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Laplacians on A Graph

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Summary. A fundamental relationship is established between the eingenvalues of the Laplacian of a closed Riemannian manifold and those of a finite graph which approximates the manifold.

0. INTRODUCTION

In this paper, we will study Laplacians on a graph whose edges have variable length. Let Γ be a (finite or infinite) graph. We asume on Γ that there are at most finitely many vertices adjacent to each vertex $x \in \Gamma$, and that there is at most one edge, if it exists, joining two distinct vertices and no edge joining a vertex with itself. Let $V(\Gamma)$ and $E(\Gamma)$ denote the set of vertices of Γ and the set of directed edges of Γ. We write $x \sim y$ if $x, y \in V(\Gamma)$ are adjacent, and use the notations $[x, y]$ or $-[y, x]$ to denote the directed edge from x to y.

To define a Laplacian on Γ, we introduce a *length function* and a *weight function*

$$l : E(\Gamma) \longrightarrow R_+$$

$$m : V(\Gamma) \longrightarrow R_+$$

satisfying $l([x, y]) = l([y, x])$ and

$$\inf_{e \in E(\Gamma)} l(e) > 0. \tag{0.1}$$

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and define the inner products for $f, g \in L^2(V)$ and $\phi, \psi \in L^2(E)$ by

$$(f,g) = \sum_x m(x)f(x)g(x), \quad (\phi,\psi) = \frac{1}{2}\sum_e l(e)\phi(e)\psi(e).$$

As an analogue of the exterior derivative, we define a operator $d : L^2(V) \longrightarrow L^2(E)$ by

$$df([x,y]) = \frac{f(x) - f(y)}{l([x,y])}.$$

From the assumption (0.1), this operator turns out to be bounded. The adjoint operator $\delta : L^2(E) \longrightarrow L^2(V)$ is given by

$$\delta\phi(x) = \frac{1}{m(x)}\sum_{x\sim y}\phi([x,y]).$$

We define a Laplacian on (Γ, l, m) by

$$\Delta f(x) = \delta df(x).$$

Then we obtain

$$(\Delta f, f) = (df, df), \qquad \Delta f(x) = \frac{1}{m(x)}\sum_{x\sim y}\frac{f(x) - f(y)}{l([x,y])}.$$

This gives a generalization of the Laplacian which was given in [D,K] for the case where $l \equiv 1$ and $m = m_l$ in our setting.

In section 1, we show a relation between the bottom of the spectrum of the Laplacian and an isoperimetric constant on a graph. To recall what is known for Riemannian manifolds, let M be a noncompact Riemannian manifold of dimension $\geqq 2$. The *Cheeger constant* of M, $h(M)$, is defined by

$$h(M) = \inf_\Omega \frac{A(\partial\Omega)}{V(\Omega)},$$

where Ω ranges over all open submanifolds of M with compact closure in M and with smooth boundary. The *bottom of the spectrum* of the Laplacian, $\lambda(M)$, is defined by

$$\lambda(M) = \inf\{(\Delta f, f) | (f,f) = 1\}.$$

In this setup, Cheeger has shown the following result in [C].

Theorem A.

$$\lambda(M) \geqq \frac{1}{4}h^2(M).$$

And Buser has given an upper bound in [Bu].

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Theorem B. *If the Ricci curvature of M^n is bounded below by $-(n-1)\delta^2 (\delta \geqq 0)$, then*

$$\lambda(M) \leqq c\delta h(M),$$

where c is a constant depending only on the dimension.

In 1.1, we will show a counterpart of above results for an infinite graph. Let (Γ, l, m) be an infinite graph with a length function l and a weight function m. We define the *bottom* of the *spectrum* of Δ, $\lambda(\Gamma)$, by

$$\lambda = \inf\{(\Delta f, f) | (f, f) = 1\}.$$

Remark. It suffices to take the infimum only over functions with finite support from the assumption (0.1).

Let S be a subset of $V(\Gamma)$. Put

$$\partial S = \{[x, y] \in E(\Gamma) | x \in S, y \notin S\},$$

and call it the *boundary* of S. The cardinality of ∂S is denoted as

$$L(\partial S) = \sharp \partial S$$

and is called the *length* of the boundary of S. We define

$$A(S) = \sum_{x \in S} m(x)$$

and call it the *area* of S. The *isoperimetric constant* α of (Γ, l, m) is defined by

$$\alpha = \inf\left\{\frac{L(\partial S)}{A(S)} \;\middle|\; S \subset V(\Gamma),\ \sharp S < \infty\right\}. \tag{0.2}$$

Remark. (1) If Γ is the Caylay graph of Z^2 with respect to the canonical generators, then $\alpha = 0$. (2) If Γ is an infinite planar graph such that each vertex has seven adjacent vertices, then $\alpha > 0$. See [D,K] for the proof .

Dodziuk and Kendall have shown the following result in [D,K].

Theorem C. *Let (Γ, l, m_l) be an infinite graph with $l \equiv 1$. Then,*

$$\lambda \geqq \frac{1}{2}\alpha^2.$$

We will extend this result to a general graph as follows.

Theorem 1. *Let (Γ, l, m_l) be an infinite graph with $\inf_{e \in E(\Gamma)} l(e) > 0$. Then we have*

$$\frac{\alpha}{l_0} \geqq \lambda \geqq \frac{1}{2}\alpha^2,$$

where $l_0 = \inf_{e \in E(\Gamma)} l(e)$.

On the other hand, for a compact manifold M, the *Cheeger constant* of M, $h(M)$, is defined by

$$h(M) = \inf_S \frac{A(S)}{\min\{V(M_1), V(M_2)\}},$$

where S ranges over all compact $(n-1)$-dimensional submanifolds of M, which divide M into 2 open submanifolds M_1, M_2 satisfying $\partial M_1 = \partial M_2 = S$. Cheeger has also shown the next theorem in [C].

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Theorem D. *We have*

$$\lambda_1 \geqq \frac{h^2}{4},$$

where λ_1 *is the smallest positive eigenvalue of the Laplacian on* M.

And Buser has given an upper bound in [Bu].

