Research Report

KSTS/RR-93/001
Feb. 17, 1993

On Standard L-functions attached to $\mathrm{Alt}^{n-1}(\mathbb{C}^n)$-valued Siegel modular forms

by

Hideshi Takayanagi

Hideshi Takayanagi
Department of Mathematics
Keio University

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku, Yokohama, 223 Japan

KSTS/RR-93/001
February 17, 1993

On standard L-functions attached to $\mathrm{Alt}^{n-1}(\mathbb{C}^n)$-valued Siegel modular forms

HIDESHI TAKAYANAGI

Dept. of Mathematics, Keio University

Introduction

In [23], we studied some properties of standard L-functions attached to $\mathrm{sym}^l(V)$-valued Siegel modular forms of weight $\det^k \otimes \mathrm{sym}^l$. More precisely, let $\det^k \otimes \mathrm{sym}^l$ be an irreducible rational representation of $GL(n, \mathbb{C})$ with the representation space $\mathrm{sym}^l(V)$, where V is isomorphic to $\mathbb{C}^n$ and $\mathrm{sym}^l(V)$ is the l-th symmetric tensor product of V. Let f be a $\mathrm{sym}^l(V)$-valued holomorphic cusp form of weight $\det^k \otimes \mathrm{sym}^l$ for $Sp(n, \mathbb{Z})$ (size $2n$). Suppose f is an eigenform, i.e., a non-zero common eigenfunction of the Hecke algebra. Then we define the standard L-function attached to f by

$$(0.1) \qquad L(s, f, \underline{\mathrm{St}}) := \prod_p \left\{ (1-p^{-s}) \prod_{j=1}^{n} (1-\alpha_j(p)^{-1}p^{-s})(1-\alpha_j(p)p^{-s}) \right\}^{-1} ,$$

where p runs over all prime numbers and $\alpha_j(p)$ $(1 \le j \le n)$ are the Satake p-parameters of f. The right-hand side of (0.1) converges absolutely and locally uniformly for $\mathrm{Re}(s) > n+1$. We put

$$\Lambda(s, f, \underline{\mathrm{St}}) := \Gamma_{\mathbb{R}}(s+\varepsilon)\Gamma_{\mathbb{C}}(s+k+l-1) \prod_{j=2}^{n} \Gamma_{\mathbb{C}}(s+k-j) L(s, f, \underline{\mathrm{St}}) \quad ,$$

with

$$\Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) \ , \quad \Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s) \ ,$$

and

$$\varepsilon := \begin{cases} 0 & \text{for } n \text{ even,} \\ 1 & \text{for } n \text{ odd.} \end{cases}$$

Then we have the following [23, Theorem 2, 3]:

Theorem. *For $k, l \in 2\mathbb{Z}$, $k > 0$ and $l \ge 0$, $\Lambda(s, f, \underline{\mathrm{St}})$ has a meromorphic continuation to the whole s-plane and satisfies the functional equation*

$$\Lambda(s, f, \underline{\mathrm{St}}) = \Lambda(1-s, f, \underline{\mathrm{St}}) \ .$$

KSTS/RR-93/001
February 17, 1993

Suppose $k > n$. Then $\Lambda(s, f, \underline{\mathrm{St}})$ is holomorphic except for possible simple poles at $s = 0$ and $s = 1$; it has a pole at $s = 0$ (or equivalently, $s = 1$) if and only if f belongs to the $\mathbb{C}$-vector space spanned by certain theta series in [24] which is invariant under the action of the Hecke algebra.

If we note that the signature of $\det^k \otimes \mathrm{sym}^l$ is $(k + l, k, \cdots, k) \in \mathbb{Z}^n$, we speculate the following [23, §3.1 Remark] :

(C). *Let ρ be an irreducible rational representation of $GL(n, \mathbb{C})$ with the representation space $\mathcal{V}$ whose signature is $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Let f be a $\mathcal{V}$-valued holomorphic cusp form of weight ρ for $Sp(n, \mathbb{Z})$. Suppose that f is an eigenform. Then, it is expected that the completed Dirichlet series*

$$\Lambda(s, f, \underline{\mathrm{St}}) := \Gamma_{\mathbb{R}}(s + \varepsilon) \prod_{j=1}^{n} \Gamma_{\mathbb{C}}(s + \lambda_j - j) L(s, f, \underline{\mathrm{St}}) \quad ,$$

should satisfy a functional equation.

Unfortunately, within our knowledge it is not verified so far whether **(C)** holds or not except $\det^k$ and $\det^k \otimes \mathrm{sym}^l$ cases. We will give another example satisfying **(C)**.

For $l \in \mathbb{Z}$, $0 \leq l \leq n$, let $\det^k \otimes \mathrm{alt}^l$ be an irreducible rational representation of $GL(n, \mathbb{C})$ with the representation space $\mathrm{alt}^l(V)$, where V is isomorphic to $\mathbb{C}^n$ and $\mathrm{alt}^l(V)$ is the l-th alternating tensor product of V. Let $M_k^n(\mathrm{alt}^l(V))$ (resp. $S_k^n(\mathrm{alt}^l(V))$) be the $\mathbb{C}$-vector space consisting of $\mathrm{alt}^l(V)$-valued holomorphic modular (resp. cusp) forms of weight $\det^k \otimes \mathrm{alt}^l$ for $Sp(n, \mathbb{Z})$.

Suppose that $f \in S_k^n(\mathrm{alt}^{n-1}(V))$ is an eigenform. We note that the signature of $\det^k \otimes \mathrm{alt}^{n-1}$ is $(k + 1, \cdots, k + 1, k)$. We put

$$\Lambda(s, f, \underline{\mathrm{St}}) := \Gamma_{\mathbb{R}}(s + 1) \prod_{j=1}^{n-1} \Gamma_{\mathbb{C}}(s + k + 1 - j) \Gamma_{\mathbb{C}}(s + k - n) L(s, f, \underline{\mathrm{St}}) \quad .$$

Then the main result of this paper is:

Theorem 1. *Let k be an even integer, n an odd integer and $k > n > 2$. Then $\Lambda(s, f, \underline{\mathrm{St}})$ is continued analytically as an entire function and satisfies the functional equation*

$$\Lambda(s, f, \underline{\mathrm{St}}) = \Lambda(1 - s, f, \underline{\mathrm{St}}) \ .$$

(cf. Piatetski-Shapiro, Rallis [21], Weissauer [24])

Notations

1°. As usual, $\mathbb{Z}$ is the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers.

2°. Let $m, n \in \mathbb{Z}$, $m, n > 0$. If A is an $m \times n$-matrix, then we write it also as $A^{(m,n)}$, and as $A^{(m)}$ if $m = n$. The identity matrix of size n is denoted by 1_n.

KSTS/RR-93/001
February 17, 1993

3°. For $m, n \in \mathbb{Z}$, $m, n > 0$, and a commutative ring R containing 1, let $R^{(m,n)}$ (resp. $R^{(n)}$) be the R-module of all $m \times n$ (resp. $n \times n$) matrices with entries in R.

4°. For a real symmetric positive definite matrix S, $S^{1/2}$ is the unique real symmetric positive definite matrix such that $(S^{1/2})^2 = S$.

5°. For matrices $A^{(m)}$, $B^{(m,n)}$, we define $A[B] := {}^t\bar{B}AB$, where ${}^tB$ is the transpose of B and $\bar{B}$ is the complex conjugate of B.

6°. For a matrix $A^{(m)} = (a_{jh})_{1\le j,h\le m}$, $\widetilde{a_{jh}}$ is the cofactor of a_{jh} and $\widetilde{A} = (\widetilde{a_{jh}})$.

