-stable (strictly α-stable, resp.) distribution at $t = 1$, then every finite-dimensional distribution of it is α-stable (strictly α-stable, resp.) and $\{X(t)\}$ is called an α-stable (strictly α-stable, resp.) Lévy process. Likewise, if $\{X(t)\}$ is a Lévy process and $X(1)$ has an α-semi-stable (strictly α-semi-stable, resp.) distribution with a span b, then every finite-dimensional distribution of $\{X(t)\}$ is α-semi-stable (strictly α-semi-stable, resp.) with the span b, and we call $\{X(t)\}$ an α-semi-stable (strictly α-semi-stable, resp.) Lévy process with a span b.

Proof of Theorem 4.1. Let μ be the distribution of $X(1)$. Since $\{X(t)\}$ is a Lévy process, the characteristic function of $X(t)$ is $\widehat{\mu}(z)^t$. We first prove the "only if" part. By (1.3) we have $X(a) \overset{d}{\sim} bX(1) + c(1)$, namely, $\widehat{\mu}(z)^a = \widehat{\mu}(bz)e^{i\langle z, c(1)\rangle}$, implying that μ is semi-stable.

We next show the "if" part. It follows from (4.1) that, for each $t \geq 0$, $\widehat{\mu}(z)^{at} = \widehat{\mu}(bz)^t e^{i\langle z, tc\rangle}$, namely $X(at) \overset{d}{\sim} bX(t) + tc$. Since $\{X(t)\}$ has independent increments, this assures that $\{X(at)\} \overset{d}{=} \{bX(t) + tc\}$. Thus, $\{X(t)\}$ is wide-sense semi-selfsimilar. $\square$

Is a wide-sense H-selfsimilar in fact H-selfsimilar, if it is H-semi-selfsimilar? In case of a Lévy process, we shall answer this question affirmatively in the following. See Remark 5.1 for the general case.

Theorem 4.2. Let $\{X(t)\}$ be an $\mathbf{R}^d$-valued, nontrivial, α-stable Lévy process with $0 < \alpha \leq 2$. If it is strictly α-semi-stable, then it is strictly α-stable.

Proof. Use the explicit representation of the characteristic functions of α-stable distributions. Let $\alpha = 1$. Then

$$E[e^{i\langle z,X(t)\rangle}] = \exp\left[t\left\{-\int_S\Big(|\langle z,\xi\rangle| + i\frac{2}{\pi}\langle z,\xi\rangle\log|\langle z,\xi\rangle|\Big)\lambda(d\xi) + i\langle\gamma,z\rangle\right\}\right]$$

with a finite measure λ on S and $\gamma \in \mathbf{R}^d$. If $\{X(t)\}$ is strictly 1-semi-stable, then $\{X(at)\} \stackrel{d}{=} \{aX(t)\}$ for some $a > 1$, and hence

$$\int_S \langle z,\xi\rangle\log|\langle z,\xi\rangle|\lambda(d\xi) = \int_S \langle z,\xi\rangle\log|\langle az,\xi\rangle|\lambda(d\xi).$$

This means that $\int_S \xi\lambda(d\xi) = 0$, which is exactly the condition for the strict 1-stability. The case $\alpha \neq 1$ is easier and omitted. □

5. REDUCTION OF WIDE-SENSE SEMI-SELFSIMILAR PROCESSES

It is well-known that, for a stable Lévy process $\{X(t)\}$ on $\mathbf{R}^d$, there does not necessarily exist a function $k\colon [0,\infty) \to \mathbf{R}^d$ such that $\{X(t) - k(t)\}$ is a strictly stable Lévy process. This happens in case $\{X(t)\}$ has index 1. In this sense stable Lévy processes are not always reduced to strictly stable Lévy processes. Similarly, semi-stable Lévy processes are not always reduced to strictly semi-stable Lévy processes. See Sato(15). The situation is radically different for wide-sense selfsimilar processes. Namely, Sato(14) shows that if $\{X(t)\}$ is a stochastically continuous wide-sense H-selfsimilar process on $\mathbf{R}^d$ with $H > 0$, then, for some $k(t)$, $\{X(t) - k(t)\}$ is H-selfsimilar. We shall show a similar fact for wide-sense semi-selfsimilar processes. Proof is harder than in the case of wide-sense selfsimilar processes.

Theorem 5.1. Let $\{X(t)\}$ be stochastically continuous, nontrivial, wide-sense H-semi-selfsimilar on $\mathbf{R}^d$ with $H > 0$. Then there exists a nonrandom continuous function $k\colon [0,\infty) \to \mathbf{R}^d$ such that $\{X(t) - k(t), t \geq 0\}$ is H-semi-selfsimilar.

Proof. We have, for some $a > 1$ and $c\colon [0,\infty) \to \mathbf{R}^d$,

(5.1) $$\{X(t)\} \stackrel{d}{=} \{a^H X(t) + c(t)\}.$$

The function $c(t)$ is continuous on $[0,\infty)$, since $\{X(t)\}$ is stochastically continuous. If a continuous function $k(t)$ makes $\{Y(t)\} := \{X(t)-k(t)\}$ H-semi-selfsimilar, then

$$\{X(t)\} = \{Y(at)+k(at)\} \stackrel{d}{=} \{a^H Y(t)+k(at)\} = \{a^H X(t)+k(at)-a^H k(t)\},$$

and hence, by Lemma 2.2,

$$c(t) = k(at) - a^H k(t). \tag{5.2}$$

Conversely, if we can find a continuous function $k(t)$ satisfying (5.2), then it is easy to see that $\{X(t)-k(t)\}$ is stochastically continuous H-semi-selfsimilar. Let us construct such a function $k(t)$.

In general, given a continuous function $c(t)$, $a>1$, and $H>0$, we define $c_n(t)$ for $n \in \mathbf{Z}$ as follows: $c_0(t)=0$ and, for $n \geq 1$,

$$c_n(t) = \sum_{j=0}^{n-1} a^{jH} c(a^{n-j-1}t) \quad \text{and} \quad c_{-n}(t) = -a^{-nH} c_n(a^{-n}t). \tag{5.3}$$

In the notation of the proof of Theorem 2.1, $c_n(t) = c(t,a^n)$ for $n \in \mathbf{Z}$. This implies

$$c_{n+m}(t) = c_n(a^m t) + a^{nH} c_m(t), \quad \forall n,m \in \mathbf{Z}. \tag{5.4}$$

It is also easy to prove (5.4) directly from (5.3). Now we claim that

$$\lim_{n\to\infty} c_{-n}(t) = -c(0)(a^H-1)^{-1} \text{ uniformly in } [0,t_0],\ \forall t_0>0. \tag{5.5}$$

