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A Note on Satake parameters of Siegel modular forms of degree 2

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Introduction

For a positive integer k , let S_k be the space of all Siegel cusp forms of weight k on $Sp(2,\mathbb{Z})$. Suppose $f \in S_k$ is an eigenform, i.e., a non-zero common eigenfunction of the Hecke algebra. Then we define the spinor L-function attached to f by

$$(0.1) \qquad L(s,f,\underline{\mathrm{spin}})$$
$$:= \prod_p \Big\{(1-\alpha_{0,p}p^{-s})(1-\alpha_{0,p}\alpha_{1,p}p^{-s})(1-\alpha_{0,p}\alpha_{2,p}p^{-s})(1-\alpha_{0,p}\alpha_{1,p}\alpha_{2,p}p^{-s})\Big\}^{-1}$$

and the standard L-function attached to f by

$$(0.2) \qquad L(s,f,\underline{\mathrm{st}}) := \prod_p \left\{(1-p^{-s})\prod_{j=1}^{2}(1-\alpha_{j,p}^{-1}p^{-s})(1-\alpha_{j,p}p^{-s})\right\}^{-1} ,$$

where p runs over all prime numbers and $\alpha_{j,p}$ $(0 \le j \le 2)$ are the Satake p-parameters of f. The right-hand sides of (0.1) and (0.2) converge absolutely and locally uniformly for $\mathrm{Re}(s)$ sufficiently large.

For an indeterminate t , we put

$$H_p(t,f,\underline{\mathrm{spin}}) := (1-\alpha_{0,p}t)(1-\alpha_{0,p}\alpha_{1,p}t)(1-\alpha_{0,p}\alpha_{2,p}t)(1-\alpha_{0,p}\alpha_{1,p}\alpha_{2,p}t)$$

and

$$H_p(t,f,\underline{\mathrm{st}}) := (1-t)\prod_{j=1}^{2}\left(1-\alpha_{j,p}^{-1}t\right)\left(1-\alpha_{j,p}t\right) ,$$

where $H_p(t,f,\underline{\mathrm{spin}})$, $H_p(t,f,\underline{\mathrm{st}}) \in \mathbb{R}[t]$.

Definition. (cf. Kurokawa [7]) We say that $f \in S_k$ satisfies the Ramanujan–Petersson conjecture if the absolute values of the zeros of $H_p(t,f,\underline{\mathrm{spin}})$ are all equal to $p^{-\left(k-\frac{3}{2}\right)}$ for all p.

KSTS/RR-94/001
March 31, 1994

Since the Satake p-parameters satisfy $\alpha_{0,p}^2\alpha_{1,p}\alpha_{2,p} = p^{2k-3}$, this is equivalent to saying that

$$|\alpha_{1,p}| = |\alpha_{2,p}| = 1 \quad \text{for all } p \ ,$$

that is, the absolute values of the zeros of $H_p(t, f, \underline{\mathrm{st}})$ are all equal to 1 for all p, or to saying that

$$\underline{\mathrm{st}}_p(f) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a_p & b_p & 0 & 0 \\ 0 & -b_p & a_p & 0 & 0 \\ 0 & 0 & 0 & c_p & d_p \\ 0 & 0 & 0 & -d_p & c_p \end{pmatrix} \in SO(5, \mathbb{R}) \quad \text{for all } p \ ,$$

where $a_p = \dfrac{\alpha_{1,p} + \alpha_{1,p}^{-1}}{2}$, $b_p = \dfrac{\alpha_{1,p} - \alpha_{1,p}^{-1}}{2i}$, $c_p = \dfrac{\alpha_{2,p} + \alpha_{2,p}^{-1}}{2}$, $d_p = \dfrac{\alpha_{2,p} - \alpha_{2,p}^{-1}}{2i}$ (cf. Langlands [9]).

For an even integer k, let S_k^* be the Maaß subspace of S_k (cf. Maaß [11, 12, 13], Andrianov [3], Zagier [18]). We know that if f belongs to the Maaß space S_k^*, f doesn't satisfy the Ramanujan–Petersson conjecture. Now, our conjecture takes the following form:

Conjecture. (cf. Kurokawa [7, Conjecture 3])
If k is an even integer, any cusp eigenform of weight k in the orthogonal complement of the Maaß space satisfies the Ramanujan–Petersson conjecture. If k is an odd integer, any cusp eigenform of weight k satisfies the Ramanujan–Petersson conjecture.

