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# SEMI-SELFDECOMPOSABLE DISTRIBUTIONS AND A NEW CLASS OF LIMIT THEOREMS

(Running head : Semi-selfdecomposable distributions)

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**Summary.** Self-decomposable distributions are given as limits of normalized sums of independent random variables. We define semi-selfdecomposable distributions as limits of subsequences of normalized sums. More generally, we introduce a way of making a new class of limiting distributions derived from a class of distributions by taking the limits through subsequences of normalized sums, and define the class of semi-selfdecomposable distributions and a decreasing sequence of subclasses of it. We give two kinds of necessary and sufficient conditions for distributions belonging to those classes, one is in terms of the decomposability of random variables and another is in terms of Lévy measures.

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## 1 Introduction

Stable distributions are characterized as limiting distributions of normalized partial sums of independent and identically distributed random variables. Self-decomposable distributions (also called $L$ distributions) are natural extensions of stable distributions and are given by limiting distributions of normalized partial sums of independent random variables, which are not necessarily identically distributed but satisfy the infinitesimal condition, (see Remark 1.1 below). For details about stable and self-decomposable distributions, the readers can refer to, e.g., Gnedenko-Kolmogorov [4]. Urbanik [10], [11] and Sato [8] studied a sequence of decreasing subclasses $\{L_m, 1 \leq m \leq \infty\}$ of self-decomposable distributions containing all stable distributions.

On the other hand, semi-stable distributions have also been well studied as extensions of stable distributions for long time. (See, e.g., Meerschaert-Scheffler [7] and the references therein. Also see Choi [2].) As is well known, semi-stable distributions are characterized as limiting distributions of subsequences of normalized partial sums of independent and identically distributed random variable. However, although self-decomposable distributions are natural extensions of stable distributions, it seems that the distributions extending semi-stable distributions in the same way have not been well recognized.

In this paper, we shall define *semi-selfdecomposable* distributions as limiting distributions of subsequences of normalized partial sums of independent random variables, which are not necessarily identically distributed but satisfy the infinitesimal condition. We shall introduce a way of making a new class of limiting distributions derived from a class of distributions, which is similar to that introduced by Sato [8]. We replace limits of full sequences in Sato [8] by that of subsequences. This idea leads us to constructing not only the class of semi-selfdecomposable but also new decreasing classes corresponding to $\{L_m\}$ by Urbanik and Sato.

Throughout this paper, we shall use the following notation. $\mathcal{P}(\mathbf{R}^d)$ is the class of all probability distributions on $\mathbf{R}^d$, $I(\mathbf{R}^d)$ is the class of all infinitely divisible distributions on $\mathbf{R}^d$, $L(\mathbf{R}^d)$ is the class of all self-decomposable (or $L$) distributions on $\mathbf{R}^d$, $SS(\mathbf{R}^d)$ is the class of all semi-stable distributions on $\mathbf{R}^d$, $\widehat{\mu}(z)$ is the characteristic function of $\mu \in \mathcal{P}(\mathbf{R}^d)$, $\mu^{*t}$ is $t$-th convolution of $\mu$, $t \geq 0$, $\mathcal{L}(X)$ is the law of $X$, $\langle \ , \ \rangle$ is the Euclidean inner product in $\mathbf{R}^d$, and $\| \cdot \|$ is the norm induced by $\langle \ , \ \rangle$ in $\mathbf{R}^d$.

**Definition 1.1.** Let $H \subset \mathcal{P}(\mathbf{R}^d)$ and $0 < b < 1$. $\mu \in \mathcal{P}(\mathbf{R}^d)$ is said to belong to the class $Q(H, b)$ if there exist independent $\mathbf{R}^d$-valued random vectors $\{X_j\}, a_n > 0, \uparrow \infty, c_n \in \mathbf{R}^d, \{k_n\} \subset \mathbf{N}, k_n \uparrow \infty$ such that

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = b, \tag{1.1}$$

$$\mathcal{L}(X_j) \in H, \tag{1.2}$$

$$\mathcal{L}\left(\frac{1}{a_n}\sum_{j=1}^{k_n} X_j + c_n\right) \to \mu, \tag{1.3}$$

$$\lim_{n\to\infty} \max_{1\leq j\leq k_n} P\left\{\left\|\frac{1}{a_n}X_j\right\| > \varepsilon\right\} = 0, \quad \forall\varepsilon > 0. \tag{1.4}$$

**Remark 1.1.** When (1.4) is satisfied, we say that random variables $\{a_n^{-1}X_j, 1 \leq j \leq k_n, n = 1, 2, \cdots\}$ satisfy the infinitesimal condition.

**Remark 1.2.** If we can take $k_n = n$, (then automatically $b = 1$), then the above class is turned out to be the one which Sato [8] studied.

**Remark 1.3.** Although we are dealing with $\mathbf{R}^d$-valued random variables $\{X_j\}$, the normalization in (1.3) is scalar. The normalization by linear operators might be the natural procedure in higher dimensions. Generalization to the operator normalization will be studied in the forthcoming paper.

In Section 2, we shall give some basic results on $Q(H, b)$. In Section 3, the class of semi-selfdecomposable distributions will be introduced as $Q(\mathcal{P}(\mathbf{R}^d), b)$, together with a decreasing sequences of subclasses of it given through the procedure in Definition 1.1. Also a necessary and sufficient condition for a distribution belonging to those classes will be discussed. In Section 4, we shall give another necessary and sufficient condition in terms of Lévy measures.

After we completed this paper, we had a chance to see a recent paper by Bunge [1]. We want to make some comments on his paper here.

For $\mu \in \mathcal{P}(\mathbf{R}^1)$, Urbanik decomposability semigroup $D(\mu)$ is defined as the set of $c \in \mathbf{R}$ such that $\widehat{\mu}(z) = \widehat{\mu}(cz)\widehat{\rho}_c(z), \forall z \in \mathbf{R}$ for for some $\rho_c \in \mathcal{P}(\mathbf{R}^1)$. In Bunge [1], he says that $\mu$ is $C$-decomposable if $C \subset D(\mu)$, where $C$ is an arbitrary closed multiplicative subsemigroup of $[0, 1]$ containing 0 and 1, and denotes that the set of such laws by $L^C$. $L^{[0,1]}$ is the class of all self-decomposable distributions on $\mathbf{R}^1$. He also introduces the class $\mathcal{L}^C(H)$ for $H \subset \mathcal{P}(\mathbf{R}^1)$ as follows:

$$\begin{aligned}\mathcal{L}^C(H) = &\{\mu \in \mathcal{P}(\mathbf{R}^1) : \forall c \in C \setminus \{0, 1\},\\ &\text{there exist independent random variables } \{X_j\} \subset H\\ &\text{and } \{a_n\} \subset (0, \infty), \{c_n\} \subset \mathbf{R} \text{ such that}\\ &\lim_{n\to\infty} a_n/a_{n+1} = c \text{ and } \mathcal{L}\left(a_n{}^{-1}\sum_{j=1}^{n} X_j + c_n\right) \to \mu\}. \end{aligned} \tag{1.5}$$

Then he studies

$$L_m^C = \mathcal{L}^C(L_{m-1}^C), m = 0, 1, 2, \cdots \quad \text{with } L_{-1}^C = \mathcal{P}(\mathbf{R}^1).$$

Among other results he gives a necessary and sufficient condition for that $\mu \in L_m^C$, which has a similar form as one of our results below.