The limit is in fact equal to $X(0)$ a. s., as we have seen in the proof of Theorem 2.1. To prove (5.5), let $K = \sup_{s \leq t_0/a} |c(s)|$. Given $\varepsilon > 0$, choose ℓ such that $K\sum_{j=\ell+1}^{\infty} a^{-(j+1)H} < \varepsilon$. Then, for any $n > \ell$, we have

$$\begin{aligned}
|c_{-n}(t)+c(0)(a^H-1)^{-1}| &= \left| -\sum_{j=0}^{n-1} a^{-(j+1)H} c(a^{j-n}t) + c(0)\sum_{j=0}^{\infty} a^{-(j+1)H} \right| \\
&\leq \left| -\sum_{j=0}^{\ell} a^{-(j+1)H} c(a^{j-n}t) + c(0)\sum_{j=0}^{\ell} a^{-(j+1)H} \right| \\
&\quad + \left| \sum_{j=\ell+1}^{n-1} a^{-(j+1)H} c(a^{j-n}t) \right| + \left| c(0) \sum_{j=\ell+1}^{\infty} a^{-(j+1)H} \right| \\
&= I_1 + I_2 + I_3, \text{ say.}
\end{aligned}$$

We have $I_2 + I_3 < 2\varepsilon$ for $t \in [0, t_0]$. Furthermore, $I_1 \to 0$ as $n \to \infty$ uniformly in $t \in [0, t_0]$, since c is continuous. We thus conclude (5.5). Now we proceed in two steps.

Step 1. We consider the case where $c(1) = 0$. Define $k(t), 0 \le t < \infty$, by

$$k(t) = c_n(a^{-n}t) \quad \text{for } a^n \le t < a^{n+1}, \quad n \in \mathbf{Z}$$

and $k(0) = -c(0)(a^H - 1)^{-1}$. We claim that $k(t)$ is continuous. For that, it is enough to see that $k(t) \to k(0)$, $t \downarrow 0$, and that $c_n(a^{-n}t) \to c_{n+1}(1)$, $t \uparrow a^{n+1}$. The latter is evident from $c_{n+1}(1) = c_n(a) + a^{nH}c_1(1) = c_n(a)$ by (5.4) and by the assumption $c(1) = 0$. We have

$$\sup_{t \in (0, a^{n+1})} |k(t) - k(0)| = \sup_{m \le n} \sup_{t \in [a^m, a^{m+1})} |c_m(a^{-m}t) - k(0)| = \sup_{m \le n} \sup_{t \in [1, a)} |c_m(t) - k(0)|,$$

which tends to 0 as $n \to -\infty$ by (5.5). Thus $k(t)$ is continuous. We have

$$\begin{aligned} k(at) - c(t) &= k(0) I_{\{0\}}(at) + \sum_{n=-\infty}^{\infty} I_{[a^n, a^{n+1})}(at) c_n(a^{-n}at) - c(t) \\ &= (k(0) - c(0)) I_{\{0\}}(t) + \sum_{n=-\infty}^{\infty} I_{[a^n, a^{n+1})}(t)(c_{n+1}(a^{-n}t) - c(t)). \end{aligned}$$

Since $k(0) - c(0) = a^H k(0)$ and since $c_{n+1}(a^{-n}t) - c(t) = a^H c_n(a^{-n}t)$ by (5.4), we see that $k(at) - c(t) = a^H k(t)$. That is, $k(t)$ satisfies (5.2).

Step 2. Let us consider the general case. Define $h(t) = \frac{t-a}{a-1} a^{-H} c(1)$ and $\widetilde{c}(t) = c(t) - h(at) + a^H h(t)$. This corresponds to considering the wide-sense H-semi-selfsimilar process $\{X(t) - h(t)\}$. Then $\widetilde{c}(1) = c(1) + a^H h(1) = 0$. Let $\widetilde{k}(t)$ be such that $\widetilde{k}(at) - a^H \widetilde{k}(t) = \widetilde{c}(t)$, as in Step 1. Now let $k(t) = h(t) + \widetilde{k}(t)$. Then, clearly, $k(at) - a^H k(t) = c(t)$. □

Corollary 5.1. Let $\{Y(t)\}$ be stochastically continuous, nontrivial, H-semi-selfsimilar on $\mathbf{R}^d$ with $H > 0$. For any continuous function $c\colon [0, \infty) \to \mathbf{R}^d$, we can find a continuous function $k\colon [0, \infty) \to \mathbf{R}^d$ such that $\{X(t)\} := \{Y(t) + k(t)\}$ is a wide-sense H-semi-selfsimilar process satisfying (5.1) with that $c(t)$.

This is seen from the proof of Theorem 5.1.

Corollary 5.2. Let $\{X(t)\}$ be an $\mathbf{R}^d$-valued, stochastically continuous, nontrivial, wide-sense H-semi-selfsimilar process with $H > 0$ with independent increments. Then there exists a nonrandom continuous function $k\colon [0,\infty) \to \mathbf{R}^d$ such that $\{X(t) - k(t), t \geq 0\}$ is H-semi-selfsimilar with independent increments.

This is an obvious consequence of Theorem 5.1. The statement in Corollary 5.2 is not true if we replace "process with independent increments" by "Lévy process", as was mentioned at the beginning of this section.

Remark 5.1. Let $\{X(t)\}$ be an $\mathbf{R}^d$-valued, stochastically continuous, nontrivial, wide-sense H-selfsimilar process with $H > 0$. Then $\{X(t)\}$ is H-semi-selfsimilar with epoch $p(> 1)$ if and only if there exist an H-selfsimilar process $\{Y(t)\}$ and a function $g : \mathbf{R} \to \mathbf{R}^d$ which is continuous and periodic with period $\log p$ such that $\{X(t)\} \stackrel{d}{=} \{Y(t) + t^H g(\log t)\}$. For the proof, use the fact in Sato[(14)] that there exist an H-selfsimilar process $\{Y(t)\}$ and a continuous function $k(t)$ such that $\{X(t)\} \stackrel{d}{=} \{Y(t) + k(t)\}$. The details are omitted. Thus, the general answer to the question raised immediately before Theorem 4.2 is no.

6. SEMI-SELFSIMILAR PROCESSES WITH INDEPENDENT INCREMENTS

As we have seen in Section 4, semi-stable Lévy processes are wide-sense semi-selfsimilar with independent and stationary increments. Let us discuss the class of processes that are (wide-sense) semi-selfsimilar, stochastically continuous and have independent increments but do not necessarily have stationary increments. In the case of stochastically continuous selfsimilar processes with independent increments, an intimate connection with the class L, that is, the class of selfdecomposable distributions, is found by Sato[(14)]. Namely, when $H > 0$ is fixed, we have a one-to-one correspondence between the class of such processes with exponent H and the class L, by looking at marginal distributions at $t = 1$. Using the class $L(b)$, an extension of the class L introduced by Maejima and Naito[(10)], we shall see the correspondence is preserved in a weaker sense in our case.

Definition 6.1. Let $0 < b < 1$. $\mu \in \mathcal{P}(\mathbf{R}^d)$ is said to belong to the class $L(b)$ if there exists $\rho \in I(\mathbf{R}^d)$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}(z), \quad \forall z \in \mathbf{R}^d. \tag{6.1}$$

Any probability distribution that belongs to $L(b)$ for some $b \in (0, 1)$ is called *semi-selfdecomposable*.