Theorem E. *If the Ricci curvature of* M^n *is bounded below by* $-(n-1)\delta^2 (\delta \geqq 0)$, *then*

$$\lambda_1(M) \leqq c_1(\delta h(M) + h^2(M)),$$

where c_1 *is a constant depending only on the dimension.*

We will show a couterpart of above results in 1.2. Let (Γ, l, m) be a finite graph, and S a subset of $V(\Gamma)$. We define

$$\alpha(S;\Gamma) = \inf_T \left\{ \frac{L(\partial T)}{A(T)} \Big| T \subset S \right\},$$

and

$$\alpha(\Gamma) = \min_{(S_1,S_2); S_1 \neq \emptyset, S_2 \neq \emptyset, S_1 \cap S_2 = \emptyset} \{\max(\alpha(S_1;\Gamma), \alpha(S_2;\Gamma))\}.$$

Example. Let C_n denote a circle graph with n vetices. The circle graph is a graph which is homeomorphic to S^1. Take $l \equiv 1$ and $m = m_l$. Then,

$$\alpha(C_n) = 2/[n/2].$$

We denote the smallest positive eigenvalue of the Laplacian on (Γ, l, m) as λ_1.

Theorem 3. *Let* (Γ, l, m_l) *be a finite graph with a length function* l. *Then,*

$$\frac{2}{l_0}\alpha \geqq \lambda_1 \geqq \frac{1}{2}\alpha^2,$$

where $l_0 = \min_{e \in E(\Gamma)} l(e)$.

In section 2, we study a spectral convergence among a class of finite graphs. For compact Riemannian manifolds, Fukaya [F] has obtained a convergence theorem for Laplacians. To recall a result from what is shown by him, let $\mathcal{M}(n, D)$ denote the class of all Riemannian manifolds whose volumes are 1 and whose sectional curvatures are not bigger than $D^2/\text{diameter}^2$ and not smaller than $-D^2/\text{diameter}^2$. $\mathcal{DM}(n, D)$ denotes the closure of $\mathcal{M}(n, D)$ with respect to the measured-Hausdorff-topology (which is defined in [F]) in the class of all compact metric space X with a Borel measure μ. Then, he showed

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Theorem F. *If* $\lim_{m.H.} M_i = (X,\mu) \in \mathcal{DM}(n,D)$ *for* $\{M_i\}_i \subset \mathcal{M}(n,D)$. *Then, there exists a self adjoint operator* P *on* $L^2(X,\mu)$ *such that*

$$\lambda_k(P) = \lim_i \lambda_k(\Delta_{M_i}),$$

where λ_k *denotes the* k*-th eigenvalue of the each operator.*

For finite graphs, we will show a kind of counterpart of above result. Let (Γ, l, m) be a finite graph with a length function l and a weight function m. Γ turns out to be a metric space by the path metric induced by l where we assume l is linear on each $e \in E(\Gamma)$. We define the *total weight* $m(\Gamma)$ of (Γ, l, m) by

$$m(\Gamma) = \sum_{x \in V(\Gamma)} m(x),$$

and the symbol $\mathcal{G}(C)$ denotes the class of finite graphs whose total weights are not bigger than C. We say $\{(\Gamma_n, l_n)\}_{n=1,2,\dots}$ converges to (Γ, l) with respect to the *Hausdorff distance on graphs* and write $\lim_H(\Gamma_n, l_n) = (\Gamma, l)$ if there exist simplicial maps

$$\phi_n : \Gamma_n \to \Gamma$$

and positive number ε_n such that

$$\lim_n \varepsilon_n = 0, \tag{0.3}$$

$$\varepsilon_n \text{ -neighborhood of } \phi_n(\Gamma_n) \text{ is equal to } \Gamma, \tag{0.4}$$

for each $x, y \in \Gamma_n$, we have

$$|d(\phi_n(x), \phi_n(y)) - d(x,y)| < \varepsilon_n, \tag{0.5}$$

where we assume the map ϕ_n is linear on each $e \in E(\Gamma_n)$. Let $\lambda_k(\Gamma, l, m)$, or simply $\lambda_k(\Gamma)$, denote the k-th eigenvalue of the Laplacian on (Γ, l, m). We have the following theorem.

Theorem 4. *Let* $\{(\Gamma_i, l_i, m_{l_i})\}_{i=1}^{\infty}$ *be a sequence of elements of* $\mathcal{G}(C)$ *and* (Γ, l) *a finite graph such that* $\lim_H(\Gamma_i, l_i) = (\Gamma, l)$. *Then there exists a weight function* $\tilde{m}$ *on* Γ *and a subsequence* $\{(\Gamma_j, l_j, m_{l_j})\}_j$ *such that for* $k = 1, 2, \dots, \sharp V(\Gamma) - 1$,

$$\lim_j \lambda_k(\Gamma_j, l_j, m_{l_j}) = \lambda_k(\Gamma, l, \tilde{m}), \tag{2.1}$$

and for $k \geqq \sharp V(\Gamma)$,

$$\lim_j \lambda_k(\Gamma_j, l_j, m_{l_j}) = \infty. \tag{2.2}$$

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In section 3, which is the main part of this paper, we study a relation between the eigenvalues of the Laplacian of a closed manifold and those of its approximating graph. For example, let S^1 be the unit circle, and $\lambda_k(S^1)$ denote the k-th eigenvalue of the Laplacian on S^1. It is known that

$$\{\lambda_k(S^1)\}_{k=1}^{\infty} = \{0,1,1,4,4,9,9,...\}.$$

Let (C_n, l_n) be a circle graph of n-vertices with the length function $l_n \equiv 2\pi/n$. Then the sequence $\{(C_n, l_n)\}_n$ converges to S^1 as a metric space. We denote the eigenvalues of the Laplacian on (C_n, l_n, m_{l_n}) by $spec(C_n)$. Then if n is odd,

$$\mathrm{spec}(C_n) = \left(\frac{n}{2\pi}\right)^2 \times \{0, \underbrace{2(1-\cos\frac{2\pi}{n}), ..., (1-\cos\frac{n-1}{n}\pi)}_{\textbf{mult. =2}}\}.$$

If n is even,

$$\mathrm{spec}(C_n) = \left(\frac{n}{2\pi}\right)^2 \times \{0, \underbrace{2(1-\cos\frac{2\pi}{n}), ..., (1-\cos\frac{n-2}{n}\pi)}_{\textbf{mult. =2}}, 4\}.$$

Since $\lim_n(\frac{n}{2\pi})^2 2(1-\cos\frac{2k}{n}\pi) = k^2$, we have

$$\lim_n \lambda_k(C_n) = \lambda_k(S^1),$$

for each k.