However, a big difference between his paper and ours is that we are dealing with random variables satisfying the infinitesimal condition, but it is not necessarily the case in Bunge [1]. Thus, the distributions studied in Bunge [1] are not necessarily infinitely divisible. Nevertheless, he shows that $\mu \in L_\infty^C = \bigcap_{m=0}^{\infty} L_m^C$

is infinitely divisible. Therefore it might be interesting to characterize the class $L^C_\infty$ by Lévy measure and compare it with our classes in this paper.

Finally, we must mention the name of Loève as a pioneer of the decomposable problem. (This history is also mentioned in Bunge [1].) Loève [5] (also see [6], page 352) defined the $c$-decomposability in one dimension. Namely, $\mu$ is called $c$-decomposable for $0 < c < 1$, if $\mu$ is in $L^C(\mathcal{P}(\mathbf{R}^1))$, where $C$ in (1.5) is $\{c^n\}_{n=0}^\infty \cup \{0\}$, and he proved results similar to our Theorems 2.1, 3.1, 3.2 and 3.5 below in his setting. Also, when $c$-decomposable $\mu$ is infinitely divisible, he gave one-dimensional version of our Lemma 4.1 below.

## 2 Some basic results for $Q(H, b)$

**Proposition 2.1.** $Q(H, b) \subset I(\mathbf{R}^d)$.

*Proof.* Obvious from (1.3) and (1.4). □

**Proposition 2.2.** Let $\nu \in \mathcal{P}(\mathbf{R}^d)$. Then $Q(\{\nu\}, b) \subset SS(\mathbf{R}^d)$.

*Proof.* Sine $\mathcal{L}(X_j) = \nu$ in Definition 1.1, $\{X_j\}$ are i.i.d. random vectors. So the requirements in Definition 1.1 are well known as those for that the limiting distribution $\mu$ is semi-stable.

**Remark 2.1.** In the ordinary definition of semi-stable distributions, it is assumed that $\lim_{n\to\infty} k_n/k_{n+1} (=: p,$ say) exists, instead of (1.1). However, as a result, (1.1) follows. Furthermore it is known that there exists $\alpha \in (0, 2]$ uniquely so that $b^\alpha = p$.

**Proposition 2.3.** Let $0 < b < 1$. Then $Q(H, b) \subset Q(H, b^m), \quad m = 1, 2, , \cdots$ .

*Proof.* Let $\mu \in Q(H, b)$. Then there exist $\{X_j\}, \{a_n\}, \{c_n\}, \{k_n\}$ as in Definition 1.1. Define

$$\widetilde{a}_n = a_{nm}, \quad \widetilde{c}_n = c_{nm}, \quad \widetilde{k}_n = k_{nm}.$$

Then

$$\lim_{n\to\infty}\frac{\widetilde{a}_n}{\widetilde{a}_{n+1}}=\lim_{n\to\infty}\frac{a_{nm}}{a_{(n+1)m}}=b^m,$$

and (1.3) and (1.4) hold with the replacements of $a_n, k_n, c_n$ by $\widetilde{a}_n, \widetilde{k}_n, \widetilde{c}_n$. Hence $\mu \in Q(H, b^m)$. $\square$

**Definition 2.1.** $H \subset \mathcal{P}(\mathbf{R}^d)$ is said to be completely closed if $H$ is closed under convergence, convolution and type equivalence. Here $H$ is said to be closed under type equivalence if $\mathcal{L}(X) \in H$ implies $\mathcal{L}(aX + c) \in H$ for any $a > 0$ and $c \in \mathbf{R}^d$.

**Definition 2.2.** $H \subset \mathcal{P}(\mathbf{R}^d)$ is said to be completely closed in the strong sense if $H \subset I(\mathbf{R}^d)$, $H$ is completely closed and $H$ is closed under $t$-th convolution for any $t > 0$.

**Proposition 2.4.** Let $0 < b < 1$ and suppose that $H$ is completely closed. Then $Q(H, b) \subset H$.

*Proof.* If $\mu \in Q(H, b)$, then it follows from the complete closedness of $H$ and (1.3) that $\mu \in H$. $\square$

**Theorem 2.1.** Let $0 < b < 1$.

(i) Suppose that $H \subset \mathcal{P}(\mathbf{R}^d)$ is completely closed. Then a necessary condition for that $\mu \in Q(H, b)$ is that there exists $\rho_0 \in H \cap I(\mathbf{R}^d)$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_0(z), \quad \forall z \in \mathbf{R}^d. \tag{2.1}$$

(ii) Suppose that $H$ is completely closed in the strong sense. Then the existence of $\rho_0 \in H \cap I(\mathbf{R}^d)(= H)$ satisfying (2.1) is also sufficient for that $\mu \in Q(H, b)$.

(iii) If $H$ is completely closed in the strong sense, so is $Q(H, b)$.

*Proof.* (i) Suppose $\mu \in Q(H, b)$. Then there exist $\{X_j\}, \{a_n\}, \{c_n\}$ and $\{k_n\}$

satisfying the conditions in Definition 1.1. We have

$$\frac{1}{a_n}\sum_{j=1}^{k_n} X_j + c_n = \left(\frac{1}{a_n}\sum_{j=1}^{k_{n-1}} X_j + \frac{a_{n-1}}{a_n}c_{n-1}\right) + \left(\frac{1}{a_n}\sum_{j=k_{n-1}+1}^{k_n} X_j + c_n - \frac{a_{n-1}}{a_n}c_{n-1}\right) \tag{2.2}$$

and denote the characteristic functions of the left hand side of (2.2) and of the first and the second terms on the right hand side of (2.2) by $\varphi_n(z), \varphi_{n,1}(z)$ and $\varphi_{n,2}(z)$, respectively.

