Research Report

KSTS/RR-97/003
June 20, 1997

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by

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POINCARÉ-CARTAN CLASS AND DEFORMATION QUANTIZATION OF KÄHLER MANIFOLDS

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ABSTRACT. We introduce a complete invariant for Weyl manifolds, called a Poincaré-Cartan class. Applying the constructions of Weyl manifold to complex manifolds via the Poincaré-Cartan class, we propose the notion of noncommutative Kähler manifold. For a given Kähler manifold, the necessary and sufficient condition for a Weyl manifold to be a noncommutative Kähler manifold is given. In particular, there exists a noncommutative Kähler manifold for any Kähler manifold. We also construct the noncommutative version of S^1-principal bundle over a quantizable Weyl manifold.

INTRODUCTION

The construction of deformation quantization of symplectic manifolds has been extensively studied in recent works. The purpose of this paper is to present a cohomological invariant of Weyl manifolds appeared in the construction of the star products on a symplectic manifold. As introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in [BFL], a deformation quantization, or more precisely a star-product on a symplectic manifold M is an associative product $*$ on $C^\infty(M)[[\nu]]$, the space of formal power series in ν with coefficients in $C^\infty(M)$, such that

(D1) $f * g = fg + \frac{\nu}{2}\{f,g\} + \cdots$, for $f,g \in C^\infty(M)$, where $\{\,,\,\}$ stands for the Poisson bracket on M.
(D2) $1 * f = f = f * 1, \quad \nu \in$ center.
(D3) Complex conjugation $f \to \bar{f}$ is an anti-automorphism of $(C^\infty(M)[[\nu]], *)$, where $\overline{\nu} = -\nu$.

By the localization theorem (cf. [OMY2], [O, p.312]), we may always assume that the star-product $*$ has the locality; i.e. $\operatorname{supp} f*g \subset \operatorname{supp} f \cap \operatorname{supp} g$ as $\mathbf{C}[[\nu]]$-valued functions.

One construction of star-products for symplectic manifolds was first showed by Vey [V] and Lichnerowicz [L] via a torsion free flat connection. Using different approaches, De Wilde-Lecomte [DL], Fedosov [F] and Omori-Maeda-Yoshioka [OMY1] have proved the existence of a star-product for an arbitrary symplectic manifold. De Wilde-Lecomte worked algebraically via careful cohomological arguments, while Fedosov and Omori-Maeda-Yoshioka used geometric method on the Weyl bundle (cf. [W]). Fedosov's crucial idea is to construct a flat connection on the sections of the Weyl bundle. [OMY1] built a noncommutative version of manifolds, called *Weyl manifolds* from a given symplectic manifold. Thus, it is natural to ask how the constructions by [DL], [F] and [OMY1] relate to each other. Deligne [D] studied relationship between the construction of the star-product by De Wilde-Lecomte and by Fedosov and showed that these constructions are equivalent to each other. Recently, there has been interesting work on the equivalence of star products by Xu [X] and Bertelson-Cahen-Gutt [BCG].

In this paper, we first remark the equivalence of the star-product constructed in [OMY1]; The Weyl manifold, by definition (cf.§3.1), is constructed by patching 'noncommutative coordinates', and constructions of the star product are built on that of Weyl manifolds. The quantum version of Darboux's theorem (cf. [O], p.317) combined with the inverse Moyal product formula (1.2) easily gives that all star-products are obtained as the algebra of Weyl functions on a Weyl manifold (cf. remark after Theorem 3.2). Fedosov's flat connection is the connection on a Weyl algebra bundle for which all Weyl functions are characterized as parallel sections.

We show in this paper that *there is a bijective correspondence between the equivalence class of Weyl manifolds and the second cohomology group* $H^2(M, \nu^2\mathbf{C}[[\nu^2]])$. (Theorem 3.5.) The correspondence is indeed given by a characteristic Čech 2-cohomology class (cf. Definition 3.4) called the *Poincaré-Cartan class* which comes from a patching of 'quantized Darboux coordinates' to make a noncommutative manifold. The Poincaré-Cartan class has been proposed previously by Karasev and Maslov in [KM] to be an invariant for their asymptotic quantization theory. It is remarked that its integration on a circle coincides with the original Poincaré-Cartan invariant (cf. [O]).

On the other hand, a characteristic class was defined by Nest-Tsygan [NT] in terms of the curvature of the connection for the Weyl bundle, which distinguishes Fedosov star-products up to equivalence (cf. [F]). The characteristic class defined in [NT] may equal to the Poincaré-Cartan class for the Weyl manifold.

The main purpose of this paper is to apply the Poincaré-Cartan class to complex manifolds and to propose the notion of a noncommutative Kähler manifold. A Kähler manifold is a special type of symplectic manifolds with the option that their coordinate transformations are not only symplectic but also holomorphic. For a given Kähler manifold M, we give a necessary and sufficient condition for a Weyl manifold over M to be a noncommutative Kähler manifold in terms of its Poincaré-Cartan class (Theorem 4.6.). We also show that there exists a noncommutative Kähler manifold for every Kähler manifold (Theorem 5.2.).

The second subject of this paper discussed in §6 is, as an application of the construction of star products via Weyl manifold, a construction of a quantum S^1-bundle over a symplectic manifold with the quantization condition. In patching up the (noncommutative) local coordinates to obtain the Weyl manifold, we use a derivation which generates a noncommutative version of the circle action on the S^1-bundle of a symplectic manifold satisfying the quantization condition.

Furthermore, if the base manifold M has a Kähler structure, then the noncommutative version of the associated line bundle has the structure which one may call 'holomorphic line bundle'. This structure naturally gives the notion of holomorphic sections, and the space of all holomorphic sections is a maximal commutative subalgebra. It should be remarked that the construction of star-products in [OMY1] has an advantage of yielding naturally such constructions.

1. WEYL FUNCTIONS

We first review briefly Weyl functions and Weyl diffeomorphisms on a Weyl algebras. In [OMY1] we treated Weyl algebras over $\mathbf{C}$ and in [OMY3] we gave several remarks on Weyl algebras over $\mathbf{R}$. Here we start with a Weyl algebra over $\mathbf{R}$.

A Weyl algebra $\boldsymbol{W}$ is the algebra generated formally by

$$\nu, X_1, \cdots, X_n, Y_1, \cdots, Y_n$$

over $\mathbf{R}$ with the fundamental relations $[\nu, X_i] = [\nu, Y_i] = 0$, $[X_i, X_j] = [Y_i, Y_j] = 0$, $[X_i, Y_j] = -\nu\delta_{ij}$. The multiplication of the algebra is denoted by $*$. Then, the Weyl algebra $\boldsymbol{W}$ can be identified with the algebra $\mathbf{R}[[X, Y, \nu]]$ of formal power series with the following product, called the *Moyal product*;

$$a * b = a \exp\{-\frac{\nu}{2}\overleftarrow{\partial_X} \dot{\wedge} \overrightarrow{\partial_Y}\} b, \tag{1.1}$$

where $a\overleftarrow{\partial_X} \dot{\wedge} \overrightarrow{\partial_Y} b = \sum_{j=1}^n \{\partial_{X_j} a \cdot \partial_{Y_j} b - \partial_{Y_j} a \cdot \partial_{X_j} b\}$. We put the usual adic-topology on $\boldsymbol{W}$. The formula (1.1) can be inverted to recapture the commutative product as follows:

$$a \cdot b = a \exp\{\frac{\nu}{2}\overleftarrow{\partial_X} \overset{*}{\wedge} \overrightarrow{\partial_Y}\} b, \tag{1.2}$$

where $a\overleftarrow{\partial_X} \overset{*}{\wedge} \overrightarrow{\partial_Y} b = \sum_{j=1}^n \{\partial_{X_j} a * \partial_{Y_j} b - \partial_{Y_j} a * \partial_{X_j} b\}$. This can be viewed as a method of construction of a commutative product from the $*$-product. This idea appears in §3 to make a model space of a Weyl manifold, and in §6 to solve an equation given by using $*$-product. We define an involutive anti-automorphism $a \to \bar{a}$ by setting $\overline{X}_i = X_i, \quad \overline{Y}_j = Y_j, \quad \bar{\nu} = -\nu$.

Note that there are other systems of elements $(X_1', \cdots, X_n', Y_1', \cdots, Y_n')$ of $\boldsymbol{W}$ with the same fundamental relations which topologically generate the same $\boldsymbol{W}$. We call such $X_1', \cdots, X_n', Y_1', \cdots, Y_n'$ *quantum canonical generators* (QC-generators).

1.1 Weyl function.

Let U be an open set of $\mathbf{R}^{2n}$ with linear coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$, and $\boldsymbol{W}_U$ the trivial algebra bundle $U \times \boldsymbol{W}$. Let $\Gamma(\boldsymbol{W}_U)$ be the space of all continuous sections of $\boldsymbol{W}_U$ with respect to the compact open topology. $\Gamma(\boldsymbol{W}_U)$ is an associative algebra over $\mathbf{R}$ under the pointwise $*$-product. Define the sections ξ_i, η_i of $\boldsymbol{W}_U$ by

$$\xi_i(p) = x_i(p) + X_i, \qquad \eta_i(p) = y_i(p) + Y_i, \quad i = 1, \cdots, n. \tag{1.3}$$

Then, we have $[\xi_i, \eta_j] = -\nu\delta_{ij}$, $[\xi_i, \xi_j] = [\eta_i, \eta_j] = 0$. Consider a polynomial of the form $\mathfrak{p}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum a_{\lambda\mu}\boldsymbol{\xi}^\lambda \cdot \boldsymbol{\eta}^\mu$, $a_{\lambda\mu} \in \mathbf{R}$, $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n)$, $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_n)$, which can be viewed as a section of $\boldsymbol{W}_U$, and we get

$$\mathfrak{p}(\boldsymbol{\xi}, \boldsymbol{\eta})(p) = \sum_{\lambda\mu} \frac{1}{\lambda!\mu!} {\partial_x}^\lambda {\partial_y}^\mu \mathfrak{p}(x(p), y(p)) X^\lambda \cdot Y^\mu.$$

Keeping this formula in mind for any $\mathbf{R}[[\nu]]$-valued C^∞ function f, we define a section $f^\sharp(\boldsymbol{\xi},\boldsymbol{\eta})$, called a *Weyl function*, by the formula

$$f^\sharp(\boldsymbol{\xi},\boldsymbol{\eta})(p)=\sum_{\lambda\mu}\tfrac{1}{\lambda!\mu!}\partial_x{}^\lambda\partial_y{}^\mu f(p)X^\lambda\cdot Y^\mu. \tag{1.4}$$

For $f\in C^\infty(U)[[\nu]]$ we call $f^\sharp$ the *Weyl continuation* of f. Obviously $\xi_i=x_i^\sharp$, and $\eta_i=y_i^\sharp$. We define $\mathcal{F}(\boldsymbol{W}_U)$ to be the set of all Weyl functions. $\mathcal{F}(\boldsymbol{W}_U)$ is a closed subalgebra of $\Gamma(\boldsymbol{W}_U)$ (cf.[OMY1]).

It is easily seen that the $*$-product $f^\sharp * g^\sharp$ is given by the same formula (1.1), i.e.

$$f^\sharp * g^\sharp(p)=(f\exp-\frac{\nu}{2}\{\overleftarrow{\partial_x}\dot{\wedge}\overrightarrow{\partial_y}\}g)^\sharp,\qquad \text{(cf. [OMY1]).} \tag{1.5}$$

Moreover, the involutive anti-automorphism $a\mapsto\bar{a}$ extends naturally on $\Gamma(\boldsymbol{W}_U)$ and $\overline{\mathcal{F}(\boldsymbol{W}_U)}=\mathcal{F}(\boldsymbol{W}_U)$. We have $\overline{f^\sharp}=(\bar{f})^\sharp$.

1.2 Integration on $\boldsymbol{W}_U$.

For a Weyl function $f^\sharp\in\mathcal{F}(\boldsymbol{W}_U)$ with f integrable on U, we define the *integral* of $f^\sharp$ by

$$\int_U f^\sharp=\int_U f dV\in\mathbf{R}[[\nu]],$$

where $dV=dx_1\cdots dx_n dy_1\cdots dy_n$ is the usual volume element on U. Integration by parts shows that if one of f, g has a compact support, then $\int_U f\{\overleftarrow{\partial_y}\dot{\wedge}\overrightarrow{\partial_x}\}^k g dV=0$. Hence, we have

$$\int_U f^\sharp * g^\sharp=\int_U f\cdot g dV. \tag{1.6}$$

In particular, we have

$$\int_U f^\sharp * g^\sharp=\int_U g^\sharp * f^\sharp,\qquad \overline{\int_U f^\sharp}=\int_U\overline{f^\sharp}. \tag{1.7}$$

1.3 The contact Weyl Lie algebra.

We define a derivation L_0 as follows:

$$L_0\nu=2\nu^2,\quad L_0X_i=\nu X_i,\quad L_0Y_i=\nu Y_i.$$

Together with a formal symbol τ, we define a Lie algebra, called a *contact Weyl Lie algebra*, $\mathfrak{g}=\mathbf{R}\tau\oplus\boldsymbol{W}$ with the bracket:

$$[a\tau+f,b\tau+g]=aL_0g-bL_0f+[f,g]. \tag{1.8}$$

We easily see that $[\mathfrak{g},\mathfrak{g}]\subset\nu*\boldsymbol{W}$. We set also $\bar{\tau}=\tau$ to define an involutive anti-automorphism.

Definition 1.1. A linear mapping $A:\mathfrak{g}\to\mathfrak{g}$ is called a *ν-isomorphism*, if A is a Lie algebra isomorphism satisfying (i) $A(\nu)=\nu$, (ii) $A\boldsymbol{W}=\boldsymbol{W}$ and (iii) the restriction $A|_{\boldsymbol{W}}$ is an algebra isomorphism.

$D:\mathfrak{g}\to\mathfrak{g}$ is called a *ν-derivation* if D is a Lie algebra derivation satisfying (i)$D(\nu)=0$, (ii)$D\boldsymbol{W}\subset\boldsymbol{W}$ and (iii) the restriction $D|_{\boldsymbol{W}}$ is an algebra derivation. (Cf.[OMY1] Definition 4.2.)

Although L_0 and hence τ depends on the choice of QC-generators, it is easy to see that the ν-isomorphism class of $\mathfrak{g}$ is determined only by $\boldsymbol{W}$.

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Lemma 1.2. *Every ν-derivation $D : \mathfrak{g} \to \mathfrak{g}$ can be written in the form $D = \mathrm{ad}(\nu^{-1} * f) + c\ \mathrm{ad}(\log \nu)$, where $f \in \boldsymbol{W}$, $c \in \mathbf{R}$.*

If $D(\tau) \in \nu * \boldsymbol{W}$, then there are $g \in \boldsymbol{W}$ and $c \in \mathbf{R}$ such that $D = ad(g) + c\, ad(\log \nu)$, and c is determined uniquely by D. g is determined only up to constant.

If $D(\tau) \in \nu^2 * \boldsymbol{W}$, then $D = ad(\nu * g)$ where g is determined uniquely by D.

Here, we first remark that $\mathrm{ad}(\nu^{-1} * f)$ and $\mathrm{ad}(\log \nu)$ are defined by only symbolic use of $\nu^{-1} * f$ and $\log \nu$. Note that the above lemma is proved in [OMY1, Proposition 4.3] in the case of complex coefficients, but the proof works also for the real coefficients. Though the second statement was not given there, it can be seen easily by the proof.

