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The Mean Curvature of Gauge Orbits

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Abstract

We discuss the formal analogue of the trace of the second fundamental form for gauge orbits within the space of connections, endowed with its L^2 metric, for a principal G-bundle over a compact Riemannian manifold M. We compute the singularity of the formal analogue, and show that the formal analogue is finite for gauge orbits of flat connections. There is an alternative zeta function regularization for the trace, which is finite for Yang-Mills connections. This leads to an infinite dimensional version of Hsiang's theorem. For an $SU(2)$ bundle over a simply connected 4-manifold with positive definite intersection form, the gauge orbits of reducible selfdual connections are minimal submanifolds of the space of all selfdual connections in this regularized sense. Minimal

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gauge orbits of flat $SU(2)$ connections over certain Seifert fibered homology spheres are also shown to exist by this version of Hsiang's theorem.

1 Introduction

Let M be a Riemannian manifold with an isometric action of a compact Lie group G. It is natural to suspect that minimal submanifolds of M occur among the orbits of G. In fact, it is a theorem of W.-Y. Hsiang that if an orbit $\mathcal{O}$ is an extremal for volume among all nearby diffeomorphic orbits, then $\mathcal{O}$ is a minimal submanifold. In particular, orbits of isolated diffeomorphism type are automatically minimal submanifolds.

Gauge theory provides an infinite dimensional analogue of this setup. The manifold M is replaced by $\mathcal{A}$, the space of connections on a fixed principal G-bundle P over a compact Riemannian manifold, with its L^2 metric, and G is replaced by $\mathcal{G}$, the gauge group of the bundle. However, the gauge orbits have both infinite dimension and codimension, so it is problematic to define minimal submanifolds in this context.

In this paper, we attempt to rigorously define the formal analogue in gauge theory of $Tr\ II$, the trace of the second fundamental form, in order to define when a gauge orbit is minimal. For gauge orbits, the component of $Tr\ II$ in the direction of a vector N normal to the orbit is formally the trace of a linear operator H_N on $\Omega^0(Ad\ P)$, the space of zero-forms with values in the associated bundle $Ad\ P$. However, we cannot expect H_N to be trace class in general, as its definition involves Green's operators which are well known not to be trace class. Indeed, our main technical result (Theorem 3.7) is the calculation of the asymptotics of the kernel h_N of H_N near the diagonal. We show that the highest order pole of $h_N(x,x)$ has residue $tr(N(x) \wedge *d_A^*F_A(x))$. Here $N \in \Omega^1(Ad\ P)$ is a normal vector at the connection A to the orbit of A, F_A is the curvature of A, and d_A^* is the adjoint of the exterior derivative d_A associated to A. Since the variation of the Yang-Mills functional in the direction of N satisfies $\delta_N YM(A) = \int_M tr(N(x) \wedge *d_A^*F_A(x))\, dx$, we certainly will not be able to integrate the kernel over the diagonal if A is not Yang-Mills. However, we are able to show that $Tr\ II$ exists for gauge orbits of flat connections, in the sense that $tr\ h_N(x,x) < \infty$ (Theorem 3.19).

Since the direct approach to generalizing $Tr\ II$ does not work very well except at flat connections, we turn to a zeta function regularization of $Tr\ II$. This has the effect of replacing the many pointwise obstructions to the existence of $h_N(x,x)$ by one global obstruction to defining $Tr\ H_N$, namely the nonvanishing of $\int_M \tilde{W}$, where $\tilde{W}$ is a certain asymptotic coefficient for the kernel of a heat type operator associated to H_N. It turns out that this regularization is finite if the dimension of the base manifold is two or is odd, and in dimension four the regularization is finite iff the connection is Yang-Mills (Theorem 5.10).

Although the zeta function method does not always succeed, it seems to be the correct approach in that it is compatible with the Faddeev-Popov ghost determinant. More specifically, the Faddeev-Popov ghost determinant gives a regularization of the volume element of a gauge orbit, so we can define what it means for a gauge orbit to have extremal volume among nearby orbits in terms of the infinitesimal variation of the ghost determinant. This allows us to prove an infinite dimensional analogue of Hsiang's theorem (Theorem 5.14), stating that an irreducible gauge orbit with extremal volume among nearby orbits has $Tr\ H_N = 0$ for all normal vectors N, and hence is minimal in our regularized sense. Our proof involves neither the exponential map, used in Hsiang's finite dimensional proof, nor the first variation formula, which in finite dimensions relates the variation of the volume element with $Tr\ II$, as these techniques are unavailable in our case. Without these basic techniques, it is not immediate that Hsiang's theorem carries over to gauge theory. Applying these methods to the setup of Donaldson's theorem, we show that the orbits of reducible selfdual connections for an $SU(2)$ bundle over a simply connected 4-manifold with positive definite intersection form are minimal submanifolds within the space of all selfdual connections (Theorem 5.24). This is again compatible with Hsiang's theorem, as the reducible orbits are of isolated type. Moreover, it is a trivial corollary of our version of Hsiang's theorem that at least two minimal gauge orbits exist over any closed manifold component within a moduli space of irreducible connections. By work of Fintushel and Stern, this produces examples in dimension 3 of minimal gauge orbits of flat $SU(2)$ connections over certain Seifert homology spheres (Corollary 5.15).