This class $L(b)$ is the class $L_0(b)$ in Maejima and Naito(10). It is a class of limits of subsequences, satisfying uniform asymptotic negligibility condition, of usually normalized partial sums of independent random vectors. It is proved that $L(b) \subset I(\mathbf{R}^d)$. (See Proposition 2.1 of Maejima and Naito(10).) A probability distribution μ is in the class L, or selfdecomposable, if and only if it is in the class $L(b)$ for every $b \in (0, 1)$.

Theorem 6.1. Suppose that $\{X(t), t \geq 0\}$ is a nontrivial, stochastically continuous, wide-sense semi-selfsimilar, $\mathbf{R}^d$-valued process with exponent $H > 0$ and an epoch $a > 1$. Suppose further that $\{X(t)\}$ has independent increments. Then $\mathcal{L}(X(t))$ is semi-selfdecomposable for any $t \geq 0$. Actually it belongs to the class $L(a^{-H})$. For any $t > 0$, $\mathcal{L}(X(t))$ is nondegenerate.

Proof. Denote $\mu_t = \mathcal{L}(X(t))$ and $\mu_{s,t} = \mathcal{L}(X(t) - X(s))$ for $s < t$. It follows from the independent increments property and the stochastic continuity that μ_t and $\mu_{s,t}$ are infinitely divisible and $\widehat{\mu}_{at}(z) = \widehat{\mu}_t(z)\widehat{\mu}_{t,at}(z)$. The property (5.1) implies $\widehat{\mu}_{at}(z) = \widehat{\mu}_t(a^H z)e^{i\langle z, c(t)\rangle}$. Hence we have

$$\widehat{\mu}_t(z) = \widehat{\mu}_{at}(a^{-H}z)e^{-i\langle a^{-H}z, c(t)\rangle} = \widehat{\mu}_t(a^{-H}z)\widehat{\mu}_{t,at}(a^{-H}z)e^{-i\langle a^{-H}z, c(t)\rangle}.$$

This shows that $\mu_t \in L(a^{-H})$. Let us show nondegeneracy of μ_t for $t > 0$. By Theorem 2.1 (iii), μ_0 is degenerate. Hence μ_{t_0} is nondegenerate for some $t_0 > 0$ by the nontriviality of $\{X(t)\}$. It follows that μ_t is nondegenerate for any $t \geq t_0$, by the independent increments property. If $0 < t < t_0$, then recall that $\widehat{\mu}_{a^n t}(z) = \widehat{\mu}_t(a^{nH}z)e^{i\langle z, c(t, a^n)\rangle}$ and choose n so large that $a^n t \geq t_0$, to conclude nondegeneracy of μ_t. □

Theorem 6.2. Let $a > 1$ and $H > 0$. Suppose that $\{\mu_t,\ 1 \le t < a\} \subset \mathcal{P}(\mathbf{R}^d)$ is given and satisfies the following five conditions.

(1) For any $t \in [1,a)$, μ_t is nondegenerate.

(2) For any $t \in [1,a)$, $\widehat{\mu}_t(z) \ne 0$ for all z.

(3) For any s,t with $1 \le s \le t < a$, there exists $\mu_{s,t} \in \mathcal{P}(\mathbf{R}^d)$ such that $\mu_t = \mu_s * \mu_{s,t}$.

(4) μ_t is continuous with respect to $t \in [1,a)$ in the sense of weak convergence.

(5) $\lim_{t\uparrow a} \widehat{\mu}_t(z) = \widehat{\mu}_1(a^H z)$ for $z \in \mathbf{R}^d$.

Then $\mu_t \in L(a^{-H})$ for $t \in [1,a)$ and there exists a nontrivial, stochastically continuous, H-semi-selfsimilar, $\mathbf{R}^d$-valued process $\{X(t), t \ge 0\}$ with independent increments with epoch a such that $\mathcal{L}(X(t)) = \mu_t$ for $t \in [1,a)$. Such a process $\{X(t)\}$ is unique in law.

Proof. Define μ_t for $0 \le t < \infty$ as follows. $\mu_0 = \delta_0$ and, for $a^n \le t < a^{n+1}$ with $n \in \mathbf{Z}$,

$$\widehat{\mu}_t(z) = \widehat{\mu}_{a^{-n}t}(a^{nH} z). \tag{6.2}$$

Then, for $1 \le t < a$, μ_t is identical with the given one. We have $\widehat{\mu}_a(z) = \widehat{\mu}_1(a^H z)$ and $\widehat{\mu}_{a^n}(z) = \widehat{\mu}_1(a^{nH} z) = \widehat{\mu}_a(a^{(n-1)H} z)$. We claim that μ_t is continuous in t. As $t \uparrow a^{n+1}$, $\widehat{\mu}_t(z) \to \widehat{\mu}_1(a^{(n+1)H} z) = \widehat{\mu}_{a^{n+1}}(z)$ by (5) and (6.2). Suppose that μ_t does not tend to δ_0 as $t \downarrow 0$. Then, there are $t_k \downarrow 0$, $z_0 \in \mathbf{R}^d$, and $\varepsilon > 0$ such that $|\widehat{\mu}_{t_k}(z_0) - 1| > \varepsilon$. Choose $n_k \in \mathbf{Z}$ such that $a^{n_k} \le t < a^{n_k+1}$ and let $s_k = a^{-n_k} t_k$. Then there is a subsequence of $\{s_k\}$ that tends to some $s \in [1,a]$. We identify this subsequence with $\{s_k\}$. We have $n_k \downarrow -\infty$ and $\mu_{t_k}(z_0) = \widehat{\mu}_{s_k}(a^{n_k H} z_0) \to \widehat{\mu}_s(0) = 1$, which is absurd. Hence $\mu_t \to \delta_0 = \mu_0$ as $t \downarrow 0$. Thus, using (4), we see that μ_t is continuous in $t \in [0,\infty)$. Next we claim that, for $0 \le s \le t < \infty$, there is a unique $\mu_{s,t} \in \mathcal{P}(\mathbf{R}^d)$ satisfying

$$\mu_t = \mu_s * \mu_{s,t}. \tag{6.3}$$

If such $\mu_{s,t}$ exists, it is unique because $\widehat{\mu}(z) \ne 0$ by (2) and (6.2). If $1 \le s \le t < a$, then the existence of $\mu_{s,t}$ is assumed in (3). If $1 \le s < t = a$, then $\widehat{\mu}_{s,r}(z) = \widehat{\mu}_r(z)/\widehat{\mu}_s(z) \to \widehat{\mu}_a(z)/\widehat{\mu}_s(z)$ as $r \uparrow a$ and, by Lévy's continuity theorem, there exists $\mu_{s,a} \in \mathcal{P}(\mathbf{R}^d)$ such that $\widehat{\mu}_{s,a}(z) = \widehat{\mu}_a(z)/\widehat{\mu}_s(z)$. this satisfies (6.3) for $1 \le s < t = a$. We define $\mu_{a,a} = \delta_0$. Thus we have $\mu_{s,t}$ satisfying (6.3) for $1 \le s \le t \le a$. If $a^n \le s \le t \le a^{n+1}$ with