Let U be an open subset of $\mathbf{R}^{2n}$ with coordinates $x_1, \cdots, x_n, y_1, \cdots, y_n$, and $\Gamma(\mathfrak{g}_U)$ the space of all continuous sections of the trivial bundle $\mathfrak{g}_U = U \times \mathfrak{g}$ over U. We define a section by

$$\tilde{\tau}(p) = \tau - \textstyle\sum_{i=1}^{n}(y_i(p)X_i - x_i(p)Y_i). \tag{1.9}$$

The sections ξ_i, η_i given by (1.3) are contained in $\Gamma(\mathfrak{g}_U)$, and we have

$$[\tilde{\tau}, \xi_i] = \nu * \xi_i,\ [\tilde{\tau}, \eta_i] = \nu * \eta_i,\ [\xi_i, \eta_j] = -\nu\delta_{ij},\ [\tilde{\tau}, \nu] = 2\nu^2. \tag{1.10}$$

We give several remarks for the complexification. The notion of Weyl algebras and Weyl functions can be easily complexified by considering the tensor product with $\mathbf{C}$. We denote these by $\boldsymbol{W}^C$ and $\mathcal{F}(\boldsymbol{W}_U)^C$. $\boldsymbol{W}$, $\mathcal{F}(\boldsymbol{W}_U)$ are real subalgebras of $\boldsymbol{W}^C$, $\mathcal{F}(\boldsymbol{W}_U)^C$. The involutive anti-automorphism extends naturally by setting $\bar{i} = -i$ to these complexified algebras.

Here, it should be careful that for instance $\boldsymbol{W}$ is *not* the subspace $\{a \in \boldsymbol{W}^C; \bar{a} = a\}$. To avoid the confusion that might occur, we define as follows: A linear mapping $\Phi : \mathcal{F}(\boldsymbol{W}_U)^C \to \mathcal{F}(\boldsymbol{W}_U)^C$ over $\mathbf{R}$ is said to have the *hermitian property* if $\overline{\Phi(f)} = \Phi(\bar{f})$ holds for every f, and Φ is said to have the *real-to-real property* if $\Phi(\mathcal{F}(\boldsymbol{W}_U)) \subset \mathcal{F}(\boldsymbol{W}_U)$.

Notions of ν-isomorphisms and ν-derivations of $\mathfrak{g}$ extend for the complexification $\mathfrak{g}^C = \mathfrak{g} \otimes \mathbf{C}$. Lemma 1.2 and the followed remark hold for the complexified case.

2. Patching diffeomorphisms

2.1 Weyl diffeomorphisms and contact Weyl diffeomorphisms.

Let U and V be open subsets of $\mathbf{R}^{2n}$ with coordinates $x_1, \cdots, x_n, y_1, \cdots, y_n$. Consider the trivial algebra bundles $\boldsymbol{W}_U = U \times \boldsymbol{W}$, and $\boldsymbol{W}_V = V \times \boldsymbol{W}$ over U and V respectively. For a bundle isomorphism Φ:

$$\begin{array}{ccc} \boldsymbol{W}_U & \xrightarrow{\ \Phi\ } & \boldsymbol{W}_V \\ \big\downarrow & & \big\downarrow \\ U & \xrightarrow{\ \varphi\ } & V \end{array}$$

we define the pullback $\Phi^* : \Gamma(\boldsymbol{W}_V) \to \Gamma(\boldsymbol{W}_U)$ by $(\Phi^* S)(p) = \Phi^{-1} S(\varphi(p))$ where φ is the induced diffeomorphism on U.

A continuous algebra isomorphism $\Psi : \mathcal{F}(\boldsymbol{W}_V) \to \mathcal{F}(\boldsymbol{W}_U)$ such that $\Psi(\nu) = \nu$ will be called a *pre-Weyl diffeomorphism.* The following lemma is shown in [OMY1, Lemma 3.2] :

Lemma 2.1. *For any pre-Weyl diffeomorphism $\Psi : \mathcal{F}(\boldsymbol{W}_V) \to \mathcal{F}(\boldsymbol{W}_U)$, there exists a unique bundle isomorphism Φ such that $\Psi = \Phi^*$. In particular, the induced diffeomorphism $\varphi : U \to V$ is a symplectic diffeomorphism with respect to the natural symplectic 2-form $\Omega = \sum dx_i \wedge dy_i$.*

A pre-Weyl diffeomorphism $\Psi : \mathcal{F}(\boldsymbol{W}_V) \to \mathcal{F}(\boldsymbol{W}_U)$ is called a *Weyl diffeomorphism*, if Ψ has the *hermitian property* $\Psi(\bar{f}) = \overline{\Psi(f)}$.

By Lemma 2.1 and the same proof of [OMY3] Proposition 2, we see easily that any pre-Weyl diffeomorphism Ψ has the volume preserving property:

$$\int_U \Psi(f) = \int_V f. \tag{2.1}$$

Remark that the definition of Weyl diffeomorphism is slightly stronger than that defined in [OMY1, Definition 3.4]. Though (2.1) is requested in the definition of Weyl diffeomorphism in [OMY3], this holds automatically by the above observation.

Note that the notion of ν-derivations in Definition 1.1 extends naturally to $\Gamma(\mathfrak{g}_U)$. Remember that a ν-derivation induces, by definition, an algebra derivation on $\Gamma(\boldsymbol{W}_U)$.

Definition 2.2. A ν-derivation $\Xi : \Gamma(\mathfrak{g}_U) \to \Gamma(\mathfrak{g}_U)$ is called a *contact Weyl vector field* if $\Xi(\nu) = 0$, $\Xi\mathcal{F}(\boldsymbol{W}_U) \subset \mathcal{F}(\boldsymbol{W}_U)$ and $\Xi(\tilde{\tau}) \in \mathcal{F}(\boldsymbol{W}_U)$.

2.2 Contact Weyl diffeomorphisms.

We call an isomorphism $\Phi^{c*} : \Gamma(\mathfrak{g}_V) \to \Gamma(\mathfrak{g}_U)$ a *pointless contact diffeomorphism* if Φ^{c*} is a Lie algebra isomorphism such that $\Phi^{c*}(\nu) = \nu$, $\Phi^{c*}(\tilde{\tau}) \in \tilde{\tau} + \mathcal{F}(\boldsymbol{W}_U)$, $\Phi^{c*}\mathcal{F}(\boldsymbol{W}_V) = \mathcal{F}(\boldsymbol{W}_U)$, and the restriction $\Phi^{c*}|\mathcal{F}(\boldsymbol{W}_V)$ is an algebra isomorphism. Φ^{c*} is called a *contact Weyl diffeomorphism*, if the restriction to $\mathcal{F}(\boldsymbol{W}_V)$ gives a Weyl diffeomorphism.

Proposition 2.3. *Suppose U, V are diffeomorphic to the open unit disk D^{2n} of $\mathbf{R}^{2n}$. For every symplectic diffeomorphism $\varphi : U \to V$, there is a Weyl diffeomorphism $\Phi : \boldsymbol{W}_U \to \boldsymbol{W}_V$ inducing φ between base spaces. Moreover, Φ^* extends to a contact Weyl diffeomorphism $\Phi^{c*} : \Gamma(\mathfrak{g}_V) \to \Gamma(\mathfrak{g}_U)$ such that $\overline{\Phi^{c*}(f)} = \Phi^{c*}(\bar{f})$.*

Proposition 2.3 is given in [OMY1, Theorems 3.7 and 4.7] in the case of complex coefficients, but this holds also for the real case by the same proof. In the proof of [OMY1, Theorem 3.7], φ is requested to be a symplectic diffeomorphism of $\overline{U}$ onto $\overline{V}$. However this condition is easily removed by considering an exhausting family of closed subsets of U and V.

The Φ given by Proposition 2.3 is called a *lift* of φ. Note that the lift Φ of φ is not unique in general.

Let $\boldsymbol{W}_U^C$ and $\mathcal{F}(\boldsymbol{W}_U)^C$ be the complexification of $\boldsymbol{W}_U$ and $\mathcal{F}(\boldsymbol{W}_U)$ respectively. Notions of pre-Weyl diffeomorphisms and Weyl diffeomorphisms extends naturally on these complexified algebras.

Let $\Gamma(\mathfrak{g}_U)^C$ be the complexification of $\Gamma(\mathfrak{g}_U)$. As in Lemma 1.2, the notion of contact Weyl vector fields and pointless contact diffeomorphisms, etc. extends naturally to $\Gamma(\mathfrak{g}_U)^C$. By Lemma 1.2 and the remark mentioned in the last paragraph of section 1, we have:

Lemma 2.4. *For a contact Weyl vector field $\Xi : \Gamma(\mathfrak{g}_U)^C \to \Gamma(\mathfrak{g}_U)^C$ there exist $f \in \mathcal{F}(\boldsymbol{W}_U)^C$ and $c' \in \mathbf{C}$ such that $\Xi = \mathrm{ad}(\nu^{-1} * f) + c' \,\mathrm{ad}(\log \nu)$.*

(1) *If $\Xi(\tilde{\tau}) \in \nu * \mathcal{F}(\boldsymbol{W}_U)^C$, then $f \in \nu * \mathcal{F}(\boldsymbol{W}_U)^C$, $c' \in \mathbf{C}$, and c' is uniquely determined. $\nu^{-1} * f$ is determined only up to constant.*

(2) *If $\Xi(\tilde{\tau}) \in \nu^2 * \Gamma(\mathfrak{g}_U)^C$, then $c' = 0$, and f can be taken in $\nu^2 * \mathcal{F}(\boldsymbol{W}_U)^C$, and such f is unique.*

(3) *If Ξ has the real-to-real property; $\Xi \Gamma(\mathfrak{g}_U) \subset \Gamma(\mathfrak{g}_U)$, then $c' \in \mathbf{R}$ and f can be taken in $\mathcal{F}(\boldsymbol{W}_U)$.*

(4) *If Ξ has the real-to-real property, the hermitian property; $\overline{\Xi(h)} = \Xi(\bar{h})$ and $\Xi(\tilde{\tau}) \in \nu * \mathcal{F}(\boldsymbol{W}_U)$, then $c' = 0$ and f can be taken in $\nu^2 * \mathcal{F}(\boldsymbol{W}_U)$, and hence such f is unique.*

Proof. (1) and (2) are easy to see by Lemma 1.2, and (3) is given by the similar proof. For (4), we see by (1)-(3) that there are $g \in \mathcal{F}(\boldsymbol{W}_U)$ and $c' \in \mathbf{R}$ such that $\Xi = \mathrm{ad}(g) + c' \,\mathrm{ad}(\log \nu)$. By the hermitian property, we have $\overline{\Xi(\tilde{\tau})} = \Xi(\tilde{\tau})$. It follows that $[\tilde{\tau}, g + \bar{g}] = 4c'\nu$. Since $g + \bar{g} \in \sum_{k \geq 0} \nu^{2k} C^\infty(U)^\sharp$, we have $c' = 0$ and g is written in the form $g = \sum \nu^{2k+1} g_{2k+1}$. It follows $f = \nu * g \in \nu^2 * \mathcal{F}(\boldsymbol{W}_U)$. This yields $\Xi(\tilde{\tau}) \subset \nu^2 * \mathcal{F}(\boldsymbol{W}_U)$, and hence f is determined uniquely by Ξ. □

Considering formal expansion in ν^k, we see that if a pointless contact diffeomorphism $\Phi^{c*} : \Gamma(\mathfrak{g}_U)^C \to \Gamma(\mathfrak{g}_U)^C$ induces the identity on the base space U, then Φ^{c*} is written in the form

$$\Phi^{c*} = \prod_{\infty}^{0} e^{\mathrm{ad}(\nu^k h_k^\sharp)} e^{c\,\mathrm{ad}(\nu^{-1})} e^{c'\,\mathrm{ad}(\log \nu)}, \tag{2.2}$$

using $h_k \in C^\infty(U)^C$ for every integer $k \geq 0$ and $c, c' \in \mathbf{C}$, where the notation $\prod_\infty^0 I_k$ means $\cdots I_k \cdots I_2 I_1 I_0$. c' is determined uniquely by Φ^{c*} by virtue of Lemma 2.4, (1). By Lemma 2.4 we easily obtain the following:

Corollary 2.5. *If a pointless contact diffeomorphism $\Phi^{c*} : \Gamma(\mathfrak{g}_U)^C \to \Gamma(\mathfrak{g}_U)^C$ induces the identity on the base space U, then h_k, c, c' in (2.2) satisfies the following:*

(1) *If $\Phi^{c*}(\tilde{\tau}) \in \tilde{\tau} + 2c + \nu^2 * \mathcal{F}(\boldsymbol{W}_U)^C$, then $c' = 0$, $h_0 = 0$, and c, h_k $(k \geq 1)$ are unique.*

(2) *If Φ^{c*} has the real-to-real property; $\Phi^{c*}\Gamma(\mathfrak{g}_U) = \Gamma(\mathfrak{g}_U)$, then $c, c' \in \mathbf{R}$ and $h_k \in \mathcal{F}(\boldsymbol{W}_U)$.*

(3) *If Φ^{c*} has the real-to-real property and the hermitian property; $\overline{\Phi^{c*}(f)} = \Phi^{c*}(\bar{f})$, then $c' = 0$, $h_{2k} = 0$ for $k \geq 0$, and c and h_{2k+1} $(k \geq 0)$ are unique.*

(4) *If Φ^{c*} induces the identity on $\mathcal{F}(\boldsymbol{W}_U)^C$, then there are $\tilde{c} \in \mathbf{C}[[\nu]]$, $c' \in \mathbf{C}$ such that $\Phi^{c*} = e^{\tilde{c}\,\mathrm{ad}(\nu^{-1}) + c'\,\mathrm{ad}(\log \nu)}$. Furthermore, if Φ^{c*} has the real-to-real property and the hermitian property, then $c' = 0$ and $\tilde{c} \in \mathbf{R}[[\nu^2]]$.*

Proof. Here we give the proof of (4). Set $\Phi^{c*}(\tilde{\tau}) = \tilde{\tau} + g$, $g \in \mathcal{F}(\boldsymbol{W}_U)^C$. As $[\tilde{\tau}, \xi_i] = \nu \xi_i$, $[\tilde{\tau}, \eta_j] = \nu \eta_j$, and Φ^{c*} is an isomorphism, we have $[g, \xi_i] = [g, \eta_j] = 0$, hence $g \in \mathbf{C}[[\nu]]$. The second statement of (4) follows easily. □

The next lemma is given in [OMY1]:

Lemma 2.6. *For every pre-Weyl diffeomorphism* $\Phi^* : \mathcal{F}(\boldsymbol{W}_U)^C \to \mathcal{F}(\boldsymbol{W}_U)^C$ *there is a pointless contact diffeomorphism* $\Phi^{c*} : \Gamma(\mathfrak{g}_U)^C \to \Gamma(\mathfrak{g}_U)^C$ *which extends* Φ^*.

Proof. By Lemma 2.1, Φ^* induces a symplectic diffeomorphism φ on U. By Proposition 2.3, there is a Weyl diffeomorphism Ψ^* which is a lift of φ. Let Ψ^{c*} be a contact Weyl diffeomorphism which extends Ψ^*.