For general cases, let M be a closed Riemannian manifold. A subset V of M is called *ε-separated* if $d_M(x,y) \geqq \varepsilon$ for any distinct points $x, y \in V$. We construct a graph from a maximal ε-separated subset V by joining the distinct points x, y in V by a edge if and only if $d(x,y) \leqq 3\varepsilon$, and call it an $\varepsilon - net$ in M. An $\varepsilon - net$ exists in M for any $\varepsilon > 0$, [K]. We will show the following theorem.

Theorem 6. *Let M be a closed Riemannian manifold and (Γ_n, l_n, m_{l_n}) a 1/n -net in M with the length function $l_n \equiv 1/n$ for each $n \in N$. Then,*

$$\frac{1}{C}\limsup_n \lambda_k(\Gamma_n, l_n, m_{l_n}) \leqq \lambda_k(M) \leqq C \liminf_n \lambda_k(\Gamma_n, l_n, m_{l_n})$$

for each k, where $\lambda_k(M)$ is the k-th eigenvalue of the Laplacian on M and C is a number depending only on the dimension.

From this theorem, we can know a rough behavior of the eigenvalues of the Laplacian of M by ditecting that of Γ_n, which is easier since the function space over Γ_n has finite dimension. So far, the constant C strongly depends on the dimension, which grows exponentially, and the author doesn't know if the inequalities in Theorem 6 hold for a constant C' which is independent on the dimension by taking a nice sequence of graphs Γ_n.

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1. The bottom of the spectrum

1.1 The bottom of the spectrum for an infinite graph.

In this section, we show a relation between the bottom of the spectrum of the Laplacian and an isoperimetric constant of an infinite graph. Dodziuk and Kendall have shown the following result in [D,K].

Theorem C. *Let* (Γ, l, m_l) *be an infinite graph with* $l \equiv 1$*. Then,*

$$\lambda \geqq \frac{1}{2}\alpha^2.$$

In fact, this theorem is true for any l with $\inf_{e \in E(\Gamma)} l(e) > 0$.

Theorem 1. *Let* (Γ, l, m_l) *be an infinite graph with* $\inf_{e\in E(\Gamma)} l(e) > 0$*. We have*

$$\frac{\alpha}{l_0} \geqq \lambda \geqq \frac{1}{2}\alpha^2.$$

where $l_0 = \inf_{e \in E(\Gamma)} l(e)$.

Remark. From this theorem, we have

$$\alpha(\Gamma) = 0 \Longleftrightarrow \lambda(\Gamma) = 0.$$

Proof. We can prove Theorem 1 by a slight modification of Dodziuk and Kendall's proof in [D,K]. First, to show the second inequality, take $f \in L^2(V)$ of finite support with $(f,f) = 1$. Since $(df, df) \geqq (d|f|, d|f|)$ and $(|f|,|f|) = (f,f)$, we can assume $f \geqq 0$. Define

$$A = A(f) = \sum_{x\sim y} |f^2(x) - f^2(y)|,$$

where $\sum_{x\sim y}$ means to take the sum over all the ordered pairs of vertices (x,y) with $x \sim y$. Then

$$\begin{aligned} A &= \sum_{x\sim y}(f(x)+f(y))|f(x)-f(y)| \\ &= \sum_{x\sim y}(f(x)+f(y))\sqrt{l([x,y])}\,\frac{|f(x)-f(y)|}{\sqrt{l([x,y])}} \\ &\leqq \sqrt{\sum_{x\sim y} l([x,y])(f(x)+f(y))^2}\sqrt{\sum_{x\sim y}\frac{(f(x)-f(y))^2}{l([x,y])}} \\ &\leqq \sqrt{\sum_{x\sim y} l([x,y])\{2(f^2(x)+f^2(y))\}}\,\sqrt{2(df,df)} \\ &= 2\sqrt{(f,f)}\sqrt{2(df,df)} = 2\sqrt{2(df,df)}. \qquad (1.1) \end{aligned}$$

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On the other hand, we can estimate A from below. Let

$$\{f(x)|x \in V(\Gamma)\} = \{0 = \beta_0 < \beta_1 < \cdots < \beta_N\},$$

and

$$K_i = \{x \in V(\Gamma)|f(x) \geqq \beta_i\}.$$

Then

$$\partial K_i = \{[x,y] \mid f(x) \geqq \beta_i,\ f(y) < \beta_i\}.$$

From the definition of α, we have

$$\alpha A(K_i) \leqq L(\partial K_i).$$

Since

$$A = \sum_{x \sim y} |f^2(x) - f^2(y)| = 2\sum_{i=1}^{N} \sum_{f(x)=\beta_i} \sum_{\substack{x\sim y \\ f(y)<\beta_i}} (f^2(x) - f^2(y)),$$

if $x \sim y, f(x) = \beta_i, f(y) = \beta_{i-k} < \beta_i$, then

$$[x,y] \in \partial K_i \cap \partial K_{i-1} \cap \cdots \cap \partial K_{i-k+1},$$

and

$$f^2(x) - f^2(y) = (\beta_i^2 - \beta_{i-1}^2) + \cdots + (\beta_{i-k+1}^2 - \beta_{i-k}^2).$$