7°. For $n \in \mathbb{Z}$, $n > 0$, we put

$$\mathbb{T}^{(n)} := \left\{ T = \begin{pmatrix} t_1 & & & 0 \\ & t_2 & & \\ & & \ddots & \\ 0 & & & t_n \end{pmatrix} \in \mathbb{Z}^{(n)} \,\middle|\, t_j > 0 \ (1 \le j \le n),\ t_1 | \cdots | t_n \right\}.$$

8°. For $n \in \mathbb{Z}$, $n > 0$, let $\Gamma^n := Sp(n, \mathbb{Z})$ be the Siegel modular group of degree n and let $\mathfrak{H}_n$ be the Siegel upper half space of degree n, that is,

$$\mathfrak{H}_n := \{ Z = X + iY \in \mathbb{C}^{(n)} \mid {}^tZ = Z,\ Y > 0 \}.$$

For each $r \in \mathbb{Z}$ with $0 \le r \le n$, we put

$$P_{n,r} := \left\{ \begin{pmatrix} * & * \\ C^{(n)} & D^{(n)} \end{pmatrix} \in \Gamma^n \,\middle|\, C = \begin{pmatrix} 0 & 0 \\ 0 & C_4{}^{(r)} \end{pmatrix},\ D = \begin{pmatrix} * & 0 \\ * & D_4{}^{(r)} \end{pmatrix} \right\}.$$

All these are subgroups of Γ^n.

9°. For $n \in \mathbb{Z}$, $n \ge 0$, we put

$$\Gamma_n(s) := \prod_{j=1}^{n} \Gamma\left(s - \frac{j-1}{2}\right),$$

and

$$\gamma(s) := \begin{cases} \dfrac{\Gamma_n\left(\frac{s+n}{2}\right)}{\Gamma_n\left(\frac{s}{2}\right)} & \text{for } n \text{ even}, \\[2ex] \dfrac{\Gamma_{n-1}\left(\frac{s+n}{2}\right)}{\Gamma_{n-1}\left(\frac{s-1}{2}\right)} & \text{for } n \text{ odd}, \end{cases}$$

where $\Gamma(s)$ is the gamma function. We note

$$\gamma(s) = \gamma(1-s).$$

Moreover we put

$$\xi(s) := \Gamma_{\mathbb{R}}(s)\zeta(s) = \xi(1-s),$$

where $\zeta(s)$ is the Riemann zeta function.

Throughout the paper we understand that a product (resp. a sum) over an empty set is equal to 1 (resp. 0).

KSTS/RR-93/001
February 17, 1993

§1 Preliminary

Let ρ be a finite-dimensional representation of $GL(n,\mathbb{C})$ with the representation space $\mathcal{V}$. By definition, $\mathcal{V}$-valued C^∞-Siegel modular forms of weight ρ are C^∞-functions from $\mathfrak{H}_n$ to $\mathcal{V}$ satisfying

$$(f|_\rho M)(Z) = f(Z) \tag{1.1}$$

for all $Z \in \mathfrak{H}_n$ and $M = \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \in \Gamma^n$, where

$$(f|_\rho M)(Z) := \rho((CZ+D)^{-1})f(M\langle Z\rangle) \text{ and } M\langle Z\rangle := (AZ+B)(CZ+D)^{-1}.$$

The space of all such functions is denoted by $M_\rho^n(\mathcal{V})^\infty$.

We write $|_k$ for $\rho = \det^k$ and we omit subscripts ρ, k when there is no fear of confusion.

A holomorphic function f from $\mathfrak{H}_n$ to $\mathcal{V}$ is called a $\mathcal{V}$-valued Siegel modular form of weight ρ if it satisfies (1.1) and if it is holomorphic at the cusps when $n=1$. The space of $\mathcal{V}$-valued Siegel modular forms of weight ρ is denoted by $M_\rho^n(\mathcal{V})$.

We define the Siegel operator Φ on $M_\rho^n(\mathcal{V})$ by

$$(\Phi f)(Z) := \lim_{t\to\infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix}\right)$$

for $Z \in \mathfrak{H}_{n-1}$. Let $\mathcal{V}'$ be the subspace of $\mathcal{V}$ generated by the values of Φf for all $f \in M_\rho^n(\mathcal{V})$. Then $\mathcal{V}'$ is invariant under the transformations

$$\rho\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right),\ g \in GL(n-1,\mathbb{C}).$$

If we assume $\mathcal{V}' \neq \{0\}$, we get the representation ρ' of $GL(n-1,\mathbb{C})$ with the representation space $\mathcal{V}'$. Thus the operator Φ defines the map

$$\Phi : M_\rho^n(\mathcal{V}) \longrightarrow M_{\rho'}^{n-1}(\mathcal{V}').$$

Suppose $f \in M_\rho^n(\mathcal{V})$. Then it is called a cusp form if it satisfies $\Phi f = 0$, and we put

$$S_\rho^n(\mathcal{V}) := \{f \in M_\rho^n(\mathcal{V}) \mid f \text{ is a cuspform }\}.$$

If ρ is an irreducible rational representation, ρ is equivalent to an irreducible rational representation $\tilde\rho$ satisfying the following condition: Let $\tilde{\mathcal{V}}$ be the representation space of $\tilde\rho$. Then, there exists a unique one-dimensional vector subspace $\mathbb{C}\tilde v$ of $\tilde{\mathcal{V}}$ such that for any upper triangular matrix of $GL(n,\mathbb{C})$,

$$\tilde\rho\left(\begin{pmatrix} g_{11} & & * \\ & \ddots & \\ 0 & & g_{nn} \end{pmatrix}\right)\tilde v = \left(\prod_{j=1}^n {g_{jj}}^{\lambda_j}\right)\tilde v,$$

KSTS/RR-93/001
February 17, 1993

where $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}^n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Then we call $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ the signature of ρ.

Remark. Suppose the signature of ρ is $(\lambda_1, \lambda_2, \cdots, \lambda_n)$. We note that $M_\rho^n(\mathcal{V}) = \{0\}$ if $\lambda_n < 0$ and that $M_\rho^n(\mathcal{V})^\infty = \{0\}$ if $\lambda_1 + \cdots + \lambda_n \not\equiv 0 \mod 2$.

Now, we put

$$G^+Sp(n,\mathbb{Q}) := \left\{ M \in GL(2n,\mathbb{Q}) \;\middle|\; {}^tM \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} M = \mu(M) \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \;,\; \mu(M) > 0 \right\} .$$

For $g \in G^+Sp(n,\mathbb{Q})$, let $\Gamma^n g \Gamma^n = \bigcup_{j=1}^r \Gamma^n g_j$ be a decomposition of the double coset $\Gamma^n g \Gamma^n$ into left cosets. For $f \in M_\rho^n(\mathcal{V})$ (resp. $S_\rho^n(\mathcal{V})$, $M_\rho^n(\mathcal{V})^\infty$), we define the Hecke operator $(\Gamma^n g \Gamma^n)$ by

$$f|(\Gamma^n g \Gamma^n) := \sum_{j=1}^r f|g_j \;.$$

Let $f \in S_\rho^n(\mathcal{V})$ be an eigenform. We define the standard L-function attached to f by (0.1). We also define the following series:

$$D(s,f) := \sum_{T \in \mathbb{T}^{(n)}} \lambda(f,T) \det(T)^{-s} \;, \tag{1.2}$$

where $\lambda(f,T)$ is the eigenvalue on f of the Hecke operator $\left(\Gamma^n \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \Gamma^n\right)$, $T \in \mathbb{T}^{(n)}$. By Böcherer [6] , we have:

$$\zeta(s) \prod_{j=1}^n \zeta(2s-2j) D(s,f) = L(s-n, f, \underline{\mathrm{St}}) \;. \tag{1.3}$$