Hence, $\Phi^*\Psi^{*-1}$ induces the identity on the base space. It follows by Corollary 2.5, (3) that $\Phi^* = \Psi^* e^{\mathrm{ad}(h)}$, $h \in \mathcal{F}(\boldsymbol{W}_U)^C$. Note that $\mathrm{ad}(\nu^{-1})$ component is not used, since these act trivially on $\mathcal{F}(\boldsymbol{W}_U)^C$. Hence we define a pointless contact diffeomorphism Φ^{c*} by $\Psi^{c*}e^{\mathrm{ad}(h)}$. □

A contact Weyl diffeomorphism Φ^{c*} has by definition the real-to-real property and it may be assumed by Proposition 2.3 that Φ^{c*} has the hermitian property. We now remark the following:

Lemma 2.7. *If a pointless contact diffeomorphism* $\Phi^{c*} : \Gamma(\mathfrak{g}_U)^C \to \Gamma(\mathfrak{g}_U)^C$ *has the real-to-real property and the hermitian property, then* $\Phi^{c*}(\tilde{\tau})$ *is written in the form*

$$\tilde{\tau} + g_0^\sharp + \nu^2 * g_2^\sharp + \cdots + \nu^{2k} * g_{2k}^\sharp + \cdots, \quad g_{2k} \in C^\infty(U).$$

3. Poincaré-Cartan classes

3.1 Weyl manifold.

Let $\boldsymbol{W}_M$ be a locally trivial algebra bundle with the fiber isomorphic to $\mathbf{W}$. Then for an open covering $\{V_\alpha\}$ of M, there are local trivializations $\Phi_\alpha : \boldsymbol{W}_{V_\alpha} \to \boldsymbol{W}_{U_\alpha}$ associated to V_α where $\boldsymbol{W}_{V_\alpha}$ is the restriction of $\boldsymbol{W}_M$ and $\boldsymbol{W}_{U_\alpha}$ is the trivial algebra bundle over U_α ($\subset \mathbf{R}^{2n}$). Denote by $\varphi_\alpha : V_\alpha \to U_\alpha$ the induced homeomorphism.

Definition 3.1. $\boldsymbol{W}_M$ is called a (real) *Weyl manifold*, if for each V_α, V_β such that $V_\alpha \cap V_\beta \neq \emptyset$, the patching transformation

$$\Phi_{\alpha\beta} = \Phi_\beta \Phi_\alpha^{-1} : U_{\alpha\beta} \times \boldsymbol{W} \to U_{\beta\alpha} \times \boldsymbol{W}, \tag{3.1}$$

where $U_{\alpha\beta} = \varphi_\alpha(V_\alpha \cap V_\beta)$, induces a Weyl diffeomorphism $\Phi^*_{\alpha\beta}$. Each $\Phi_\alpha : \boldsymbol{W}_{V_\alpha} \to \boldsymbol{W}_{U_\alpha}$ is called a *local Weyl chart* on $\boldsymbol{W}_M$, and $\boldsymbol{W}_{U_\alpha}$ is called the *model algebra* over V_α.

If $\Phi^*_{\alpha\beta}$ are merely pre-Weyl diffeomorphisms, then $\boldsymbol{W}_M$ is called a *pre-Weyl manifold.*

By Lemma 2.1, the base manifold M of a pre-Weyl manifold $\boldsymbol{W}_M$ has a C^∞ symplectic structure.

The following was the main theorem of [OMY1]:

Theorem 3.2. *On every* C^∞ *symplectic manifold* M, *there exists a Weyl manifold* $\boldsymbol{W}_M$. *In particular, the system of trivial Weyl algebra bundles* $\{\boldsymbol{W}_{U_\alpha}\}$ *can be patched together via Weyl diffeomorphisms.*

The notions of Weyl functions, the involutive anti-automorphism $f \to \bar{f}$, and integration are naturally defined on a Weyl manifold $\boldsymbol{W}_M$.

Denote by $\mathcal{F}(\boldsymbol{W}_M)$ the algebra of all Weyl functions on $\boldsymbol{W}_M$. Two Weyl manifolds $\boldsymbol{W}_M$, $\boldsymbol{W}'_M$ are said to be *isomorphic*, if there is an algebra isomorphism $\Psi : \mathcal{F}(\boldsymbol{W}_M) \to \mathcal{F}(\boldsymbol{W}'_M)$ inducing the identity on the base manifold M.

Using the fact that $\mathcal{F}(\boldsymbol{W}_M)$ is linearly isomorphic to $C^\infty(M)[[\nu]]$, we translate the algebra structure of $\mathcal{F}(\boldsymbol{W}_M)$ over to $C^\infty(M)[[\nu]]$. In particular, $C^\infty(M)[[\nu]]$ is a noncommutative associative algebra which can be viewed as a deformation quantization of $(C^\infty(M), \cdot)$. Through this observation, we see also that complex conjugation $f \to \bar{f}$ is an involutive anti-automorphism of $(C^\infty(M)[[\nu]], *)$.

Suppose conversely that we have a deformation quantization $(C^\infty(M)[[\nu]], *)$ with an involutive anti-automorphism $f \to \bar{f}$ such that $\bar{f} = f$ for any $f \in C^\infty(M)$ and $\bar{\nu} = -\nu$. Let $\{V_\alpha\}$ be a locally finite simple open covering of M. Note that by the localization theorem [OMY2], the above $*$-product can be localized on $C^\infty(V_\alpha)[[\nu]]$.

Here we need a definition;

Definition 3.3. For $f \in C^\infty(U)[[\nu]]$, the *body part* $b(f)$ of f is an $\mathbf{R}$-valued C^∞-function on U such that $f - b(f) \in \nu C^\infty(U)[[\nu]]$. A system of elements $\xi_1, \cdots, \xi_{2n}$ $\in C^\infty(U)[[\nu]]$ are called *topological generators* (T-generators), if the body parts $b(\xi_1), \cdots, b(\xi_{2n})$ are local coordinates on U.

By the same idea of Weyl continuation, every $f \in C^\infty(U)[[\nu]]$ can be viewed as a 'function' of $\xi_1, \cdots, \xi_{2n}$, whenever $\xi_1, \cdots, \xi_{2n}$ are T-generators.

By the quantum version of Darboux's theorem [O], there are elements $\xi_1, \cdots,$ $\xi_n, \eta_1, \cdots, \eta_n$ of $C^\infty(U)[[\nu]]$ such that

$$\bar{\xi}_i = \xi_i, \quad \bar{\eta}_i = \eta_i, \quad [\xi_i, \xi_j] = [\eta_i, \eta_j] = 0, \quad [\xi_i, \eta_j] = -\nu\delta_{ij}, \tag{3.2}$$

which are T-generators. $\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_n$ are called *quantum canonical generators* (QC-generators).

As in (1.2), we use the inverse Moyal product formula to make a commutative product $\circ$. We identify V_α with $U_\alpha \subset R^{2n}$. It is not hard to see that the mapping $f \to \bar{f}$ remains as an involutive automorphism of $(C^\infty(V_\alpha)[[\nu]], \circ)$, and $f \circ g$ is decomposed for some $k \geq 1$ into

$$f \circ g = f \cdot g + \sum_{l \geq k} \nu^{2l} \varpi_{2l}(f, g), \quad \varpi_{2l}(f, g) = \varpi_{2l}(g, f) \in C^\infty(V_\alpha).$$

Since the first component ϖ_{2k} is a Hochschild 2-cocycle, and hence a Hochschild 2-coboundary by [OMY2, Theorem 2.2], it is easy to see that $(C^\infty(V_\alpha)[[\nu]], \circ)$ is isomorphic to $(C^\infty(V_\alpha)[[\nu]], \cdot)$ with the usual commutative product $\cdot$. Hence, there is an open subset U_α of $\mathbf{R}^n$ and $(C^\infty(V_\alpha)[[\nu]], *)$ is isomorphic to $\mathcal{F}(\boldsymbol{W}_{U_\alpha})$ through an isomorphism Ψ^*_α with the hermitian property.

On each $V_\alpha \cap V_\beta$, the identity mapping of $(C^\infty(V_\alpha \cap V_\beta)[[\nu]], *)$ onto itself regarded as $(C^\infty(V_\beta \cap V_\alpha)[[\nu]], *)$, induces a Weyl diffeomorphism $\Phi^*_{\alpha\beta} : \mathcal{F}(\boldsymbol{W}_{U_{\beta\alpha}}) \to \mathcal{F}(\boldsymbol{W}_{U_{\alpha\beta}})$. Hence, *any deformation quantization $(C^\infty(M)[[\nu]], *)$ with an involutive anti-automorphism is obtained as an algebra of Weyl functions on a Weyl manifold.*

3.2 Poincaré-Cartan classes.

For a symplectic manifold M, there are Weyl manifolds over M which are not isomorphic. We give the complete invariant for the isomorphism class of a Weyl manifold as an element of $H^2(M)[[\nu^2]]$.

Let $\{V_\alpha\}$ be a covering of M. For each α let $\varphi_\alpha : V_\alpha \to U_\alpha \subset \mathbf{R}^{2n}$ be a symplectomorphic coordinate map.

Consider the trivial Lie algebra bundle $\mathfrak{g}_{U_\alpha}$ on U_α. Recall that Theorem 3.2 was proved [OMY1] by constructing a contact Weyl diffeomorphism $\Phi^{c*}_{\alpha\beta} : \Gamma(\mathfrak{g}_{U_{\beta\alpha}}) \to \Gamma(\mathfrak{g}_{U_{\alpha\beta}})$ for $V_\alpha \cap V_\beta \neq \emptyset$, patching $\mathfrak{g}_{U_\alpha}$ and $\mathfrak{g}_{U_\beta}$ together. Let $\Phi^*_{\alpha\beta}$ be the restriction $\Phi^{c*}_{\alpha\beta}|\mathcal{F}(\boldsymbol{W}_{U_{\beta\alpha}})$.

It is clear that $\{\Phi^*_{\alpha\beta}\}$ gives a pre-Weyl manifold if and only if $\Phi^{c*}_{\alpha\beta}$ satisfy $\Phi^{c*}_{\alpha\alpha} = 1$ and $\Phi^{c*}_{\alpha\beta}\Phi^{c*}_{\beta\gamma}\Phi^{c*}_{\gamma\alpha} = e^{c_{\alpha\beta\gamma}\mathrm{ad}(\nu^{-1}) + c'_{\alpha\beta\gamma}\mathrm{ad}(\log\nu)}$ on every $V_\alpha \cap V_\beta \cap V_\gamma \neq \emptyset$, where $c_{\alpha\beta\gamma} \in \mathbf{R}[[\nu]]$ and $c'_{\alpha\beta\gamma} \in \mathbf{R}$. The necessity is given by Corollary 2.5, (4), and the sufficiency is given by that $e^{c_{\alpha\beta\gamma}\mathrm{ad}(\nu^{-1}) + c'_{\alpha\beta\gamma}\mathrm{ad}(\log\nu)}$ is the identity on each subalgebra $\mathcal{F}(\boldsymbol{W}_{U_{\alpha\beta\gamma}})$, where $U_{\alpha\beta\gamma} = \varphi_\alpha(V_\alpha \cap V_\beta \cap V_\gamma)$.

$\{\Phi^*_{\alpha\beta}\}$ gives a Weyl manifold if and only if $\Phi^{c*}_{\alpha\beta}$ has the hermitian property furthermore. If this is the case, we see that $c'_{\alpha\beta\gamma} = 0$ and $c_{\alpha\beta\gamma} \in \mathbf{R}[[\nu^2]]$. Under these situations the family $\{\mathcal{F}(\boldsymbol{W}_{U_\alpha})\}$ of algebras is patched together to give an algebra sheaf on M.

It is easily seen that $\{c_{\alpha\beta\gamma}\}$ and $\{c'_{\alpha\beta\gamma}\}$ are Čech 2-cocycles on M. (Cf.[O, p353], [OMY1, Lemma 5.6].)

In what follows Weyl manifolds are our main concern, but pre-Weyl manifolds are occasionally used for a supplementary role.

Definition 3.4. For a family $\{\Gamma(\mathfrak{g}_{U_\alpha})\}$ constructed on a Weyl manifold $\boldsymbol{W}_M$, $\{c_{\alpha\beta\gamma}\}$ is called the *Poincaré-Cartan 2-cocycle* of $\{\mathfrak{g}_{U_\alpha}\}$.

If we set

$$c_{\alpha\beta\gamma} = c^{(0)}_{\alpha\beta\gamma} + \nu^2 c^{(2)}_{\alpha\beta\gamma} + \cdots, \tag{3.3}$$

then $\{c^{(0)}_{\alpha\beta\gamma}\}$ is cohomologous to a Čech cocycle given by the symplectic 2-form Ω on M (cf. [O, p357], [KM]). We call the cohomology class of $\{c_{\alpha\beta\gamma}\}$ the *Poincaré-Cartan class* of $\{\boldsymbol{W}_{U_\alpha}\}$ and denote it by $c(\boldsymbol{W}_M) = \sum_{k\geq 0} \nu^{2k} c^{(2k)}(\boldsymbol{W}_M)$.

In [OMY1], we constructed a Weyl manifold on M such that $c_{\alpha\beta\gamma} = c^{(0)}_{\alpha\beta\gamma} \in \mathbf{R}$.

The following is essentially the same result of [D] and [NT]:

Theorem 3.5. *The equivalence of Weyl manifolds $\boldsymbol{W}_M$ up to isomorphism is determined by the Poincaré-Cartan class. Moreover, for every $c = \sum_{k\geq 0} \nu^{2k} c^{(2k)} \in H^2(M)[[\nu^2]]$ such that $c^{(0)}$ is the class of symplectic 2-form, there exists a Weyl manifold $\boldsymbol{W}_M$ whose Poincaré-Cartan class $c(\boldsymbol{W}_M)$ is c.*

Proof. Let $\{c_{\alpha\beta\gamma}\}$, $\{c'_{\alpha\beta\gamma}\}$ be Poincaré-Cartan cocycles of $\{\mathfrak{g}_{U_\alpha}\}$ and $\{\mathfrak{g}'_{U_\alpha}\}$ respectively. Suppose the Poincaré-Cartan classes coincide. Then, there exists $b_{\alpha\beta} \in \mathbf{R}[[\nu^2]]$ on every $V_\alpha \cap V_\beta \neq \emptyset$ such that $b_{\alpha\beta} = -b_{\beta\alpha}$ and

$$c'_{\alpha\beta\gamma} - c_{\alpha\beta\gamma} = b_{\alpha\beta} + b_{\beta\gamma} + b_{\gamma\alpha}.$$

Note that $b_{\alpha\beta}$ can be replaced by $b_{\alpha\beta}+c_{\alpha\beta}$ such that $c_{\alpha\beta}+c_{\beta\gamma}+c_{\gamma\alpha}=0$. Since $e^{b_{\alpha\beta}\mathrm{ad}(\nu^{-1})}$ is an automorphism, we can replace $\Phi^{c*}_{\alpha\beta}$ by $\acute{\Phi}^{c*}_{\alpha\beta}=\Phi^{c*}_{\alpha\beta}e^{b_{\alpha\beta}\mathrm{ad}(\nu^{-1})}$.