Thus

$$\begin{aligned} A &= 2\sum_{i=1}^{N} \sum_{[x,y]\in\partial K_i} (\beta_i^2 - \beta_{i-1}^2) \\ &= 2\sum_{i=1}^{N} L(\partial K_i)(\beta_i^2 - \beta_{i-1}^2) \geqq 2\alpha \sum_{i=1}^{N} A(K_i)(\beta_i^2 - \beta_{i-1}^2) \\ &= 2\alpha\{\sum_{i=1}^{N} A(K_i)\beta_i^2 - \sum_{i=1}^{N} A(K_i)\beta_{i-1}^2\} \\ &= 2\alpha\{\sum_{i=1}^{N} \sum_{x\in K_i/K_{i-1}} m(x)\beta_i^2\} = 2\alpha \sum_{x} m(x) f^2(x) \end{aligned}$$

$$= 2\alpha(f,f) = 2\alpha. \tag{1.2}$$

Combining the two estimates (1.1), (1.2), we have

$$2\alpha \leqq A \leqq 2\sqrt{2(df,df)}.$$

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Thus $\alpha \leqq \sqrt{2\lambda}$,

$$\frac{1}{2}\alpha^2 \leqq \lambda. \tag{1.3}$$

To prove the other inequality in the theorem, let S be a subset of $V(\Gamma)$, and f_S the characteristic function of S. Then we have

$$(f_S, f_S) = A(S) \quad \text{and} \quad (df_S, df_S) \leqq \frac{L(\partial S)}{l_0}.$$

Thus

$$\lambda \leqq \frac{(df_S, df_S)}{(f_S, f_S)} \leqq \frac{L(\partial S)}{l_0 A(S)},$$

$$\lambda \leqq \frac{\alpha}{l_0}. \tag{1.4}$$

We have Theorem 1 from (1.3) and (1.4). □

Corollary 2. *Let (Γ, l, m_l) be an infinite graph with $l \equiv 1$. We have*

$$\alpha \geqq \lambda \geqq \frac{1}{2}\alpha^2.$$

Furthermore, if the equality holds for the second inequality, then λ is not an eigenvalue; namely, there is no function $f \in L^2(V)$ with $(f, f) = 1$ and $(\Delta f, f) = \lambda$.

Proof. Put $l_0 = 1$ in Theorem 1, we obtain $\alpha \geqq \lambda \geqq \frac{1}{2}\alpha^2$. The latter part is proved by the maximum principle. Namely, assume there exists a function $f \in L^2(V)$ with $(f, f) = 1$ and $(df, df) = \lambda$ in the case $\lambda = \frac{1}{2}\alpha^2$. Let x_0 be a point where f takes its maximum. Then, it is seen that x_0 is an isolated maximum point of f or f is a constant function from the equality condition of Schwarz inequality which is used to show (1.1). Since Γ is infinite and $(f, f) = 1$, f can not be a constant function, thus x_0 is an isolated maximum point of f. Then the set K_N in the proof of Theorem 1 consists of isolated maximum points and it follows $L(\partial K_N) = A(K_N)$. Then $\alpha = 1$ from the equality condition of the inequality (1.2). But, taking $S = \{x, y\}$ in (0.2) for $x, y \in V(\Gamma)$ with $x \sim y$, we can show $\alpha < 1$, since we have $L(\partial S) \leqq A(S) - 2$. It contradicts $\alpha = 1$. Thus there is no function $f \in L^2(V)$ with $(f, f) = 1$ and $(df, df) = (\Delta f, f) = \lambda$. □

Remark. As is stated before, Dodziuk and Kendall [D,K] have already shown the same inequality in this case.

1.2. The bottom of the spectrum for a finite graph.

In this section, we show a relation between λ_1 and α for a finite graph.

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Theorem 3. *Let (Γ, l, m_l) be a finite graph with a length function l. Then,*

$$\frac{2}{l_0}\alpha \geqq \lambda_1 \geqq \frac{1}{2}\alpha^2,$$

where $l_0 = \min_{e \in E(\Gamma)} l(e)$.

Proof. Let f be the eigenfunction for λ_1 with $(f, f) = 1$. Put

$$f_+(x) = \max(f(x), 0),$$

$$f_-(x) = \min(f(x), 0),$$

and put

$$S_+ = \{x \in V(\Gamma) | f(x) > 0\}, S_- = \{x \in V(\Gamma) | f(x) < 0\}.$$

Since $(f, 1)_\Gamma = 0$, we have $f_+ \not\equiv 0$, $f_- \not\equiv 0$. As $\lambda_1 f_+(x) \geqq \Delta f_+(x)$ and $f_+ \geqq 0$, we have

$$\lambda_1 = \frac{(\lambda_1 f_+, f_+)}{(f_+, f_+)} \geqq \frac{(\Delta f_+, f_+)}{(f_+, f_+)} = \frac{(df_+, df_+)}{(f_+, f_+)} \geqq \frac{1}{2}\alpha^2(S_+; \Gamma),$$

where the last inequality is shown by the same argument as the second inequality in Theorem 1. Also,

$$\lambda_1 \geqq \frac{1}{2}\alpha^2(S_-; \Gamma).$$

Therefore

$$\lambda_1 \geqq \frac{1}{2}(\max\{\alpha(S_+; \Gamma), \alpha(S_-; \Gamma)\})^2 \geqq \frac{1}{2}\alpha^2,$$

since $S_+ \neq \emptyset, S_- \neq \emptyset, S_- \cap S_+ = \emptyset$. To show $\frac{2}{l_0}\alpha \geqq \lambda_1$, let S_1 and S_2 be subsets of $V(\Gamma)$ with $S_1 \neq \emptyset, S_2 \neq \emptyset, S_1 \cap S_2 = \emptyset$. Define a function f on $V(\Gamma)$ by

$$f(x) = \begin{cases} 1, & x \in S_1, \\ -1, & x \in S_2, \\ 0, & x \notin S_1 \cup S_2. \end{cases}$$

Then, since $(f, f) = A(S_1) + A(S_2), (df, df) \leqq \frac{2}{l_0}(L(\partial S_1) + L(\partial S_2))$, we have

$$\frac{(df, df)}{(f, f)} \leqq \frac{2}{l_0}\frac{L(\partial S_1) + L(\partial S_2)}{A(S_1) + A(S_2)} \leqq \frac{2}{l_0}\max\left\{\frac{L(\partial S_1)}{A(S_1)}, \frac{L(\partial S_2)}{A(S_2)}\right\}.$$

Therefore, $\lambda_1 \leqq \frac{2}{l_0}\alpha$. □

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2. Spectral convergence in a class of finite graphs

In this section we discuss a spectral convergence in a class of finite graphs. Let (Γ, l, m) be a finite graph with a length function l and a weight function m. Γ turns out to be a metric space by the path metric induced by l. We have the following theorem.