$n \in \mathbf{Z} \setminus \{0\}$, then it follows from the existence of $\mu_{a^{-n}s,a^{-n}t}$ that

$$\widehat{\mu}_t(z) = \widehat{\mu}_{a^{-n}t}(a^{nH}z) = \widehat{\mu}_{a^{-n}s}(a^{nH}z)\widehat{\mu}_{a^{-n}s,a^{-n}t}(a^{nH}z) = \widehat{\mu}_s(z)\widehat{\mu}_{a^{-n}s,a^{-n}t}(a^{nH}z)$$

and hence $\mu_{s,t}$ exists. If $a^m \le s \le a^{m+1} \le a^n \le t \le a^{n+1}$, then $\mu_{s,t}$ satisfying (6.3) is given by

$$\mu_{s,t} = \mu_{s,a^{m+1}} * \mu_{a^{m+1},a^{m+2}} * \cdots * \mu_{a^{n-1},a^n} * \mu_{a^n,t}.$$

Finally define $\mu_{0,t} = \mu_t$. Now the existence of $\mu_{s,t}$ satisfying (6.3) is proved for all $0 \le s \le t < \infty$. Obviously

$$\mu_{s,t} * \mu_{t,u} = \mu_{s,u} \quad \text{for } 0 \le s \le t \le u < \infty.$$

The standard argument based upon Kolmogorov's extension theorem now gives a stochastically continuous process $\{X(t)\}$ with independent increments such that $\mathcal{L}(X(t)) = \mu_t$. It is nontrivial by the assumption (1). We have $X(at) \overset{d}{\sim} a^H X(t)$ since, for $a^n \le t < a^{n+1}$, $\widehat{\mu}_{at}(z) = \widehat{\mu}_{a^{-n}t}(a^{(n+1)H}z) = \widehat{\mu}_t(a^H z)$ by (6.2). Combined with the independent increments property, this implies $\{X(at)\} \overset{d}{=} \{a^H X(t)\}$, that is, H-semi-selfsimilarity with epoch a. Now it follows from Theorem 6.1 that $\mu_t \in L(a^{-H})$ for every t. Conversely, if $\{X(t)\}$ is a process having the desired properties, then $\mathcal{L}(X(t)) = \mu_t$ must satisfy (6.2). Hence $\{X(t)\}$ is unique in law. This completes the proof. □

Application of Theorems 6.1 and 6.2 gives a new characterization of the class $L(b)$.

Theorem 6.3. Let $0 < b < 1$ and $H > 0$. Let $\mu \in \mathcal{P}(\mathbf{R}^d)$ and suppose μ is nondegenerate. Then the following two statements are equivalent.

(i) $\mu \in L(b)$.

(ii) There exists $\{X(t), t \ge 0\}$, a nontrivial, stochastically continuous, H-semi-selfsimilar process with independent increments with epoch $b^{-1/H}$ such that $\mathcal{L}(X(1)) = \mu$.

Proof. Let $a = b^{-1/H}$. Theorem 6.1 says that (ii) implies (i). Conversely, let us assume (i). For $1 \le t < a$, define μ_t by

$$\widehat{\mu}_t(z) = \widehat{\mu}(z)^{(a-t)/(a-1)}\widehat{\mu}(a^H z)^{(t-1)/(a-1)}.$$

Since μ is infinitely divisible, so is μ_t, and hence $\widehat{\mu}_t(z) \neq 0$. This system $\{\mu_t,\, 1 \le t < a\}$ satisfies all conditions in Theorem 6.2. Indeed, $\mu_1 = \mu$ and $\lim_{t\uparrow a} \widehat{\mu}_t(z) = \widehat{\mu}(a^H z)$. Thus Conditions (1), (2), and (5) are satisfied. Condition (4) also follows from our definition of μ_t. To see (3), note that, for $1 \le s \le t < a$,

$$\widehat{\mu}_t(z) = \widehat{\mu}(z)^{(a-t)/(a-1)} \widehat{\mu}(a^H z)^{(s-1)/(a-1)} \widehat{\mu}(a^H z)^{(t-s)/(a-1)},$$

$$\widehat{\mu}_s(z) = \widehat{\mu}(z)^{(a-t)/(a-1)} \widehat{\mu}(a^H z)^{(s-1)/(a-1)} \widehat{\mu}(z)^{(t-s)/(a-1)},$$

and $\widehat{\mu}(z) = \widehat{\mu}(a^{-H} z)\widehat{\rho}(z)$ with some $\rho \in I(\mathbf{R}^d)$. Thus $\widehat{\mu}_t(z) = \widehat{\mu}_s(z)\widehat{\rho}(a^H z)^{(t-s)/(a-1)}$, which shows Condition (3). Now Theorem 6.2 says that there is a process $\{X(t)\}$ as asserted in the statement (ii). □

We can generalize Theorem 6.2 to wide-sense semi-selfsimilar case.

Theorem 6.4. Let $a > 0$ and $H > 0$ and let $c\colon [0,\infty) \to \mathbf{R}^d$ be a continuous function. Suppose that $\{\mu_t,\, 1 \le t < a\} \subset \mathcal{P}(\mathbf{R}^d)$ is given and satisfies Conditions (1)—(4) of Theorem 6.2 and

(5′) $\lim_{t\uparrow a} \widehat{\mu}_t(z) = \widehat{\mu}_1(a^H z) e^{i\langle z, c(1)\rangle}$ for $z \in \mathbf{R}^d$.

Then $\mu_t \in L(a^{-H})$ for $t \in [1, a)$ and there exists, uniquely in law, a nontrivial, stochastically continuous, H-semi-selfsimilar, $\mathbf{R}^d$-valued process $\{X(t), t \ge 0\}$ with independent increments satisfying $\{X(at)\} \overset{d}{=} \{a^H X(t) + c(t)\}$ such that $\mathcal{L}(X(t)) = \mu_t$ for $t \in [1, a)$.

Note that the function $c(t)$ for the process $\{X(t)\}$ in Theorem 6.1 is necessarily continuous owing to the stochastic continuity of $\{X(t)\}$.

Proof. As in the proof of Theorem 5.1, we can construct, from the given function $c(t)$, a continuous function $k(t)$ satisfying the relation (5.2). Let $\rho_t = \mu_t * \delta_{-k(t)}$ for $t \in [1, a)$. Then the system $\{\rho_t, 1 \le t < a\}$ satisfies Conditions (1)—(5) of Theorem 6.2 replacing $\{\mu_t, 1 \le t < a\}$. Indeed, (1)—(4) are obvious and (5) is shown as follows:

$$\widehat{\rho}_t(z) = \widehat{\mu}_t(z) e^{-i\langle z, k(t)\rangle} \to \widehat{\mu}_1(a^H z) e^{i\langle z, c(1) - k(a)\rangle} = \widehat{\mu}_1(a^H z) e^{-i\langle z, a^H k(1)\rangle} = \widehat{\rho}_1(a^H z)$$

by (5) and by (5.2) as $t \uparrow a$. Therefore, Theorem 6.2 says that there exists a nontrivial, stochastically continuous, H-semi-selfsimilar process $\{Y(t), t \ge 0\}$ with independent increments with epoch a such that $\mathcal{L}(Y(t)) = \rho_t$ for $t \in [1, a)$. Let $X(t) = Y(t) + k(t)$. Then $\{X(t)\}$ is a desired process.