By (1.3), $\varphi_n(z) \to \widehat{\mu}(z)$. Next if we use (1.1), we have

$$\varphi_{n,1}(z) = E\left[\exp\left\{i\left\langle z, \frac{a_{n-1}}{a_n}\left(\frac{1}{a_{n-1}}\sum_{j=1}^{k_{n-1}} X_j + c_{n-1}\right)\right\rangle\right\}\right] \to \widehat{\mu}(bz).$$

By Proposition 2.1, we know that $\mu \in I(\mathbf{R}^d)$, and hence $\widehat{\mu}(z) \neq 0, \forall z \in \mathbf{R}^d$. Therefore, the limit $\chi(z) := \lim_{n\to\infty} \varphi_{n,2}(z)$ exists and

$$\chi(z) = \frac{\widehat{\mu}(z)}{\widehat{\mu}(bz)}. \tag{2.3}$$

The right hand side of (2.3) is continuous at $z = 0$. Hence $\chi(z)$ is the characteristic function of a probability distribution ($\rho_0$, say). Thus $\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_0(z), \rho_o \in \mathcal{P}(\mathbf{R}^d)$.

Since $\rho_0$ is the limiting distribution of the normalized sums of independent random vectors $\{X_j\}$ satisfying (1.4), we have $\rho_0 \in I(\mathbf{R}^d)$. Furthermore, the assumption that $\mathcal{L}(X_j) \in H$ and the complete closedness of $H$ imply that $\rho_0 \in H$.

(ii) We first note that (2.1) implies that $\widehat{\mu}(z) \neq 0, \forall z \in \mathbf{R}^d$. If not, there exists $z_0 \in \mathbf{R}^d$ such that $\widehat{\mu}(z_0) = 0, \widehat{\mu}(z) \neq 0$ when $\|z\| < \|z_0\|$. Then by (2.1), $0 = \widehat{\mu}(z_0) = \widehat{\mu}(bz_0)\widehat{\rho}(z_0)$. Since $0 < b < 1, \widehat{\mu}(bz_0) \neq 0$, and so $\widehat{\rho}(z_0) = 0$, which contradicts the assumption that $\rho_0 \in H \cap I(\mathbf{R}^d)$.

We construct $\{X_j\}, \{a_n\}.\{c_n\}, \{k_n\}$ satisfying (1.1)–(1.4) in Definition 1.1. First choose $\{k_n\} \subset \mathbf{N}$ $(k_0 = 0)$ such that

$$\lim_{n\to\infty} \max_{1\le i\le n} \left|\widehat{\rho}_0(b^{n-i}z)^{1/(k_i-k_{i-1})} - 1\right| = 0. \tag{2.4}$$

It is enough to choose $\{k_n\}$ such that $k_n - k_{n-1} \to \infty$ as $n \to \infty$. Define a sequence of independent random vectors $\{X_j\}$ by

$$\widehat{\mathcal{L}(X_j)}(z) = \widehat{\rho}_0(b^{-i}z)^{1/(k_i-k_{i-1})}, \quad k_{i-1} < j \le k_i$$

and let

$$Y_n = b^n \sum_{j=1}^{k_n} X_j.$$

Then we have

$$\begin{aligned}\widehat{\mathcal{L}(Y_n)}(z) &= \prod_{j=1}^{k_n} \widehat{\mathcal{L}(X_j)}(b^n z) = \prod_{i=1}^{n} \prod_{j=k_{i-1}+1}^{k_i} \widehat{\mathcal{L}(X_j)}(b^n z) \\ &= \prod_{i=1}^{n} \widehat{\rho}_0(b^{n-i}z) = \frac{\widehat{\mu}(b^{n-1}z)}{\widehat{\mu}(b^n z)} \cdots \frac{\widehat{\mu}(z)}{\widehat{\mu}(bz)} = \frac{\widehat{\mu}(z)}{\widehat{\mu}(b^n z)},\end{aligned}$$

where we have used the fact that $\widehat{\mu}(z) \ne 0, \forall z \in \mathbf{R}^d$. Thus

$$\lim_{n\to\infty} \widehat{\mathcal{L}(Y_n)}(z) = \widehat{\mu}(z),$$

which assures (1.1) and (1.3) with $a_n = b^{-n}$.

It follows from (2.4) that

$$\lim_{n\to\infty} \max_{1\le j\le k_n} |\widehat{\mathcal{L}(b^n X_j)}(z) - 1| = 0,$$

implying (1.4). (See, e.g., Chung [3], Theorem 7.1.1.) Furthermore, the assumption that $\rho_0 \in H \cap I(\mathbf{R}^d)$ and the complete closedness in the strong sense of $H$ assure that $\rho_0^{*t} \in H$, implying that $\mathcal{L}(X_j) \in H$. This is (1.2). This completes the proof of the statement (ii) of the theorem.

(iii) We only show that $Q(H,b)$ is closed under convergence. Let $\mu_n \in Q(H,b)$ and suppose $\mu_n \to \mu_\infty$. By (2.1), for each $n \geq 1$,

$$\widehat{\mu}_n(z) = \widehat{\mu}_n(bz)\widehat{\rho}_{n,0}(z), \quad \rho_{n,0} \in H \cap I(\mathbf{R}^d).$$

Since $\mu_n$ converges, $\lim_{n\to\infty} \rho_{n,0} (=: \rho_{\infty,0}$ , say) exists. Since $H$ and $I(\mathbf{R}^d)$ are completely closed, $\rho_{\infty,0} \in H \cap I(\mathbf{R}^d)$. Therefore

$$\widehat{\mu}_\infty(z) = \widehat{\mu}_\infty(bz)\widehat{\rho}_{\infty,0}(z), \quad \rho_{\infty,0} \in H \cap I(\mathbf{R}^d),$$

which means that $\mu_\infty \in Q(H,b)$ by the statement (ii) of this theorem.

Closedness under convolution, type equivalence and $t$-convolution can be shown similarly. □

## 3 The class of semi-selfdecomposable distributions and its subclasses

In this section, we define the class of semi-selfdecomposable distributions and its subclasses given through the procedure in Definition 1.1 staring from $H = \mathcal{P}(\mathbf{R}^d)$, and give some characterizations for these classes.

For each $b \in (0,1)$, define

$$L_0(b) = Q(\mathcal{P}(\mathbf{R}^d), b)$$

and

$$L_m(b) = Q(L_{m-1}(b), b), \quad m = 1, 2, , \cdots . \tag{3.1}$$

**Definition 3.1.** Let $0 < b < 1$. A probability distribution $\mu \in L_0(b)$ is called $b$-semi-selfdecomposable, or simply semi-selfdecomposable on $\mathbf{R}^d$.