Theorem 4. *Let $\{(\Gamma_i, l_i, m_{l_i})\}_{i=1}^{\infty}$ be a sequence of elements of $\mathcal{G}(C)$ and (Γ, l) a finite graph such that $\lim_H(\Gamma_i, l_i) = (\Gamma, l)$. Then there exists a weight function $\tilde{m}$ on Γ and a subsequence $\{(\Gamma_j, l_j, m_{l_j})\}_j$ such that for $k = 1, 2, ..., \sharp V(\Gamma) - 1$,*

$$\lim_j \lambda_k(\Gamma_j, l_j, m_{l_j}) = \lambda_k(\Gamma, l, \tilde{m}), \tag{2.1}$$

and for $k \geqq \sharp V(\Gamma)$,

$$\lim_j \lambda_k(\Gamma_j, l_j, m_{l_j}) = \infty. \tag{2.2}$$

To prove Theorem 4, we will need the following Lemma (see Chapter 1 of [Ch]) called *minimax principle*. We write $L^2(V(\Gamma))$ just as $L^2(\Gamma)$.

Lemma.

$$\lambda_k(\Gamma) = \inf_{\mathcal{F}_{k+1}} \sup_{f \in \mathcal{F}} \frac{(df, df)}{(f, f)}$$

where $\mathcal{F}_{k+1}$ runs over linear subspaces of $L^2(\Gamma)$ of dimension $k+1$.

The expression $(df, df)/(f, f)$ is called the *Rayleigh quotient* of f. Suppose simplicial maps $\phi_i : \Gamma_i \to \Gamma$ satisfy (0.3) $\sim$ (0.5). We will show Claim 1 - Claim 5 in the following.

Claim 1. For sufficiently large i, we have

$$\phi_i \text{ is surjective,} \tag{2.3}$$

$$\sharp(\phi_i^{-1}(e)) = 1, \tag{2.4}$$

for any $e \in E(\Gamma)$.

Proof of Claim 1. Let $l_0 = \min_{e \in E(\Gamma)} l(e) > 0$. If i is large enough to satisfy $\varepsilon_i < \frac{l_0}{10}$, then we have that ϕ_i satisfies (2.3) and (2.4). $\square$

Since $(\Gamma_i, l_i, m_{l_i}) \in \mathcal{G}(C)$, taking a subsequence if necessary, $\lim_i m_{l_i}(\phi_i^{-1}(x))$ exists for each $x \in V(\Gamma)$. We define a positive function $\tilde{m}$ on Γ by

$$\tilde{m}(x) = \lim_i m_{l_i}(\phi_i^{-1}(x)),$$

and take it as a weight function on Γ.

Claim 2.

$$\lambda_k(\Gamma, l, \tilde{m}) \geqq \limsup_i \lambda_k(\Gamma_i, l_i, m_{l_i}), \tag{2.5}$$

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for $k = 1, 2, ..., \sharp V(\Gamma) - 1$.

Proof of Claim 2. We define a linear operator

$$\Psi_i : L^2(\Gamma) \to L^2(\Gamma_i)$$

by $\Psi_i(f) = f \circ \phi_i$ for $f \in L^2(\Gamma)$. By a calculation using Claim 1, we have

$$(df, df)_\Gamma = \lim_i (d\Psi_i(f), d\Psi_i(f))_{\Gamma_i},$$

and

$$(f, f)_{(\Gamma, \tilde{m})} = \lim_i (\Psi_i(f), \Psi_i(f))_{\Gamma_i}.$$

Thus, for $f \in L^2(\Gamma)$ with $f \neq 0$,

$$\frac{(df, df)_\Gamma}{(f, f)_{(\Gamma, \tilde{m})}} = \lim_i \frac{(d\Psi_i(f), d\Psi_i(f))_{\Gamma_i}}{(\Psi_i(f), \Psi_i(f))_{\Gamma_i}}. \tag{2.6}$$

Since $\Psi_i(\mathcal{F}_{k+1})$ is a linear subspace of $L^2(\Gamma_i)$ of dimension $k+1$ if $\mathcal{F}_{k+1}$ is a linear subspace of $L^2(\Gamma)$, we have $\lambda_k(\Gamma, l, \tilde{m}) \geqq \limsup_i \lambda_k(\Gamma_i, l_i, m_{l_i})$, for $k = 1, 2, ..., \sharp V(\Gamma) - 1$, by Lemma. □

Claim 3. If f is a function on a finite graph (G, l, m), then we have

$$|f(x) - f(y)| \leqq \sqrt{(df, df)}\sqrt{d_G(x, y)},$$

for $x, y \in V(G)$, where $d_G(\ , \ )$ is the distance induced from l on G.

Proof of Claim 3. Take vertices $x_0, x_1, .., x_n$ of $V(G)$ with $x_0 = x, x_n = y, x_i \sim x_{i+1} (i = 0, 1, ..n-1)$, and $d_G(x, y) = \sum_{i=0}^{n-1} l([x_i, x_{i-1}])$. Then

$$\begin{aligned} |f(x) - f(y)| &\leqq \sum_i |f(x_i) - f(x_{i+1})| \leqq \sqrt{\sum_i \frac{(f(x_i) - f(x_{i+1}))^2}{l(x_i, x_{i+1})}} \sqrt{\sum_i l(x_i, x_{i+1})} \\ &\leqq \sqrt{(df, df)}\sqrt{d_G(x, y)}. \end{aligned}$$

□

Claim 4.

$$\liminf_i \lambda_k(\Gamma_i, l_i, m_{l_i}) \geqq \lambda_k(\Gamma, l, \tilde{m}),$$

for $k = 1, 2, ..., \sharp V(\Gamma) - 1$.