The paper is organized as follows. In §2, the geometry of the space of connections is reviewed, and the second fundamental form operator H_N is defined. §3 states the main local theorem on the asymptotics of h_N, the kernel of H_N. The proof of this theorem is given in §4. The proof, which follows standard techniques, is unavoidably long; while it is easy to write down which curvature terms from the base manifold and the bundle connection may appear in the highest order pole of the pointwise obstruction, we must explicitly show that the terms from the base manifold vanish in the adjoint representation. In contrast, it is much simpler to check that no base curvature terms appear in the global obstruction. In any case, many of the details of the proof of Theorem 3.7 are collected in two appendices. The zeta function regularization of $Tr\ H_N$ is given in §5.

We would like to thank Tom Parker very much for pointing out an error in an earlier version of this paper and for providing us with a simpler proof of Proposition A.36.

It was a very pleasant surprise to learn from Dick Palais that Hsiang's theorem can be derived from the symmetric criticality principle. As such, we hope that this paper will be viewed as an outgrowth of Palais' important work on transformation groups.

2 Geometric Preliminaries

2.1 The Riemannian structure on $\mathcal{M}$

Let (M, g) be a closed Riemannian n-manifold and let P be a principal G-bundle over M, where G is a compact Lie group. Let $\mathcal{A}$ be the affine space of smooth connections on P, $\mathcal{G}$ the gauge group of automorphisms of P covering the identity, and $\Omega^k(Ad\ P)$ the space of k-forms with values in the vector bundle $Ad\ P = P \times_{Ad} \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Fix a bi-invariant metric on G; this metric together with g determines an inner product on $\Omega^k(Ad\ P)$. This inner product in turn induces the L^2 Riemannian metric on $\mathcal{A}$ from the canonical identification of $\Omega^1(Ad\ P)$ with $T_A\mathcal{A}$ for each $A \in \mathcal{A}$: given $\tau, \eta \in T_A\mathcal{A}$, we set

$$\langle \tau, \eta \rangle = \int_M \langle \tau(x), \eta(x) \rangle_x \, d_g x \tag{2.1}$$

where $\langle , \rangle_x$ is the pointwise inner product on $\Omega^1(Ad\ P)$. The L^2 inner product is invariant under the action of $\mathcal{G}$ on $\mathcal{A}$, so the L^2 Riemannian metric descends to a metric on the moduli space $\mathcal{A}/\mathcal{G}$, which has been studied in several papers [1], [2]; in particular, [2] is a good reference for §2.1, §2.2. In this paper, we are more interested in the geometry of $\mathcal{A}$, considered as a Riemannian manifold with an isometric action of the Lie group $\mathcal{G}$.

Let $d_A : \Omega^0(Ad\ P) \to \Omega^1(Ad\ P)$ be the covariant derivative associated to the connection A. Up to a Z_2 factor, the action of $\mathcal{G}$ is free on the open subset of irreducible non-flat connections, i.e. connections with $Ker\ d_A = 0$. Let $\mathcal{O}_A$ be the $\mathcal{G}$-orbit of A. A standard argument in elliptic theory gives:

Proposition 2.2 *(i) For $A \in \mathcal{A}$, we have the splitting*

$$T_A\mathcal{A} \simeq Ker\ d_A^* \oplus Im\ d_A \tag{2.3}$$

where d_A^* is the adjoint of d_A with respect to the inner product (2.1).
(ii) Let $\mathcal{O}_A^\perp$ be the normal space to $\mathcal{O}_A$ in $T_A\mathcal{A}$. Then $T_A\mathcal{O}_A \simeq Im\ d_A$ and $\mathcal{O}_A^\perp \simeq Ker\ d_A^*$.

2.2 The Riemannian connection on $\mathcal{A}$

Let $\mathcal{X}(\mathcal{A})$ be the set of smooth vector fields on $\mathcal{A}$. By Proposition 2.2, $X \in \mathcal{X}(\mathcal{A})$ has a decomposition

$$X(A) = X^T(A) + X^\perp(A) \tag{2.4}$$

with $X^T(A) \in T_A\mathcal{O}_A$ and $X^\perp(A) \in \mathcal{O}_A^\perp$.