**Remark 3.1.** According to the terminology by Loève [5], our $\mu \in L_0(b)$ may should be called $b$-decomposable. However, $b$-decomposable distributions in the sense of Loève are not necessarily infinitely divisible. Our $L_0(b)$ is a subclass of $I(\mathbf{R}^d)$. So, in parallel to naming for semi-stable random variables, we want to call our $\mu \in L_0(b)$ semi-selfdecomposable.

**Theorem 3.1.** Let $0 < b < 1$.

(i) A necessary and sufficient condition for that $\mu \in L_0(b)$ is that for some $\rho_0 \in I(\mathbf{R}^d)$, (2.1) holds.

(ii) $L_0(b)$ is completely closed in the strong sense.

*Proof.* First note that since $H = I(\mathbf{R}^d)$ is completely closed in the strong sense, it follows from Theorem 2.1 that Theorem 3.1 is true if we replace $L_0(b)$ by $Q(I(\mathbf{R}^d), b)$. So, to prove the theorem, it is enough to show that

$$Q(\mathcal{P}(\mathbf{R}^d), b) = Q(I(\mathbf{R}^d), b). \tag{3.2}$$

However, since $Q(\mathcal{P}(\mathbf{R}^d), b) \supset Q(I(\mathbf{R}^d), b)$, it is enough to show that $Q(\mathcal{P}(\mathbf{R}^d), b) \subset Q(I(\mathbf{R}^d), b)$.

Suppose that $\mu \in Q(\mathcal{P}(\mathbf{R}^d), b)$. Since $H = \mathcal{P}(\mathbf{R}^d)$ is completely closed, it follows from (i) of Theorem 2.1 that for some $\rho_0 \in I(\mathbf{R}^d)$, $\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_0(z)$, $\forall z \in \mathbf{R}^d$. Since this is the condition in (ii) of Theorem 2.1 with $H = I(\mathbf{R}^d)$, we see that $\mu \in Q(I(\mathbf{R}^d), b)$. This concludes (3.2), completing the proof of the theorem. □

**Theorem 3.2.**

$$L(\mathbf{R}^d) = \bigcap_{0<b<1} L_0(b).$$

*Proof.* Theorem 2.1 and Corollary 2.4 in Sato [8] assure that $\mu \in L(\mathbf{R}^d)$ if and only if for any $b \in (0,1)$, there exists $\rho_b \in I(\mathbf{R}^d)$ such that $\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_b(z), \forall z \in \mathbf{R}^d$. Hence by (i) of Theorem 3.1 we conclude that $\mu \in L(\mathbf{R}^d)$ if only if $\mu \in L_0(b)$ for any $b \in (0,1)$. □

**Theorem 3.3.** Let $0 < b < 1$ and $m = 1, 2, \cdots$.

(i) $L_m(b)$ is completely closed in the strong sense.

(ii) A necessary and sufficient condition for that $\mu \in L_m(b)$ is that for some $\rho_m \in L_{m-1}(b)$,

$$\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_m(z), \quad \forall z \in \mathbf{R}^d.$$

*Proof.* By (ii) of Theorem 3.1, $L_0(b)$ is completely closed in the strong sense. So, by the definition (3.1) and (iii) of Theorem 2.1, $L_m(b), m = 1, 2, \cdots$, are completely closed in the strong sense, which is the statement (i). Then the statement (ii) is given by (i) and (ii) of Theorem 2.1 and Proposition 2.1. □

**Theorem 3.4.** Let $0 < b < 1$ and put

$$L_\infty(b) = \bigcap_{m=0}^{\infty} L_m(b).$$

Then

(i) $L_\infty(b)$ is completely closed in the strong sense.

(ii) $L_\infty(b) = Q(L_\infty(b), b)$. Namely, $L_\infty(b)$ is invariant under the $Q(\cdot, b)$-operation and hence a necessary and sufficient condition for that $\mu \in L_\infty(b)$ is that for some $\rho_\infty \in L_\infty(b), \widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_\infty(z), \forall z \in \mathbf{R}^d$.

(iii) $L_\infty(b)$ is the largest class among the classes which are invariant under the $Q(\cdot, b)$-operation.

*Proof.* (i) We have seen that $L_m(b), m \geq 0$, are completely closed in the strong sense. So is $L_\infty(b)$.

(ii) Since $L_\infty(b)$ is completely closed, it follows from Proposition 2.4 that $L_\infty(b) \supset Q(L_\infty(b), b)$. So, it is enough to show the converse inclusion.

Let $\mu \in L_\infty(b)$. Then $\mu \in L_m(b), \forall m \geq 0$. By (i) of Theorem 3.1 and (ii) of Theorem 3.3, for each $m \geq 0$ there exists $\rho_m \in L_{m-1}(b)$ (with the convention $L_{-1}(b) = I(\mathbf{R}^d)$) such that $\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_m(z)$. Since $\widehat{\mu}(z) \neq 0, \forall z \in \mathbf{R}^d$, we have

$$\widehat{\rho}_m(z) = \frac{\widehat{\mu}(z)}{\widehat{\mu}(bz)},$$

which is independent of $m$. If we write it $\widehat{\rho}_\infty$, then $\rho_\infty \in L_{m-1}(b), \forall m \geq 0$, equivalently, $\rho_\infty \in L_\infty(b)$ and

$$\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_\infty(z), \quad \forall z \in \mathbf{R}^d.$$

Since $L_\infty(b)$ is completely closed in the strong sense, it follows from (ii) of Theorem 2.1 that $\mu \in Q(L_\infty(b), b)$.

(iii) Suppose that $H(\subset \mathcal{P}(\mathbf{R}^d))$ satisfies that $Q(H, b) = H$. Obviously for each $m \geq 0$

$$H = Q^m(H, b) \subset Q^m(\mathcal{P}(\mathbf{R}^d), b) = L_{m-1}(b),$$

where

$$Q^m(H, b) = \overbrace{Q(Q(\cdots(Q(Q}^{m}(H, b), b), \cdots), b).$$

Thus

$$H \subset \bigcap_{m=0}^{\infty} L_{m-1}(b) = L_\infty(b).$$

The proof of the theorem is thus completed. □

**Corollary 3.1.** Let $0 < b < 1$.

$$I(\mathbf{R}^d) \supset L_0(b) \supset \cdots \supset L_m(b) \supset \cdots \supset L_\infty(b).$$

*Proof.* Obvious from Proposition 2.1, the complete closedness of $L_m(b)$, $m \geq 0$, and Proposition 2.4. □

**Theorem 3.5.** Let $\mu \in SS(\mathbf{R}^d)$ and suppose that for some $a, b \in (0, 1)$ and $c \in \mathbf{R}^d$,

$$\widehat{\mu}(z)^a = \widehat{\mu}(bz)e^{i\langle z,c\rangle}, \quad \forall z \in \mathbf{R}^d. \tag{3.3}$$

Then $\mu \in L_\infty(b)$.