Proof of Claim 4. We define a linear operator

$$\Phi_i : L^2(\Gamma_i) \to L^2(\Gamma)$$

by

$$\Phi_i(g)(x) = \frac{\sum_{y \in \phi_i^{-1}(x)} m_{l_i}(y) g(y)}{\sum_{y \in \phi_i^{-1}(x)} m_{l_i}(y)},$$

for $g \in L^2(\Gamma_i)$ and $x \in V(\Gamma)$. Let g_i be the k -th eigenfunction on Γ_i with $(g_i, g_i) = 1$. From (2.5), we can assume there exists some constant $C(k) < \infty$ such that $(dg_i, dg_i)_{\Gamma_i} \leqq C(k)$ for any i. Then we can show

$$\lim_i (\Phi_i(g_i), \Phi_i(g_i))_{(\Gamma, \tilde{m})} = 1, \tag{2.7}$$

and

$$\liminf_i (d\Phi_i(g_i), d\Phi_i(g_i))_\Gamma \leqq \liminf_i (dg_i, dg_i)_{\Gamma_i}. \tag{2.8}$$

In fact, (2.8) is shown as follows. By Claim 3, for any $x \in V(\Gamma)$,

$$|g_i(y_1) - g_i(y_2)| \leqq \sqrt{(dg_i, dg_i)_{\Gamma_i}} \sqrt{d_{\Gamma_i}(y_1, y_2)} \leqq \sqrt{(dg_i, dg_i)_{\Gamma_i}} \sqrt{\varepsilon_i}$$

for any $y_1, y_2 \in \phi_i^{-1}(x)$. Thus,

$$|\Phi_i(g_i)(x) - g_i(y)| \leqq \sqrt{\varepsilon_i} \sqrt{(dg_i, dg_i)_{\Gamma_i}} \leqq \sqrt{\varepsilon_i} \sqrt{C(k)}, \tag{2.9}$$

for any $y \in \phi_i^{-1}(x)$. It is seen from (2.9) and Claim 1 that

$$\liminf_i \{(d\Phi_i(g_i), d\Phi_i(g_i))_\Gamma - (dg_i, dg_i)_{\Gamma_i}\} \leqq 0,$$

thus, we have (2.8). (2.7) follows from (2.9) and the definitions of $\Phi_i(g)$ and $\tilde{m}$. From (2.7) and (2.8), we have

$$\liminf_i \frac{(d\Phi_i(g_i), d\Phi_i(g_i))_\Gamma}{(\Phi_i(g_i), \Phi_i(g_i))_{(\Gamma, \tilde{m})}} \leqq \liminf_i \frac{(dg_i, dg_i)_{\Gamma_i}}{(g_i, g_i)_{\Gamma_i}}. \tag{2.10}$$

Since the dimension of a linear subspace of $L^2(\Gamma_i)$ may decrease when we map it into $L^2(\Gamma)$ by Φ_i, we cannot immediately conclude $\liminf_i \lambda_k(\Gamma_i, l_i, m_{l_i}) \geqq \lambda_k(\Gamma, l, \tilde{m})$ from (2.10) by the same argument as Claim 2. However, for any functions $g_1, ..., g_k$ in $L^2(\Gamma_i)$ which are linearly independent, there exist functions $\tilde{g}_1, ..., \tilde{g}_k$, in $L^2(\Gamma_i)$ such that the Rayleigh quotient of $\tilde{g}_j$ is arbitrarily near to that of g_j for each j and that $\Phi_i(\tilde{g}_1), ..., \Phi_i(\tilde{g}_k)$ are linearaly independent in $L^2(\Gamma)$. Thus, we have Claim 4 from (2.10) using Lemma. □

Claim 5.

$$\lim_i \lambda_k(\Gamma_i, l_i, m_{l_i}) = \infty, \tag{2.2}$$

for $k \geqq \sharp V(\Gamma)$.

Proof of Claim 5. If the claim does not hold, then, taking a subsequence if necessary, we have

$$\lim_i \lambda_k(\Gamma_i, l_i, m_{l_i}) < \infty, \tag{2.11}$$

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for $k = \sharp V(\Gamma)$. From (2.11) and Lemma, there are a positive number $C < \infty$ and functions

$$g_{i,0}, ..., g_{i,k}$$

in $L^2(\Gamma_i)$ such that

$$(g_{i,a}, g_{i,b})_{\Gamma_i} = \delta_{ab}, 0 \leqq a, b \leqq k \tag{2.12}$$

and

$$(dg_{i,a}, dg_{i,a})_{\Gamma_i} \leqq C, 0 \leqq a \leqq k. \tag{2.13}$$

From (2.12),(2.13) and Claim 3, we have

$$\lim_i (\Phi_i(g_{i,a}), \Phi_i(g_{i,b}))_{(\Gamma,\tilde{m})} = \delta_{ab}, \tag{2.14}$$

for $0 \leqq a, b \leqq k$. It follows from (2.14) that for large i,

$$\Phi_i(g_{i,0}), ..., \Phi_i(g_{i,k})$$

are linearly independent functions in $L^2(\Gamma)$, which contradicts that $\dim(L^2(\Gamma)) = \sharp V(\Gamma) - 1$. Therefore, we showed the claim. □

We have Theorem 4 from Claim 2, Claim 4, and Claim 5.

Corollary 5. *Let $\{(\Gamma_i, l_i)\}_{i=1}^{\infty}$ be a sequence of finite graphs and (Γ, l) a finite graph, such that $\lim_H(\Gamma_i, l_i) = (\Gamma, l)$ and there exists a positive number C with $\sharp V(\Gamma_i) < C$ for any i. Then, taking a subsequence $\{(\Gamma_j, l_j, m_{l_j})\}_j$, we have*

$$\lim_j \lambda_k(\Gamma_j, l_j, m_{l_j}) = \lambda_k(\Gamma, l, m_l)$$

for $k = 1, 2, ..., \sharp V(\Gamma) - 1$, and

$$\lim_j \lambda_k(\Gamma_j, l_j, m_{l_j}) = \infty. \tag{2.5}$$

for $k \geqq \sharp V(\Gamma)$.

Proof. From $\sharp V(\Gamma_i) < C$, and $\lim_H(\Gamma_i, l_i) = (\Gamma, l)$, there is a positive number C' with $(\Gamma_i, l_i, m_{l_i}) \in \mathcal{G}(C')$ for any i. Therfore, we can apply Theorem 4 with $\tilde{m} = m_l$. Corollary 5 is shown. □

3. A relation between a Riemannian manifold and its net on the eigenvalues of the Laplacians

In this section, we discuss a relation between a closed Riemannin manifold and its net on the eigenvalues of the Laplacians. We have the following theorem.