To see the asserted uniqueness in law, suppose that $\{\widetilde{X}(t)\}$ satisfies all required conditions in the theorem. Define $\widetilde{Y}(t) = \widetilde{X}(t) - k(t)$. Then we can use the uniqueness assertion in Theorem 6.2 for $\{\widetilde{Y}(t)\}$. □

If, in Theorems 6.2 and 6.4, we assume the infinite divisibility of μ_t, $t \in [1, a)$, beforehand, then we can reformulate them by using Lévy-Khintchine representation (A, ν, β) of $\mu \in I(\mathbf{R}^d)$. Recall that

$$\widehat{\mu}(z) = \exp\left\{-\frac{1}{2}\langle Az, z\rangle + \int_{\mathbf{R}^d}\left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2}\right)\nu(dx) + i\langle \beta, z\rangle\right\}, \tag{6.4}$$

where A is a symmetric nonnegative-definite matrix, ν is a measure on $\mathbf{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int |x|^2(1+|x|^2)^{-1}\nu(dx) < \infty$, and $\beta \in \mathbf{R}^d$. The representation (A, ν, β) of μ is unique and called the (Lévy-Khintchine) generating triplet of μ. The measure ν is called the Lévy measure of μ. Let us denote by C_* the class of bounded continuous functions f on $\mathbf{R}^d$ such that $f(x) = 0$ on a neighborhood of the origin.

Theorem 6.5. Let $a > 1$ and $H > 0$. Suppose that we are given $\{\mu_t, 1 \le t < a\} \subset I(\mathbf{R}^d)$ such that the generating triplet (A_t, ν_t, β_t) of μ_t satisfies the following conditions.

(1) For any $t \in [1, a)$, $A_t \neq 0$ or $\nu_t \neq 0$.

(2) For any $1 \le s < t < a$, $\langle A_s z, z\rangle \le \langle A_t z, z\rangle$ for $z \in \mathbf{R}^d$ and $\nu_s(B) \le \nu_t(B)$ for $B \in \mathcal{B}(\mathbf{R}^d)$.

(3) As $s \to t \in [1, a)$, $\langle A_s z, z\rangle \to \langle A_t z, z\rangle$ for $z \in \mathbf{R}^d$, $\int f(x)\nu_s(dx) \to \int f(x)\nu_t(dx)$ for $f \in C_*$, and $\beta_s \to \beta_t$.

(4) As $t \uparrow a$, $\langle A_t z, z\rangle \to a^{2H}\langle A_1 z, z\rangle$ for $z \in \mathbf{R}^d$, $\int f(x)\nu_t(dx) \to \int f(a^H x)\nu_1(dx)$ for $f \in C_*$, and

$$\beta_t \to a^H\beta_1 + a^H \int x\left(\frac{1}{1+a^{2H}|x|^2} - \frac{1}{1+|x|^2}\right)\nu_1(dx).$$

Then the conclusion in Theorem 6.2 is true.

Theorem 6.6. Let $a > 0$ and $H > 0$ and let $c\colon [0, \infty) \to \mathbf{R}^d$ be continuous. Suppose that $\{\mu_t,\ 1 \le t < a\} \subset I(\mathbf{R}^d)$ is given and that the generating triplet (A_t, ν_t, β_t) of μ_t

satisfies (1)—(4) of Theorem 6.5 with the limit in (4) having an additional term $c(1)$. Then the conclusion in Theorem 6.4 is true.

Proofs of Theorems 6.5 and 6.6 use the well-known condition for convergence in terms of generating triplets. We omit the details.

7. EXAMPLES OF SEMI-SELFSIMILAR PROCESSES WITH INDEPENDENT INCREMENTS

Let us construct, using Theorem 6.5, examples of semi-selfsimilar processes with independent increments. We use the following representation of the characteristic functions of distributions of class $L(b)$. For $B \subset \mathbf{R}^d$ and $b > 0$ we denote $bB = \{bx \colon x \in B\}$. The words "increase" and "decrease" are used in the wide sense allowing flatness.

Theorem 7.1 (Maejima and Naito[10])**.** Fix $0 < b < 1$. Let $\mu \in I(\mathbf{R}^d)$ with Lévy measure ν. Then the following statements are equivalent.

(i) μ is of class $L(b)$.

(ii) $\nu(bB) \geq \nu(B)$ for $B \in \mathcal{B}(\mathbf{R}^d)$.

(iii) ν is either identically zero or of the form

$$\nu(B) = -\int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) dN_\xi(r) \quad \text{for } B \in \mathcal{B}(\mathbf{R}^d),$$

where λ is a finite measure on the unit sphere S and $N_\xi(r)$ is Borel measurable in $\xi \in S$, right-continuous and decreasing in $r > 0$, and

$$N_\xi(br) - N_\xi(br') \geq N_\xi(r) - N_\xi(r') \quad \text{for } 0 < r < r'.$$

Notice that the class $L(b)$ imposes no condition on the Gaussian part A.

Example 7.1. (A distribution of class $L(b)$ on $\mathbf{R}^d$) Fix $0 < b < 1$. Let $S_n(b) = \{x \in \mathbf{R}^d : b^{-n} < |x| \leq b^{-n-1}\}$. Let ν_0 be an arbitrary finite measure on $S_0(b)$ and let $\{k_n, n \in \mathbf{Z}\}$ be a nonnegative decreasing sequence such that $\sum_{n\geq 0} k_n < \infty$ and $\sum_{n\leq -1} b^{-2n} k_n < \infty$. We construct a measure ν by letting $\nu(\{0\}) = 0$ and $\nu(B) = k_n \nu_0(b^n B)$ for Borel sets $B \subset S_n(b)$. Then, $\int_{|x|>1} \nu(dx) = \sum_{n\geq 0} k_n \nu_0(S_0(b)) < \infty$. Since

$\int_{S_n(b)} f(x)\nu(dx) = k_n \int_{S_0(b)} f(b^{-n}x)\nu_0(dx)$ for any nonnegative Borel function f, we see that $\int_{|x|\le 1} |x|^2\nu(dx) = \sum_{n\le -1} k_n b^{-2n} \int_{S_0(b)} |x|^2\nu_0(dx) < \infty$. Since $\nu(B) = \sum_{n\in\mathbf{Z}} \nu(B\cap S_n(b)) = \sum_{n\in\mathbf{Z}} k_n\nu_0(b^nB)$, it follows from the decrease of $\{k_n\}$ that $\nu(bB) \ge \nu(B)$. Therefore, by Theorem 7.1, ν is the Lévy measure of a distribution of class $L(b)$. If $k_n = cb^{n\alpha}$ with $0 < \alpha < 2$ and $c = \text{const.} > 0$, then this ν and $A = 0$ give an α-semi-stable distribution. We see that Lévy measures of distributions of class $L(b)$ can be discrete, continuous singular, absolutely continuous, or mixture of them. A simplest example is given by $\nu(dx) = \sum_{n\in\mathbf{Z}} k_n\delta_{b^{-n}x_0}(dx)$ with $x_0 \in S_0(b)$.