For $k \in 2\mathbb{Z}$, $k > 0$, $s \in \mathbb{C}$ and $Z = (z_{jh}) \in \mathfrak{H}_n$, we define the Eisenstein series by

$$E_k^n(Z,s) := \sum_{M = \begin{pmatrix} * & * \\ C^{(n)} & D^{(n)} \end{pmatrix} \in P_{n,0} \backslash \Gamma^n} \det(CZ+D)^{-k} \det(\mathrm{Im}(M\langle Z \rangle))^s \;.$$

Then $E_k^n(Z,s) \in M_k^{n\,\infty}$, where $M_k^{n\,\infty}$ is the space of C^∞-Siegel modular forms of weight k. The function $E_k^n(Z,s)\det(\mathrm{Im}(Z))^{-s}$ converges absolutely and locally uniformly for $k + 2\mathrm{Re}(s) > n+1$. Moreover we have the following by Mizumoto [19], [20]. (see also Andrianov-Kalinin [2], Kalinin [13], Langlands [18]):

KSTS/RR-93/001
February 17, 1993

Theorem 2. *Let $n, k \in \mathbb{Z}$, $n, k > 0$, k : even. Then for $Z \in \mathfrak{H}_n$,*

$$\mathbb{E}_k^n(Z,s) := \frac{\Gamma_n\left(s+\frac{k}{2}\right)}{\Gamma_n(s)}\xi(2s)\prod_{j=1}^{[\frac{n}{2}]}\xi(4s-2j)E_k^n\left(Z, s-\frac{k}{2}\right)$$

is invariant under $s \mapsto \frac{n+1}{2} - s$ and it is an entire function in s.

It is also known that every partial derivative (in z_{jh}'s) of the Eisenstein series $E_k^n(Z,s)$ is slowly increasing(locally uniformly in s). That is, by Mizumoto [20] we have:

Theorem 3. *Let $n, k \in \mathbb{Z}$, $n, k > 0$, k : even. For each $s_0 \in \mathbb{C}$, we can take $d \in \mathbb{Z}$, $d > 0$ and a suitable neighborhood $\mathcal{U}$ of s_0 depending only on n, k and s_0 such that $(s-s_0)^d E_k^n(Z,s)$ is holomorphic in s on $\mathcal{U}$. Then, for $s \in \mathcal{U}$, $l \in \mathbb{Z}$, $l \geq 0$, $\mathrm{Im}(Z) \geq \varepsilon 1_n$ ($\varepsilon > 0$), there exist positive constants α, β depending only on n, k, l, ε, s_0, d and $\mathcal{U}$ such that*

$$\left|(s-s_0)^d \frac{\partial^l}{\partial z_{j_1h_1} \cdots \partial z_{j_lh_l}} E_k^n(Z,s)\right| \leq \alpha \det(\mathrm{Im}(Z))^\beta$$

$(1 \leq j_\nu\ ,\ h_\nu \leq n)$.

§2 Differential operators

In what follows, we put

$$\begin{aligned} V_1 &= \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n\ , & x_1 &= (e_1, \cdots, e_n)\ , \\ V_2 &= \mathbb{C}e_{n+1} \oplus \cdots \oplus \mathbb{C}e_{2n}\ , & x_2 &= (e_{n+1}, \cdots, e_{2n})\ . \end{aligned}$$

Let $\mathrm{alt}^{n-1}(V_1)$ (resp. $\mathrm{alt}^{n-1}(V_2)$) be the $(n-1)$-th alternating tensor product of V_1 (resp. V_2). If we put

$$\begin{aligned} t_j &:= (-1)^{j-1} e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_n\ , \\ t_{n+j} &:= (-1)^{j-1} e_{n+1} \wedge \cdots \wedge e_{n+j-1} \wedge e_{n+j+1} \wedge \cdots \wedge e_{2n}\ (1 \leq j \leq n)\ , \end{aligned}$$

we can write

$$\mathrm{alt}^{n-1}(V_1) = \mathbb{C}t_1 \oplus \cdots \oplus \mathbb{C}t_n \quad \text{and} \quad \mathrm{alt}^{n-1}(V_2) = \mathbb{C}t_{n+1} \oplus \cdots \oplus \mathbb{C}t_{2n}\ .$$

Moreover, we put

$$y_1 := (t_1, \cdots, t_n) \quad \text{and} \quad y_2 := (t_{n+1}, \cdots, t_{2n})\ .$$

If for each $g \in GL(n, \mathbb{C})$, g acts on x_j $(j = 1, 2)$ by $x_j g$, then $\det^k \otimes \mathrm{alt}^{n-1}(g)$ acts on y_j $(j = 1, 2)$ by

$$\det{}^k \otimes \mathrm{alt}^{n-1}(g) y_j := \det(g)^k y_j \widetilde{g} = \det(g)^{k+1} y_j {}^t g^{-1}\ .$$

KSTS/RR-93/001
February 17, 1993

If we put $\alpha = (a_1, \cdots, a_n) \in \mathbb{C}^n$, $\det^k \otimes \mathrm{alt}^{n-1}(g)$ acts on $\sum_{j=1}^n a_j t_j = y_1 {}^t\alpha \in \mathrm{alt}^{n-1}(V_1)$ and $y_2 {}^t\alpha \in \mathrm{alt}^{n-1}(V_2)$ by

$$\det{}^k \otimes \mathrm{alt}^{n-1}(g)(y_j {}^t\alpha) := \det(g)^k y_j \widetilde{g}^t \alpha = \det(g)^{k+1} y_j {}^t g^{-1} \, {}^t\alpha \ (j = 1, 2) \, .$$

Thus we get the action of $\det^k \otimes \mathrm{alt}^{n-1}$ on $\mathrm{alt}^{n-1}(V_j)$ $(j = 1, 2)$.

Let ι be the isomorphism from V_1 to V_2 defined by $\iota(e_j) = e_{n+j}$ $(1 \leq j \leq n)$. It induces the isomorphism (also denoted by ι) from $\mathrm{alt}^{n-1}(V_1)$ to $\mathrm{alt}^{n-1}(V_2)$. For a $\mathrm{alt}^{n-1}(V_1)$-valued function f on $\mathfrak{H}_n$ and for $Z \in \mathfrak{H}_n$, we define $\iota(f)$ by

$$(\iota(f))(Z) := \iota(f(Z)) \, .$$

For a function f on $\mathfrak{H}_{2n}$, $\begin{pmatrix} Z^{(n)} & U^{(n)} \\ {}^tU^{(n)} & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$, we define the pullback d^* by

$$(d^* f)\left(\begin{pmatrix} Z & U \\ {}^tU & W \end{pmatrix}\right) := f\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}\right) \, .$$

We consider $\Gamma^n \times \Gamma^n$ imbedded in Γ^{2n} by

$$\begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \times \begin{pmatrix} A'^{(n)} & B'^{(n)} \\ C'^{(n)} & D'^{(n)} \end{pmatrix} \longmapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ C & 0 & D & 0 \\ 0 & C' & 0 & D' \end{pmatrix} ,$$

and when convenient will identify $\Gamma^n \times \Gamma^n$ with its image in Γ^{2n}.

We summarize some facts on differential operators obtained from invariant pluri-harmonic polynomials in Ibukiyama [12] .