*Proof.* If we put

$$\widehat{\rho}(z) = \widehat{\mu}(z)^{1-a} e^{i\langle z,c\rangle}, \tag{3.4}$$

then we have from (3.3) that

$$\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}(z). \tag{3.5}$$

So, to have that $\mu \in L_\infty(b)$, it is enough to show that $\rho \in L_\infty(b)$ by (ii) of Theorem 3.4.

Since $\mu \in SS(\mathbf{R}^d) \subset I(\mathbf{R}^d)$, $\rho$ in (3.4) is in $I(\mathbf{R}^d)$. Thus by (3.5), $\mu \in L_0(b)$. Since $L_0(b)$ is closed under $t$-th convolution and under type equivalence, we have that $\rho \in L_0(b)$. Thus again by (3.5), we see that $\mu \in L_1(b)$. Similarly we conclude that $\mu \in L_m(b), \forall m \geq 0$. □

## 4 Characterizations of $L_m(b)$ by Lévy measures

Any probability distribution $\mu \in L_m(b), m \geq 0$, is infinitely divisible and its characteristic function $\widehat{\mu}(z)$ has the following Lévy-Khintchine representation:

$$\widehat{\mu}(z) = \exp\left\{ i\langle z,\gamma\rangle - \frac{1}{2}\langle Az,z\rangle + \int_{\mathbf{R}^d\setminus\{0\}} g(z,x)\nu(dx) \right\}, \tag{4.1}$$

where $\nu$ (called Lévy measure of $\widehat{\mu}(z)$) is a measure on $\mathbf{R}^d \setminus \{0\}$ satisfying $\int_{\mathbf{R}^d\setminus\{0\}}(1 \wedge \|x\|^2)\nu(dx) < \infty$, $A$ is a symmetric nonnegative definite matrix, $\gamma \in \mathbf{R}^d$,

$$g(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle 1_D(x),$$

$D = \{x \in \mathbf{R}^d : \|x\| \leq 1\}$ and $1_D(x)$ is the indicator function of $D$. In this section, we give a necessary and sufficient condition for that $\mu \in L_m(b)$ in terms of its Lévy measure $\nu$.

**Definition 4.1.** $f : \mathbf{R} \to \mathbf{R}$ is said to be wide-sense convex with period $a > 0$, if for any $s \in \mathbf{R}$ and $\delta > 0$,

$$f(s+\delta) - f(s) \leq f(s+\delta+a) - f(s+a).$$

Let $S = \{x \in \mathbf{R}^d : \|x\| = 1\}$ and for any $E \in \mathcal{B}((0,\infty))$ and $B \in \mathcal{B}(S)$, write

$$EB := \{x \in \mathbf{R}^d \setminus \{0\} : x = r\xi, r \in E, \xi \in B\}.$$

**Lemma 4.1.** Let $\mu \in I(\mathbf{R}^d)$ and $\nu$ be its Lévy measure. Let $0 < b < 1$. A necessary and sufficient condition for that $\mu \in L_0(b)$ is that for any $G \in \mathcal{B}(\mathbf{R}^d \setminus \{0\}), \nu(bG) \geq \nu(G)$, where $bG = \{bx : x \in G\}$. Furthermore, this condition is equivalent to that for each $B \in \mathcal{B}(S)$, $f(s) = \nu((e^{-s},\infty)B)$ is wide-sense convex with the period $-\log b$.

*Proof.* Recall from (i) of Theorem 3.1 that $\mu$ belong to $L_0(b)$ if and only if there exists $\rho_0 \in I(\mathbf{R}^d)$ such that

$$\widehat{\rho}_0(z) = \frac{\widehat{\mu}(z)}{\widehat{\mu}(bz)}.$$

If we write the Lévy-Khintchine representation of $\widehat{\rho}_0(z)$ formally, it is

$$\exp\left\{ i\langle z, (1-b)\gamma + c\rangle - \frac{1}{2}\langle (1-b^2)Az, z\rangle + \int_{\mathbf{R}^d\setminus\{0\}} g(z,x)\left(\nu(dx) - \nu(\frac{dx}{b})\right)\right\}, \quad (4.2)$$

where

$$c = \int_{b<\|bx\|\leq 1} bx\nu(dx).$$

Thus, for that $\rho_0 \in I(\mathbf{R}^d)$, namely for that (4.2) is actually the Lévy-Khintchine representation of an infinitely divisible distribution function, it is necessary and sufficient that

$$\nu(G) - \nu(\frac{1}{b}G) \geq 0, \quad \forall G \in \mathcal{B}(\mathbf{R}^d \setminus \{0\}).$$

As to the second half of the lemma, it is enough to notice that if $\nu(G) \leq \nu(bG)$, then

$$\begin{aligned} f(s+\delta) - f(s) &= \nu((e^{-s-\delta}, e^{-s}]B) \\ &\leq \nu((be^{-s-\delta}, be^{-s}]B) = f(s+\delta-\log b) - f(s-\log b), \end{aligned}$$

and its converse argument is also verified. □

**Theorem 4.1.** Let $0 < b < 1$. For that $\mu \in L_0(b)$, it is necessary and sufficient that $\mu \in I(\mathbf{R}^d)$ and its Lévy measure is $\nu \equiv 0$ or

$$\nu(EB) = -\int_B \lambda(d\xi) \int_E dF_\xi(r), \quad E \in \mathcal{B}((0,\infty)), B \in \mathcal{B}(S). \tag{4.3}$$

Here $\lambda$ is a probability measure on $S$, for each $r > 0$, $F_\xi(r)$ is $\xi$-measurable, and for each $\xi \in S$, $F_\xi(r)$ is right continuous and nonincreasing, $\lim_{r\to\infty} F_\xi(r) = 0$,

$$F_\xi(b(r+\delta)) - F_\xi(br) \le F_\xi(r+\delta) - F_\xi(r) \le 0 \tag{4.4}$$

for every $\delta > 0$ and $r > 0$, and for each $\xi \in S$

$$0 < -\int_0^\infty (1 \wedge r^2) dF_\xi(r) = \int_{\mathbf{R}^d\setminus\{0\}} (1 \wedge \|x\|^2)\nu(dx) =: K < \infty, \tag{4.5}$$

where the value $K$ is independent of $\xi$. This representation is unique in the following sense. If $\nu \not\equiv 0$ and two pairs $(\lambda, F_\xi)$ and $(\widetilde{\lambda}, \widetilde{F}_\xi)$ satisfy the above conditions, then $\lambda = \widetilde{\lambda}$ and $F_\xi = \widetilde{F}_\xi$ for $\lambda$-a.e. $\xi$. (We call $F_\xi$, uniquely determined in this sense, the $F$-function of $\mu \in L_0(b)$.)