We will analyze this conjecture from elementary properties of L-functions.

Although several authors make numerical researches on our conjecture, so far we don't know even the existence of f which satisfies the Ramanujan–Petersson conjecture (cf. Kurokawa [7], Skoruppa [17]).

Notation

1°. As usual, $\mathbb{Z}$ is the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers.

2°. Let $m, n \in \mathbb{Z}$, $m, n > 0$. If A is an $m \times n$-matrix, then we write it also as $A^{(m,n)}$, and as $A^{(m)}$ if $m = n$. The identity matrix of size n is denoted by 1_n.

3°. For $n \in \mathbb{Z}$, $n > 0$, let $A^{(n)}$ be a diagonal matrix with diagonal entries $a_1, \cdots, a_n$. We denote it by $\mathrm{d}(a_1, \cdots, a_n)$.

4°. For $n \in \mathbb{Z}$, $n > 0$, let $\Gamma^n := Sp(n, \mathbb{Z})$ be the Siegel modular group of degree n and let $\mathfrak{H}_n$ be the Siegel upper half space of degree n, that is,

$$\mathfrak{H}_n := \{Z = X + iY \in \mathbb{C}^{(n)} | {}^t Z = Z, \ Y > 0\}.$$

5°. We put

$$\Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) \quad \text{and} \quad \Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) \ ,$$

where $\Gamma(s)$ is the gamma function.

KSTS/RR-94/001
March 31, 1994

§1 Preliminaries

Let k be a positive integer. A holomorphic function f on $\mathfrak{H}_n$ is called a Siegel modular form of weight k if it satisfies

$$(f|M)(Z) := \det(CZ+D)^{-k} f((AZ+B)(CZ+D)^{-1}) = f(Z)$$

for all $Z \in \mathfrak{H}_n$ and $M = \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \in \Gamma^n$ and if it is holomorphic at the cusps when $n=1$. The space of Siegel modular forms of weight k is denoted by M_k^n.

We define the Siegel operator Φ on M_k^n by

$$(\Phi f)(Z) := \lim_{t\to\infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix}\right)$$

for $Z \in \mathfrak{H}_{n-1}$. Then the operator Φ defines the map $\Phi : M_k^n \longrightarrow M_k^{n-1}$. Suppose $f \in M_k^n$. Then it is called a cusp form if it satisfies $\Phi f = 0$.

In what follows, we restrict ourselves to the case $n=2$ and we omit subscripts concerning the case $n=2$ when there is no fear of confusion.

We define $G^+ := G^+Sp(2,\mathbb{Q})$ by

$$G^+ := \left\{ M \in GL(4,\mathbb{Q}) \;\middle|\; {}^tM \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} M = \mu(M) \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} \;,\; \mu(M) > 0 \right\} ,$$

and for a prime number p, $G_p^+ := G^+ \cap GL(4, \mathbb{Z}[p^{-1}])$.

Let $\mathcal{H}$ (resp. $\mathcal{H}_p$) be the free $\mathbb{C}$-module generated by the double cosets $\Gamma g \Gamma$, $g \in G^+$ (resp. G_p^+). Then $\mathcal{H}$ is a commutative algebra and we call it the Hecke algebra (over $\mathbb{C}$). We get $\mathcal{H} = \otimes_p \mathcal{H}_p$, where the tensor product is the restricted one. Moreover, the structure of $\mathcal{H}_p$ is known: For $0 \le j \le 2$, let w_j be an automorphism of $\mathbb{C}[X_0{}^{\pm1}, X_1{}^{\pm1}, X_2{}^{\pm1}]$ such that

$$\begin{aligned} &w_0(X_0) = X_0, \quad w_0(X_1) = X_2, \quad w_0(X_2) = X_1, \\ &w_1(X_0) = X_0X_1, \; w_1(X_1) = X_1^{-1}, \; w_1(X_2) = X_2, \\ &w_2(X_0) = X_0X_2, \; w_2(X_1) = X_1, \quad w_1(X_2) = X_2^{-1} . \end{aligned}$$

The automorphisms w_j $(0 \le j \le 2)$ generate a finite group W. We call it the Weyl group. We get

$$\Psi : \mathcal{H}_p \xrightarrow{\;\cong\;} \mathbb{C}[X_0{}^{\pm1}, X_1{}^{\pm1}, X_2{}^{\pm1}]^W ,$$

where $\mathbb{C}[X_0{}^{\pm1}, X_1{}^{\pm1}, X_2{}^{\pm1}]^W$ is the W-invariant subalgebra of $\mathbb{C}[X_0{}^{\pm1}, X_1{}^{\pm1}, X_2{}^{\pm1}]$.