Example 7.2. (A distribution of class $L(b)$ on $\mathbf{R}$ with absolutely continuous Lévy measure) Let $0 < b < 1$ and let ν be zero on $(-\infty, 0)$ and $\nu(B) = \int_B k(x)g(\log x)\frac{dx}{x}$ on $(0,\infty)$, where $g(x)$ is a nonnegative, bounded, Borel function on $\mathbf{R}$ periodic with period $-\log b$ and $k(x)$ is a nonnegative decreasing function on $(0,\infty)$ satisfying $\int_0^\infty x(1+x^2)^{-1}k(x)dx < \infty$. Then ν is the Lévy measure of a distribution of class $L(b)$. In fact,

$$\int_{br}^{br'} k(x)g(\log x)\frac{dx}{x} = \int_r^{r'} k(bx)g(\log x)\frac{dx}{x} \ge \int_r^{r'} k(x)g(\log x)\frac{dx}{x}$$

for $0 < r < r'$, and Theorem 7.1 applies. This example is not covered by Example 7.1. The Lévy measures thus defined on $(0,\infty)$ can serve as $-dN_\xi(r)$ in (iii) of Theorem 7.1 to construct Lévy measures of distributions in $L(b)$ on $\mathbf{R}^d$. If $k(x) = x^{-\alpha}$ with $0 < \alpha < 2$ and $A = 0$, we obtain an α-semi-stable distribution (Sato[15]).

In the four examples that follow, we give semi-selfsimilar processes on $\mathbf{R}$ with independent increments.

Example 7.3. Let $a > 1$ and $H > 0$. For $t \in [1,a)$, let ν_t be zero on $(-\infty,0)$ and $\nu_t(B) = \int_B k_t(x)g_t(\log x)\frac{dx}{x}$ on $(0,\infty)$, where $g_t(x)$ and $k_t(x)$ satisfy the following conditions:

(1) For any fixed $t \in [1,a)$, $g_t(x)$ is nonnegative, bounded, Borel, periodic with period $H\log a$, and not identically zero.

(2) For any fixed $t \in [1,a)$, $k_t(x)$ is nonnegative, decreasing with $\int_0^\infty x(1+x^2)^{-1}k_t(x)dx < \infty$, and not identically zero.

(3) For any fixed $x > 0$, the function $k_t(x)g_t(\log x)$ is continuous and increasing in

$t \in [1, a)$ and tends to $k_1(a^{-H}x)g_1(\log x)$ as $t \uparrow a$.

Then, there exists $\{X(t), t \geq 0\}$, a stochastically continuous, H-semi-selfsimilar process having epoch a with independent increments such that, for $t \in [1, a)$, $\mu_t = \mathcal{L}(X(t))$ has Lévy measure ν_t. In fact, we can check all conditions in Theorem 6.5, letting $A_t = 0$ and choosing β_t appropriately. Using Example 7.2, we see that μ_t is of class $L(a^{-H})$.

Example 7.4. Let $a > 1$ and $H = 1/\alpha$ with $0 < \alpha < 2$. In Example 7.3 let $g_t(x) = g_1(x)$ and $k_t(x) = tx^{-\alpha}$. Then Conditions (2) and (3) are automatic. The resulting process with $A_t = 0$ is an α-semi-stable Lévy process.

Example 7.5. Choose $a > 1$, $H > 0$, and $0 < \alpha < 2$ arbitrarily. We do not assume $H = 1/\alpha$. Let $k_t(x) = tx^{-\alpha}$ in Example 7.3. There is freedom of choice of the function $g_t(x)$. Let $g_1(x) = c > 0$, a constant function, and, for every $t \in (1, a)$, let $g_t(x)$ be a nonconstant function. We can find such $g_t(x)$ satisfying Conditions (1) and (3). Condition (3) is satisfied if, for any $x > 0$, $tg_t(\log x)$ continuously increases to $a^{\alpha H}c$ as $t \uparrow a$. If $A_t = 0$, the resulting process has α-stable distribution at $t = a^n$, $n \in \mathbf{Z}$, but has α-semi-stable, not α-stable distribution at every other t. It has exponent H.

Example 7.6. Again choose $a > 1$, $H > 0$, and $0 < \alpha < 2$ arbitrarily. Let $g_t(x) = 1$ for all $x \in \mathbf{R}$ and $t \in [1, a)$. Let $k_t(x) = h(t)x^{-\alpha}$, where $h(t)$ is a positive continuous function on $[1, a)$, increasing to $a^{\alpha H}h(1)$ as $t \uparrow a$. Then Conditions (1), (2), (3) in Example 7.3 are satisfied. The process with $A_t = 0$ has exponent H and has α-stable distribution at every $t > 0$. This is a process obtained from an α-stable Lévy process through a nonrandom time change.

A stochastically continuous H-selfsimilar process, $H > 0$, with independent increments is uniquely determined by its distribution at $t = 1$, while the distribution at $t = 1$ can be chosen arbitrarily from the class L (Sato[(14)]). Properties of such a process are studied by Sato[(14)], Watanabe[(18)], and Sato and Yamamuro[(16)]. But, a stochastically continuous H-semi-selfsimilar process with independent increments is not determined by its distribution at $t = 1$, as the examples above show. We can choose an arbitrary distribution of class $L(a^{-H})$ as its distribution at $t = 1$ (Theorem 6.3), and there remains some freedom of choice of μ_t for $1 < t < a$.

8. CONNECTION TO PERIODICALLY STATIONARY PROCESSES

Definition 8.1. An $\mathbf{R}^d$-valued stochastic process $\{Y(t), t \in \mathbf{R}\}$ is said to be *periodically stationary with period* $p(>0)$ if

$$\{Y(t+p), t \in \mathbf{R}\} \stackrel{d}{=} \{Y(t), t \in \mathbf{R}\}. \tag{8.1}$$

This notion is introduced by Hurd[5] as a generalization of strict stationarity under the phrase "periodically nonstationary." In the literature, we can also find periodically correlated processes which are second order processes with the property (8.1) only in their correlations. Such processes have several names in the literature. See, e.g., Hurd[6] and the references therein. The Lamperti transformation linking between selfsimilar processes and stationary processes (see Lamperti[8], also see Burnecki et al.[2]) also links between semi-selfsimilar processes and periodically stationary processes, and gives a new class of periodically stationary semi-stable processes as we will show below in Example 8.1.

Theorem 8.1. (i) Let $\{Y(t), t \in \mathbf{R}\}$ be a periodically stationary process with period p, and fix $H > 0$. Define $\{X(t), t \geq 0\}$ by

$$X(t) = \begin{cases} t^H Y(\log t), & t > 0, \\ 0, & t = 0. \end{cases}$$

Then $\{X(t), t \geq 0\}$ is H-semi-selfsimilar with epoch e^p.