Since $e^{b_{\alpha\beta}\mathrm{ad}(\nu^{-1})}$ is the identity on $\mathcal{F}(\boldsymbol{W}_{U_{\alpha\beta}})$, this replacement does not change the isomorphism class of $\mathcal{F}(\boldsymbol{W}_M)$, but it changes the Poincaré-Cartan cocycle from $\{c_{\alpha\beta\gamma}\}$ to $\{c'_{\alpha\beta\gamma}\}$. Hence we can assume that we have two families $\{\Phi^{c*}_{\alpha\beta}\}$ and $\{\acute{\Phi}^{c*}_{\alpha\beta}\}$ of patching transformations such that

$$\Phi^{c*}_{\alpha\beta}\Phi^{c*}_{\beta\gamma}\Phi^{c*}_{\gamma\alpha}=\acute{\Phi}^{c*}_{\alpha\beta}\acute{\Phi}^{c*}_{\beta\gamma}\acute{\Phi}^{c*}_{\gamma\alpha}=e^{c_{\alpha\beta\gamma}\mathrm{ad}(\nu^{-1})}.$$

Since $\Phi^{c*}_{\alpha\beta}(\acute{\Phi}^{c*}_{\alpha\beta})^{-1}$ induces the identity on the base space, we see by Corollary 2.5, (3) that there is a unique $h_{\alpha\beta}$ such that $\acute{\Phi}^{c*}_{\alpha\beta}=\Phi^{c*}_{\alpha\beta}e^{\mathrm{ad}(\nu h_{\alpha\beta})}$. $e^{c\,\mathrm{ad}(\nu^{-1})}$-terms can be removed by using the ambiguity of $b_{\alpha\beta}$ mentioned above. By a standard argument of Čech cohomology, we see that $\acute{\Phi}^{c*}_{\alpha\beta}=e^{\mathrm{ad}(\nu h_\alpha)}\Phi^{c*}_{\alpha\beta}e^{-\mathrm{ad}(\nu h_\beta)}$. (See also Lemmas 5.4 through 5.6.) This implies that two families are isomorphic.

Conversely suppose there is a Weyl diffeomorphism $\Psi:\boldsymbol{W}'_M\to\boldsymbol{W}_M$ which induces the identity on the base manifold. That is, Ψ^* defines an algebra isomorphism of $\mathcal{F}(\boldsymbol{W}_M)$ onto $\mathcal{F}(\boldsymbol{W}'_M)$ with the hermitian property such that $\Psi^*(\nu)=\nu$. The isomorphism Ψ^* is equivalently given by a family $\{\Psi^*_\alpha\}$ of isomorphisms:

$$\Psi^*_\alpha:\mathcal{F}(\boldsymbol{W}_{U\alpha})\to\mathcal{F}(\boldsymbol{W}'_{U\alpha}) \tag{3.4}$$

each of which induces the identity map on the base space U_α such that

$$\Psi^*_\alpha\Phi^*_{\alpha\beta}(\Psi^*_\beta)^{-1}=\acute{\Phi}^*_{\alpha\beta}. \tag{3.5}$$

If we extend Ψ^*_α to a contact Weyl diffeomorphism Ψ^{c*}_α, then the replacement of $\Phi^{c*}_{\alpha\beta}$ by $\Psi^{c*}_\alpha\Phi^{c*}_{\alpha\beta}(\Psi^{c*}_\beta)^{-1}$ makes no change of Poincaré-Cartan cocycle.

By (3.5) and Corollary 2.5,(4), we have $\acute{\Phi}^{c*}_{\alpha\beta}=\Psi^{c*}_\alpha\Phi^{c*}_{\alpha\beta}(\Psi^{c*}_\beta)^{-1}e^{b_{\alpha\beta}\mathrm{ad}(\nu^{-1})}$. However this type of replacements changes the Poincaré-Cartan cocycle within the same cohomology class.

Suppose $c=\sum_{k\geq 0}\nu^{2k}c^{(2k)}\in H^2(M)[[\nu^2]]$ is given. Then, we start with a Weyl manifold $\boldsymbol{W}^{(0)}_M$ with a Poincaré-Cartan cocycle $\{c^{(0)}_{\alpha\beta\gamma}\}$, and changing patching Weyl diffeomorphisms we construct a Weyl manifold with a Poincaré-Cartan class c.

Let $\Phi^*_{\alpha\beta}:\mathcal{F}(\boldsymbol{W}_{U_{\beta\alpha}})\to\mathcal{F}(\boldsymbol{W}_{U_{\alpha\beta}})$ be the patching Weyl diffeomorphism of $\boldsymbol{W}^{(0)}_M$ and let $\Phi^{c*}_{\alpha\beta}$ be its extension as a contact Weyl diffeomorphism. Let $\{c^{(2k)}_{\alpha\beta\gamma}\}$ be a Čech cocycle involved in $c^{(2k)}$. Since the sheaf cohomology of C^∞ functions $H^2(M,\mathcal{E})=\{0\}$, there is $h^{(2)}_{\alpha\beta}\in C^\infty(U_{\alpha\beta})$ on each $U_{\alpha\beta}$ such that

$$-c^{(2)}_{\alpha\beta\gamma}=h^{(2)}_{\alpha\beta}+\varphi^*_{\alpha\beta}h^{(2)}_{\beta\gamma}+\varphi^*_{\alpha\gamma}h^{(2)}_{\gamma\alpha}. \tag{3.6}$$

For a function $r_{\alpha\beta}\in C^\infty(U_{\alpha\beta})[[\nu^2]]$, we set $\tilde{h}_{\alpha\beta}=(h^{(2)}_{\alpha\beta})^\sharp+\nu^2 r^\sharp_{\alpha\beta}$. If we use $\grave{\Phi}^{c*}_{\alpha\beta}=\Phi^{c*}_{\alpha\beta}e^{\mathrm{ad}(\nu\tilde{h}_{\beta\alpha})}$ as patching diffeomorphisms for every $V_\alpha\cap V_\beta\neq\emptyset$, then we see

$$\grave{\Phi}^{c*}_{\alpha\beta}\grave{\Phi}^{c*}_{\beta\gamma}\grave{\Phi}^{c*}_{\gamma\alpha}=\Phi^{c*}_{\alpha\beta}\Phi^{c*}_{\beta\gamma}\Phi^{c*}_{\gamma\alpha}e^{\mathrm{ad}(\nu\Phi^*_{\alpha\beta}\tilde{h}_{\beta\alpha})}e^{\mathrm{ad}(\nu\Phi^*_{\alpha\gamma}\tilde{h}_{\gamma\beta})}e^{\mathrm{ad}(\nu\Phi^*_{\alpha\alpha}\tilde{h}_{\alpha\gamma})}.$$

Here we used the general formula

$$\Phi^{c*}_{\alpha\beta} e^{\mathrm{ad}(h)} = e^{\mathrm{ad}(\Phi^*_{\alpha\beta} h)} \Phi^{c*}_{\alpha\beta} \tag{3.7}$$

for $h \in \mathcal{F}(\boldsymbol{W}_{U_{\alpha\beta}})$, proved by the uniqueness of solution of ordinary differential equations. By (3.6), we have

$$e^{\mathrm{ad}(\nu\Phi^*_{\alpha\beta}\tilde{h}_{\beta\alpha})} e^{\mathrm{ad}(\nu\Phi^*_{\alpha\gamma}\tilde{h}_{\gamma\beta})} e^{\mathrm{ad}(\nu\Phi^*_{\alpha\alpha}\tilde{h}_{\alpha\gamma})} = e^{\nu^2 c^{(2)}_{\alpha\beta\gamma}\mathrm{ad}\,\nu^{-1}} \quad \mathrm{mod}\ \nu^4, \tag{3.8}$$

By working on the term ν^4, $\nu^6, \cdots$, we can tune up $r_{\alpha\beta}$ by the same manner as in [OMY1] so that

$$e^{\mathrm{ad}(\nu\Phi^*_{\alpha\beta}\tilde{h}_{\beta\alpha})} e^{\mathrm{ad}(\nu\Phi^*_{\alpha\gamma}\tilde{h}_{\gamma\beta})} e^{\mathrm{ad}(\nu\Phi^*_{\alpha\alpha}\tilde{h}_{\alpha\gamma})} = e^{\nu^2 c^{(2)}_{\alpha\beta\gamma}\mathrm{ad}\,\nu^{-1}}.$$

It follows that $\{\grave{\Phi}^{c*}_{\alpha\beta}\}$ defines a Weyl manifold $\grave{\boldsymbol{W}}_M$ with the Poincaré Cartan class $c^{(0)} + \nu^2 c^{(2)}$. Replacing $\Phi^{c*}_{\alpha\beta}$ by $\grave{\Phi}^{c*}_{\alpha\beta}$ and repeating a similar argument as above, we can replace the condition $\mathrm{mod}\,\nu^4$ in (3.8) by $\mathrm{mod}\,\nu^6$. Repeating this procedure, we have a Weyl manifold $\boldsymbol{W}_M$ such that $c(\boldsymbol{W}_M) = c \in H^2(M)[[\nu^2]]$. □

4. Noncommutative Kähler manifolds

In this section, we introduce a restricted notion of deformation quantization for Kähler manifolds, which we call a *noncommutative Kähler manifold.*

4.1 Paracoordinates.

Let us first review the calculus of complex variables, which differs crucially from the real case. Let U be an open subset of $\mathbf{R}^m$ with coordinate functions $x_1, \cdots, x_m$, and $C^\infty(U)^C$ the space of all $\mathbf{C}$-valued C^∞ functions on U. Consider a set $\{z_1, \cdots, z_m\}$ in $C^\infty(U)^C$. Set

$$\tilde{U} = \{\psi_z(p); p \in U\}, \quad \psi_z(p) = (z_1(p), \cdots, z_m(p)).$$

$z_1, \cdots, z_m$ are called *paracoordinates* of U, if the following conditions are satisfied:

(1) $\tilde{U}$ is a real m dimensional C^∞ submanifold of $\mathbf{C}^m$ such that the complex span of the tangent space $T_p\tilde{U}$ equals $\mathbf{C}^m$, i.e. $T_p\tilde{U} + \sqrt{-1}T_p\tilde{U} = \mathbf{C}^m$.

(2) $\psi_z : U \to \tilde{U}$ is a diffeomorphism.

ψ_z is called the *coordinate map* of paracoordinates.

The inverse matrix of $(\frac{\partial z_i}{\partial x_j})$ is occasionally denoted by $(\frac{\partial x_j}{\partial z_i})$, though we do not define the derivative $\frac{\partial x_j}{\partial z_i}$. Moreover, a C^∞ mapping f from $\tilde{U}$ into $\mathbf{C}$ is written in the form $f(z_1, \cdots, z_m)$, even though $z_1, \cdots, z_m$ are not necessarily independent complex variables on $\tilde{U}$. Since z_i are C^∞ functions of $x_1, \cdots, x_m$, $f(\psi_z(\boldsymbol{x}))$ is a C^∞ function of $x_1, \cdots, x_m$. We define $\frac{\partial f}{\partial z_i}$ as follows:

$$\frac{\partial f}{\partial z_i} = \sum_{k=1}^{m} \frac{\partial x_k}{\partial z_i}\frac{\partial f}{\partial x_k}. \tag{4.1}$$

Note that the right hand side of (4.1) is computed as elements of $C^\infty(U)^C$ or of $C^\infty(\tilde{U})^C$. Higher order derivatives are defined similarly.

At each point $\tilde{p} \in \tilde{U}$, $\{dz_1, \cdots, dz_m\}$ forms a real basis in real coefficients of the cotangent space $T^*_{\tilde{p}}\tilde{U}$. Let $u_1, \cdots, u_m \in C^\infty(U)^C$ be other paracoordinates with $\tilde{U}' = \{(u_1(p), \cdots, u_m(p)); p \in U\}$. We set:

$$\frac{\partial z_i}{\partial u_j} = \sum_{k=1}^{m} \frac{\partial z_i}{\partial x_k} \frac{\partial x_k}{\partial u_j} \tag{4.2}$$

although z_i is not a genuine function of $u_1, \cdots, u_m$. Note that $\psi_u \psi_z^{-1}$ is a C^∞ diffeomorphism of $\tilde{U}$ onto $\tilde{U}'$. The chain rule also holds for these.

4.2 Kähler manifolds.

Let M be a smooth symplectic manifold with a symplectic form Ω. For a function $f \in C^\infty(M)^C$, we denote by $\bar{f}$ the complex conjugate of f. The Poisson bracket $\{\ ,\ \}$ defined by Ω satisfies $\overline{\{f, g\}} = \{\bar{f}, \bar{g}\}$ for any $f, g \in C^\infty(M)^C$.

Note that a Kähler manifold M is characterized as a real symplectic manifold covered by open subsets $\{V_\alpha\}$ such that for each V_α there is a homeomorphism $\varphi_\alpha : V_\alpha \to U_\alpha \subset \mathbf{C}^n$ with the following properties:

(1) The coordinate functions $z_1^\alpha, \cdots z_n^\alpha$ on $\mathbf{C}^n$ satisfy $\{z_i^\alpha, z_j^\alpha\} = \{\bar{z}_i^\alpha, \bar{z}_j^\alpha\} = 0$,
(2) The matrix $(\{z_i^\alpha, \bar{z}_j^\alpha\})$ is nondegenerate,
(3) On each intersection $V_\alpha \cap V_\beta$, setting $U_{\alpha\beta} = \varphi_\alpha(V_\alpha \cap V_\beta)$, the coordinate transformation $\varphi_{\alpha\beta} = \varphi_\beta \varphi_\alpha^{-1} : U_{\alpha\beta} \to U_{\beta\alpha}$ is holomorphic.

$z_1^\alpha, \cdots, z_m^\alpha$ are called *Kähler coordinates* (K-coordinates) on V_α.

We can assume that $\{V_\alpha\}$ is a locally finite, simple open Stein covering: i.e. a locally finite open covering such that $V_{\alpha_1} \cap \cdots \cap V_{\alpha_k}$ is a contractible Stein manifold.

Let V be one of V_α. As is known in [KN], there exist K-coordinates $z_1, \cdots, z_n$ on V. We can assume that there is a Kähler potential F on V, i.e. a real valued C^∞ function $F(\boldsymbol{z}, \bar{\boldsymbol{z}})$ such that the symplectic form equals

$$\Omega = \sum \Omega_{k,l} dz_k \wedge d\bar{z}_l, \qquad \Omega_{k,l} = \frac{\sqrt{-1}}{2} \frac{\partial^2 F}{\partial z_k \partial \bar{z}_l}. \tag{4.3}$$

Setting $z_i^* = \frac{\sqrt{-1}}{2}\frac{\partial F}{\partial z_i}$, we have $\Omega = \sum dz_i \wedge dz_i^*$, $\{z_i, z_j^*\} = \delta_{ij}$ and $\{z_i^*, z_j^*\} = 0$. $z_1, \cdots, z_n, z_1^*, \cdots, z_n^*$ are called *complex canonical coordinates* (CC-coordinates).

Since $\frac{\partial^2 F}{\partial z_k \partial \bar{z}_l}$ is nondegenerate, the CC-coordinates are paracoordinates of V. Note that the *canonical conjugate variables* $z_1^*, \cdots, z_n^*$ are not uniquely defined, as z_i^* can be replaced by $\tilde{z}_i^* = z_i^* + \frac{\partial h}{\partial z_i}$ for any holomorphic function h. In using CC-coordinates, the Poisson bracket becomes

$$\{f, g\} = \sum \left(\frac{\partial f}{\partial z_i}\frac{\partial g}{\partial z_i^*} - \frac{\partial g}{\partial z_i}\frac{\partial f}{\partial z_i^*}\right). \tag{4.4}$$

We consider the relationship between

$$z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n \quad \text{and} \quad z_1, \cdots, z_n, z_1^*, \cdots, z_n^* \tag{4.5}$$

on V. Let φ_z, φ'_z be the coordinate maps of

$$(z_1(p), \cdots, z_n(p), \bar{z}_1(p), \cdots, \bar{z}_n(p)), \quad (z_1(p), \cdots, z_n(p), z_1^*(p), \cdots, z_n^*(p))$$

respectively. We set $\tilde{V} = \{\varphi_z(p) \in \mathbf{C}^{2n}; p \in V\}$, $\tilde{V}' = \{\varphi'_z(p) \in \mathbf{C}^{2n}; p \in V\}$.