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Theorem 6. *Let M be a closed Riemannian manifold and (Γ_n, l_n, m_{l_n}) a $1/n$-net in M with the length function $l_n \equiv 1/n$ for each $n \in N$. Then,*

$$\frac{1}{C} \limsup_n \lambda_k(\Gamma_n, l_n, m_{l_n}) \leqq \lambda_k(M) \leqq C \liminf_n \lambda_k(\Gamma_n, l_n, m_{l_n})$$

for each k, where $\lambda_k(M)$ is the k-th eigenvalue of the Laplacian on M and C is a number which depends only on the dimension.

The proof consists of two parts. First, to show $\lambda_k(M) \leqq C \liminf_n \lambda_k(\Gamma_n)$, we construct a linear operator

$$S_n : L^2(\Gamma_n) \to C^\infty(M)$$

for each n, which satisfies

$$\frac{(dS_n(f), dS_n(f))_M}{(S_n(f), S_n(f))_M} \leqq C \frac{(df, df)_{\Gamma_n}}{(f, f)_{\Gamma_n}},$$

for sufficiently large n. Next, to show $\limsup_n \lambda_k(\Gamma_n) \leqq C \lambda_k(M)$, we construct a linear operator

$$T_n : C^\infty(M) \to L^2(\Gamma_n)$$

for each n with the following property. Let $\mathcal{F}$ be a finite dimensional linear subspace of $C^\infty(M)$, we denote the set $\{f \in \mathcal{F} | (f, f) = 1\}$ by $\mathcal{F}(1)$. Then for any $\varepsilon > 0$, taking sufficiently large n, we have

$$\frac{(dT_n(f), dT_n(f))_{\Gamma_n}}{(S_n(f), S_n(f))_{\Gamma_n}} \leqq C \frac{(df, df)_M + \varepsilon}{(f, f)_M - \varepsilon},$$

for each $f \in \mathcal{F}(1)$.

Notations. For a point $x \in M$, we put $B(x, r) = \{y \in M | d(x, y) < r\}$ and denote its volume in M by $vol(B(x, r))$.

Constants. We introduce several constants which we will use to prove the theorem. It is easily seen that there exist constants $C_1, C_2, ..., C_8$ which depend only on the dimension d such that taking sufficiently large n, we have for any $x_i \in \Gamma_n$,

$$C_1 \leqq \sharp\{x_j \in \Gamma_n; x_i \sim x_j\} \leqq C_2,$$

$$n^d vol(B(x_i, \frac{1}{n})) \leqq C_3,$$

$$C_4 \leqq n^d vol(B(x_i, \frac{1}{3n})),$$

$$C_5 \leqq n^d vol(B(x_i, \frac{1}{2n})) \leqq C_6,$$

$$vol(B(x_i, \frac{1}{n})) \leqq C_7 vol(B(x_i, \frac{1}{2n})),$$

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and

$$\sharp(\Gamma_n) \leqq C_8 n^d vol(M).$$

Proof of Theorem 6. Fix n and let $\{x_j\}_{j=1}^{\sharp V(\Gamma_n)} = V(\Gamma_n)$. Take a partition of unity $\{u_{n,j}\}_j$ on M with the following properties.

$$supp(u_{n,j}) \subset B(x_j, \frac{2}{n}) \qquad for \ each \ j,$$

$$u_{n,j} = 1 \quad on \quad B(x_j, \frac{1}{3n}),$$

$$(du_{n,j}(x), du_{n,j}(x)) \leqq n^2, \qquad for \ any \ x \in M.$$

Since $\sum_j u_{n,j} = 1$,

$$\sum_j du_{n,j} = 0. \tag{3.1}$$

If $d(x, x_j) > \frac{2}{n}$ for $x \in M$, then

$$du_{n,j}(x) = 0. \tag{3.2}$$

We define a linear operator for each n,

$$S_n : L^2(\Gamma_n) \to C^\infty(M)$$

by

$$S_n(f)(x) = \sum_{x_j \in \Gamma_n} f(x_j) u_{n,j}(x)$$

for $f \in L^2(\Gamma_n)$.

Claim 1. Taking sufficiently large n,

$$(dS_n(f), dS_n(f))_M \leqq \frac{2C_2C_3}{n^{d-1}}(df, df)_{\Gamma_n}$$

for any $f \in L^2(\Gamma_n)$.

Proof of Claim 1. For each $x \in M$, take $x_k \in \Gamma_n$ with $d(x, x_k) \leqq \frac{1}{n}$, then

$$\begin{aligned} dS_n(f)(x) &= \sum_{x_j \in \Gamma_n} f(x_j) du_{n,j}(x) \\ &= \sum_j (f(x_k) - f(x_j)) du_{n,j}(x) + f(x_k) \sum_j du_{n,j}(x), \end{aligned}$$

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using (3.1)

$$= \sum_j (f(x_k) - f(x_j)) du_{n,j}(x),$$

using (3.2)

$$= \sum_{x_j \in \Gamma_n; d(x,x_j) \leqq \frac{2}{n}} (f(x_k) - f(x_j)) du_{n,j}(x),$$

$$|dS_n(f)(x)| \leqq \sum_{x_j; d(x_j,x_k) \leqq \frac{3}{n}} |(f(x_k) - f(x_j))| n$$
$$= \sum_{x_j; x_j \sim x_k} |(f(x_k) - f(x_j))| n.$$

Thus,

$$(dS_n(f)(x), dS_n(f)(x)) \leqq n^2 \Big(\sum_{i; x_i \sim x_k} |f(x_i) - f(x_k)| \Big)^2$$
$$\leqq n^2 C_2 \sum_{i; x_i \sim x_k} (f(x_i) - f(x_k))^2.$$

Therfore,

$$(dS_n(f), dS_n(f)) \leqq n^2 C_2 \sum_{x_k \in \Gamma_n} \{ \sum_{i; x_i \sim x_k} (f(x_i) - f(x_k))^2 vol(B(x_k, \frac{1}{n}) \}$$
$$\leqq C_2 C_3 \frac{1}{n^{d-1}} \sum_{x_k \in \Gamma_n} \sum_{i; x_i \sim x_k} \frac{(f(x_i) - f(x_k))^2}{n} = \frac{2 C_2 C_3}{n^{d-1}} (df, df)_{\Gamma_n}$$

Claim 2. For sufficiently large n, we have

$$(S_n(f), S_n(f))_M \geqq \frac{C_4}{C_2 n^{d-1}} (f, f)_{\Gamma_n}.$$

Proof of Claim 2.

$$(f, f)_{\Gamma_n} = \sum_{x_j \in \Gamma_n} f^2(x_j) m_{l_n}(x_j) \leqq \frac{C_2}{n} \sum_{x_j \in \Gamma_n} f^2(x_j)$$
$$\leqq \frac{C_2}{n} \frac{n^d}{C_4} \sum_{x_j \in \Gamma_n} f^2(x_j) vol B(x_i, \frac{1}{3n})$$
$$\leqq \frac{C_2}{C_4} n^{d-1} \int_M (S_n(f), S_n(f)) dM = \frac{C_2}{C_4} n^{d-1} (S_n(f), S_n(f))_M$$

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From Claim 1 and Claim 2, we have the next claim.