Let ρ_0 (resp. $\rho_0{}'$) be an irreducible rational representation of $GL(n, \mathbb{C})$ with the representation space $\mathcal{V}$ (resp. $\mathcal{V}'$) , where ρ_0 is equivalent to $\rho_0{}'$. For $n, k \in \mathbb{Z}$, $n, k > 0$, let $X = (x_{j\nu})$ be a variable on $\mathbb{C}^{(n,2k)}$. We put

$$\Delta_{jh} := \sum_{\nu=1}^{2k} \frac{\partial^2}{\partial x_{j\nu} \partial x_{h\nu}} \, .$$

A polynomial $P(X)$ on $\mathbb{C}^{(n,2k)}$ is called pluri-harmonic if $\Delta_{jh} P = 0$ for each j, h with $1 \leq j \leq h \leq n$.

From now on, we assume that $k \geq n$.

Suppose that a polynomial map

$$P : \mathbb{C}^{(n,2k)} \times \mathbb{C}^{(n,2k)} \longrightarrow \mathcal{V} \otimes \mathcal{V}'$$

satisfies the following three conditions:

(2.1) $\qquad P(X_1, X_2)$ is pluri-harmonic for each X_j $(j = 1, 2)$,

KSTS/RR-93/001
February 17, 1993

(2.2) $$P(X_1 g, X_2 g) = P(X_1, X_2) \text{ for each } g \in O(2k)\ ,$$

(2.3) $$P(a_1 X_1, a_2 X_2) = (\rho_0(a_1) \otimes \rho_0'(a_2)) P(X_1, X_2) \text{ for each } a_j \in GL(n, \mathbb{C})\ (j = 1, 2)\ .$$

Then there exists a unique polynomial map Q on $\mathbb{C}^{(2n)}$ such that

$$P(X_1, X_2) = Q \begin{pmatrix} X_1{}^t X_1 & X_1{}^t X_2 \\ X_2{}^t X_1 & X_2{}^t X_2 \end{pmatrix}\ .$$

Let $\mathfrak{Z} = (z_{jh})$ be a variable on $\mathfrak{H}_{2n}$. We put

$$\frac{\partial}{\partial \mathfrak{Z}} := \left(\frac{1 + \delta_{jh}}{2} \frac{\partial}{\partial z_{jh}} \right)_{1 \le j,h \le 2n},$$

where, for $z_{jh} = x_{jh} + i y_{jh}$,

$$\frac{\partial}{\partial z_{jh}} = \frac{1}{2} \left(\frac{\partial}{\partial x_{jh}} - i \frac{\partial}{\partial y_{jh}} \right)\ , \quad \frac{\partial}{\partial \bar{z}_{jh}} = \frac{1}{2} \left(\frac{\partial}{\partial x_{jh}} + i \frac{\partial}{\partial y_{jh}} \right)\ .$$

If we put

$$\mathbb{D} := d^* Q \left(\frac{\partial}{\partial \mathfrak{Z}} \right)\ ,$$

we have the following by Ibukiyama [12]:

Theorem 4. (i) *Let F be any $\mathbb{C}$-valued C^∞-function on $\mathfrak{H}_{2n}$. If we put $\rho = \det^k \otimes \rho_0$ and $\rho' = \det^k \otimes \rho_0'$, then for each $(g, g') \in \Gamma^n \times \Gamma^n$ and $\mathfrak{Z} = \begin{pmatrix} Z^{(n)} & U^{(n)} \\ {}^t U^{(n)} & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$, we get the following commutation relation:*

$$\left((\mathbb{D}F)\big|_\rho (g)_Z \big|_{\rho'} (g')_W \right)(\mathfrak{Z}) = \left(\mathbb{D} \left(F\big|_k (g, g') \right) \right)(\mathfrak{Z})\ ,$$

where $(\ )_Z$ (resp. $(\ )_W$) denotes the action on Z (resp. W).

(ii) *The operator $\mathbb{D}$ sends modular forms to modular forms:*

$$\mathbb{D} : M_k^{2n\,\infty} \longrightarrow M_\rho^n(\mathcal{V})^\infty \otimes M_{\rho'}^n(\mathcal{V}')^\infty\ .$$

Moreover, $\mathbb{D}$ is a holomorphic operator and it satisfies

$$\mathbb{D} : M_k^{2n} \longrightarrow M_\rho^n(\mathcal{V}) \otimes M_{\rho'}^n(\mathcal{V}')\ .$$

Now we apply it to $\det^k \otimes \mathrm{alt}^{n-1}$ cases.

Let $\rho_0 = \mathrm{alt}^{n-1}$ (resp. $\rho_0' = \mathrm{alt}^{n-1}$) be the representation of $GL(n, \mathbb{C})$ with the representation space $\mathrm{alt}^{n-1}(V_1)$ (resp. $\mathrm{alt}^{n-1}(V_2)$). For a variable $\mathfrak{Z} = (z_{jh})$ on $\mathfrak{H}_{2n}$, we put $u_{jh} := z_{j\ n+h}$ $(1 \le j, h \le n)$, $U^{(n)} := (u_{jh})$ and $\frac{\partial}{\partial U} := \left(\frac{\partial}{\partial u_{jh}} \right)_{1 \le j,h \le n}$. For functions on $\mathfrak{H}_{2n}$, we define the differential operator $\mathcal{D}$ by

$$\mathcal{D} := d^* \left(y_1 \widetilde{\frac{\partial}{\partial U}}{}^t y_2 \right)\ .$$

Then we have:

KSTS/RR-93/001
February 17, 1993

Proposition 1. *Let $k \geq n$.*

(i) *Let F be any $\mathbb{C}-$valued $C^\infty-$function on $\mathfrak{H}_{2n}$. Then for each $(g,g') \in \Gamma^n \times \Gamma^n$ and $\mathfrak{Z} = \begin{pmatrix} Z & U \\ {}^tU & W \end{pmatrix} \in \mathfrak{H}_{2n}$, we get the following commutation relation:*

$$\Big((\mathcal{D}F)\big|_\rho(g)_Z\big|_{\rho'}(g')_W\Big)(\mathfrak{Z}) = \Big(\mathcal{D}\left(F\big|_k(g,g')\right)\Big)(\mathfrak{Z}) \ .$$

(ii) *The operator $\mathcal{D}$ sends modular forms to modular forms:*

$$\mathcal{D} : M_k^{2n^\infty} \longrightarrow M_k^n(\mathrm{alt}^{n-1}(V_1))^\infty \otimes M_k^n(\mathrm{alt}^{n-1}(V_2))^\infty \ .$$

Moreover, $\mathcal{D}$ is a holomorphic operator and it satisfies

$$\mathcal{D} : M_k^{2n} \longrightarrow M_k^n(\mathrm{alt}^{n-1}(V_1)) \otimes M_k^n(\mathrm{alt}^{n-1}(V_2)) \ .$$

Proof. Let X_j $(j=1,2)$ be a variable on $\mathbb{C}^{(n,2k)}$. If we put

$$\frac{\partial}{\partial U} = X_1{}^tX_2 \ ,$$

the polynomial $y_1\widetilde{X_1{}^tX_2}{}^ty_2$ satisfies the three conditions (2.1), (2.2), (2.3). Therefore we get Proposition 1 by Theorem 4. □

§3 Proof of Theorem 1

We prove Theorem 1 according to Bčherer's method in [5].