Let $\{t_1, \cdots t_{2n}\}$ be a real coordinate system of V. Note that $\varphi'_z\varphi_z^{-1}$ is a C^∞ diffeomorphism of $\tilde{V}$ onto $\tilde{V}'$. Then, $\varphi'_z\varphi_z^{-1}$ can be written as $\varphi'_z\varphi_z^{-1}(\boldsymbol{z}, \bar{\boldsymbol{z}}) = (\boldsymbol{z}, \boldsymbol{z}^*)$. If we consider the inverse mapping of $\varphi'_z\varphi_z^{-1}$, $\bar{\boldsymbol{z}}$ can be viewed as a 'function' of $\boldsymbol{z}, \boldsymbol{z}^*$, which can be understood as a sort of *implicit function theorem.*

Note that $\{dz_i,\ d\bar{z}_i\}$, $\{dz_i,\ dz_i^*\}$ are real bases of the cotangent spaces $T^*_{\tilde{V}}$ and $T^*_{\tilde{V}'}$ respectively, and we have that there are relations:

$$dz_i^* = \frac{\sqrt{-1}}{2}\sum\left(\frac{\partial^2 F}{\partial z_i \partial \bar{z}_k} d\bar{z}_k + \frac{\partial^2 F}{\partial z_i \partial z_k} dz_k\right), \quad 1 \le i \le n. \tag{4.6}$$

Since $\frac{\partial^2 F}{\partial z_i \partial \bar{z}_k}$ is non-singular, the above equality can be inverted to solve $d\bar{z}_k$.

We consider the exterior algebra $\Lambda^*(V) = \sum \Lambda^{p,q}(V)$ consisting of elements of the form:

$$\omega = \sum \omega_{i_1\cdots i_p, j_1\cdots j_q}(\boldsymbol{z}, \boldsymbol{z}^*) dz_{i_1}^* \wedge \cdots \wedge dz_{i_p}^* \wedge dz_{j_1} \wedge \cdots dz_{j_q}.$$

Define the partial exterior derivatives $\partial\omega$, $\partial^*\omega$ by:

$$\begin{aligned} \partial\omega &= \sum -\{z_{j_0}^*, \omega_{i_1\cdots i_p, j_1\cdots j_q}\} dz_{j_0} \wedge dz_{i_1}^* \wedge \cdots \wedge dz_{i_p}^* \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_q} \\ \partial^*\omega &= \sum \{z_{i_0}, \omega_{i_1\cdots i_p, j_1\cdots j_q}\} dz_{i_0}^* \wedge dz_{i_1}^* \wedge \cdots \wedge dz_{i_p}^* \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_q}. \end{aligned} \tag{4.7}$$

Thus, $\partial^* z_i^* = \sum (ad(z_k) z_i^*) dz_k^* = dz_i^*$, $\quad \partial z_i = \sum -(ad(z_k^*) z_i) dz_k^* = dz_i$

Hence we may set

$$\{z_i,\ \} = \frac{\partial}{\partial z_i^*}, \quad -\{z_i^*,\ \} = \frac{\partial}{\partial z_i}.$$

Using the Jacobi identity for the Poisson bracket, we have

$$(\partial^*)^2 = \partial^2 = 0, \quad \partial\partial^* + \partial^*\partial = 0.$$

We set $d = \partial + \partial^*$. The following is a slight modification of the Poincaré lemma:

Lemma 4.1. *Let V be an open contractible subset of M with CC-coordinates $(z_1, \cdots, z_n, z_1^*, \cdots, z_n^*)$. If $d\omega = 0$ for $\omega \in \Lambda^{p,q}(V)$, then there exist $\theta_1 \in \Lambda^{p-1,q}(V)$ and $\theta_2 \in \Lambda^{p,q-1}(V)$ such that $\omega = \partial^*\theta_1 + \partial\theta_2$.*

By Lemma 4.1, we have

Lemma 4.2. *On a Kähler manifold M, every holomorphic coordinate transformation $\varphi_{\alpha\beta} : U_{\alpha\beta} \to U_{\beta\alpha}$ induces a Poisson algebra isomorphism of the form:*

$$\varphi_{\alpha\beta}^*(z_i^\beta) = \varphi_{\alpha\beta}^i(\boldsymbol{z}^\alpha), \quad \varphi_{\alpha\beta}^*(z_i^{*\beta}) = \sum_k ((d\varphi_{\alpha\beta})^{-1})_i^k \cdot (z_k^{*\alpha} + \frac{\partial g_{\alpha\beta}}{\partial z_k^\alpha}) \tag{4.8}$$

where $g_{\alpha\beta}$ is a holomorphic function.

Proof. Let $z_1, \cdots, z_n$ and $w_1, \cdots, w_n$ be K-coordinates on $U_{\alpha\beta}$ and $U_{\beta\alpha}$ respectively. Let $z_1, \cdots, z_n, z_1^*, \cdots, z_n^*$ and $w_1, \cdots, w_n, w_1^*, \cdots, w_n^*$ are associated CC-coordinates respectively. Since $w_i = w_i(\boldsymbol{z})$; holomorphic function of $z_1, \cdots, z_n$, we have also $\bar{w}_i = \bar{w}_i(\overline{\boldsymbol{z}})$. Since $w_i^* = w_i^*(\boldsymbol{z}, \bar{\boldsymbol{z}})$, and $\bar{z}_i = \bar{z}_i(\boldsymbol{z}, \boldsymbol{z}^*)$, we see that $w_i = w_i(\boldsymbol{z})$, $w_i^* = w_i^*(\boldsymbol{z}, \boldsymbol{z}^*)$. The Poisson isomorphism which induced by the coordinate transformation is written as $\varphi_{\alpha\beta}^*(w_i) = w_i(\boldsymbol{z})$, $\varphi_{\alpha\beta}^*(w_i^*) = w_i^*(\boldsymbol{z}, \boldsymbol{z}^*)$.

We define another Poisson isomorphism $\tilde{\varphi}_{\alpha\beta}^*$ by setting

$$\tilde{\varphi}_{\alpha\beta}^*(w_i) = w_i(\boldsymbol{z}), \quad \tilde{\varphi}_{\alpha\beta}^*(\tilde{w}_i^*) = \sum \frac{\partial z_k}{\partial w_i} z_k^*$$

using the correspondence similar to transition functions of cotangent bundle.

Since both $(w_1, \cdots, w_n, w_1^*, \cdots, w_n^*)$ and $(w_1, \cdots, w_n, \tilde{w}_1^*, \cdots, \tilde{w}_n^*)$ are CC-coordinates, we have $\{w_i, w_j^* - \tilde{w}_j^*\} = 0$. It follows that $g_j = w_j^* - \tilde{w}_j^*$ is holomorphic.

By $\{w_i^*, w_j^*\} = \{\tilde{w}_i^*, \tilde{w}_j^*\} = 0$, we have $\{\tilde{w}_i^*, g_j\} - \{\tilde{w}_j^*, g_i\} = 0$, which implies $d(\sum g_i(\boldsymbol{w}) dw_i) = 0$. By Lemma 4.1, we have $g_i = \frac{\partial g}{\partial w_i}$.

Put $g = g_{\alpha\beta}(\boldsymbol{z})$. Since $\varphi_{\alpha\beta}$ is a holomorphic diffeomorphism, we have

$$w_i^* = \tilde{w}_i^* + \frac{\partial g}{\partial w_i} = \sum_k \frac{\partial z_k}{\partial w_i}(z_k^{*\alpha} + \frac{\partial g_{\alpha\beta}}{\partial z_k^\alpha}).$$

□

It is obvious that $\varphi_{\alpha\alpha}^* = 1$, $g_{\alpha\alpha}$ =const, and on every $V_\alpha \cap V_\beta \cap V_\gamma \neq \emptyset$, we see that

$$\varphi_{\alpha\beta}^* \varphi_{\beta\gamma}^* \varphi_{\gamma\alpha}^* = 1, \quad \varphi_{\alpha\beta}^* g_{\beta\alpha} + \varphi_{\alpha\gamma}^* g_{\gamma\beta} + g_{\alpha\gamma} = \text{const.}. \tag{4.9}$$

4.3 Noncommutative Kähler manifold.

Let M be a Kähler n-manifold. In the following we give a noncommutative version of K-coordinates. Viewing M as a real symplectic manifold, we construct a real Weyl manifold $\boldsymbol{W}_M$ as a locally trivial Weyl algebra bundle over M and a noncommutative algebra $\mathcal{F}(\boldsymbol{W}_M)$ of the Weyl functions of $\boldsymbol{W}_M$.

We now consider the complexifications $\boldsymbol{W}_M^C$ and $\mathcal{F}(\boldsymbol{W}_M)^C$. The complexification $\mathcal{F}(\boldsymbol{W}_M)^C$ is viewed as a subalgebra of the sections of the complex Weyl algebra bundle $\boldsymbol{W}_M^C$.

Let U be a contractible open subset of $\mathbf{R}^{2n}$.

Definition 4.3. (Cf. Definition 3.3.) For $f \in \mathcal{F}(\boldsymbol{W}_U)^C$, the *body part* $b(f)$ of f is a C^∞-function on U such that $f - b(f)^\sharp \in \nu\Gamma(\boldsymbol{W}_U)^C$. A system of elements $\xi_1, \cdots, \xi_{2n} \in \mathcal{F}(\boldsymbol{W}_U)^C$ are called *topological complex generators* (TC-generators), if the body parts $b(\xi_1), \cdots, b(\xi_{2n})$ are paracoordinates on U.

If $\xi_1, \cdots, \xi_{2n}$ are TC-generators, then these elements together with ν generate a dense subalgebra of $\mathcal{F}(\boldsymbol{W}_U)^C$.

On a local coordinate neighborhood U, TC-generators $\zeta_1, \cdots, \zeta_n, \bar{\zeta}_1, \cdots, \bar{\zeta}_n \in \mathcal{F}(\boldsymbol{W}_U)^C$ are called *quantum Kähler coordinates* (QK-coordinates), if $[\zeta_i, \zeta_j] = [\bar{\zeta}_i, \bar{\zeta}_j] = 0$, and the body part of the matrix $(-\frac{1}{\nu}[\zeta_i, \bar{\zeta}_j])$ is non-degenerate.

The following is easy to see:

Proposition 4.4. *Let $U \subset \mathbf{C}^n$ be a domain which is a Stein manifold. Suppose $\zeta_1, \cdots, \zeta_n \in \mathcal{F}(\boldsymbol{W}_U)^C$ satisfy $[\zeta_i, \zeta_j] = 0$. Then, for any holomorphic function $f(t_1, \cdots, t_n)$ on a domain U, $f(\zeta_1, \cdots, \zeta_n)$ can be defined by using a polynomial approximation, to be an element of $\mathcal{F}(\boldsymbol{W}_U)^C$.*

Definition 4.5. A complexified pre-Weyl manifold $\boldsymbol{W}_M^C$ is called a *noncommutative Kähler manifold*, if there is an open covering $\{V_\alpha\}$ with QK-coordinates $z_1^\alpha, \cdots, z_n^\alpha$, $\bar{z}_1^\alpha, \cdots, \bar{z}_n^\alpha$ of each $\mathcal{F}(\boldsymbol{W}_{U_\alpha})^C$, the model algebra over V_α, satisfying the following: On every $U_{\alpha\beta} = \varphi_\alpha(V_\alpha \cap V_\beta)$, two systems of the generators are related through a pre-Weyl diffeomorphism $\Phi_{\alpha\beta}^* : \mathcal{F}(\boldsymbol{W}_{U_{\beta\alpha}})^C \to \mathcal{F}(\boldsymbol{W}_{U_{\alpha\beta}})^C$ such that there is a holomorphic mapping $\varphi_{\alpha\beta} = (\varphi_{\alpha\beta}^1, \cdots, \varphi_{\alpha\beta}^n)$ of $U_{\alpha\beta}$ onto $U_{\beta\alpha}$ with

$$\Phi_{\alpha\beta}^*(z_i^\beta) = \varphi_{\alpha\beta}^i(\boldsymbol{z}^\alpha). \tag{4.10}$$

$\varphi_{\alpha\beta}$ is called a *holomorphic coordinate change.*

By the above definition, it is easily seen that the base manifold M of a noncommutative Kähler manifold $\boldsymbol{W}_M^C$ is a Kähler manifold (cf.§4.2).

A function of QK-coordinates $\boldsymbol{z}^\alpha$ remains a function of QK-coordinates $\boldsymbol{z}^\beta$ after any patching transformation $\Phi_{\beta\alpha}^*$. Hence on the noncommutative Kähler manifold $\boldsymbol{W}_M^C$, the notion of *quantum holomorphic function* is well-defined as a function of $\boldsymbol{z}^\alpha$ on each $\boldsymbol{W}_{U_\alpha}^C$.

We now consider a Weyl manifold $\boldsymbol{W}_M$ over M and its complexification $\boldsymbol{W}_M^C$.

Let $\{c_{\alpha\beta\gamma}\}$ be the Poincaré-Cartan cocycle of $\{\mathfrak{g}(U_\alpha)\}$. Since constant functions can be viewed as holomorphic functions, there is a natural homomorphism π of $H^2(M)$ into $H^2(M, \mathcal{O})$, the sheaf cohomology group of holomorphic functions.

The following is the main theorem of this paper:

Theorem 4.6. *A Weyl manifold $\boldsymbol{W}_M$ constructed on a Kähler manifold M is a noncommutative Kähler manifold, if and only if $\pi(c^{(2k)}(\boldsymbol{W}_M)) = 0$ for every $k \geq 1$. In particular, if $H^2(M, \mathcal{O}) = \{0\}$, then any Weyl manifold constructed on M is a noncommutative Kähler manifold.*

5. Proof of Theorem 4.6

5.1 Quantum complex coordinates.

Let M be a Kähler manifold. According to [OMY1], there exists a Weyl manifold $\boldsymbol{W}_M^C$.

Theorem 5.1. *There is an open covering $\{V_\alpha\}$ of M such that on every V_α there are QK-coordinates $\zeta_1^\alpha, \cdots, \zeta_n^\alpha$, $\bar{\zeta}_1^\alpha, \cdots, \bar{\zeta}_n^\alpha$ and the quantum canonical conjugate $\zeta_1^{*\alpha}, \cdots, \zeta_n^{*\alpha}$ with $[\zeta_i^\alpha, \zeta_j^{*\alpha}] = -\nu\delta_{ij}$, $[\zeta_i^{*\alpha}, \zeta_j^{*\alpha}] = 0$.*

Proof. We first take K-coordinates $z_1^\alpha, \cdots, z_n^\alpha$ and we make CC-coordinates on U_α;

$$z_1^\alpha, \cdots, z_n^\alpha, z_1^{*\alpha}, \cdots, z_n^{*\alpha}.$$

Set $\zeta_i^\alpha = (z_i^\alpha)^\sharp$, $\zeta_i^{*\alpha} = (z_i^{*\alpha})^\sharp$. Then, we get

$$[\zeta_i^\alpha, \zeta_j^\alpha] = [\zeta_i^{*\alpha}, \zeta_j^{*\alpha}] = 0 \pmod{\nu^2}, \quad [\zeta_i^\alpha, \zeta_j^{*\alpha}] = -\nu\delta_{ij} \pmod{\nu^2}. \tag{5.1}$$

Set as follows:

$$(5.2)\qquad \begin{aligned}&[\zeta_i^\alpha, \zeta_j^\alpha] = \nu^2 a_{ij} \pmod{\nu^3}, \quad [\zeta_i^{*\alpha}, \zeta_j^{*\alpha}] = \nu^2 c_{ij} \pmod{\nu^3},\\ &[\zeta_i^\alpha, \zeta_j^{*\alpha}] = \nu\delta_{ij} + \nu^2 b_{ij} \pmod{\nu^3}.\end{aligned}$$

Define a 2-form ω as follows:

$$(5.3)\qquad \omega = \sum (a_{ij} dz_i^{*\alpha} \wedge dz_j^{*\alpha} - b_{ij} dz_i^{*\alpha} \wedge dz_j^\alpha + c_{ij} dz_i^\alpha \wedge dz_j^\alpha)$$

By the Jacobi identity, we have $d\omega = 0$. Thus, by Lemma 4.1, there exists a 1-form

$$(5.4)\qquad \theta = \sum \lambda_i dz_i^{*\alpha} + \sum \kappa_j dz_j^\alpha$$

such that $d\theta = \omega$.