Lemma 2.5 *For $X \in \mathcal{X}(\mathcal{A})$ and $A \in \mathcal{A}$, we have*

$$X^T(A) = d_A G_A d_A^* X(A) \quad (2.6)$$
$$X^\perp(A) = X(A) - d_A G_A d_A^* X(A) \quad (2.7)$$

where G_A is the Green's operator for $d_A^* d_A : \Omega^0(ad\ P) \to \Omega^0(Ad\ P)$.

PROOF. Note that G_A is well defined on $Im\ d_A^*$. It is immediate that $X(A) - d_A G_A d_A^* X(A) \in Ker\ d_A^*$. The Lemma then follows from the uniqueness of the decomposition (2.4).

We now define a connection D on $\mathcal{A}$ by

$$D_X Y(A) = \frac{d}{dt} Y(A + tX(A))|_{t=0} \quad (2.8)$$

for $X, Y \in \mathcal{X}(\mathcal{A})$, $A \in \mathcal{A}$. It is elementary to verify that D is the Levi-Civita connection for the L^2 metric–i.e. we have

$$\begin{aligned} 2\langle D_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ &\quad + \langle [X,Y], Z\rangle + \langle [Z,X], Y\rangle - \langle [Y,Z], X\rangle \\ D_X Y - D_Y X &= [X, Y] \end{aligned}$$

2.3 The second fundamental form on $\mathcal{O}_A$

Let M be a submanifold of a finite dimensional Riemannian manifold $\bar{M}$ with Levi-Civita connection $\bar{\nabla}$. The second fundamental form on M is $II : TM \otimes TM \to \nu$ (ν is the normal bundle of M in $\bar{M}$) given by $II(X,Y) = (\bar{\nabla}_X Y)^\perp$, where $\perp$ denotes projection into the normal bundle [3, Ch. I]. M is a minimal submanifold in $\bar{M}$ if $Tr\ II \equiv 0$. Let $\{e_i\}$ be an orthonormal frame of TM near a point p. Pick a normal vector N_p at p and extend it to a normal vector field, also denoted N, near p. Then $Tr\ II_p = 0$ iff for all choices of N_p

$$0 = \langle \sum_i (\bar{\nabla}_{e_i} e_i)^\perp, N\rangle_p = \langle \sum_i \bar{\nabla}_{e_i} e_i, N\rangle_p = \sum_i \langle e_i, \bar{\nabla}_{e_i} N\rangle \quad (2.9)$$

since $d\langle e_i, N\rangle = 0$. If Y^T denotes the M-tangential component of $Y \in T_p\bar{M}$, the last term in (2.9) equals $\sum \langle e_i, (\bar{\nabla}_{e_i} N)^T\rangle = Tr(X \mapsto (\bar{\nabla}_X N)^T)$. This shows that the last expression is independent of the extension N. In particular:

Lemma 2.10 *Let M be a submanifold of the finite dimensional Riemannian manifold $\bar{M}$. M is a minimal submanifold iff for every $p \in M$ and $N \in \nu_p$, $Tr(X \mapsto (\bar{\nabla}_X N)^T) = 0$ for $X \in T_pM$.*

We now carry this computation formally over to $\mathcal{O}_A \subset \mathcal{A}$. Take a normal vector field $\tilde{N}(\tilde{A})$ to $\mathcal{O}_A$ with $\tilde{N}(A) = N \in \nu_A\mathcal{O}_A$. As in (2.9), the trace of the second fundamental form will be independent of the extension $\tilde{N}$ of N, so we may put $\tilde{N}(\tilde{A}) = N - d_{\tilde{A}}G_{\tilde{A}}d^*_{\tilde{A}}N$. The tangential part of $D_X\tilde{N}$ at A, for $X \in T_A\mathcal{O}_A$, is given by

$$(D_X\tilde{N})^T(A) = [\frac{d}{dt}(N - d_{A(t)}G_{A(t)}d^*_{A(t)}N)|_{t=0}]^T \qquad (2.11)$$

where $A(t) = A + tX$. Putting $(D_X\tilde{N})^T(A) = H_N$, we have

$$H_N(X) = -d_AG_AP^*_XN \qquad (2.12)$$

where $P^*_XN = \frac{d}{dt}d^*_{A(t)}N|_{t=0}$ and we have used $d^*_AN = 0$.

Lemma 2.13 $P^*_XN = - * [X, *N] = -P^*_NX$.