For $g \in G^+$, let $\Gamma g \Gamma = \bigcup_{j=1}^r \Gamma g_j$ be a decomposition of the double coset $\Gamma g \Gamma$ into left cosets. For $f \in M_k$ (resp. S_k) , we define the Hecke operator $(\Gamma g \Gamma)$ by

$$f|(\Gamma g \Gamma) := \mu(g)^{2k-3} \sum_{j=1}^r f|g_j .$$

KSTS/RR-94/001
March 31, 1994

Then we get a homomorphism $\mathcal{H} \longrightarrow \mathrm{End}(M_k)$ (resp. $\mathrm{End}(S_k)$) .

For $\delta \in \mathbb{Z}$, $\delta > 0$ and a prime number p, we put

$$T(p^\delta) := \sum_{\mu(g)=p^\delta} (\Gamma g \Gamma) ,$$

where $g = \mathrm{d}(p^{d_1}, p^{d_2}, p^{e_1}, p^{e_2}) \in G_p^+$, $d_j, e_j \in \mathbb{Z}$ $(j = 1, 2)$ and $0 \le d_1 \le d_2 \le e_2 \le e_1$.

Suppose $f \in M_k$ is an eigenform. We denote the eigenvalue of $(\Gamma g \Gamma)$ on f by $\lambda_f(\Gamma g \Gamma)$ and that of $T(p^\delta)$ on f by $\lambda_f(p^\delta)$.

If the homomorphism $\lambda_f : \mathcal{H}_p \longrightarrow \mathbb{C}$ coincides with the composite map of the isomorphism Ψ and the evaluation map

$$\mathbb{C}[X_0{}^{\pm 1}, X_1{}^{\pm 1}, X_2{}^{\pm 1}]^W \xrightarrow{(X_0, X_1, X_2) \mapsto (\alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p})} \mathbb{C} ,$$

then the numbers $\alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p} \in \mathbb{C}^*$, the Satake p-parameters of f, are uniquely determined modulo W. In this case, they are uniquely determined by

$$\lambda_f(\Gamma p 1_4 \Gamma) = p^{-3} \alpha_{0,p}^2 \alpha_{1,p} \alpha_{2,p} , \quad \lambda_f(p) = \alpha_{0,p}(1 + \alpha_{1,p})(1 + \alpha_{2,p})$$

and

$$\lambda_f \left(\Gamma \mathrm{d}(1, p, p^2, p) \Gamma \right) = p^{-1} \alpha_{0,p}^2 (\alpha_{1,p} + \alpha_{2,p})(1 + \alpha_{1,p}\alpha_{2,p}) + (p^{-1} - p^{-3}) \alpha_{0,p}^2 \alpha_{1,p} \alpha_{2,p} ,$$

up to the action of the Weyl group W.

We summarize some facts on Siegel modular forms and on L-functions attached to them.

In what follows, we suppose that $f \in S_k$ is an eigenform.

(I) It follows from the hermiteness of Hecke operators $(\Gamma g \Gamma)$, $g \in G^+$, that we have $\lambda_f(\Gamma g \Gamma) \in \mathbb{R}$. In fact, by Kurokawa [8], we know that the eigenvalues on f of the Hecke algebra over $\mathbb{Q}$ generate a totally real finite extension of $\mathbb{Q}$.

(II) We put

$$\Lambda(s, f, \underline{\mathrm{st}}) := \Gamma_{\mathbb{R}}(s) \prod_{j=1}^{2} \Gamma_{\mathbb{C}}(s + k - j) L(s, f, \underline{\mathrm{st}}) \quad .$$

Andrianov–Kalinin [4] and Böcherer [5] (cf. Piatetski-Shapiro and Rallis [16]) have discovered that $\Lambda(s, f, \underline{\mathrm{st}})$ has a meromorphic continuation to the whole s-plane and satisfies the functional equation

$$\Lambda(s, f, \underline{\mathrm{st}}) = \Lambda(1 - s, f, \underline{\mathrm{st}}) .$$

Moreover, Mizumoto [14] has shown that it is entire.