**Remark 4.1.** If $\mu \in SS(\mathbf{R}^d)$ and for some $a, b \in (0,1)$ and $c \in \mathbf{R}^d$, $\widehat{\mu}(z)^a = \widehat{\mu}(bz)e^{i<z,c>}$, then it is known that there exists $\alpha \in (0,2]$ uniquely such that $b = a^{1/\alpha}$. (See, e.g., Choi [2], Proposition 2.5.) If $\alpha \neq 2$, $\mu$ is purely non-Gaussian and $A = O$, $\nu \not\equiv 0$ in (4.1). Then it is known that $\nu$ have the form

$$\nu(EB) = -\int_B \lambda(d\xi) \int_E d\left\{\frac{H_\xi(r)}{r^\alpha}\right\}, \quad E \in \mathcal{B}((0,\infty)), B \in \mathcal{B}(S),$$

where $\lambda$ is a probability measure on $S$, for each $r > 0$, $H_\xi(r)$ is $\xi$-measurable, for $\xi \in S$, $H_\xi$ is right continuous nonnegative function of $r > 0$, $H_\xi(r)/r^\alpha$ is nonincreasing in $r > 0$ and $H_\xi(br) = H_\xi(r)$. Also

$$0 < -\int (1 \wedge r^2) d\left\{\frac{H_\xi(r)}{r^\alpha}\right\} = \text{const.} < \infty,$$

independent of $\xi$. (See Choi [2], Proposition 2.3.) Therefore $H_\xi(r)/r^\alpha$ is the $F$-function of this $\mu$. If we put

$$F_\xi(r) = \frac{H_\xi(r)}{r^\alpha},$$

then

$$\begin{aligned} F_\xi(b(r+\delta)) - F_\xi(br) &= \frac{H_\xi(b(r+\delta))}{a(r+\delta)^\alpha} - \frac{H_\xi(br)}{ar^\alpha} \\ &= \frac{1}{a}\left\{\frac{H_\xi(r+\delta)}{(r+\delta)^\alpha} - \frac{H_\xi(r)}{r^\alpha}\right\} \le F_\xi(r+\delta) - F_\xi(r) \quad (\le 0), \end{aligned}$$

which is (4.4). Thus the $F$-function of this semi-stable $\mu$ satisfies the condition in Theorem 4.1.

**Remark 4.2.** We have seen in Theorem 3.2 that $L(\mathbf{R}^d) \subset L_0(b)$, for any $b \in (0,1)$. This can also be checked by the Lévy measure in Theorem 4.1. We know that the Lévy measure of $\mu \in L(\mathbf{R}^d)$ is

$$\nu(EB) = \int_B \lambda(d\xi) \int_E \frac{k_\xi(r)}{r} dr, \quad E \in \mathcal{B}((0,\infty)), B \in \mathcal{B}(S),$$

where $\lambda$ is a probability measure, for each $r > 0, k_\xi(r)$ is $\xi$-measurable, for each $\xi \in S$, $k_\xi(r)$ is a nonnegative nonincreasing and right continuous function of $r > 0$, and

$$0 < \int_0^\infty (1 \wedge r^2) k_\xi(r) dr = \text{const.} < \infty$$

independent of $\xi$. (See Sato [8].) Therefore $\int_r^\infty \{k_\xi(u)/u\} du$ is the $F$-function of $\mu$. If we put

$$F_\xi(r) = \int_r^\infty \frac{k_\xi(u)}{u} du,$$

then for any $b \in (0,1)$,

$$\begin{aligned} F_\xi(b(r+\delta)) - F_\xi(br) &= \int_{br}^{b(r+\delta)} dF_\xi(u) \\ &= -\int_{br}^{b(r+\delta)} \frac{k_\xi(u)}{u} du = -\int_r^{r+\delta} \frac{k_\xi(bv)}{v} dv. \end{aligned}$$

Since $k_\xi(bv) \geq k_\xi(v)$, the above is

$$\leq -\int_r^{r+\delta} \frac{k_\xi(v)}{v} dv = F_\xi(r+\delta) - F_\xi(r).$$

Thus the $F$-function of $L$ distribution satisfies the condition in Theorem 4.1 for each $0 < b < 1$ and hence $L(\mathbf{R}^d) \subset L_0(b), \forall b \in (0,1)$.

*Proof of Theorem 4.1.*

(*Necessity.*) Let $\mu \in L_0(b)$ and $\nu \not\equiv 0$. Put $N(u,B) = \nu([u,\infty)B)$. Define a probability measure $\lambda$ on $S$ by

$$\begin{aligned} \lambda(B) &= \frac{1}{K}\int_{(0,\infty)B} (1 \wedge \|x\|^2)\nu(dx) \qquad (4.6) \\ &= -\frac{1}{K}\int_0^\infty (1 \wedge u^2) dN(u,B), \quad B \in \mathcal{B}(S). \end{aligned}$$

For each $u > 0$, $N(u, \cdot)$ is absolutely continuous with respect to $\lambda$. Hence for each $s \in \mathbf{R}$, there exists a nonnegative $\xi$-measurable function $h_\xi(u)$ such that

$$N(e^{-s}, B) = \int_B h_\xi(s)\lambda(d\xi), \quad B \in \mathcal{B}(S).$$

If $s_1 < s_2$, then for any $B \in \mathcal{B}(S)$,

$$N(e^{-s_1}, B) \leq N(e^{-s_2}, B),$$

implying that for any $B \in \mathcal{B}(S)$,

$$\int_B (h_\xi(s_1) - h_\xi(s_2))\lambda(d\xi) \leq 0.$$

Thus

$$h_\xi(s_1) \leq h_\xi(s_2), \quad \lambda - a.e.\xi, \qquad (4.7)$$

where the set of $\lambda$-measure one depends on $s_1$ and $s_2$.

Now, from Lemma 4.1, $N(e^{-s}, B)$ is wide-sense convex with the period $-\log b$, and so for any $B \in \mathcal{B}(S)$, any $s \in \mathbf{R}$ and any $\delta > 0$,

$$N(e^{-(s+\delta)}, B) - N(e^{-s}, B) \leq N(e^{-(s+\delta-\log b)}, B) - N(e^{-(s-\log b)}, B).$$

Therefore for each $B \in \mathcal{B}(S)$,

$$\int_B (h_\xi(s+\delta-\log b) - h_\xi(s-\log b) - h_\xi(s+\delta) + h_\xi(s))\lambda(d\xi) \geq 0.$$

Hence for $\lambda$-a.e. $\xi$, (where the set of $\lambda$-measure one depends on $s$ and $\delta$,)

$$h_\xi(s+\delta) - h_\xi(s) \leq h_\xi(s+\delta-\log b) - h_\xi(s-\log b). \tag{4.8}$$

If we put

$$\begin{aligned} S_1 = \{\xi \in S :& (4.7) \text{ holds for all rational numbers } s_1 \text{ and } s_2 \text{ with } s_1 < s_2, \\ & \text{and (4.8) holds for all rational numbers } s \\ & \text{and for all positive rational numbers } \delta\}, \end{aligned}$$

then $\lambda(S_1) = 1$.