PROOF. Let δ_X indicate a variation in the direction X. Then $P^*_XN = \delta_Xd^*_AN = \delta_X(- * d_A*)N$, where $*$ is the Hodge star operator on M extended to $\Omega^0(Ad\ P)$. This last expression locally equals

$$\delta_X(- * (d + [A, \cdot])*)N = - * \delta_X[A, *N] = - * \frac{d}{dt}|_{t=0}[A + tX, *N] = - * [X, *N].$$

Moreover,

$$\begin{aligned} [X, *N] &= X \wedge *N - *N \wedge X = (-1)^{n-1}(* * X \wedge *N - *N \wedge * * X) \\ &= (-1)^{n-1}(*X \wedge N - N \wedge *X) = (-1)^{n-1}[*X, N] = -[N, *X] \end{aligned}$$

which shows that $P^*_XN = -P^*_NX$.

Since $T_A\mathcal{O}_A \simeq Im\ d_A$,

$$H_N : Im\ d_A \subset \Omega^1(Ad\ P) \to Im\ d_A \subset \Omega^1(Ad\ P) \qquad (2.14)$$

is a linear operator which we call the *second fundamental form in the direction* N for $\mathcal{O}_A$ at A. When N is understood, we just call H_N the second fundamental form. If A is irreducible, $Im\ d_A \simeq \Omega^0(Ad\ P)$, so (2.14) is conjugate to the linear operator

$$\tilde{H}_N = G_AP^*_Nd_A : \Omega^0(Ad\ P) \to \Omega^0(Ad\ P). \qquad (2.15)$$

where we have used Lemma 2.13.

We now check that formally the trace of the second fundamental form H_N equals the trace of $\tilde{H}_N$, even if A is reducible. Let $\{e_i\}$ be an orthonormal basis of $\Omega^0(Ad\ P)$

consisting of eigensections of $d_A^* d_A$ with eigenvalues $\{\lambda_i\}$. For $\lambda_i \neq 0$, $\{d_A e_i/\sqrt{\lambda_i}\}$ forms an orthonormal basis of $Im\ d_A \subset \Omega^1(Ad\ P)$, and

$$\begin{aligned} Tr\ H_N &= \sum_i \langle H_N(d_A e_i), d_A e_i\rangle \cdot \frac{1}{\lambda_i} = -\sum_i \langle G_A P^*_{d_A e_i} N, d_A^* d_A e_i\rangle \cdot \frac{1}{\lambda_i} \\ &= -\sum_i \langle G_A P^*_{d_A e_i} N, e_i\rangle = \sum_i \langle G_A P^*_N d_A e_i, e_i\rangle \\ &= Tr\ \tilde{H}_N \end{aligned}$$

From now on, we will always work with the operator $\tilde{H}_N$ on $\Omega^0(Ad\ P)$, and denote it just by H_N. We will call a gauge orbit a *minimal submanifold* of $\mathcal{A}$ if $Tr\ H_N = 0$ for all $N \in Ker\ d_A^*$.

For future reference, we rewrite Lemma 2.13 in local coordinates. We may assume that the group G is contained in $U(N)$. Choose local coordinates $(x_1, ..., x_n)$ on a neighborhood U such that $Ad\ P|_U \simeq U \times \mathfrak{u}(N)$, where $\mathfrak{u}(N)$ is the Lie algebra of $U(N)$. For a fixed basis of $\mathfrak{u}(N)$, an element $Y \in \mathfrak{u}(N)$ may be written as $Y = (Y_d^c)$, $c, d = 1, ..., N$. Thus $X \in \Omega^1(Ad\ P)$ has the local expression $X = X_{id}^{\ c} dx^i$, with summation convention, and there are similar expressions for elements of $\Omega^k(Ad\ P)$. We leave the following corollary to the reader.

Corollary 2.16 *In local coordinates,*

$$(P_X^* N)_d^c = -g^{ij}(X_{ir}^{\ c} N_{jd}^{\ r} - N_{jr}^{\ c} X_{id}^{\ r}).$$

3 Asymptotics of the trace of the second fundamental form and flat connections

Although we want to take the trace of the operator $H_N: \Omega^0(Ad\ P) \to \Omega^0(Ad\ P)$ to obtain the mean curvature, it is not clear that the trace converges. To determine if the trace exists, we must study the asymptotic behavior of the kernel function of H_N near the diagonal. We first show:

Proposition 3.1 *$H_N: \Omega^0(Ad\ P) \to \Omega^0(Ad\ P)$ in (2.12) has a distribution kernel $h_N(x,y) \in \Omega^0(Ad\ P \otimes (Ad\ P)^*)$; i.e. for any $\phi \in \Omega^0(Ad\ P)$, we have*

$$H_N(\phi)(x) = \int_M h_N(x,y)\phi(y) d_g y \qquad (3.2)$$

PROOF: Denote by $G_{da}^{cb}(x,y) = G(x,y) = G_A(x,y)$ the kernel function of the Green's operator G_A of $d_A^* d_A: \Omega^0(Ad\ P) \to \Omega^0(Ad\ P)$; i.e. G_A satisfies $d_A^* d_A G_A = Id - P_A$, where P_A is the projection onto $Ker\ d_A$, and the differentiation is in the x variable. Here the indices c, d (resp. b, a) give the coordinates of the fiber at x (resp. of y), thought of as a subspace of the matrix group $\mathfrak{u}(N)$, with respect to a fixed basis. Assuming without loss of generality that ϕ has support in a coordinate chart over which P is trivial, we have

$$[H_N(\phi)]^c_d(x) = \int_M G^{cb}_{da}(x,y)[{}^y\nabla_i\phi^a_e(y)N_{jb}^{\ e}(y) - {}^y\nabla_i\phi^e_b N_{je}^{\ a}(y)]g^{ij}(y)\ d_g y \tag{3.3}$$

where $\nabla = \nabla_A = d_A$ and ${}^y\nabla$ denotes the covariant derivative in y induced by A. Using integration by parts and $d_A^* N = 0$, we have

$$[H_N(\phi)]^c_d(x) = \int_M -g^{ij}(y)[{}^y\nabla_i G^{ce}_{da}(x,y)N_{je}^{\ b} - {}^y\nabla_i G^{cb}_{de}(x,y)N_{ja}^{\ e}(y)]\phi^a_b\ d_g y \tag{3.4}$$

So, putting

$$(h_N)^{cb}_{da}(x,y) = -g^{ij}(y)[{}^y\nabla_i G^{ce}_{da}(x,y)N_{je}^{\ b}(y) - {}^y\nabla_i G^{cb}_{de}(x,y)N_{ja}^{\ e}(y)] \tag{3.5}$$

we have (3.2).

Of course, there is a coordinate free proof of the Proposition, which yields $h_N(x,y) = *_y[*_y d_A G(x,y), N(y)]$, where d_A acts in the y variable. We leave the details to the reader.

By standard properties of $G(x,y)$, $h_N(x,y)$ is smooth outside the diagonal of $M \times M$. The following result gives the asymptotic properties for $h_N(x,y)$ near the diagonal. To be precise, the expansion (3.8) below is obtained by substituting a formal solution for the Green's operator into (3.5); as recalled in Appendix B, it is an asymptotic solution in the usual sense only for the singular terms.

To set the notation, let $r = r(x,y) = dist(x,y)$, $\Gamma = r^2$ and $\tau = (n-2)/2$. We denote the pointwise Yang-Mills Lagrangian by $ym(x) = tr(F_A(x) \wedge *F_A(x))$, where F_A is the curvature of the connection A and $*$ is the Hodge star. The Yang-Mills action is of course $YM(A) = \int_M ym(x)\ d_g x$, and the formula $F_A = dA + [A,A]$ and integration by parts easily leads to the formula

$$\delta_N\ YM(A) = \int_M tr(N \wedge *d_A^* F_A)\ d_g x \tag{3.6}$$

We set $\delta(N, ym)(x) = tr(N(x) \wedge *d_A^* F_A(x))$. Note that $\delta(N, ym)(x)$ does not equal the variation $\delta_N\ ym(x)$, but we do have that A is a Yang-Mills connection iff $\delta(N, ym)(x) = 0$ for all x and N.

Theorem 3.7 *$h_N(x,y)$ in (3.2) has the following asymptotic expansion near the diagonal in $M \times M$:*

(i) dim M odd:

$$h_N(x,y) \sim \sum_{j=0}^{\infty} \Gamma^{j-\tau-1} \tilde{U}_j(x,y) + \sum_{j=1}^{\infty} \Gamma^{j-1} \tilde{V}_j(x,y) \tag{3.8}$$

and

$$tr\ \tilde{U}_0(x,x) = 0, \quad tr\ \tilde{U}_1(x,x) = 0, \quad tr\ \tilde{V}_1(x,x) = 0 \tag{3.9}$$

$$tr\ \tilde{U}_2(x,x) = -\frac{n}{3(4-n)} \delta(N, ym)(x) \tag{3.10}$$

(ii) dim M=2;

$$h_N(x,y) \sim \sum_{j=0}^{\infty} \Gamma^{j-1} \tilde{U}_j(x,y) + \sum_{j=0}^{\infty} \Gamma^{j} \log \Gamma \cdot \tilde{V}_j(x,y) \tag{3.11}$$

with

$$tr\ \tilde{U}_0(x,x) = 0, \quad tr\ \tilde{U}_1(x,x) = 0 \tag{3.12}$$

$$tr\ \tilde{V}_0(x,x) = -\frac{1}{6} \delta(N, ym)(x) \tag{3.13}$$

(iii) dim M even, dim M $\geq$ 4:

$$h_N(x,y) \sim \sum_{j=0}^{\infty} \Gamma^{j-\tau-1} \tilde{U}_j(x,y) + \sum_{j=0}^{\infty} \Gamma^{j} \log \Gamma \cdot \tilde{V}_j(x,y) \tag{3.14}$$