Claim 3. For sufficiently large n, we have

$$\frac{(dS_n(f), dS_n(f)_M}{(S_n(f), S_n(f))_M} \leqq \frac{2C_2^2C_3}{C_4}\frac{(df, df)_{\Gamma_n}}{(f, f)_{\Gamma_n}}.$$

We define a linear operator for each n

$$T_n : C^\infty(M) \to L^2(\Gamma_n)$$

by

$$T_n(f)(x_i) = \frac{\int_{B(x_i, \frac{1}{n})} f dM}{volB(x_i, \frac{1}{n})},$$

for $f \in C^\infty(M)$ and $x_i \in \Gamma_n$. Then, we have

Claim 4. Let $\mathcal{F}$ be a finite dimensional linear subspace of $C^\infty(M)$. Then for any small $\varepsilon > 0$, taking sufficiently large n, we have

$$\frac{(dT_n(f), dT_n(f))_{\Gamma_n}}{(T_n(f), T_n(f))_{\Gamma_n}} \leqq \frac{18C_2C_3}{C_1C_5}\frac{(df, df)_M + \varepsilon}{(f, f)_M - \varepsilon},$$

for each $f \in \mathcal{F}(1)$.

To prove Claim 4, we first show the next two claims for each $f \in \mathcal{F}(1)$ under the same conditions as Claim 4.

Claim 5.

$$(f, f)_M \leqq \frac{2C_3}{C_1 n^{d-1}}(T_n(f), T_n(f))_{\Gamma_n} + \varepsilon C_7 vol(M),$$

and

Claim 6.

$$(dT_n(f), dT_n(f))_{\Gamma_n} \leqq n^{d-1}\left\{\frac{9C_2}{C_5}(df, df)_M + \varepsilon\frac{9C_2}{2C_5}vol(M)\right\}.$$

Proof of Claim 5. For any $\varepsilon > 0$, taking n large, we have

$$\int_{B(x_i, \frac{1}{n})}(f, f)dM \leqq \{2(T_n(f)(x_i))^2 + \varepsilon\}volB(x_i, \frac{1}{n}),$$

for any $f \in \mathcal{F}(1)$ since $\mathcal{F}(1)$ is compact. Therefore,

$$\begin{aligned}(f,f)_M &\leqq \sum_i \int_{B(x_i,\frac{1}{n})} (f,f)dM \\ &\leqq 2\sum_i (T_n(f)(x_i))^2 volB(x_i,\frac{1}{n}) + \varepsilon \sum_i volB(x_i,\frac{1}{n}) \\ &\leqq \frac{2C_3}{n^d}\sum_i (T_n(f)(x_i))^2 + \varepsilon C_7 \sum_i volB(x_i,\frac{1}{2n}) \\ &\leqq \frac{2C_3}{C_1 n^d}\sum_i (T_n(f)(x_i))^2 m_{l_n}(x_i) + \varepsilon C_7 vol(M) \\ &= \frac{2C_3}{C_1 n^d}(T_n(f),T_n(f))_{\Gamma_n} + \varepsilon C_7 vol(M).\end{aligned}$$

□

Proof of Claim 6. Since $\mathcal{F}(1)$ is compact, taking n sufficiently large, for any $x_i, x_j \in \Gamma_n$ with $x_i \sim x_j$, we have

$$(T_n(f)(x_i) - T_n(f)(x_j))^2 \leqq \left\{ \frac{2\int_{B(x_i,\frac{1}{2n})}(df,df)dM}{volB(x_i,\frac{1}{2n})} + \varepsilon \right\} d^2(x_i,x_j),$$

since $d^2(x_i,x_j) \leqq \frac{9}{n^2}$,

$$\leqq \frac{18}{n^2}\frac{n^d}{C_5}\int_{B(x_i,\frac{1}{2n})}(df,df)dM + \frac{9}{n^2}\varepsilon.$$

Therefore,

$$\begin{aligned}(dT_n(f),dT_n(f))_{\Gamma_n} &= \frac{1}{2}\sum_{x_i\sim x_j}(T_n(f)(x_i) - T_n(f)(x_j))^2 n \\ &\leqq \frac{9C_2 n^{d-1}}{C_5}\sum_i \int_{B(x_i,\frac{1}{2n})}(df,df)dM + \frac{9C_2}{2n}\sharp(\Gamma_n)\varepsilon \\ &\leqq \frac{9C_2 n^{d-1}}{C_5}\int_M (df,df)dM + \frac{9C_2 n^{d-1}}{2C_5}vol(M)\varepsilon \\ &= \frac{9C_2 n^{d-1}}{C_5}(df,df)_M + \frac{9C_2 n^{d-1}}{2C_5}vol(M)\varepsilon.\end{aligned}$$

□

From Claim 5 and Claim 6, we have Claim 4. □We are now in the position to complete the proof of Theorem 6. From Claim 3, we can conclude

$$\lambda_k(M) \leqq \frac{2C_2^2 C_3}{C_4}\liminf_n \lambda_k(\Gamma_n, l_n)$$

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for each k by the same argument we did when we proved Theorem 4 in section 2. Also, we have from Claim 4 that

$$\limsup_n \lambda_k(\Gamma_n, l_n) \leqq \frac{18C_2C_3}{C_1C_5}\lambda_k(M)$$

for each k. Thus We showed Theorem 6. □

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