(III) We put

$$\Lambda(s, f, \underline{\mathrm{spin}}) := \Gamma_{\mathbb{C}}(s) \Gamma_{\mathbb{C}}(s - k + 2) L(s, f, \underline{\mathrm{spin}}) \quad .$$

Andrianov [1] has shown that $\Lambda(s, f, \underline{\mathrm{spin}})$ has a meromorphic continuation to the whole s-plane and satisfies the functional equation

$$\Lambda(s, f, \underline{\mathrm{spin}}) = (-1)^k \Lambda(2k - 2 - s, f, \underline{\mathrm{spin}}) .$$

For an odd integer k, it is entire.

For an even integer k, Evdokimov [6] and Oda [15] have shown that $\Lambda(s, f, \underline{\mathrm{spin}})$ has a simple pole at $s = k$ (or equivalently, at $s = k - 2$) if and only if $f \in S_k^*$.

KSTS/RR-94/001
March 31, 1994

§2 Results

First we note that the set $\{\alpha_{1,p}\ ,\ \alpha_{1,p}^{-1}\ ,\ \alpha_{2,p}\ ,\ \alpha_{2,p}^{-1}\}$ is invariant under the action of the Weyl group W.

Lemma. *For an eigenform $f \in S_k$, the set $\{\alpha_{1,p}\ ,\ \alpha_{1,p}^{-1}\ ,\ \alpha_{2,p}\ ,\ \alpha_{2,p}^{-1}\}$ is one of the following types:*

Type I. $\{p^{a_1}\ ,\ p^{-a_1}\ ,\ p^{a_2}\ ,\ p^{-a_2}\}$ *or* $\{-p^{a_1}\ ,\ -p^{-a_1}\ ,\ -p^{a_2}\ ,\ -p^{-a_2}\}$ *, where* $a_1, a_2 \in \mathbb{R}$ *and* $0 < a_2 < a_1$.

Type II. $\{e^{i\theta}\ ,\ e^{-i\theta}\ ,\ p^a\ ,\ p^{-a}\}$ *or* $\{-1\ ,\ -1\ ,\ -p^a\ ,\ -p^{-a}\}$ *, where* $a \in \mathbb{R}$ *,* $0 < a$ *and* $0 \leq \theta < 2\pi$.

Type III. $\{p^a e^{i\theta}\ ,\ p^{-a}e^{-i\theta}\ ,\ p^a e^{-i\theta}\ ,\ p^{-a}e^{i\theta}\}$ *, where* $a \in \mathbb{R}$ *,* $0 < a$ *and* $0 \leq \theta < 2\pi$.

Type RP. $\{e^{i\theta_1}\ ,\ e^{-i\theta_1}\ ,\ e^{i\theta_2}\ ,\ e^{-i\theta_2}\}$ *, where* $0 \leq \theta_1, \theta_2 < 2\pi$.

Proof. By $H_p(t, f, \underline{\mathrm{st}}) \in \mathbb{R}[t]$, we have

$$\{\alpha_{1,p}\ ,\ \alpha_{1,p}^{-1}\ ,\ \alpha_{2,p}\ ,\ \alpha_{2,p}^{-1}\} = \{\overline{\alpha_{1,p}}\ ,\ \overline{\alpha_{1,p}}^{-1}\ ,\ \overline{\alpha_{2,p}}\ ,\ \overline{\alpha_{2,p}}^{-1}\}\ .$$

From this fact , Lemma is proved except for signatures in type I and in type II.

In type I , it follows from $\lambda_f(p) \in \mathbb{R}$ that we have $\alpha_{0,p} \in \mathbb{R}$. Combining this with $\alpha_{0,p}^2 \alpha_{1,p}\alpha_{2,p} = p^{2k-3}$, we obtain $\alpha_{1,p}\alpha_{2,p} > 0$.