If n=4

$$tr\ \tilde{U}_0(x,x) = 0, \quad tr\ \tilde{U}_1(x,x) = 0 \tag{3.15}$$

$$tr\ \tilde{V}_0(x,x) = -\frac{2}{3} \delta(N, ym)(x) \tag{3.16}$$

and if $n \geq 6$,

$$tr\ \tilde{U}_0(x,x) = 0, \quad tr\ \tilde{U}_1(x,x) = 0 \tag{3.17}$$

$$tr\ \tilde{U}_2(x,x) = -\frac{n}{3(4-n)} \delta(N, ym)(x) \tag{3.18}$$

The proof will be given in §4.

We now consider the space $\mathcal{F}$ of flat connections on P. If nonempty, the corresponding moduli space satisfies

$$\mathcal{F}/\mathcal{G} \simeq Hom(\pi_1(M), G)/G$$

where G acts by conjugation on the right hand side. The correspondence is given by sending a flat connection to its holonomy around loops in M; the inverse map sends a representation to the associated flat bundle P.

Theorem 3.19 *For any flat connection and any normal direction N, tr $h_N(x,x)$ is finite for all $x \in M$. Thus $Tr\ H_N$ exists for all flat connections.*

The proof of Theorem 3.19 will be given in §4.4.

4 The Green's operator for $d_A^* d_A$

4.1 Asymptotics of the Green's operator

Let $G(x,y) = G_A(x,y)$ be the Green's operator for $\Delta = \Delta_A = d_A^* d_A$. Set $\omega(x,y) = \sum_\lambda \phi_\lambda(x) \otimes \phi_\lambda(y)$, where $\{\phi_\lambda\}$ is a finite orthonormal basis of $Ker\ \Delta$. For x close to y, set $\theta(x,y) = \det(d\exp_y(\exp_y^{-1} x))$, where $\exp_y : T_yM \to M$ is the exponential map.

The following asymptotic expansion for $G(x,y)$ near the diagonal is more or less standard (cf. [4]). For the sake of completeness, we give a proof in Appendix B.

Proposition 4.1 *For (x,y) near the diagonal in $M \times M$, $G(x,y)$ has the following asymptotics:*
(i) dim M odd:

$$G(x,y) \sim \sum_{j=0}^{\infty} \Gamma^{j-\tau} u_j(x,y) + \sum_{j=0}^{\infty} \Gamma^j v_j(x,y) \tag{4.2}$$

where

$$\begin{cases} u_0(x,y) = \theta^{-\frac{1}{2}}(x,y)P(x,y) \\ u_j(x,y) = \frac{u_0(x,y)}{4(j-\tau)r^j} \int_0^r s^{j-1} u_0^{-1}(x(s),y)\Delta u_{j-1}(x(s),y)\ ds & j \geq 1 \\ v_0(x,y) = 0 \\ v_1(x,y) = -\frac{u_0(x,y)}{4r^{2n}} \int_0^r s^{2n-1} u_0^{-1}(x(s),y)\omega(x(s),y)\ ds \\ v_j(x,y) = -\frac{u_0(x,y)}{jr^{2n}} \int_0^r s^{2n+4j-5} u_0^{-1}(x(s),y)\Delta v_{j-1}(x(s),y)\ ds & j \geq 2 \end{cases} \tag{4.3}$$

(ii) dim M = 2:

$$G(x,y) \sim \sum_{j=0}^{\infty} \Gamma^{j-\tau} u_j(x,y) + \sum_{j=0}^{\infty} \Gamma^j \log\Gamma \cdot \hat{u}_j(x,y) \tag{4.4}$$

where

$$\begin{cases} \hat{u}_0(x,y) = \theta^{-\frac{1}{2}}(x,y)P(x,y) \\ \hat{u}_j(x,y) = \frac{\hat{u}_0(x,y)}{4jr^j} \int_0^r s^{j-1} \hat{u}_0^{-1}(x(s),y)\Delta \hat{u}_{j-1}(x(s),y)\ ds \end{cases} \tag{4.5}$$

and

$$\begin{cases} u_0(x,y) = 0 \\ u_1(x,y) = -\frac{\hat{u}_0(x,y)}{4r} \int_0^r s^{-1}\hat{u}_0^{-1}(x(s),y)[-\omega(x(s),y) - 4\hat{u}_1(x(s),y) \\ \qquad\qquad -\Delta\hat{u}_0(x(s),y)]\, ds \\ u_j(x,y) = -\frac{\hat{u}_0(x,y)}{4r^j} \int_0^r s^{j-1}\hat{u}_0^{-1}(x(s),y)[-4j\hat{u}_j(x(s),y) - \frac{1}{j}\Delta\hat{u}_{j-1}(x(s),y)]\, ds \end{cases} \tag{4.6}$$