If we put, for each $\xi \in S_1$,

$$\widetilde{h}_\xi(s) = \sup_{s'<s, s':\text{rationals}} h_\xi(s'),$$

then $\widetilde{h}_\xi(s)$ is, for each $\xi \in S_1$, left continuous and nondecreasing with respect to $s$, and for each $s > 0$, $\xi$-measurable, and it satisfies (4.8) with the replacement of $h_\xi(s)$ by $\widetilde{h}_\xi(s)$, and then

$$N(e^{-s}, B) = \int_B \widetilde{h}_\xi(s)\lambda(d\xi).$$

If we put

$$F_\xi(r) = \widetilde{h}_\xi(-\log r),$$

then $F_\xi(r)$ is, for each $\xi \in S$, right continuous and nonincreasing with respect to $r$, and

$$N(e^{-s}, B) = -\int_B \lambda(d\xi) \int_{e^{-s}}^\infty dF_\xi(r).$$

Namely, we have

$$\nu([u,\infty)B) = N(u,B) = -\int_B \lambda(d\xi) \int_u^\infty dF_\xi(r) \tag{4.9}$$

and we see that

$$\nu(EB) = -\int_B \lambda(d\xi) \int_E dF_\xi(r). \tag{4.10}$$

On the other hand, it follows from (4.6) and (4.9) that for each $B \in \mathcal{B}(S)$,

$$\lambda(B) = -\frac{1}{K}\int_B \lambda(d\xi) \int_0^\infty (1 \wedge r^2) dF_\xi(r),$$

and hence

$$-\int_0^\infty (1 \wedge r^2) dF_\xi(r) = K, \quad \lambda - a.e.\xi, \tag{4.11}$$

which is (4.5). We denote the set of $\lambda$-measure one for which (4.11) holds by $S_2$.

To end the proof for the necessity, we show that $F_\xi(r)$ constructed above satisfies (4.4) and (4.5). For any $\xi \in (S_1 \cap S_2)^c$, we define $F_\xi(r)$ suitably so that (4.4) and (4.5) are satisfied. Since $\lambda((S_1 \cap S_2)^c) = 0$, this construction does not change the representation (4.10). For any $\xi \in S_2$, the property (4.5) is nothing but (4.11). For each $\xi \in S_1$, we check (4.4). We have

$$\begin{aligned}
&F_\xi(b(r+\delta)) - F_\xi(br) \\
&\quad = \widetilde{h}_\xi(-\log b(r+\delta)) - \widetilde{h}_\xi(-\log br) \\
&\quad = \widetilde{h}_\xi(-\log(r+\delta) - \log b) - \widetilde{h}_\xi(-\log r - \log b) \\
&\quad = -\{\widetilde{h}_\xi(-\log(r+\delta) + \delta' - \log b) - \widetilde{h}_\xi(-\log(r+\delta) - \log b)\},
\end{aligned}$$

where $\delta' = \log(r+\delta) - \log r > 0$. Then it follows from (4.8) that

$$\begin{aligned}
&F_\xi(b(r+\delta)) - F_\xi(br) \\
&\quad \le -\{\widetilde{h}_\xi(-\log(r+\delta) + \delta') - \widetilde{h}_\xi(-\log(r+\delta))\} \\
&\quad = \widetilde{h}_\xi(-\log(r+\delta)) - \widetilde{h}_\xi(-\log r) \\
&\quad = F_\xi(r+\delta) - F_\xi(r),
\end{aligned}$$

concluding (4.4).

(*Sufficiency.*) By Lemma 4.1, it is enough to show that $\nu(bG) \geq \nu(G)$ for any $G \in (\mathbf{R}^d \setminus \{0\})$. By (4.3), for any $E \in \mathcal{B}((0,\infty))$ and $B \in \mathcal{B}(S)$,

$$\nu(bEB) = -\int_B \lambda(d\xi) \int_{bE} dF_\xi(r) = -\int_B \lambda(d\xi) \int_E dF_\xi(br).$$

Thus by (4.3),

$$\nu(bEB) - \nu(EB) = -\int_B \lambda(d\xi) \int_E d(F_\xi(br) - F_\xi(r)) \geq 0.$$

(*The uniqueness of the representation.*) Suppose $\nu$ is represented by $\lambda$ and $F_\xi(r)$ as in (4.3). By (4.6), $\lambda$ is unique. Hence $F_\xi(r)$ is also unique for $\lambda$-a.e. $\xi$ by (4.3). This completes the proof of Theorem 4.1. □

**Definition 4.2.** Let $0 < b < 1$. For a function $F : (0,\infty) \to \mathbf{R}$, define

$$\mathcal{E}_b F(s) = F(bs) - F(s)$$

and its $m$-th iteration

$$\mathcal{E}_b^m F(s) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} F(b^j s).$$

Also define, for $\delta > 0$,

$$\Delta_\delta f(s) = f(s+\delta) - f(s).$$

Then we say that $F$ has *the property* $(m, b)$ if

$$\Delta_\delta \mathcal{E}_b^j F(s) \leq 0, \quad 1 \leq \forall j \leq m, \forall s > 0, \forall \delta > 0.$$

When $F$ has the property $(m, b)$ for any $m \geq 1$, then we say that it has *the property* $(\infty, b)$.

**Theorem 4.2.** Let $0 < b < 1$ and $m = 0, 1, 2, \cdots, \infty$. A necessary and sufficient condition for that $\mu \in L_m(b)$ is that $\mu \in L_0(b)$, and if $\nu \not\equiv 0$, then the $F$-function, $F_\xi$, of $\mu$ has the property $(m+1, b)$ for $\lambda$-a.e. $\xi$.

*Proof.* The property $(1, b)$ is nothing but (4.4) in Theorem 4.1. Thus the statement for $m = 0$ is that of Theorem 4.1. For general $m$, we show the statement by induction. Suppose that Theorem 4.2 is true for $m = m_0$.