Replacing ζ_i^α, $\zeta_i^{*\alpha}$ by $\tilde{\zeta}_i = \zeta_i^\alpha - \nu\lambda_i$, $\tilde{\zeta}_i^* = \zeta_i^{*\alpha} + \nu\kappa_i$, we obtain

$$[\tilde{\zeta}_i, \tilde{\zeta}_j] = [\tilde{\zeta}_i^*, \tilde{\zeta}_j^*] = 0 \quad (\bmod\ \nu^3), \qquad [\tilde{\zeta}_i, \tilde{\zeta}_j^*] = -\nu\delta_{ij} \quad (\bmod\ \nu^3).$$

Repeating this procedure yields Theorem 5.1. □

$\zeta_1^\alpha, \cdots, \zeta_n^\alpha, \zeta_1^{*\alpha}, \cdots, \zeta_n^{*\alpha}$ will be called *quantum complex canonical generators* (QCC-generators).

5.2 Standard noncommutative Kähler manifold.

On a Kähler manifold M, we take a coordinate covering $\{V_\alpha\}$ and K-coordinates $z_1^\alpha, \cdots, z_n^\alpha$ on U_α, where $U_\alpha = \varphi_\alpha(V_\alpha)$. By using the argument in §4.2 on each U_α, there are CC-coordinates $z_1^\alpha, \cdots, z_n^\alpha$, $z_1^{*\alpha}, \cdots, z_n^{*\alpha}$. Identifying V_α with U_α, we use above CC-coordinates on V_α.

Let $\psi_z^\alpha : V_\alpha \to \mathbf{C}^{2n}$ be the coordinate map of these paracoordinates and let $\tilde{V}_\alpha = \{\psi_z^\alpha(p); p \in V_\alpha\}$. We define a star-product $*_\alpha$ on $C^\infty(\tilde{V}_\alpha)^C[[\nu]]$ by

$$(5.5)\qquad f *_\alpha g = f \exp\{-\frac{\nu}{2} \overleftarrow{\partial_{z^\alpha}} \dot{\wedge} \overrightarrow{\partial_{z^{*\alpha}}}\} g.$$

$(C^\infty(\tilde{V}_\alpha)^C[[\nu]], *_\alpha)$ can be viewed as the algebra of Weyl functions $\mathcal{F}(\hat{\boldsymbol{W}}^C_{\tilde{V}_\alpha})$ of the trivial complex Weyl algebra bundle $\hat{\boldsymbol{W}}^C_{\tilde{V}_\alpha}$ over $\tilde{V}_\alpha$. Since $\tilde{V}_\alpha$ is diffeomorphic to V_α through the coordinate map ψ_z^α, $\mathcal{F}(\hat{\boldsymbol{W}}^C_{\tilde{V}_\alpha})$ may be written as $\mathcal{F}(\hat{\boldsymbol{W}}^C_{V_\alpha})$. We now identify $C^\infty(\tilde{V}_\alpha)^C[[\nu]]$ with $\mathcal{F}(\hat{\boldsymbol{W}}^C_{V_\alpha})$.

Let $\varphi_{\alpha\beta} = \varphi_\beta\varphi_\alpha^{-1}$ be classical holomorphic coordinate transformations. By (4.8), (4.9), we see the following: Under the same notations as in Lemma 4.2, we set $\hat{\Phi}^*_{\alpha\beta}(\nu) = \nu$ and

$$(5.6)\qquad \hat{\Phi}^*_{\alpha\beta}(z_i^\beta) = \varphi^i_{\alpha\beta}(\boldsymbol{z}^\alpha), \quad \hat{\Phi}^*_{\alpha\beta}(z_i^{*\beta}) = \sum_k ((d\varphi_{\alpha\beta})^{-1})_i^k \cdot (z_k^{*\alpha} + \frac{\partial g^{\alpha\beta}}{\partial z_k^\alpha}).$$

Theorem 5.2. (i) *The mapping* $\hat{\Phi}^*_{\alpha\beta}$ *extends to a pre-Weyl diffeomorphism of* $(\mathcal{F}(\hat{\boldsymbol{W}}^C_{V_{\beta\alpha}}), *_\beta)$ *onto* $(\mathcal{F}(\hat{\boldsymbol{W}}^C_{V_{\alpha\beta}}), *_\alpha)$ *such that*

$$\hat{\Phi}^*_{\alpha\alpha} = 1, \qquad \hat{\Phi}^*_{\alpha\beta}\hat{\Phi}^*_{\beta\gamma}\hat{\Phi}^*_{\gamma\alpha} = 1. \tag{5.7}$$

Thus, we obtain a noncommutative Kähler manifold $\hat{\boldsymbol{W}}^C_M$ *in which* $z^\alpha_1, \cdots, z^\alpha_n$, $z^{*\alpha}_1, \cdots, z^{*\alpha}_n$ *are local QCC-generators.*
(ii) *Moreover,* $\hat{\Phi}^*_{\alpha\beta}$ *extends to a pointless contact diffeomorphism* $\hat{\Phi}^{c*}_{\alpha\beta}$ *such that*

$$\hat{\Phi}^{c*}_{\alpha\beta}\hat{\Phi}^{c*}_{\beta\gamma}\hat{\Phi}^{c*}_{\gamma\alpha} = e^{c^{(0)}_{\alpha\beta\gamma}\mathrm{ad}(\nu^{-1})}, \quad c^{(0)}_{\alpha\beta\gamma} \in \mathbf{C}, \tag{5.8}$$

and $c^{(0)}_{\alpha\beta\gamma}$ *defines a cohomology class in the coefficients* $\mathbf{C}$ *of the symplectic 2-form on* M.

Proof. Omitting subscripts α, β, we denote by $z'_i = \varphi^i(\boldsymbol{z})$. By (5.5), we have

$$\begin{aligned} &[\sum_k \frac{\partial z_k}{\partial z'_i}\cdot(z^*_k + \frac{\partial g}{\partial z_k}),\ z'_j] = \nu\frac{\partial z_k}{\partial z'_i}\frac{\partial z'_j}{\partial z_k} = \nu\delta_{ij} \\ &[z'_i, z'_j] = 0, \qquad [\sum_k \frac{\partial z_k}{\partial z'_i}\cdot(z^*_k + \frac{\partial g}{\partial z_k}),\ \sum_m \frac{\partial z_m}{\partial z'_j}\cdot(z^*_m + \frac{\partial g}{\partial z_m})] = 0 \end{aligned} \tag{5.9}$$

Thus, setting $z'^*_i = \sum \frac{\partial z_k}{\partial z'_i}\cdot(z^*_k + \frac{\partial g}{\partial z_k})$, we see $z'_1, \cdots, z'_n, z'^*_1, \cdots, z'^*_n$ are QCC-generators of $C^\infty(V_{\alpha\beta})^C[[\nu]]$.

We show that $\hat{\Phi}^*_{\alpha\beta}$ extends to a pre-Weyl diffeomorphism. Since $\varphi_{\alpha\beta}$ is a symplectic diffeomorphism, Proposition 2.3 gives a lift $\Psi^*_{\alpha\beta}$ of $\varphi_{\alpha\beta}$. By Theorem 3.2, we may assume that $\Psi^*_{\alpha\beta}$ are patching Weyl diffeomorphisms of a Weyl manifold $\boldsymbol{W}_M$.

We consider $(\Psi^*_{\alpha\beta})^{-1}\hat{\Phi}^*_{\alpha\beta}$ on the above QCC-generators. Set

$$(\Psi^*_{\alpha\beta})^{-1}\hat{\Phi}^*_{\alpha\beta}(z'_i) = z'_i + h_i, \quad (\Psi^*_{\alpha\beta})^{-1}\hat{\Phi}^*_{\alpha\beta}(z'^*_i) = z'^*_i + h^*_i.$$

By (5.9) together with Lemma 4.1, we easily see that there are elements $h_{\alpha\beta} \in C^\infty(V_{\alpha\beta})^C[[\nu]]$ such that $\hat{\Phi}^*_{\alpha\beta} = \Psi^*_{\alpha\beta}e^{\mathrm{ad}(h_{\alpha\beta})}$. Since $\Psi^*_{\alpha\beta}$ and $e^{\mathrm{ad}(h_{\alpha\beta})}$ are pre-Weyl diffeomorphisms, $\hat{\Phi}^*_{\alpha\beta}$ extends to a pre-Weyl diffeomorphism. (5.7) follows directly from (4.9). Thus, we get a noncommutative Kähler manifold $\hat{\boldsymbol{W}}_M$.

Though we can make, by Lemma 2.6, a pointless contact diffeomorphism $\hat{\Phi}^{c*}_{\alpha\beta}$ which extends $\hat{\Phi}^*_{\alpha\beta}$, we construct $\hat{\Phi}^{c*}_{\alpha\beta}$ directly in two ways to obtain (5.8).

We define a contact Weyl Lie algebra $\Gamma(\mathfrak{g}^C_{U_\beta})$ by joining τ^β with the relations

$$[\tau^\beta, \nu] = 2\nu^2, \quad [\tau^\beta, z^\beta_i] = \nu z^\beta_i, \quad [\tau^\beta, z^{*\beta}_i] = \nu z^{*\beta}_i. \tag{5.10}$$

To obtain the extension $\hat{\Phi}^{c*}_{\alpha\beta}$ of $\hat{\Phi}^*_{\alpha\beta}$ we have only to know the function $f_{\alpha\beta}$ given by $\hat{\Phi}^{c*}_{\alpha\beta}(\tau^\beta) = \tau^\alpha + f_{\alpha\beta}$. By (5.6), we set $z'_i = \varphi^*_{\alpha\beta}(\boldsymbol{z}^\beta_i)$, $z'^*_i = \varphi^*_{\alpha\beta}(\boldsymbol{z}^{*\beta}_i)$ on $U_{\alpha\beta}$. Then, by (5.9) and (5.10), $f_{\alpha\beta}$ must satisfy

$$[\tau^\alpha + f_{\alpha\beta}, z'_i] = \nu z'_i, \quad [\tau^\alpha + f_{\alpha\beta}, z'^*_i] = \nu z'^*_i. \tag{5.11}$$

Note that $[\tau^\alpha, h] = \nu E^\alpha h$, where E^α is the Euler operator given by $E^\alpha = \sum_{i=1}^n (z_i^\alpha \cdot \partial_{z_i^\alpha} + z_i^{*\alpha} \cdot \partial_{z_i^{*\alpha}})$. (5.11) can be rewritten by using the usual commutative product, as

$$\frac{\partial}{\partial z_i'^*} f_{\alpha\beta} = -(I - E^\alpha) z_i', \quad \frac{\partial}{\partial z_i'} f_{\alpha\beta} = (I - E^\alpha) z_i'^*. \tag{5.12}$$

Thus, by solving (5.12) via Lemma 4.1, we found $f_{\alpha\beta}$. Since the right hand side of (5.12) does not involve ν, $f_{\alpha\beta}$ does not involve ν. Put

$$c^{(0)}_{\alpha\beta\gamma} = \tfrac{1}{2}(f_{\alpha\beta} + \varphi^*_{\alpha\beta} f_{\beta\gamma} + \varphi^*_{\alpha\gamma} f_{\gamma\alpha}).$$

Then, we have $c^{(0)}_{\alpha\beta\gamma} \in \mathbf{C}$ and $\hat{\Phi}^{c*}_{\alpha\beta} \hat{\Phi}^{c*}_{\beta\gamma} \hat{\Phi}^{c*}_{\gamma\alpha} = e^{c^{(0)}_{\alpha\beta\gamma} \mathrm{ad}(\nu^{-1})}$.

To find the de Rham cohomology class corresponding to $c^{(0)}_{\alpha\beta\gamma}$ through the isomorphism between Čech cohomology and de Rham cohomology, we recall another recipe of constructions of $f_{\alpha\beta}$. That is, we find a 1-form $\tilde{\theta}_\alpha$ on every V_α such that $\varphi^*_{\alpha\beta}\tilde{\theta}_\beta - \tilde{\theta}_\alpha = \frac{1}{2} df_{\alpha\beta}$ on $V_{\alpha\beta} = V_\alpha \cap V_\beta$, because $\{d\tilde{\theta}_\alpha\}$ defines a global closed 2-form.

To find $\tilde{\theta}_\alpha$, we remark that there is a one parameter family ψ_t of symplectic diffeomorphisms $\psi_t : V_{\alpha\beta,0} \to V_{\alpha\beta,t}$ such that $V_{\alpha\beta,0} = V_{\alpha\beta}$, $\psi_0 = 1$ and $V_{\alpha\beta,1} = V_{\alpha\beta}$, $\psi_1 = (\varphi^1_{\alpha\beta}, \cdots, \varphi^{2n}_{\alpha\beta})$. (Cf. [OMY3, Lemma A] and [KNz].) Define an infinitesimal symplectic transformation H_t by $H_t = (\frac{d}{dt}\psi_t)\psi_t^{-1}$. Since H_t is a Hamiltonian vector field, there is a C^∞ function h_t such that $\Omega \lrcorner H_t = -dh_t$.

Recall that a lift Ψ_t^{c*} is given by solving $\frac{d}{dt}\Psi_t^{c*} = \Psi_t^{c*}\mathrm{ad}(\frac{1}{\nu}h_t)$. If we set $\Psi_t^{c*}(\tau^\alpha) = \tau^\alpha + f_t$, then f_t must satisfy the differential equation

$$\frac{d}{dt} f_t = \frac{1}{\nu}[h_t, f_t] + 2h_t - \frac{1}{\nu}[\tau^\alpha, h_t]. \tag{5.13}$$

By the first construction of $f_{\alpha\beta}$, we may set $f_1 = f_{\alpha\beta} \bmod \nu$.

Note that setting $\tilde{\theta}_\alpha = \frac{1}{2}\sum(z_i^\alpha dz_i^{*\alpha} - z_i^{*\alpha} dz_i^\alpha)$, we have $\frac{1}{\nu}[\tau^\alpha, h_t] = E^\alpha h_t = 2\tilde{\theta}_\alpha \lrcorner H_t$. Solving the equation of ν^0-component of (5.13), we have

$$f_{\alpha\beta} = \psi_1^* \int_0^1 \psi_t^{*-1}(2h_t - 2\tilde{\theta}_\alpha \lrcorner H_t) dt. \tag{5.14}$$

Note that $\Omega = d\tilde{\theta}_\alpha$ on V_α (cf. 4.2). By Cartan's formula of Lie derivatives, we see that

$$df_{\alpha\beta} = 2\psi_1^* \int_0^1 \psi_t^{*-1}(dh_t - d(\tilde{\theta}_\alpha \lrcorner H_t)) dt = -2\psi_1^* \int_0^1 \psi_t^{*-1} \mathcal{L}_{H_t} \tilde{\theta}_\alpha dt.$$

Hence we have $df_{\alpha\beta} = 2(\varphi^*_{\alpha\beta}\tilde{\theta}_\beta - \tilde{\theta}_\alpha)$, by remarking $\psi_1^{*-1}\tilde{\theta}_\alpha = \varphi^*_{\alpha\beta}\tilde{\theta}_\beta$. Thus, we see that $d\tilde{\theta}_\alpha = 2\sum dz_i^\alpha \wedge dz_i^{*\alpha}$. The last assertion is proved. □

The noncommutative Kähler structure obtained by Theorem 5.2 seems to be isomorphic to that of Karabegov [Ka].