In the same way, we can determine signatures in type II. □

Theorem. *Conjecture holds if any eigenform $f \in S_k$ satisfies the following conditions:*

(A) *For any prime p, the logarithms of the absolute values of the zeros of $H_p(t, f, \underline{\mathrm{spin}})$ (or equivalently, of $H_p(t, f, \underline{\mathrm{st}})$) to the base p are independent of p.*

(B) *For all but a finite number of primes p , $H_p(t, f, \underline{\mathrm{spin}})$ and $H_p(t, f, \underline{\mathrm{st}})$ have no negative real zeros.*

With the use of the Satake parameters, we can replace the condition **(A)** by the following form:

(A′) *For any primes p and q, by the suitable action of the Weyl group,*

$$\log_p |\alpha_{1,p}| = \log_q |\alpha_{1,q}| \quad \textit{and} \quad \log_p |\alpha_{2,p}| = \log_q |\alpha_{2,q}|$$

hold.

If we note that our L-functions $L(s, f, \underline{\mathrm{spin}})$ and $L(s, f, \underline{\mathrm{st}})$ are unramified at all p in the sense of Langlands [9] , we can understand that "any prime" in **(A)** is "any unramified prime". If so, **(A)** is true for many L-functions which have the Euler product expansions, e.g., the Riemann zeta function, the Diriclet L-functions, the Hasse–Weil L-functions and so on.

KSTS/RR-94/001
March 31, 1994

Proof. For a positive integer k , let $f \in S_k$ be an eigenform.

We note that , under the condition **(A)** , the types of $\{\alpha_{1,p}\ ,\ \alpha_{1,p}^{-1}\ ,\ \alpha_{2,p}\ ,\ \alpha_{2,p}^{-1}\}$ are the same for all p. If $\{\alpha_{1,p}\ ,\ \alpha_{1,p}^{-1}\ ,\ \alpha_{2,p}\ ,\ \alpha_{2,p}^{-1}\}$ is of type I (resp. type II, type III or type RP) for any prime p , we say that f is of type I (resp. type II, type III or type RP).

In what follows , we assume that f satisfies the condition **(B)** .

If f is of type I , then for almost all p , $H_p(t,f,\underline{\mathrm{st}})$ is the following form:

$$H_p(t,f,\underline{\mathrm{st}}) = (1-t)(1-p^{a_1}t)(1-p^{-a_1}t)(1-p^{a_2}t)(1-p^{-a_2}t)\ ,$$

where a_1 and a_2 are independent of p . Then $L(s,f,\underline{\mathrm{st}})$ has a pole at $s = 1 + a_1$. This contradicts the fact **(II)**.

If f is of type II , then for almost all p , $H_p(t,f,\underline{\mathrm{st}})$ is the following form:

$$H_p(t,f,\underline{\mathrm{st}}) = (1-t)(1-\mathrm{e}^{i\theta_p}t)(1-\mathrm{e}^{-i\theta_p}t)(1-p^{a}t)(1-p^{-a}t)\ ,$$

where $0 \le \theta_p < 2\pi$ and a is independent of p . Then $L(s,f,\underline{\mathrm{st}})$ has a pole at $s = 1 + a$. This contradicts the fact **(II)**.

If f is of type III , then for almost all p , $H_p(t,f,\underline{\mathrm{spin}})$ is the following form:

$$H_p(t,f,\underline{\mathrm{spin}}) = \left(1-p^{k-\frac{3}{2}+a}t\right)\left(1-p^{k-\frac{3}{2}-a}t\right)\left(1-p^{k-\frac{3}{2}}\mathrm{e}^{i\theta_p}t\right)\left(1-p^{k-\frac{3}{2}}\mathrm{e}^{-i\theta_p}t\right)\ ,$$

where $0 \le \theta_p < 2\pi$ and a is independent of p . Then $L(s,f,\underline{\mathrm{spin}})$ has a pole at $s = k - \frac{1}{2} + a$. If k is an odd integer , this contradicts the fact **(III)**. If k is an even integer, we have $f \in S_k^*$ and $a = \frac{1}{2}$. □

For an odd integer k , we put $S_k^* = \{0\}$ when there is no fear of confusion.

Suppose that any eigenform $f \in S_k$ satisfies the condition **(A)** .

If f of type I occurs , then there exist infinitely many prime numbers such that $\lambda_f(p) < 0$ and $\lambda_f\left(\Gamma \mathrm{d}(1,p,p^2,p)\Gamma\right) < 0$.

If f of type II occurs , then there exist infinitely many prime numbers such that $\lambda_f(p) = 0$.