(ii) Let $\{X(t), t \geq 0\}$ be semi-selfsimilar with exponent H and epoch $a(>1)$. Define

$$Y(t) = e^{-Ht} X(e^t), \quad t \in \mathbf{R}. \tag{8.2}$$

Then $\{Y(t), t \in \mathbf{R}\}$ is periodically stationary with period $p = \log a$.

(iii) In (i) and (ii) above, $\{X(t), t \geq 0\}$ is stochastically continuous if and only if $\{Y(t), t \in \mathbf{R}\}$ is stochastically continuous.

Proof. (i) Let p be a period of $\{Y(t)\}$. Then we have

$$\begin{aligned}\{X(e^p t), t \geq 0\} &= \{e^{pH} t^H Y(\log e^p t), t \geq 0\} \\ &\stackrel{d}{=} \{e^{pH} t^H Y(\log t), t \geq 0\} = \{e^{pH} X(t), t \geq 0\}.\end{aligned}$$

Thus $\{X(t), t \geq 0\}$ is H-semi-selfsimilar and e^p is an epoch of it.

(ii) Since $\{X(t)\}$ is H-semi-selfsimilar with epoch a, we have

$$\begin{aligned}\{Y(t+\log a), t \in \mathbf{R}\} &= \{e^{-H(t+\log a)} X(e^{t+\log a}), t \in \mathbf{R}\} \\ &= \{e^{-Ht} a^{-H} X(ae^t), t \in \mathbf{R}\} \stackrel{d}{=} \{e^{-Ht} X(e^t), t \in \mathbf{R}\} = \{Y(t), t \in \mathbf{R}\},\end{aligned}$$

completing the proof of assertion (ii).

(iii) The assertion is trivial except the fact that $X(t) \to 0$ in probability as $t \downarrow 0$ whenever $\{Y(t), t \in \mathbf{R}\}$ is stochastically continuous. Let us show this fact. In any subclass of $\{Y(t) : t \in [0,p]\}$, we can find a sequence that converges in probability. Thus, $\{\mathcal{L}(Y(t)) \colon t \in [0,p]\}$ is tight. Hence we conclude that for any $\varepsilon > 0$,

$$P(|X(t)| > \varepsilon) = P(|Y(\log t)| > t^{-H}\varepsilon) \leq \sup_{s \in [0,p]} P(|Y(s)| > t^{-H}\varepsilon) \to 0$$

as $t \downarrow 0$. □

Example 8.1. Let $0 < \alpha \leq 2$ and let $\{X(t), t \geq 0\}$ be a strictly α-semi-stable Lévy process, and define a periodically stationary process $\{Y(t), t \in \mathbf{R}\}$ by (8.2) with $H = 1/\alpha$. We call this new process *α-semi-stable Ornstein-Uhlenbeck process.* (When $\alpha = 2$, it is nothing but the ordinary Ornstein-Uhlenbeck process.)

Definition 8.2. Let $\{X(t), t \in T\}, T = [0,\infty)$ or $\mathbf{R}$, be an $\mathbf{R}^d$-valued stochastic process and let $0 < \alpha \leq 2$. $\{X(t), t \in T\}$ is said to be (strictly) *α-semi-stable process* if for any $n \geq 1$ and for any $t_1. \cdots, t_n \in T$, $(X(t_1), \cdots, X(t_n))$ is (strictly) α-semi-stable in $\mathbf{R}^{d\times n}$. (When $\alpha = 2$, it is (mean 0) Gaussian.)

Theorem 8.2. The α-semi-stable Ornstein-Uhlenbeck process $\{Y(t), t \in \mathbf{R}\}$ is strictly α-semi-stable in the sense of Definition 8.2.

Proof. Since $\{X(t)\}$ is a strictly α-semi-stable Lévy process, the characteristic function $\widehat{\mu}(z)$ of $X(1)$ satisfies $\widehat{\mu}(z)^a = \widehat{\mu}(a^{1/\alpha}z)$ for some $a \in (0,1) \cup (1,\infty)$. Furthermore, the characteristic function of $X(t) - X(s), 0 \le s < t$, is $\widehat{\mu}(z)^{t-s}$. We show that $Y := (Y(t_1), \cdots, Y(t_n))$ is α-semi-stable in $\mathbf{R}^{d\times n}$.

Let $z = (z_1, \cdots, z_n) \in \mathbf{R}^{d\times n}, z_\ell \in \mathbf{R}^d, 1 \le \ell \le n$. Then we have, with the convention $t_0 = -\infty$, $\varphi(z) := E[e^{i\langle z,Y\rangle}] = E[\exp\{i\sum_{\ell=1}^n \langle z_\ell, Y(t_\ell)\rangle\}]$ and

$$\sum_{\ell=1}^n \langle z_\ell, Y(t_\ell)\rangle = \sum_{\ell=1}^n \langle z_\ell, e^{-Ht_\ell} X(e^{t_\ell})\rangle = \sum_{k=1}^n \langle p_k, X(e^{t_k}) - X(e^{t_{k-1}})\rangle,$$

where $p_k = \sum_{\ell=k}^n e^{-Ht_\ell} z_\ell \ \in \mathbf{R}^d$. Thus

$$\varphi(z)^a = \left(\prod_{k=1}^n \widehat{\mu}(p_k)^{e^{t_k} - e^{t_{k-1}}}\right)^a = \prod_{k=1}^n \widehat{\mu}(a^{1/\alpha} p_k)^{e^{t_k} - e^{t_{k-1}}} = \varphi(a^{1/\alpha} z),$$

concluding that Y is strictly α-semi-stable. □

9. SEMI-SELFSIMILAR PROCESSES WITH STATIONARY INCREMENTS

In this section, we shall define integrals with respect to the random measure induced by semi-stable Lévy processes, and give some examples of semi-selfsimilar processes with stationary, but not necessarily independent, increments.

Let $0 < \alpha \le 2$. Denote by $\{S_\alpha(t), t \ge 0\}$ a nontrivial $\mathbf{R}^d$-valued symmetric α-semi-stable Lévy process and extend the time parameter set to $\mathbf{R}$ by making

$$\{S_\alpha(t), t < 0\} \stackrel{d}{=} \{-S_\alpha(-t), t < 0\}$$

and letting $\{S_\alpha(t), t < 0\}$ and $\{S_\alpha(t), t \ge 0\}$ be independent. We furthermore assume that the distribution of $S_\alpha(1)$ is full in the sense that it is not concentrated in any proper hyperplane of $\mathbf{R}^d$. Let $L^\alpha(\mathbf{R})$ be the class of $f\colon \mathbf{R} \to \mathbf{R}$ with $|f(u)|^\alpha$ integrable.

Theorem 9.1 (Rajput and Rama-Murthy[(11)]). Let f be a nonrandom function in $L^\alpha(\mathbf{R})$. Then the stochastic integral

$$I(f) = \int_{-\infty}^{\infty} f(u) dS_\alpha(u)$$

can be defined in the sense of convergence in probability and $I(f)$ is also symmetric α-semi-stable.