5.3 Proof of Theorem 4.6.

Suppose we have a Weyl manifold $\boldsymbol{W}_M$ with the Poincaré-Cartan class $\{c_{\alpha\beta\gamma}\}$. Let $\Phi^*_{\alpha\beta}$ be Weyl diffeomorphisms giving patching transformations, and let $\Phi^{c*}_{\alpha\beta}$ be the lifts of $\Phi^*_{\alpha\beta}$ given in §3.2. Let $\varphi_{\alpha\beta} : U_{\alpha\beta} \to U_{\beta\alpha}$ be the coordinate transformation induced by $\Phi^*_{\alpha\beta}$. $\varphi_{\alpha\beta}$ is a symplectomorphism and a holomorphic diffeomorphism at the same time.

By the assumption of Theorem 4.6, we have that for every $k \geq 1$, $\{c^{(2k)}_{\alpha\beta\gamma}\}$ can be written in the form $c^{(2k)}_{\alpha\beta\gamma} = g^{(2k)}_{\alpha\beta} + \varphi^*_{\alpha\beta} g^{(2k)}_{\beta\gamma} + \varphi^*_{\alpha\gamma} g^{(2k)}_{\gamma\alpha}$, $(k \geq 1)$ where $g^{(2k)}_{\alpha\beta}$ is a holomorphic function on $U_{\alpha\beta} = \varphi_\alpha(V_\alpha \cap V_\beta)$.

Beside $\Phi^{c*}_{\alpha\beta}$, we define another family of pre-Weyl diffeomorphisms

$$\breve{\Phi}^{c*}_{\alpha\beta} = \hat{\Phi}^{c*}_{\alpha\beta} \exp \sum_{k\geq 1} \frac{1}{2k-1} \mathrm{ad}(\nu^{2k-1} g^{(2k)}_{\alpha\beta}) \tag{5.15}$$

by using $g^{(2k)}_{\alpha\beta}$ given above. By (3.7), we see that $\{\breve{\Phi}^{c*}_{\alpha\beta}\}$ satisfies

$$\breve{\Phi}^{c*}_{\alpha\beta}\breve{\Phi}^{c*}_{\beta\gamma}\breve{\Phi}^{c*}_{\gamma\alpha} = \hat{\Phi}^{c*}_{\alpha\beta}\hat{\Phi}^{c*}_{\beta\gamma}\hat{\Phi}^{c*}_{\gamma\alpha} \exp \textstyle\sum_k \frac{1}{2k-1} \mathrm{ad}(\nu^{2k-1}(g^{(2k)}_{\alpha\beta} + \hat{\Phi}^*_{\alpha\beta} g^{(2k)}_{\beta\gamma} + \hat{\Phi}^*_{\alpha\gamma} g^{(2k)}_{\gamma\alpha})).$$

Hence by (5.6), (5.8), we have

Lemma 5.3. $\breve{\Phi}^{c*}_{\alpha\beta}\breve{\Phi}^{c*}_{\beta\gamma}\breve{\Phi}^{c*}_{\gamma\alpha} = \Phi^{c*}_{\alpha\beta}\Phi^{c*}_{\beta\gamma}\Phi^{c*}_{\gamma\alpha} = e^{c_{\alpha\beta\gamma}\mathrm{ad}(\nu^{-1})}$.

The above lemma also shows that the system $\{\breve{\Phi}^{c*}_{\alpha\beta}\}$ defines a pre-Weyl manifold $\breve{\boldsymbol{W}}_M$. However, we see also the following:

Lemma 5.4. *For each α, β such that $V_\alpha \cap V_\beta \neq \emptyset$, there exist a unique $h_{\alpha\beta} \in \mathcal{F}(\boldsymbol{W}_{U_{\alpha\beta}})^C$ and a unique $c_{\alpha\beta} \in \mathbf{C}$ such that $\Phi^{c*}_{\alpha\beta} = \breve{\Phi}^{c*}_{\alpha\beta} e^{\mathrm{ad}(\nu h_{\alpha\beta})} e^{c_{\alpha\beta}\mathrm{ad}(\nu^{-1})}$*

Proof. We already know that $f_{\alpha\beta}$ dose not involve ν. Hence by (5.15) we see that $\breve{\Phi}^{c*}_{\alpha\beta}(\tau^\beta)$ is written in the form $\tau^\alpha + \sum_{k\geq 0} \nu^{2k} * h^\sharp_{2k}$, $h_{2k} \in C^\infty(U_{\alpha\beta})$. Apply now Lemma 2.7 to $\Phi^*_{\alpha\beta}$. Since both $\Phi^*_{\alpha\beta}$ and $\hat{\Phi}^*_{\alpha\beta}$ induce the same $\varphi_{\alpha\beta}$ on the base spaces, we see by Corollary 2.5, (1) that $\Phi^*_{\alpha\beta}(\hat{\Phi}^*_{\alpha\beta})^{-1}(\tau^\alpha)$ is written uniquely in the form $\tau^\alpha + 2c_{\alpha\beta} + \nu^2 * h'_{\alpha\beta}$, where $h'_{\alpha\beta} \in \mathcal{F}(\boldsymbol{W}_{U_{\beta\alpha}})^C$. Hence, we have $\Phi^{c*}_{\alpha\beta} = \breve{\Phi}^{c*}_{\alpha\beta} e^{\mathrm{ad}(h_{\alpha\beta})} e^{c_{\alpha\beta}\mathrm{ad}(\nu^{-1})}$. We remark also that $c_{\alpha\beta} + c_{\beta\gamma} + c_{\gamma\alpha} = 0$. □

The identities $\Phi^{c*}_{\alpha\beta}\Phi^{c*}_{\beta\alpha} = 1$, $\breve{\Phi}^{c*}_{\alpha\beta}\breve{\Phi}^{c*}_{\beta\alpha} = 1$ together with (3.7) yield $h_{\beta\alpha} = -\breve{\Phi}^*_{\alpha\beta} h_{\alpha\beta}$ and $c_{\alpha\beta} = -c_{\beta\alpha}$.

In what follows we use

$$\breve{\Psi}^*_{\alpha\beta} = \breve{\Phi}^*_{\alpha\beta} e^{c_{\alpha\beta}\mathrm{ad}(\nu^{-1})} \tag{5.16}$$

instead of $\breve{\Phi}^*_{\alpha\beta}$, since this replacement (5.16) does not change the Poincaré-Cartan cocycle by the above remark.

The identity in Lemma 5.3 gives the following cocycle property for $\{h_{\alpha\beta}\}$:

Lemma 5.5. *On $\Gamma(\mathfrak{g}_{U_{\alpha\beta\gamma}})$, we have $e^{\mathrm{ad}(\nu h_{\alpha\beta})} e^{\mathrm{ad}(\nu \breve{\Psi}^*_{\alpha\beta} h_{\beta\gamma})} e^{\mathrm{ad}(\nu \breve{\Psi}^*_{\alpha\gamma} h_{\gamma\alpha})} = 1$.*

The next lemma shows that this cocycle is a coboundary, and hence $\boldsymbol{W}_M \cong \breve{\boldsymbol{W}}_M$.

Lemma 5.6. *For each α, there exists $h_\alpha \in \mathcal{F}(\boldsymbol{W}_{U_\alpha})^C$ such that*

$$\Phi^{c*}_{\alpha\beta} = e^{\mathrm{ad}(\nu h_\alpha)}\check{\Psi}^{c*}_{\alpha\beta}e^{-\mathrm{ad}(\nu h_\beta)}. \tag{5.17}$$

Proof. Let $\varphi_{\alpha\beta}$ be the induced symplectic diffeomorphism by $\Phi^*_{\alpha\beta}$. Using Lemma 5.5, we have by identifying $h_{\alpha\beta}$ with an ordinary function that

$$h_{\alpha\beta} + \varphi^*_{\alpha\beta}h_{\beta\gamma} + \varphi^*_{\alpha\gamma}h_{\gamma\alpha} = 0 \mod \nu. \tag{5.18}$$

Taking a partition of unity $\{\phi_\alpha\}$ subordinate to the covering $\{V_\alpha\}$, we set

$$h_\alpha = \sum_\gamma \varphi^*_{\alpha\gamma}\phi_\gamma h_{\alpha\gamma} \in \mathcal{F}(\boldsymbol{W}_{U_\alpha})^C. \tag{5.19}$$

Using Lemma 5.5 again, we get $h_{\alpha\beta} = h_\alpha - \varphi^*_{\alpha\beta}h_\beta$. Setting

$$\acute{\Psi}^{c*}_{\alpha\beta} = e^{\mathrm{ad}(h_\alpha)}\check{\Psi}^{c*}_{\alpha\beta}e^{-\mathrm{ad}(h_\beta)},$$

we see that $\Phi^{c*}_{\alpha\beta} = \acute{\Psi}^{c*}_{\alpha\beta} \mod \nu^3$. By Corollary 2.5, (1), there exists a unique $\tilde{h}_{\alpha\beta}$ such that

$$\Phi^{c*}_{\alpha\beta} = \acute{\Psi}^{c*}_{\alpha\beta}e^{\mathrm{ad}(\nu^2\tilde{h}_{\alpha\beta})} \tag{5.20}$$

without $e^{c\,\mathrm{ad}(\nu^{-1})}$-term. Repeating this procedure yields the Lemma 5.6. □

We now show Theorem 4.6. We first show the necessity; $\pi(c^{(2k)}(\boldsymbol{W}_M)) = 0$ implies M is a noncommutative Kähler manifold. We may assume by Lemma 5.6 that $\boldsymbol{W}_M$ is a pre-Weyl manifold with a system of patching diffeomorphisms $\check{\Psi}^*_{\alpha\beta}$. Since $[\nu^{-1}, z^\beta_i] = 0$ and $[g^{(2k)}_{\alpha\beta}(\boldsymbol{z}^\beta), z^\beta_i] = 0$, we have by (5.15), (5.16) and Theorem 5.2 that

$$\check{\Psi}^*_{\alpha\beta}z^\beta_i = \hat{\Phi}^*_{\alpha\beta}z^\beta_i = \varphi^i_{\alpha\beta}(\boldsymbol{z}^\alpha). \tag{5.21}$$

This means the patching transformations are holomorphic.

Note that $[z^\alpha_i, \bar{z}^\alpha_j] = -\nu\{b(z^\alpha_i), b(\bar{z}^\alpha_j)\} \mod \nu$. Hence we see the body part of the matrix $(-\frac{1}{\nu}[z^\alpha_i, \bar{z}^\alpha_j])$ is nondegenerate. Thus, $\boldsymbol{W}_M$ is a noncommutative Kähler manifold.

To prove the sufficiency in Theorem 4.6, let $\boldsymbol{W}_M$ be a Weyl manifold with the Poincaré-Cartan class $c = \sum_{k\geq 0} c^{(2k)}$. By definition of Weyl manifold $\boldsymbol{W}_M$ for a Kähler manifold M, there are a simple open Stein covering $\{V_\alpha\}$, a system of trivial Lie algebra bundle $\{\mathfrak{g}_{U_\alpha}\}$ and a system of patching transformations $\{\Phi^{c*}_{\alpha\beta}\}$. Suppose that $\boldsymbol{W}_M$ is a noncommutative Kähler manifold over M. Then, we may assume that on each $\mathcal{F}(\boldsymbol{W}_{U_\alpha})^C$ there are QK-coordinates $z^\alpha_1, \cdots, z^\alpha_n, \bar{z}^\alpha_1, \cdots, \bar{z}^\alpha_n$ with the pre-Weyl diffeomorphisms $\Psi^*_{\alpha\beta} : \mathcal{F}(\boldsymbol{W}_{U_{\beta\alpha}})^C \to \mathcal{F}(\boldsymbol{W}_{U_{\alpha\beta}})^C$ satisfying the property (4.10). Since $\{\Phi^*_{\alpha\beta}\}$ and $\{\Psi^*_{\alpha\beta}\}$ are patching diffeomorphisms of the same Weyl manifold $\boldsymbol{W}_M$, there is a pre-Weyl diffeomorphism Ψ^*_α for each α such that $\Phi^*_{\alpha\beta}\Psi^*_\beta = \Psi^*_\alpha\Psi^*_{\alpha\beta}$. Hence $\boldsymbol{W}_M$ can be viewed as a pre-Weyl manifold with patching diffeomorphisms $\Psi^*_{\alpha\beta}$. Let $\Psi^{c*}_{\alpha\beta}$ be a pointless contact diffeomorphism which extends $\Psi^*_{\alpha\beta}$.

On the other hand, remark that holomorphic coordinate change $\varphi_{\alpha\beta}$ in (4.10) can be viewed as usual holomorphic coordinate transformations on the base manifold M. By Theorem 5.1, there are QCC-generators $z_1^\alpha, \cdots, z_n^\alpha$, $z_1^{*\alpha}, \cdots, z_n^{*\alpha}$ on each $\mathcal{F}(\boldsymbol{W}_{U_\alpha})^C$. By Theorem 5.2, we have pre-Weyl diffeomorphisms $\hat{\Phi}^*_{\alpha\beta}$ and pointless contact diffeomorphisms $\hat{\Phi}^{c*}_{\alpha\beta}$ which extend $\hat{\Phi}^*_{\alpha\beta}$.

Since $\Psi^{c*}_{\alpha\beta}(\hat{\Phi}^{c*}_{\alpha\beta})^{-1}$ induces the identity on the base space $U_{\alpha\beta}$, there is by (2.2) $h_{\alpha\beta} \in \mathcal{F}(\boldsymbol{W}_{U_{\alpha\beta}})^C$ such that $\Psi^*_{\alpha\beta} = \hat{\Phi}^*_{\alpha\beta}e^{\mathrm{ad}(h_{\alpha\beta})}$. The terms $e^{c\,\mathrm{ad}(\nu^{-1})}$, $e^{c'\mathrm{ad}(\log\nu)}$ need not be used because these are identities on $\mathcal{F}(\boldsymbol{W}_{U_{\alpha\beta}})^C$.

Note that $\Psi^*_{\alpha\beta}(z_i) = \hat{\Phi}^*_{\alpha\beta}(z_i)$ for every z_i. We see that $[z_i, h_{\alpha\beta}] = 0$. It follows that $h_{\alpha\beta}$ does not involve z_i^* variables, that is 'holomorphic'.