If f of type III occurs , then $f \in S_k^*$ or there exist infinitely many prime numbers such that $\lambda_f(p) < 0$.

So we have:

Corollary 1. *Let $f \in S_k$ be an eigenform.*

If f satisfies the condition **(A)** *and if $\lambda_f(p) > 0$ for almost all p , then f satisfies the Ramanujan–Petersson conjecture or f belongs to the Maaß space S_k^* .*

Now we define some L-functions attached to Siegel modular forms.

Let sym^2 be the symmetric square representation of $GL(n,\mathbb{C})$, i.e. ,

$$\mathrm{sym}^2 : GL(n,\mathbb{C}) \longrightarrow GL\left(\tfrac{n(n+1)}{2},\mathbb{C}\right)\ .$$

For an eigenform $f \in S_k$, we put

$$L(s, f, \mathrm{sym}^2(\underline{\mathrm{st}})) := \prod_p \det\left(1_{15} - \mathrm{sym}^2(\underline{\mathrm{st}}_p(f))p^{-s}\right)^{-1} \tag{2.1}$$

and

$$L(s, f, \mathrm{sym}^2(\underline{\mathrm{spin}})) := \prod_p \det\left(1_{10} - \mathrm{sym}^2(\underline{\mathrm{spin}}_p(f))p^{-(s+2k+3)}\right)^{-1} , \tag{2.2}$$

where $\underline{\mathrm{spin}}_p(f) := \mathrm{d}(\alpha_{0,p}, \alpha_{0,p}\alpha_{1,p}, \alpha_{0,p}\alpha_{2,p}, \alpha_{0,p}\alpha_{1,p}\alpha_{2,p})$. The right-hand sides of (2.1) and (2.2) converge absolutely and locally uniformly for $\mathrm{Re}(s)$ sufficiently large.

For $\sigma = \mathrm{sym}^2(\underline{\mathrm{st}})$ or $\sigma = \mathrm{sym}^2(\underline{\mathrm{spin}})$, we put

$$\Lambda(s, f, \sigma) := \Gamma(s, f, \sigma)L(s, f, \sigma) ,$$

where $\Gamma(s, f, \sigma)$ is the suitable Γ-factor of $L(s, f, \sigma)$.

Then we expect the following:

(C) *Let* $\sigma = \mathrm{sym}^2(\underline{\mathrm{st}})$ *or* $\sigma = \mathrm{sym}^2(\underline{\mathrm{spin}})$.

For any eigenform $f \in S_k$, $\Lambda(s, f, \sigma)$ *has a meromorphic continuation to the whole* s*-plane and satisfies the functional equation*

$$\Lambda(s, f, \sigma) = \varepsilon(f, \sigma)\Lambda(1 - s, f, \sigma) ,$$

where $\varepsilon(f, \sigma)$ *is a constant. Moreover,* $\Gamma(s, f, \sigma)$ *has no poles and zeros at* $s = r \in \mathbb{R}$, $r > 1$ *and if* $\Lambda(s, f, \sigma)$ *has a pole at* $s = r \in \mathbb{R}$, $r > 1$, *then* $f \in S_k^*$.

The following is proved in the same way as Theorem.

Corollary 2. *If the condition* **(C)** *holds, Conjecture is equivalent to saying that any eigenform* $f \in S_k$ *satisfies the condition* **(A)**.

Remarks. (i) For an even integer k , let $S_k^{*\perp}$ be the orthogonal complement of the Maaß space S_k^*, that is , $S_k = S_k^{*\perp} \oplus S_k^*$.

For $k = 20,\ 22,\ 24,\ 26$, let $f \in S_k^{*\perp}$ be an eigenform. Then, if $p = 2,\ 3,\ 5$, $H_p(t, f, \underline{\mathrm{st}})$ and $H_p(t, f, \underline{\mathrm{spin}})$ have no negative real zeros (cf. Skoruppa [17, Table 4]).

(ii) If $f \in S_k^*$, then $L(s, f, \mathrm{sym}^2(\underline{\mathrm{spin}}))$ diverges at $s = 2$.

(iii) In **(C)**, we don't assert absolute convergence of $L(s, f, \sigma)$ for $\mathrm{Re}(s) > 1$.

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KSTS/RR-94/001
March 31, 1994

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