We first apply the differential operator $\mathcal{D}$ to the Eisenstein series $E_k^{2n}(\mathfrak{Z},s)$. For this, we use the coset decomposition by Garrett [9] (cf. Mizumoto [19]):

Lemma 1. (i) *The double coset $P_{2n,0}\backslash\Gamma^{2n}/\Gamma^n \times \Gamma^n$ has an irredundant set of coset representatives*

$$g_{\widehat{T}} = \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 \\ 0 & \widehat{T}^{(n)} & 1_n & 0 \\ \widehat{T}^{(n)} & 0 & 0 & 1_n \end{pmatrix} ,$$

where $\widehat{T} = \begin{pmatrix} 0 & 0 \\ 0 & T^{(r)} \end{pmatrix}$, $T \in \mathbb{T}^{(r)}$ $(0 \leq r \leq n)$.

(ii) *The left coset $P_{2n,0}\backslash P_{2n,0}\ g_{\widehat{T}}(\Gamma^n \times \Gamma^n)$ has an irredundant set of coset representatives $g_{\widehat{T}}\widehat{g}_1 g_2 \widehat{g}'_1 g'_2$,*

$$\widehat{g}_1 \in G_{n,r} \ , \ g_2 \in P_{n,r}\backslash\Gamma^n \ , \ \widehat{g}'_1 \in \Gamma^r(T)\backslash G_{n,r} \ , \ g'_2 \in P_{n,r}\backslash\Gamma^n \ ,$$

where

$$G_{n,r} := \left\{ \begin{pmatrix} \widehat{A}^{(n)} & \widehat{B}^{(n)} \\ \widehat{C}^{(n)} & \widehat{D}^{(n)} \end{pmatrix} = \begin{pmatrix} 1_{n-r} & 0 & 0 & 0 \\ 0 & A^{(r)} & 0 & B^{(r)} \\ 0 & 0 & 1_{n-r} & 0 \\ 0 & C^{(r)} & 0 & D^{(r)} \end{pmatrix} \in \Gamma^n \ \middle|\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^r \right\}$$

and for $T \in \mathbb{T}^{(r)}$,

$$\Gamma^r(T) := \left\{ g \in \Gamma^r \ \middle|\ \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} g \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} \in \Gamma^r \right\} .$$

Now we prove the following (cf. Böcherer [4, Satz 9], [5, Satz 3]):

KSTS/RR-93/001
February 17, 1993

Proposition 2. *Let k be an even integer, n an odd integer and s a complex number such that $k+2\mathrm{Re}(s)>2n+1$. Suppose $k>n>2$. For $\mathfrak{Z}=\begin{pmatrix} Z^{(n)} & U^{(n)} \\ {}^tU^{(n)} & W^{(n)} \end{pmatrix}\in\mathfrak{H}_{2n}$, $\mathfrak{Z}_0=\begin{pmatrix} Z^{(n)} & 0 \\ 0 & W^{(n)} \end{pmatrix}\in\mathfrak{H}_{2n}$, we get*

$$
\begin{aligned}
&(\mathcal{D}E_k^{2n})(\mathfrak{Z},s)\\
&=\frac{\Gamma(2k+2s+1)}{\Gamma(2k+2s-n+2)}\sum_{T\in\mathbb{T}^{(n)}}\left(\mathcal{P}(Z,W,s)\Big|\left(\Gamma^n\begin{pmatrix} T & 0 \\ 0 & T^{-1}\end{pmatrix}\Gamma^n\right)_W\right)\det(T)^{-k-2s}\\
&+\frac{\Gamma(2k+2s+1)}{\Gamma(2k+2s-n+2)}\mathcal{R}(Z,W,s)\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathcal{P}(Z,W,s)\\
&:=\sum_{g\in\Gamma^n}\left\{\det(\mathrm{Im}(Z))^s\det(\mathrm{Im}(W))^s\left|\det(W+Z)\right|^{-2s}\rho\left((W+Z)^{-1}\right)\left(y_1\ {}^ty_2\right)\right\}\Big|(g)_Z\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{R}(Z,W,s):=&\sum_{T\in\mathbb{T}^{(n-1)}}\ \sum_{g_2\in P_{n,n-1}\backslash\Gamma^n}\ \sum_{g_2'\in P_{n,n-1}\backslash\Gamma^n}\ \sum_{\widehat{g}_1\in G_{n,n-1}}\ \sum_{\widehat{g}_1'\in\Gamma^{n-1}(T)\backslash G_{n,n-1}}\\
&\cdot\left\{\det(\mathrm{Im}(Z))^s\det(\mathrm{Im}(W))^s\left|\det(1_n-\widehat{T}W\widehat{T}Z)\right|^{-2s}\right.\\
&\left.\cdot\rho\left((1_n-\widehat{T}W\widehat{T}Z)^{-1}\right)\left(y_1\widehat{\widehat{T}}\ {}^ty_2\right)\right\}\Big|(\widehat{g}_1')_W\Big|(\widehat{g}_1)_Z\Big|(g_2')_W\Big|(g_2)_Z\ .
\end{aligned}
$$

Proof. It follows from Proposition 1 and Lemma 1 that

$$
\begin{aligned}
(\mathcal{D}E_k^{2n})(\mathfrak{Z},s)=&\sum_{r=0}^{n}\sum_{T\in\mathbb{T}^{(r)}}\ \sum_{g_2\in P_{n,r}\backslash\Gamma^n}\ \sum_{g_2'\in P_{n,r}\backslash\Gamma^n}\ \sum_{\widehat{g}_1\in G_{n,r}}\ \sum_{\widehat{g}_1'\in\Gamma^r(T)\backslash G_{n,r}}\\
&\left\{\mathcal{D}\left(\det(\mathrm{Im}\mathfrak{Z})^s\Big|_k g_{\widehat{T}}\right)\right\}\Big|(\widehat{g}_1')_W\Big|(\widehat{g}_1)_Z\Big|(g_2')_W\Big|(g_2)_Z\ .
\end{aligned}
$$

If for each $\widehat{T}$ we put $g_{\widehat{T}}=\begin{pmatrix} * & * \\ \mathfrak{C}^{(2n)} & \mathfrak{D}^{(2n)}\end{pmatrix}$, we get

$$
\mathcal{D}\left(\det(\mathrm{Im}\mathfrak{Z})^s\Big|_k g_{\widehat{T}}\right)=\det\left(\mathfrak{C}\overline{\mathfrak{Z}_0}+\mathfrak{D}\right)^{-s}\mathcal{D}\left(\det\left(\mathfrak{C}\mathfrak{Z}+\mathfrak{D}\right)^{-k-s}\det\left(\mathrm{Im}(\mathfrak{Z})\right)^s\right)\ ,
$$

by the form of $\mathcal{D}$ and that of $\det(\mathrm{Im}(\mathfrak{Z}))$,

$$
=\det\left(\mathfrak{C}\overline{\mathfrak{Z}_0}+\mathfrak{D}\right)^{-s}\det\left(\mathrm{Im}(\mathfrak{Z}_0)\right)^s\mathcal{D}\left(\det\left(\mathfrak{C}\mathfrak{Z}+\mathfrak{D}\right)^{-k-s}\right)\ .
$$

KSTS/RR-93/001
February 17, 1993

As an example, we compute

$$d^*\widetilde{\frac{\partial}{\partial u_{nn}}}\left(\det(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})^{-k-s}\right) .$$

Let $\mathfrak{S}_m$ be the symmetric group of degree m. We put

$$\delta := \det(\mathfrak{C}\mathfrak{Z}+\mathfrak{D}) \ , \ \delta_0 := \det(\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D}) \ , \ \partial_{jh} := \frac{\partial}{\partial u_{jh}} \quad (1 \le j, h \le n)$$

and , for $m,\ q \in \mathbb{Z}$, $0 < m$ and $0 \le q < m$,

$$L_m^q := \left\{ (l_1, \cdots, l_m) \in \mathbb{Z}^m \;\middle|\; l_\nu \ge 0 \ (1 \le \nu \le m) \ , \ \sum_{\nu=1}^{m} l_\nu = m-q, \ \sum_{\nu=1}^{m} \nu l_\nu = m \right\} .$$