We define $\Psi^{c*}_{\alpha\beta}$ by $\hat{\Phi}^{c*}_{\alpha\beta}e^{\mathrm{ad}(h_{\alpha\beta})}$. Then, $\Psi^{c*}_{\alpha\beta}$ is an extension of $\Psi^*_{\alpha\beta}$. The Poincaré-Cartan cocycle of $\boldsymbol{W}_M$ is given as

$$e^{c_{\alpha\beta\gamma}\mathrm{ad}(\nu^{-1})} = \Psi^{c*}_{\alpha\beta}\Psi^{c*}_{\beta\gamma}\Psi^{c*}_{\gamma\alpha} = \hat{\Phi}^{c*}_{\alpha\beta}\hat{\Phi}^{c*}_{\beta\gamma}\hat{\Phi}^{c*}_{\gamma\alpha}e^{\mathrm{ad}(\hat{\Phi}^*_{\alpha\beta}h_{\alpha\beta})}e^{\mathrm{ad}(\hat{\Phi}^*_{\alpha\gamma}h_{\beta\gamma})}e^{\mathrm{ad}(h_{\gamma\alpha})}.$$

By (5.8), we have $e^{c_{\alpha\beta\gamma}\mathrm{ad}(\nu^{-1})} = e^{c^{(0)}_{\alpha\beta\gamma}\mathrm{ad}(\nu^{-1})}e^{\mathrm{ad}(\hat{\Phi}^*_{\alpha\beta}h_{\alpha\beta})}e^{\mathrm{ad}(\hat{\Phi}^*_{\alpha\gamma}h_{\beta\gamma})}e^{\mathrm{ad}(h_{\gamma\alpha})}$. Let $h_{\alpha\beta} = \sum_{k\geq 0}\nu^k h^{(k)}_{\alpha\beta}$. Since $c_{\alpha\beta\gamma} = \sum_{k\geq 0}\nu^{2k}c^{(2k)}_{\alpha\beta\gamma}$, we have

$$\hat{\Phi}^*_{\alpha\beta}h_{\alpha\beta} + \hat{\Phi}^*_{\alpha\gamma}h_{\beta\gamma} + h_{\gamma\alpha} = 0 \mod \nu.$$

By identifying $h^{(0)}_{\alpha\beta}$ with an ordinary function, we get

$$0 = \varphi^*_{\alpha\beta}h^{(0)}_{\alpha\beta} + \varphi^*_{\alpha\gamma}h^{(0)}_{\beta\gamma} + h^{(0)}_{\gamma\alpha}.$$

Consider $\check{\Phi}^{c*}_{\alpha\beta} = \hat{\Phi}^{c*}_{\alpha\beta}e^{-\mathrm{ad}(h^{(0)}_{\alpha\beta})}$ instead of $\hat{\Phi}^{c*}_{\alpha\beta}$ in the above arguments. Since $h^{(0)}_{\alpha\beta}$ are holomorphic, $\check{\Phi}^{c*}_{\alpha\beta}$ can be used as patching diffeomorphisms to define a noncommutative Kähler manifold.

Now by the same reason as above, there are $h_{\alpha\beta}$ such that $\Psi^*_{\alpha\beta} = \check{\Phi}^*_{\alpha\beta}e^{\mathrm{ad}(\nu h_{\alpha\beta})}$ and $h_{\alpha\beta}$ are holomorphic. Hence, we have

$$e^{c_{\alpha\beta\gamma}\mathrm{ad}(\nu^{-1})} = e^{c^{(0)}_{\alpha\beta\gamma}\mathrm{ad}(\nu^{-1})}e^{\mathrm{ad}(\nu\check{\Phi}^*_{\alpha\beta}h_{\alpha\beta})}e^{\mathrm{ad}(\nu\check{\Phi}^*_{\alpha\gamma}h_{\beta\gamma})}e^{\mathrm{ad}(\nu h_{\gamma\alpha})}.$$

Setting $h_{\alpha\beta} = \sum_{k\geq 0}\nu^k h^{(k)}_{\alpha\beta}$, we have $c^{(2)}_{\alpha\beta\gamma} = \varphi^*_{\alpha\beta}h^{(0)}_{\alpha\beta} + \varphi^*_{\alpha\gamma}h^{(0)}_{\beta\gamma} + h^{(0)}_{\gamma\alpha}$. This implies that $c^{(2)}_{\alpha\beta\gamma}$ is a coboundary in the cochain complex with coefficients $\mathcal{O}$. Repeating this procedure, we see $\{c^{(2k)}_{\alpha\beta\gamma}\} = 0$ in $H^2(M, \mathcal{O})$ for $k \geq 1$. Thus, we obtain Theorem 4.6.

6. Construction of noncommutative contact algebras

In this section we construct a certain algebra, called a noncommutative contact algebra over a quantizable symplectic manifold.

We use notations stated in §3. Let M be a symplectic manifold with the symplectic form Ω. We assume that M is quantizable, i.e., $\frac{1}{\pi}\Omega \in H^2(M;\mathbf{Z})$. We consider a Weyl manifold $\boldsymbol{W}_M$. On each coordinate U_α, we use τ given by (1.9) and we denote it by $\tilde{\tau}_\alpha$. In this section we assume that the Poincaré-Cartan class $c(\boldsymbol{W}_M)$ is $c^{(0)}(\boldsymbol{W}_M)$. Since M is quantizable, we can assume that $c^{(0)}_{\alpha\beta\gamma}$ is taken as $\pi n_{\alpha\beta\gamma}$, where $n_{\alpha\beta\gamma} \in \mathbf{Z}$. Since $[\tilde{\tau}^\alpha, \nu^{-1}] = -2$, we see that

$$\Phi^{c*}_{\alpha\beta}\Phi^{c*}_{\beta\gamma}\Phi^{c*}_{\gamma\alpha}\tilde{\tau}^\alpha = \tilde{\tau}^\alpha + 2\pi n_{\alpha\beta\gamma}. \tag{6.1}$$

This implies $\Phi^{c*}_{\alpha\beta}\Phi^{c*}_{\beta\gamma}\Phi^{c*}_{\gamma\alpha}e^{i\tilde{\tau}^\alpha} = e^{i\tilde{\tau}^\alpha}$, and hence the associative algebras $\mathcal{A}(U_\alpha)$ generated by $e^{i\tilde{\tau}^\alpha}$ and $\mathcal{F}(\boldsymbol{W}_{U_\alpha})^C$ can be patched together to form an algebra sheaf. We denote this patched algebra by $\mathcal{A}(M)$. Every element of $\mathcal{A}(U_\alpha)$ is written in the form $\sum f_m * e^{im\tilde{\tau}^\alpha}$, $f_m \in \mathcal{F}(\boldsymbol{W}_{U_\alpha})^C$ and $\Phi^{c*}_{\alpha\beta}$ are the patching transformations.

Let $\mathcal{A}_m(M)$ be the subspace consisting of elements written in the form $f_\alpha * e^{im\tilde{\tau}^\alpha}$ on each U_α, where $f_\alpha \in \mathcal{F}(\boldsymbol{W}_{U_\alpha})$. $\mathcal{A}_m(M)$ is characterized as the eigenspace of $\frac{1}{i}\mathrm{ad}(\nu^{-1})$ with the eigenvalue $2m$. Clearly, we have

$$\mathcal{A}(M) = \bigoplus_{m\in\mathbf{Z}} \mathcal{A}_m(M),$$

and $\mathcal{A}_0(M) = \mathcal{F}(\boldsymbol{W}_M)^C$ is a subalgebra of $\mathcal{A}(M)$.

Since $\Phi^{c*}_{\alpha\beta}$ is a contact Weyl diffeomorphism, we see that there exists $f_{\alpha\beta} \in \mathcal{F}(\boldsymbol{W}_{U_{\alpha\beta}})^C$ on every $V_\alpha \cap V_\beta \neq \emptyset$ such that

$$\Phi^{c*}_{\alpha\beta}(e^{i\tilde{\tau}^\beta}) = e^{i\Phi^{c*}_{\alpha\beta}(\tilde{\tau}^\beta)} = e^{i(\tilde{\tau}^\alpha + f_{\alpha\beta})}. \tag{6.2}$$

Lemma 6.1. *There is $F_{\alpha\beta} \in \mathcal{F}(\boldsymbol{W}_{U_{\alpha\beta}})^C$ such that $e^{i(\tilde{\tau}^\alpha+f_{\alpha\beta})} = F_{\alpha\beta} * e^{i\tilde{\tau}^\alpha}$. In particular, we have*

$$e^{is(\tilde{\tau}^\alpha+t\nu)} = e^{is\tilde{\tau}^\alpha} * (1+2is\nu)^{\frac{t}{2}} = (1-2is\nu)^{-\frac{t}{2}} * e^{is\tilde{\tau}^\alpha}. \tag{6.3}$$

Proof. Consider $\psi(s) = e^{is(\tilde{\tau}^\alpha+f_{\alpha\beta})} * e^{-is\tilde{\tau}^\alpha}$. Since $[\nu, \psi(s)] = 0$, $\psi(s)$ does not involve $\tilde{\tau}^\alpha$. We have

$$\frac{d}{ds}\psi(s) = e^{is(\tilde{\tau}^\alpha+f_{\alpha\beta})} * (if_{\alpha\beta}) * e^{-is\tilde{\tau}^\alpha} = i\psi(s) * e^{is\tilde{\tau}^\alpha} * f_{\alpha\beta} * e^{-is\tilde{\tau}^\alpha} \tag{6.4}$$

Put $g(s) = e^{is\tilde{\tau}^\alpha} * f_{\alpha\beta} * e^{-is\tilde{\tau}^\alpha}$. Since $g(s) = e^{is\mathrm{ad}(\tilde{\tau}^\alpha)}f_{\alpha\beta}$, we see $g(s) \in \mathcal{F}(\boldsymbol{W}_{U_\alpha})^C$. Thus, we have a differential equation $\frac{d}{ds}\psi(s) = \psi(s) * ig(s)$, where $g(s) \in \mathcal{F}(\boldsymbol{W}_{U_\alpha})^C$ is viewed as a known function. Note that $\mathcal{F}(\boldsymbol{W}_{U_\alpha})^C \cong C^\infty(U_\alpha)^C[[\nu]]$. By the Moyal product formula, the above differential equation can be rewritten as a system of differential equations on U_α. It is easy to see that (6.4) has a unique solution in $C^\infty(U_\alpha)^C[[\nu]]$.

Note that

$$e^{is\tilde{\tau}^\alpha} * t\nu * e^{-is\tilde{\tau}^\alpha} = \frac{t\nu}{1-2is\nu}.$$

(6.3) is obtained by solving the equation (6.4) inserted the above quantity. □

Lemma 6.1 shows that $\Phi^{c*}_{\alpha\beta}(g * e^{i\tilde{\tau}^\beta}) = \Phi^{*}_{\alpha\beta}(g) * F_{\alpha\beta} * e^{i\tilde{\tau}^\alpha}$ for any $g \in \mathcal{F}(\boldsymbol{W}_{U_\beta})^C$. Since $e^{t\mathrm{ad}(\nu^{-1})}e^{\tilde{\tau}^\alpha} = e^{\tilde{\tau}^\alpha + 2t}$, we see that $e^{\frac{t}{2}\mathrm{ad}(\nu^{-1})}$ gives an $S^1 = \{e^{it}\}$ action on $\mathcal{A}(M)$. Hence the relation (6.2) together with Lemma 6.1 can be viewed as a transition rule (coordinate transformation) of a 'quantum' S^1-principal bundle and the associated line bundle. Remark that the principal S^1-bundle P_M constructed on M via quantization condition is a contact manifold with a contact form as a connection form whose curvature form is the symplectic form. We denote by L_M the line bundle associated to P_M. $\mathcal{A}(M)$ can be viewed as a noncommutative contact algebra $(C^\infty(P_M)^C[[\nu]], *)$ defined on P_M.

We denote by $\tilde{P}_M$, $\tilde{L}_M$ the quantum principal bundle and its associated line bundle respectively given by the patchwork mentioned above.

We suppose that $\boldsymbol{W}_M$ is a noncommutative Kähler manifold over a Kähler manifold M. By Theorem 5.1, there exist QCC-generators $z_1^\alpha, \cdots, z_n^\alpha, z_1^{*\alpha}, \cdots, z_n^{*\alpha}$. As we did in (5.10), joining a new element $\tilde{\tau}^\alpha$ such that

$$(6.5) \qquad [\tilde{\tau}^\alpha, z_i^\alpha] = \nu z_i^\alpha, \quad [\tilde{\tau}^\alpha, z_i^{*\alpha}] = \nu z_i^{*\alpha}, \quad [\tilde{\tau}^\alpha, \nu] = 2\nu^2,$$

we construct a family of contact Lie algebras $\{\mathfrak{g}^C_{U_\alpha}\}_\alpha$. Since the ν-isomorphism class of $g^C_{U_\alpha}$ depends only on $\boldsymbol{W}^C_{U_\alpha}$, we see that the Poincaré-Cartan cocycle of $\{\mathfrak{g}^C_{U_\alpha}\}_\alpha$ gives the class $c^{(0)}(\boldsymbol{W}_M)$ in the coefficient $\mathbf{C}$, which is assumed to be integral.

Proposition 6.2. *For every V_α there is an element $H_\alpha * e^{i\tilde{\tau}^\alpha} \in \mathcal{A}(V_\alpha)$ such that $[z_i, H_\alpha * e^{i\tilde{\tau}^\alpha}] = 0$. Moreover, if $f \in \mathcal{A}(V_\alpha)$ satisfies $[z_i^\alpha, f] = 0$ for every i, $1 \le i \le n$, then there is a holomorphic function $h(z)$ such that f can be written in the form $h(\boldsymbol{z}) * H_\alpha * e^{i\tilde{\tau}^\alpha}$.*

Proof. We need only to show the first assertion. The inverse Moyal product formula (1.2) for QCC generators $z_1^\alpha, \cdots, z_n^\alpha, z_1^{*\alpha}, \cdots, z_n^{*\alpha}$ gives a commutative product $\circ$. Using the product $\circ$, the $*$-product is given by the Moyal product formula (1.1).

Take a function $H(t)$ and put $H_\alpha = H(\sum z_i^\alpha \circ z_i^{*\alpha})$. We consider the system of equations $[z_i^\alpha, H * e^{i\tilde{\tau}^\alpha}] = 0$. By the Moyal product formula for the above QCC generators, this equals

$$(6.6) \qquad (H' \circ z_i^\alpha) * e^{i\tilde{\tau}^\alpha} + H * [z_i^\alpha, e^{i\tilde{\tau}^\alpha}] = 0$$

Since $e^{i\tilde{\tau}^\alpha} * z_i^\alpha = z_i^\alpha * e^{i(\tilde{\tau}^\alpha + \nu)}$, (6.6) is reduced by using Lemma 6.1 to a differential equation

$$(6.7) \qquad \frac{\nu}{2}\left(1 + \frac{1}{\sqrt{1-2i\nu}}\right)\frac{d}{dt}H(t) + \left(1 - \frac{1}{\sqrt{1-2i\nu}}\right)H(t) = 0.$$

(6.7) can be solved in $C^\infty(\mathbf{R})[[\nu]]$ and we have $H \in C^\infty(V_\alpha)[[\nu]]$. □

By Proposition 6.4, we see that for every $V_\alpha \cap V_\beta \neq \emptyset$ there are holomorphic functions $h^{\alpha\beta}$ such that

$$(6.8) \qquad \Phi^{c*}_{\alpha\beta}(H_\beta * e^{i\tilde{\tau}^\beta}) = h^{\alpha\beta} * H_\alpha * e^{i\tilde{\tau}^\alpha}.$$

Thus, the quantum line bundle $\tilde{L}_M$ of L_M is a holomorphic line bundle over M and $\mathcal{A}(M)$ can be viewed as the algebra of sections of $\bigoplus_{m\in\mathbf{Z}} \tilde{L}_M^m$.

Let $\mathcal{H}(M)[[\nu]]$ be the commutative algebra consisting of all holomorphic sections of $\bigoplus_{m\geq 0} \tilde{L}_M^m$. If $\mathcal{H}(M)[[\nu]] \neq \{0\}$, $\mathcal{H}(M)[[\nu]]$ can be viewed as a representation space of Weyl functions on M.

As pointed out in [CGR], the multiplication operator combined with the projection to the space of all holomorphic sections is the essence of the Berezin representation [Be], which coincides with the representation produced by geometric quantization with respect to the Kähler polarization mentioned above.

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