(iii) dim M even, dim $M \geq 4$*:*

$$G(x,y) \sim \sum_{j=0}^{\infty} \Gamma^{j-\tau} u_j(x,y) + \sum_{j=0}^{\infty} \Gamma^j \log\Gamma \cdot \hat{u}_j(x,y) \tag{4.7}$$

where

$$\begin{cases} u_0(x,y) = \theta^{-\frac{1}{2}}(x,y)P(x,y) \\ u_j(x,y) = -\frac{u_0(x,y)}{4(j-\tau)r^j} \int_0^r s^{j-1}u_0^{-1}(x(s),y)\Delta u_{j-1}(x(s),y)\, ds, \quad 1 \leq j \leq \tau - 1 \\ u_\tau(x,y) = 0 \\ u_{\tau+1}(x,y) = -\frac{u_0(x,y)}{4r^{\tau+1}} \int_0^r s^{\tau-1}u_0^{-1}(x(s),y)[-r\hat{u}_1(x(s),y) + \Delta\hat{u}_0(x(s),y) \\ \qquad\qquad -\omega(x(s),y)]\, ds \\ u_j(x,y) = \frac{u_0(x,y)}{4r^j} \int_0^r s^{j-1}u_0^{-1}(x(s),y)[\frac{1}{j-\tau}\Delta u_{j-1}(x(s),y) - 4(j-\tau)\hat{u}_{j-\tau}(x(s),y) \\ \qquad\qquad +\frac{1}{j-\tau}\Delta\hat{u}_{j-\tau-1}(x(s),y)]\, ds, \quad j \geq \tau + 1 \end{cases} \tag{4.8}$$

and

$$\begin{cases} \hat{u}_0(x,y) = -\frac{u_0(x,y)}{4r^2} \int_0^r s^{\tau-1}u_0^{-1}(x(s),y)\Delta u_{\tau-1}(x(s),y)\, ds \\ \hat{u}_j(x,y) = \frac{u_0(x,y)}{4jr^{j+\tau}} \int_0^r s^{j+\tau-1}u_0^{-1}(x(s),y)\Delta\hat{u}_{j-1}(x(s),y)\, ds \end{cases} \tag{4.9}$$

4.2 The covariant derivative of $G(x,y)$

We now directly compute the covariant derivative of $G(x,y)$ near the diagonal of $M \times M$. We first note that

$$\begin{aligned} {}^y\nabla_k G^{ca}_{db}(x,y) &= \partial_{y^k} G^{ca}_{db}(x,y) + A^a_{ke}(y) G^{ce}_{db}(x,y) \\ &\quad - A^e_{kb}(y) G^{ca}_{de}(x,y) \end{aligned} \tag{4.10}$$

where we consider (4.10) at $x \neq y$. With the same notation as in §4.1, we have

Proposition 4.11 *Near* $x = y$, ${}^y\nabla G(x,y) = {}^y d_A G(x,y)$ *has the following asymptotic expansion:*
(i) dim M odd:

$${}^y\nabla G(x,y) \sim \sum_{j=0}^{\infty} \Gamma^{j-\tau-1} U_j(x,y) + \sum_{j=1}^{\infty} \Gamma^{j-1} v_j \tag{4.12}$$

with

$$U_0(x,x) = 0, \quad U_j(x,x) = {}^y\nabla u_{j-1}(x,x) \quad j \geq 1 \tag{4.13}$$

$$V_1(x,x) = {}^y\nabla v_1(x,x), \quad V_j(x,x) = (j-1)^y\nabla\Gamma \cdot v_j + {}^y\nabla V_{j-1} \quad j \geq 2 \tag{4.14}$$

(ii) dim $M = 2$:

$$ {}^y\nabla G(x,y) \sim \sum_{j=0}^{\infty} \Gamma^j \log\Gamma \cdot V_j(x,y) + \sum_{j=0}^{\infty} \Gamma^{j-1} U_j(x,y) \tag{4.15}$$

with

$$\begin{array}{ll} U_j(x,x) = 0 & j \geq 0 \\ V_j(x,x) = {}^y\nabla \hat{u}_j(x,x) & j \geq 0 \end{array} \tag{4.16}$$