For $(l_1, \cdots, l_m) \in L_m^q$, let $\Lambda(l_1, \cdots, l_m)$ be the set consisting of $J \in \mathfrak{S}_m$ such that , if $l_\gamma \ne 0 \ (1 \le \gamma \le m)$,

$$1 \le J\left(\sum_{\nu=0}^{\gamma-1} \nu l_\nu + \gamma\lambda + 1\right) < \cdots < J\left(\sum_{\nu=0}^{\gamma-1} \nu l_\nu + \gamma\lambda + \gamma\right) \le m \quad (0 \le \lambda < l_\gamma)$$

and

$$1 \le J\left(\sum_{\nu=0}^{\gamma-1} \nu l_\nu + 1\right) < J\left(\sum_{\nu=0}^{\gamma-1} \nu l_\nu + \gamma + 1\right) < \cdots < J\left(\sum_{\nu=0}^{\gamma-1} \nu l_\nu + \gamma(l_\gamma - 1) + 1\right) \le m .$$

Then we get

$$\begin{aligned} d^*\widetilde{\partial_{nn}}\left(\delta^{-k-s}\right) &= d^*\left(\sum_{\tau \in \mathfrak{S}_{n-1}} \operatorname{sgn}(\tau)\partial_{1\tau(1)} \cdots \partial_{n-1\tau(n-1)}\right)\left(\delta^{-k-s}\right) \\ &= \sum_{q=0}^{n-2}\left\{\left(\prod_{\mu=0}^{n-2-q}(-k-s-\mu)\right)\delta_0^{-k-s-(n-1-q)}\right. \\ &\quad \left. \times\, d^* \sum_{\tau \in \mathfrak{S}_{n-1}} \ \sum_{(l_1, \cdots, l_{n-1}) \in L_{n-1}^q} \ \sum_{J \in \Lambda} \operatorname{sgn}(\tau)\ \partial_\tau^J(q; (l_1, \cdots, l_{n-1}))(\delta)\right\} , \end{aligned}$$

where $\Lambda = \Lambda(l_1, \cdots, l_{n-1})$ and

$$\begin{aligned} \partial_\tau^J(q; (l_1, \cdots, l_{n-1}))(\delta) = \prod_{\gamma=1}^{n-1}\Big\{ &\left(\partial_{\tau(J(\alpha^\gamma+1))} \cdots \partial_{\tau(J(\alpha^\gamma+\gamma))}\right)(\delta) \\ &\times \cdots \\ &\times \left(\partial_{\tau(J(\alpha^\gamma+\gamma(l_\gamma-1)+1))} \cdots \partial_{\tau(J(\alpha^\gamma+1))}\right)(\delta)\Big\} \end{aligned}$$

with $\alpha^\gamma := \sum_{\nu=0}^{\gamma-1} \nu l_\nu$, $\partial_{\tau(J(\cdot))} := \partial_{J(\cdot)\ \tau(J(\cdot))}$.

For each q $(0 \le q \le n-2)$, $(l_1, \cdots, l_{n-1}) \in L_{n-1}^q$, $\tau \in \mathfrak{S}_{n-1}$ and $J \in \Lambda$, we define

$$\left(\partial_{J(j)\ \tau(J(h))}\right) := \begin{pmatrix} \left(\left(A_\tau^J\right)_{\xi\eta}\right)^{(n-1-q)} & * \\ * & \partial_{nn} \end{pmatrix} ,$$

where , for $\sum_{\nu=1}^{\gamma-1} l_\nu + 1 \le \xi \le \sum_{\nu=1}^{\gamma} l_\nu$ and $\sum_{\nu=1}^{\gamma'-1} l_\nu + 1 \le \eta \le \sum_{\nu=1}^{\gamma'} l_\nu$, $\left(A_\tau^J\right)_{\xi\eta}$ is a $\gamma \times \gamma'$ matrix.

In the same way, we define

$$\left(b_{J(j)\ \tau(J(h))}\right) := \begin{pmatrix} \left(\left(B_\tau^J\right)_{\xi\eta}\right)^{(n-1-q)} & * \\ * & b_{nn} \end{pmatrix} ,$$

where $(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}\mathfrak{C} = \begin{pmatrix} * & \mathcal{B}^{(n)} \\ * & * \end{pmatrix}$ and $\mathcal{B} = (b_{jh})$.

Then we have

$$\begin{aligned} & d^* \sum_{\tau \in \mathfrak{S}_{n-1}} \operatorname{sgn}(\tau)\ \partial_\tau^J(q; (l_1, \cdots, l_{n-1}))(\delta) \\ &= \sum_{\sigma \in \mathfrak{S}_{n-1} / \prod_{\gamma=1}^{n-1} \mathfrak{S}_\gamma^{l_\gamma}} \left\{ \operatorname{sgn}(\sigma) \prod_{\xi=1}^{n-1-q} d^* \det\left(\left(A_\sigma^J\right)_{\xi\xi}\right)(\delta) \right\} , \end{aligned}$$

by $\quad d^* \det\left(\left(A_\sigma^J\right)_{\xi\xi}\right)(\delta) = (\gamma+1)!\ \delta_0\ \det\left(\left(B_\sigma^J\right)_{\xi\xi}\right) \quad \left(\sum_{\nu=1}^{\gamma-1} l_\nu + 1 \le \xi \le \sum_{\nu=1}^{\gamma} l_\nu\right)$,

$$\begin{aligned} &= \delta_0^{n-1-q} \prod_{\gamma=1}^{n-1} \{(\gamma+1)!\}^{l_\gamma} \sum_{\sigma \in \mathfrak{S}_{n-1} / \prod_{\gamma=1}^{n-1} \mathfrak{S}_\gamma^{l_\gamma}} \left\{ \operatorname{sgn}(\sigma) \prod_{\xi=1}^{n-1-q} \det\left(\left(B_\sigma^J\right)_{\xi\xi}\right) \right\} \\ &= \delta_0^{n-1-q}\ \widetilde{b_{nn}} \prod_{\gamma=1}^{n-1} \{(\gamma+1)!\}^{l_\gamma} . \end{aligned}$$

Since the number of elements of Λ is $\left(\prod_{\gamma=1}^{n-1} \frac{1}{l_\gamma!}\right) \frac{(n-1)!}{(1!)^{l_1} \cdots ((n-1)!)^{l_{n-1}}}$, we obtain

$$d^* \widetilde{\partial_{nn}}\left(\delta^{-k-s}\right) = (-1)^{n-1} \sum_{q=0}^{n-2} \left\{ a_{n-1}(q) \prod_{\mu=0}^{n-2-q} (2s+2k+2\mu) \right\} \delta_0^{-k-s} \widetilde{b_{nn}} , \tag{3.1}$$

where

$$a_m(q) = (-1)^q 2^{-(m-q)} m! \sum_{(l_1, \cdots, l_m) \in L_m^q} \left(\prod_{\gamma=1}^{m} \frac{(\gamma+1)^{l_\gamma}}{l_\gamma!} \right) \quad (0 < m,\ 0 \le q < m) .$$