(*Necessity.*) Suppose that $\mu \in L_{m_0+1}(b)$ and $\nu \not\equiv 0$. Then by (ii) of Theorem 3.3, there exists $\rho_{m_0+1} \in L_{m_0}(b)$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_{m_0+1}(z).$$

Let $F_\xi$ be the $F$-function of $\mu$. The Lévy measure $\widetilde{\nu}$ of $\rho_{m_0+1}$ is

$$\widetilde{\nu}(EB) = -\int_B \lambda(d\xi) \int_E d(F_\xi(r) - F_\xi(\frac{r}{b}))$$

Put

$$p(\xi) = \int_0^\infty (1 \wedge r^2) d(F_\xi(r) - F_\xi(\frac{r}{b})).$$

Then $0 < p(\xi) < K$. Let

$$\widetilde{\lambda}(d\xi) = \frac{1}{\widetilde{K}} p(\xi)\lambda(d\xi)$$

and

$$\widetilde{F}_\xi(r) = \frac{\widetilde{K}}{p(\xi)}(F_\xi(r) - F_\xi(\frac{r}{b})),$$

where $\widetilde{K}$ is determined so that $\widetilde{\lambda}$ is a probability measure on $S$. Then $\widetilde{F}_\xi(r)$ is the $F$-function of $\rho_{m_0+1}$. Since $\rho_{m_0+1} \in L_{m_0}(b)$ by the assumption of the induction, $\widetilde{F}_\xi(r)$ satisfies the property $(m_0 + 1, b)$ for $\lambda$-a.e. $\xi$. Thus

$$\Delta_\delta \mathcal{E}_b^j \widetilde{F}_\xi(r) = \frac{\widetilde{K}}{p(\xi)} \Delta_\delta \mathcal{E}_b^j (F_\xi(r) - F_\xi(\frac{r}{b})) \le 0, \quad 1 \le \forall j \le m_0 + 1, \forall s > 0, \forall \delta > 0.$$

From this we see that

$$\Delta_\delta \mathcal{E}_b^{m_0+2} F_\xi(r) \le 0, \quad \forall s > 0, \forall \delta > 0. \tag{4.12}$$

Since $\mu \in L_{m_0+1}(b) \subset L_{m_0}(b)$ by the assumption of the induction, we know that

$$\Delta_\delta \mathcal{E}_b^j F_\xi(r) \le 0, \quad 1 \le \forall j \le m_0 + 1, \forall s > 0, \forall \delta > 0.$$

This together with (4.12) assures that $F_\xi$ has the property $(m_0+2, b)$.

(*Sufficiency.*) Suppose that $F_\xi$ has the property $(m_0+2, b)$. If we follow the proof above from the bottom to the top, we see that $\rho_{m_0+1} \in L_{m_0}(b)$. Thus

$$\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_{m_0+1}(z), \quad \rho_{m_0+1} \in L_{m_0}(b),$$

implying, by (ii) of Theorem 3.3, that $\mu \in L_{m_0+1}(b)$.

The statement fro $m = \infty$ is obvious from that for finite $m$'s and from the definition of $L_\infty(b)$. This completes the proof of the theorem. $\square$

In Proposition 2.3, we have seen that

$$Q(H, b) \subset Q(H, b^m), \quad m = 1, 2, \cdots,$$

where, of course, $b^m < b$. Thus it might be asked whether

$$Q(H, b_2) \subset Q(H, b_1), \quad 0 < b_1 < b_2 < 1.$$

We conclude the paper to answer this question negatively, by applying Lemma 4.1. The following example is due to Sato [9].

**Example 4.1.** Let $d = 1$ and let $\mu$ be an infinitely divisible distribution with the Lévy measure $\nu$ :

$$\nu(dx) = \sum_{n\in\mathbf{Z}} k(2^n)\delta_{2^n}(dx),$$

where $k(\cdot) > 0$ is a monotone decreasing function and $\delta_a(dx)$ is the $\delta$-measure at $a$. Then $\mu \in L_0(\frac{1}{2})$ but $\mu \notin L_0(\frac{1}{3})$ as follows. For each $0 < x < y$, we have

$$\nu((x, y]) = \sum_{\substack{x<2^n\le y\\ n\in\mathbf{Z}}} k(2^n) \le \sum_{\substack{\frac{x}{2}<2^{n-1}\le\frac{y}{2}\\ n\in\mathbf{Z}}} k(2^{n-1}) = \nu\left(\left(\frac{x}{2}, \frac{y}{2}\right]\right),$$

where we have used $k(2^n) \le k(2^{n-1})$. Thus for each $B \in \mathcal{B}((0, \infty))$,

$$\nu(B) \le \nu(\frac{1}{2}B),$$

implying $\mu \in L_0(\frac{1}{2})$ by Lemma 4.1. However,

$$\nu\left(\left(\frac{7}{2},4\right]\right) = k(2^2) > 0 = \nu\left(\left(\frac{7}{6},\frac{4}{3}\right]\right) = \nu\left(\frac{1}{3}\left(\frac{7}{2},4\right]\right).$$

Thus there exists $B \in \mathcal{B}((0,\infty))$ such that $\nu(B) \leq \nu(\frac{1}{3}B)$ is not satisfied. Hence again by Lemma 4.1, $\mu \notin L_0(\frac{1}{3})$.

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**References**

[1] Bunge, J. (1997). Nested classes of $C$-decomposable laws. *Ann. Probab.* **25** 215–229.

[2] Choi, G.S. (1994). Criteria for recurrence and transience of semistable processes. *Nagoya Math. J.* **134** 91–106.

[3] Chung, K.L. (1974). *A Course in Probability Theory.* Academic Press.

[4] Gnedenko, B.V. and Kolmogorov, A.N. (1968). *Limit Distributions for Sums of Independent Random Variables*, 2nd ed. Addison-Wesley.

[5] Loève, M. (1945). Nouvelles classes de lois limites. *Bull. Soc. Math. France* **73** 107–126. (In French.)

[6] Loève, M. (1977). *Probability Theory I*, 4th ed. Springer-Verlag.

[7] Meerschaert, M.M. and Scheffler, H.-P. (1996). Series representation for semistable laws and their domain of semistable attraction. *J. Theor. Probab.* **9** 931–959.

[8] Sato, K. (1980). Class $L$ of multivariate distributions and its subclasses. *J.*

*Multivar. Anal.* **10** 207–232.

[9] Sato, K. (1997). Private communication.

[10] Urbanik, K. (1972). Slowly varying sequences of random variables. *Bull. Acad. Pol. Sci. Sér. Sci. Math. Astronom. Phys.* **20** 679–682.

[11] Urbanik, K. (1973). Limit laws for sequences of normed sums satisfying some stability conditions. In : Karishniah, P.R. (ed.) *Multivariate Analysis-III*, 225–237, Academic Press.