This is a special case of Theorem 4.2 of Rajput and Rama-Murthy(11). The stochastic integral $I(f)$ is defined in such a way that $I(f) = \sum_{j=1}^{k} c_j\{S_\alpha(u_j) - S_\alpha(u_{j-1})\}$ whenever $f(u) = \sum_{j=1}^{k} c_j I_{(u_{j-1},u_j]}(u)$, $u_j < u_{j+1}$, and that $I(f_n) \to I(f)$ in probability as $n \to \infty$ whenever $\int |f_n(u) - f(u)|^\alpha du \to 0$.

Theorem 9.2. For each $t \geq 0$, let $f_t \in L^\alpha(\mathbf{R})$. Then the $\mathbf{R}^d$-valued stochastic process $\{X(t)\}$ defined by

$$X(t) := I(f_t) = \int_{-\infty}^{\infty} f_t(u) dS_\alpha(u), \quad t \geq 0,$$

is symmetric α-semi-stable in the sense of Definition 8.2.

Proof. Fix $n \geq 1$ and $0 \leq t_1 < t_2 < \cdots < t_n$. We show that $I := (I(f_{t_1}), \cdots, I(f_{t_n}))$ is α-semi-stable in $\mathbf{R}^{d\times n}$. For the notational simplicity, we write $f_\ell = f_{t_\ell}$.

Let $z = (z_1, \cdots, z_n) \in \mathbf{R}^{d\times n}, z_\ell \in \mathbf{R}^d, 1 \leq \ell \leq n$. Suppose that $f_1, \cdots, f_n$ are simple functions and have the forms $f_\ell(u) = \sum_{j=1}^{k} c_{\ell j} I_{(u_{j-1},u_j]}(u)$, where the subdivision $u_0 < u_1 < \cdots < u_k$ is taken commonly for all $f_\ell, 1 \leq \ell \leq n$. Then $I(f_\ell) = \sum_{j=1}^{k} c_{\ell j}\{S_\alpha(u_j) - S_\alpha(u_{j-1})\}$. Let $\chi(z) := E[e^{i\langle z, S_\alpha(1)\rangle}]$ and $\psi(z) := -\log \chi(z)$. Then $\chi(z)^a = \chi(a^{1/\alpha} z)$ for some $0 < a < 1$ and $E[e^{i\langle z, S_\alpha(t) - S_\alpha(s)\rangle}] = \chi(z)^{t-s}$ for $s < t$. We have

$$\begin{aligned}
\varphi(z) :=& E[e^{i\langle z, I\rangle}] = E\left[\exp\left\{i\sum_{\ell=1}^{n}\langle z_\ell, I(f_\ell)\rangle\right\}\right] \\
=& E\left[\exp\left\{i\sum_{\ell=1}^{n}\Big\langle z_\ell, \sum_{j=1}^{k} c_{\ell j}\{S_\alpha(u_j) - S_\alpha(u_{j-1})\}\Big\rangle\right\}\right] \\
=& E\left[\exp\left\{i\sum_{j=1}^{k}\Big\langle \sum_{\ell=1}^{n} c_{\ell j} z_\ell, S_\alpha(u_j) - S_\alpha(u_{j-1})\Big\rangle\right\}\right] \\
=& \prod_{j=1}^{k} E\left[\exp\left\{i\Big\langle \sum_{\ell=1}^{n} c_{\ell j} z_\ell, S_\alpha(u_j) - S_\alpha(u_{j-1})\Big\rangle\right\}\right]
\end{aligned}$$

$$= \prod_{j=1}^{k} \chi \left(\sum_{\ell=1}^{n} c_{\ell j} z_\ell \right)^{u_j - u_{j-1}} .$$

Thus we have

$$\varphi(z)^a = \prod_{j=1}^{k} \chi \left(\sum_{\ell=1}^{n} c_{\ell j} z_\ell \right)^{a(u_j - u_{j-1})} = \prod_{j=1}^{k} \chi \left(a^{1/\alpha} \sum_{\ell=1}^{n} c_{\ell j} z_\ell \right)^{u_j - u_{j-1}} = \varphi(a^{1/\alpha} z),$$

concluding that φ is symmetric α-semi-stable when f_ℓ's are simple functions. The approximation of general $f_\ell \in L^\alpha(\mathbf{R})$, $1 \le \ell \le n$, by sequences of simple functions can be carried out by the standard argument. Basic facts for the proof of convergence of the corresponding integrals are that there exist constants $0 < K_1 \le K_2 < \infty$ such that $K_1 |z|^\alpha \le \psi(z) \le K_2 |z|^\alpha$ and that

$$E\left[\exp\left\{ i \left\langle z, \int_{-\infty}^{\infty} f(u) dS_\alpha(u) \right\rangle \right\} \right] = \exp\left\{ - \int_{-\infty}^{\infty} \psi(f(u)z) du \right\}, \quad \forall f \in L^\alpha(\mathbf{R}).$$

Thus we can conclude that $\{I(f_t), t \ge 0\}$ is symmetric and α-semi-stable in the sense of Definition 8.2. □

Theorem 9.3. Let $0 < H < 1$ and $0 < \alpha \le 2$ with $H \neq 1/\alpha$. Then

$$X(t) = \int_{-\infty}^{\infty} (|t-u|^{H-1/\alpha} - |u|^{H-1/\alpha}) dS_\alpha(u), \quad t \ge 0,$$

is well-defined, and $\{X(t), t \ge 0\}$ is symmetric α-semi-stable, H-semi-selfsimilar, and has stationary increments.

Proof. For the well-definedness, it is enough to check $f_t \in L^\alpha(\mathbf{R})$ for $f_t(u) = |t-u|^{H-1/\alpha} - |u|^{H-1/\alpha}$, but this is easily verified. Here we use the assumption $H < 1$. The symmetricity and the α-semi-stability are from Theorem 9.2.

The H-semi-selfsimilarity is shown as follows. Since $\{S_\alpha(u), u \in \mathbf{R}\}$, is $(1/\alpha)$-semi-selfsimilar, it satisfies $\{S_\alpha(au), u \in \mathbf{R}\} \stackrel{d}{=} \{a^{1/\alpha} S_\alpha(u), u \in \mathbf{R}\}$ for some $a > 1$. Thus

$$\{X(at)\} = \left\{ \int_{-\infty}^{\infty} f_{at}(u) dS_\alpha(u) \right\} = \left\{ \int_{-\infty}^{\infty} f_{at}(au) dS_\alpha(au) \right\}$$
$$\stackrel{d}{=} \left\{ a^{H-1/\alpha} \int_{-\infty}^{\infty} (|t-u|^{H-1/\alpha} - |u|^{H-1/\alpha}) a^{1/\alpha} dS_\alpha(u) \right\} = \{a^H X(t)\}.$$

That $\{X(t)\}$ has stationary increments is given by the stationary increments property of $\{S_\alpha(u)\}$. This completes the proof. $\square$

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