KSTS/RR-93/001
February 17, 1993

In the same way, we have

$$\mathcal{D}\left(\det(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})^{-k-s}\right)$$
$$=(-1)^{n-1}\sum_{q=0}^{n-2}\left\{a_{n-1}(q)\prod_{\mu=0}^{n-2-q}(2s+2k+2\mu)\right\}\det(\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-k-s}\left(y_1\widetilde{\mathcal{B}}\,{}^t y_2\right)$$

and

$$\mathcal{D}\Big(\det(\mathrm{Im}\mathfrak{Z})^s\Big|_k g_{\widehat{T}}\Big)=(-1)^{n-1}\sum_{q=0}^{n-2}\left\{a_{n-1}(q)\prod_{\mu=0}^{n-2-q}(2s+2k+2\mu)\right\}$$
$$\times\det(\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-k}\det(\mathrm{Im}(g_{\widehat{T}}\langle\mathfrak{Z}_0\rangle))^s\left(y_1\widetilde{\mathcal{B}}\,{}^t y_2\right)\ .$$

On the other hand, we obtain

$$\det(\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-k}\det(\mathrm{Im}(g_{\widehat{T}}\langle\mathfrak{Z}_0\rangle))^s\left(y_1\widetilde{\mathcal{B}}\,{}^t y_2\right)$$
$$=\det(\mathrm{Im}(Z))^s\det(\mathrm{Im}(W))^s\left|\det(1_n-\widehat{T}W\widehat{T}Z)\right|^{-2s}\rho\left((1_n-\widehat{T}W\widehat{T}Z)^{-1}\right)\left(y_1\widetilde{\widehat{T}}\,{}^t y_2\right)\ .$$

Therefore we have only to prove

$$\sum_{q=0}^{n-2}\left\{a_{n-1}(q)\prod_{\mu=0}^{n-2-q}(2s+2k+2\mu)\right\}=\prod_{\mu=0}^{n-2}(2s+2k-\mu)\quad .$$

To prove the formula above, we put $x=2s+2k$ and $m=n-1$, that is, we have to prove

$$\sum_{q=0}^{m-1}\left\{a_m(q)\prod_{\mu=0}^{m-1-q}(x+2\mu)\right\}=\prod_{\mu=0}^{m-1}(x-\mu)\quad . \tag{3.2}$$

We put $a_m(q)=0$ if $q\geq m$, $0>q$ or $0\geq m$.

We use induction on m.

If $m=1$, the assertion is trivial.

We suppose

$$\sum_{q=0}^{m'-1}\left\{a_{m'}(q)\prod_{\mu=0}^{m'-1-q}(x+2\mu)\right\}=\prod_{\mu=0}^{m'-1}(x-\mu)$$

for any $m'<m$. Then we have

$$\prod_{\mu=0}^{m-1}(x-\mu)=\left\{\prod_{\mu=0}^{m-2}(x-\mu)\right\}(x-m+1)$$

$$= \left\{ \sum_{q=0}^{m-2} \left\{ a_{m-1}(q) \prod_{\mu=0}^{m-2-q} (x+2\mu) \right\} \right\} (x + (2m-2-2q) - (3m-3-2q))$$
$$= \sum_{q=0}^{m-2} \left\{ a_{m-1}(q) \prod_{\mu=0}^{m-1-q} (x+2\mu) \right\}$$
$$- \sum_{q=1}^{m-1} \left\{ (3m-2q-1) a_{m-1}(q-1) \prod_{\mu=0}^{m-1-q} (x+2\mu) \right\}$$
$$= \sum_{q=0}^{m-1} \left\{ \Big(a_{m-1}(q) - (3m-2q-1) a_{m-1}(q-1) \Big) \prod_{\mu=0}^{m-1-q} (x+2\mu) \right\} \quad .$$

If we note $3l_1 + \cdots + (m+1)l_{m-1} = 3m - 2q - 1$ in L_{m-1}^{q-1} , we have

$$a_{m-1}(q) - (3m-2q-1)a_{m-1}(q-1)$$
$$= \frac{1}{m}(-1)^q 2^{-(m-q)} m! \sum_{L_{m-1}^q} \left\{ (l_1+1) \frac{2^{l_1+1}}{(l_1+1)!} \left(\prod_{\gamma=2}^{m-1} \frac{(\gamma+1)^{l_\gamma}}{l_\gamma !} \right) \right\}$$
$$+ \frac{1}{m}(-1)^q 2^{-(m-q)} m!$$
$$\times \sum_{L_{m-1}^{q-1}} \left\{ \left(\prod_{\gamma=1}^{m-1} \frac{(\gamma+1)^{l_\gamma}}{l_\gamma !} \right) \left(\sum_{\gamma=1}^{m-1} (\gamma+1)(l_{\gamma+1}+1) \frac{\gamma+2}{l_{\gamma+1}+1} \frac{l_\gamma}{\gamma+1} \right) \right\}$$
$$= (-1)^q 2^{-(m-q)} m! \sum_{L_m^q} \left\{ \left(\prod_{\gamma=1}^{m} \frac{(\gamma+1)^{l_\gamma}}{l_\gamma !} \right) \frac{1}{m} \sum_{\gamma=1}^{m} \gamma l_\gamma \right\}$$
$$= a_m(q) \quad .$$

Thus we get (3.2). □

Remark. Under the notation above, we note that the formula

$$d^* \widetilde{\partial_{jh}} \left(\delta^{-k-s} \right) = (-1)^{n-1} \prod_{\mu=0}^{n-2} (2s+2k-\mu) \delta_0^{-k-s} \widetilde{b_{jh}}$$

which is obtained from (3.1) and (3.2), and the formula

$$d^* \left(\det \left(\frac{\partial}{\partial U} \right) \right) \left(\delta^{-k-s} \right) = (-1)^n \prod_{\mu=0}^{n-1} (2s+2k-\mu) \delta_0^{-k-s-1} \det \left(\widehat{T} \right)$$

in [4, Satz 9] , [5, Satz 3] have the same meaning.

KSTS/RR-93/001
February 17, 1993

For $\sum_{j=1}^{n} a_j t_{n+j}$, $\sum_{j=1}^{n} b_j t_{n+j} \in \mathrm{alt}^{n-1}(V_2)$, we define the inner product of them by

$$\left\langle \sum_{j=1}^{n} a_j t_{n+j} \ , \ \sum_{j=1}^{n} b_j t_{n+j} \right\rangle := \sum_{j=1}^{n} a_j \bar{b}_j \ .$$

Suppose $f, g \in M_k^n(\mathrm{alt}^{n-1}(V_2))^{\infty}$. The Petersson inner product of f and g is defined by

$$(f,g) := \int_{\Gamma^n \backslash \mathfrak{H}_n} \left\langle \rho'\left(\sqrt{\mathrm{Im}(W)}\right) f(W) \ , \ \rho'\left(\sqrt{\mathrm{Im}(W)}\right) g(W) \right\rangle \det(\mathrm{Im}(W))^{-n-1} dX\, dY$$

if the right-hand side is convergent. Here $W = X + iY$ with real matrices $X = (x_{jh})$ and $Y = (y_{jh})$;

$$dX := \prod_{j \le h} dx_{jh} \ , \ dY := \prod_{j \le h} dy_{jh} \ ;$$

the integral is taken over a fundamental domain of $\Gamma^n \backslash \mathfrak{H}_n$. We write $dW = dX dY$ when there is no fear of confusion.

Theorem 5. *Let k be an even integer, n an odd integer and $k > n > 2$. If $f \in S_k^n(\mathrm{alt}^{n-1}(V_2))$ is an eigenform,*

$$\left(f \ , \ (\mathcal{D}\mathbb{E}_k^{2n}) \left(\begin{pmatrix} -\overline{Z}^{(n)} & 0 \\ 0 & * \end{pmatrix}, \frac{\bar{s}+n}{2} \right) \right)$$

$$= \pi^{nk - \frac{1}{2}(n-1)^2} i^{nk+n-1} \gamma(s) \Lambda(s, f, \underline{\mathrm{St}}) (\iota^{-1}(f))(Z) \ .$$

If Theorem 5 is proved, the functional equation of $\Lambda(s, f, \underline{\mathrm{St}})$ is obtained from that of $\mathbb{E}_k^{2n}(\mathfrak{Z}, s)$. Since it follows from Theorem 3 that the location of poles of $E_k^{2n}(\mathfrak{Z}, s)$ is invariant under the operation of $\mathcal{D}$, its holomorphy is proved in the same way as that by Mizumoto [19, Theorem 1] (cf. Weissauer [24]). Thus we get Theorem 1.

Proof of Theorem 5.

It follows from Theorem 3 that $\left(f \ , \ (\mathcal{D}E_k^{2n}) \left(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}, \bar{s} \right) \right)$ converges absolutely and locally uniformly for $k + 2\mathrm{Re}(s) > 2n + 1$. We note that $\mathcal{R}(Z, W, s)$ is orthogonal to $S_k^n(\mathrm{alt}^{n-1}(V_2))$ in the variable W by the same reason as that in Klingen [15, Satz 2]. Since the Hecke operator is a Hermitian operator and f is an eigenform , we have

$$\left(f \ , \ (\mathcal{D}E_k^{2n}) \left(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}, \bar{s} \right) \right)$$
$$= \frac{\Gamma(2k+2s+1)}{\Gamma(2k+2s-n+2)} D(k+2s, f) \left(f \ , \ \mathcal{P}(-\overline{Z}, *, \bar{s}) \right)$$

by the definition (1.2) . If we compute the integral $\left(f\ ,\ \mathcal{P}(-\overline{Z},*,\bar{s})\right)$ according to Klingen [14, § 1] (see also [5], [7], [23]) , we obtain

$$\left(f\ ,\ \mathcal{P}(-\overline{Z},*,\bar{s})\right) = 2^{n(n-2s-k)+2}\, i^{nk+n-1}\ \psi\ \left(\iota^{-1}(f)\right)(Z)$$

and

$$\psi = \int_{S^n} \det(1_n - S\overline{S})^{k+s-n-1} \left(\widetilde{(1_n - S\overline{S})}\, [{}^t p_n]\right) dS \quad ,$$

where $p_n^{(1,n)} := (0,\cdots,0,1)$ and $S^n := \left\{S \in \mathbb{C}^{(n)} \mid S = {}^tS\ ,\ 1_n - \overline{S}S > 0\ \right\}$. Moreover, by Hua [10, § 2.3] (see also [5], [7], [14], [23]) , we get

$$\psi = \pi^{\frac{n(n+1)}{2}} \left(\frac{2k+2s-n+1}{2}\right) \frac{\Gamma(k+s-n)}{\Gamma(k+s+1)} \prod_{j=1}^{n-1} \frac{\Gamma(2k+2s-2n+1+2j)}{\Gamma(2k+2s-n+1+j)} \quad .$$

Thus, by (1.3) , we obtain

$$\begin{aligned}
&\left(f\ ,\ (\mathcal{D}E_k^{2n})\left(\begin{pmatrix}-\overline{Z} & 0\\ 0 & *\end{pmatrix}, \frac{\bar{s}+n-k}{2}\right)\right)\\
&= 2^{n(1-s)+2} i^{nk+n-1} \pi^{\frac{n(n+1)}{2}} \zeta(s+n)^{-1} \prod_{j=1}^{n} \zeta(2s+2n-2j)^{-1}\\
&\times \frac{\Gamma(s+k)}{\Gamma(s+k-n+1)} \prod_{j=1}^{n} \frac{\Gamma(s+k-n-2+2j)}{\Gamma(s+k-1+j)} L(s,f,\underline{\mathrm{St}})(\iota^{-1}(f))(Z)
\end{aligned}$$

and Theorem 5 is proved. □

References

1. Andrianov, A. N., *The multiplicative arithmetic of Siegel modular forms*, Russian Math. Surveys **34** (1979), 75–148; English translation.
2. Andrianov, A. N., Kalinin, V. L., *On the analytic properties of standard zeta function of Siegel modular forms*, Math. USSR-Sb. **35** (1979), 1–17; English translation.
3. Böcherer, S., *Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen*, Math. Z. **183** (1983), 21–46.
4. Böcherer, S., *Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen II*, Math. Z. **189** (1985), 81–110.
5. Böcherer, S., *Über die Funktionalgleichung automorpher L-functionen zur Siegelschen Modulgruppe*, J. Reine Angew. Math. **362** (1985), 146–168.
6. Böcherer, S., *Ein Rationalitätssatz für formale Heckereihen zur Siegelschen Modulgruppe*, Abh. Math. Sem. Univ. Hamburg **56** (1986), 35–47.
7. Böcherer, S., Satoh, T., Yamazaki, T., *On the pullback of a Differential Operator and its Application to Vector Valued Eisenstein Series*, Commentarii Math. Univ. St. Pauli. **42** (1992), 1–22.
8. Feit, P., *Poles and residues of Eisenstein series for symplectic and unitary groups*, Mem. Amer. Math. Soc. **61** no. 346, Providence, Rhode Island, 1986.
9. Garrett, P. B., *Pullbacks of Eisenstein series ; applications*, Automorphic Forms of Several Variables, Progress in Math. **46**, 114–137, Birkhäuser, Boston-Basel-Stuttgart, 1984.

KSTS/RR-93/001
February 17, 1993

10. Hua, L. K., *Harmonic analysis of functions of several complex variables in the classical domains*, Trans. Amer. Math. Soc. **6**, Providence, Rhode Island, 1963.
11. Ibukiyama, T., *Invariant harmonic polynomials on polyspheres and some related differential equations*, preprint.
12. Ibukiyama, T., *On differential operators on automorphic forms and invariant pluri-harmonic polynomials*, preprint.
13. Kalinin, V. L., *Eisenstein series on the symplectic group*, Math. USSR-Sb. **32** (1977), 449–476; English translation.
14. Klingen, H., *Über Poincarésche Reihen zur Siegelschen Modulgruppe*, Math. Ann. **168** (1967), 157–170.
15. Klingen, H., *Zum Darstellungssatz für Siegelsche Modulformen*, Math. Z. **102** (1967), 30–43.
16. Langlands, R. P., *Problems in the theory of automorphic forms*, Lecture Notes in Math. **170**, 18–86, Springer, Berlin-Heidelberg-New York, 1970.
17. Langlands, R. P., *Euler products*, Yale Univ. Press, 1971.
18. Langlands, R. P., *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Math. **544**, Springer, Berlin-Heidelberg-New York, 1976.
19. Mizumoto, S., *Poles and residues of standard L-functions attached to Siegel modular forms*, Math. Ann. **289** (1991), 589–612.
20. Mizumoto, S., *Eisenstein series for Siegel modular groups*, preprint.
21. Piatetski-Shapiro, I., Rallis, S., *L-functions for the classical groups*, Lecture Notes in Math. **1254**, Springer, Berlin-Heidelberg-New York, 1987.
22. Shimura, G., *On Eisenstein series*, Duke Math. J. **50** (1983), 417–476.
23. Takayanagi, H., *Vector valued Siegel modular forms and their L-functions; Applications of a Differential operator*, Japan. J. Math. (to appear).
24. Weissauer, R., *Stabile Modulformen und Eisensteinreihen*, Lecture Notes in Math. **1219**, Springer, Berlin-Heidelberg-New York, 1986.

3-14-1 Hiyoshi Kohoku-ku Yokohama, 223, Japan