(iii) dim M even, dim $M \geq 4$:

$$ {}^y\nabla G(x,y) \sim \sum_{j=0}^{\infty} \Gamma^{j-\tau-1} U_j(x,y) + \sum_{j=0}^{\infty} \Gamma^j \log\Gamma \cdot V_j(x,y) \tag{4.17}$$

with

$$\text{for } n = 4: \quad U_j(x,x) = 0 \quad j \geq 0, \quad V_0(x,x) = {}^y\nabla \hat{u}_0 \tag{4.18}$$

and

$$\text{for } n \geq 6: \quad U_0(x,x) = 0, \; U_j(x,x) = {}^y\nabla \hat{u}_{j-1}(x,x) \quad j \geq 1 \tag{4.19}$$

$$V_j(x,x) = {}^y\nabla \hat{u}_j(x,x) \quad j \geq 0 \tag{4.20}$$

Here U_j, V_j are in $\Gamma(T^*M, Ad\ P \otimes (Ad\ P)^*)$.

PROOF. Taking the covariant derivative in y, we have the following:
(i) dim M odd:

$$\begin{aligned} {}^y\nabla G(x,y) & \sim \textstyle\sum_{j=0}^{\infty} {}^y\nabla(\Gamma^{j-\tau} u_j) + \sum_{j=1}^{\infty} {}^y\nabla(\Gamma^{j-1} v_j) \\ & = \textstyle\sum_{j=0}^{\infty}((j-\tau+1)^y\nabla\Gamma \cdot u_j + {}^y\nabla u_j \Gamma^{j-\tau}) \\ & \quad + \textstyle\sum_{j=2}^{\infty}((j-1)\Gamma^{j-2\,y}\nabla\Gamma \cdot v_j + \Gamma^{j-1\,y}\nabla v_j) \end{aligned} \tag{4.21}$$

Thus we set

$$\left\{ \begin{array}{l} U_0(x,y) = -\tau^y\nabla\Gamma \cdot u_0(x,y) \\ U_j(x,y) = (j-\tau+1)^y\nabla\Gamma \cdot u_j + {}^y\nabla u_{j-1}, \quad j \geq 1 \\ V_1(x,y) = {}^y\nabla v_1(x,y) \\ V_j(x,y) = (j-1)^y\nabla\Gamma \cdot v_j(x,y) + {}^y\nabla v_{j-1}(x,y), \quad j \geq 2 \end{array} \right. \tag{4.22}$$

since ${}^y\nabla\Gamma = 0$ at $x = y$. Note that using ${}^y\nabla\Gamma = 2dr \cdot \Gamma^{1/2}$ gives another asymptotic expansion of G involving fractional powers of Γ, but with coefficients that are not smooth on the diagonal.
(ii) dim M = 2:

$$
\begin{aligned}
{}^{y}\nabla G(x,y) &\sim \textstyle\sum_{j=0}^{\infty} {}^{y}\nabla[\Gamma^{j} \log\Gamma \cdot \hat{u}_j(x,y)] \\
&= \textstyle\sum_{j=0}^{\infty}[{}^{y}\nabla \hat{u}_j + (j+1)\hat{u}_{j+1}{}^{y}\nabla\Gamma]\Gamma^{j}\log\Gamma \\
&\quad + \textstyle\sum_{j=0}^{\infty}\Gamma^{j-1}({}^{y}\nabla\Gamma\cdot\hat{u}_j)
\end{aligned} \tag{4.23}
$$

Thus we set

$$
U_j(x,y) = {}^{y}\nabla\Gamma\cdot\hat{u}_j, \quad V_j(x,y) = {}^{y}\nabla\hat{u}_j + (j+1){}^{y}\nabla\Gamma \tag{4.24}
$$

(iii) dim M even, dim M ≥ 4:

$$
\begin{aligned}
{}^{y}\nabla G(x,y) &\sim \textstyle\sum_{j=0}^{\tau-1} {}^{y}\nabla(\Gamma^{j-\tau}\cdot u_j) + \sum_{j=0}^{\infty} {}^{y}\nabla(\Gamma^{j}\log\Gamma\cdot\hat{u}_j) \\
&= \textstyle\sum_{j=0}^{\tau-1}(j-\tau){}^{y}\nabla\Gamma\cdot\Gamma^{j-\tau-1}u_j + \sum_{j=0}^{\tau-1}\Gamma^{j-\tau}\,{}^{y}\nabla\hat{u}_j \\
&\quad + \textstyle\sum_{j=0}^{\infty}[j{}^{y}\nabla\Gamma\cdot\Gamma^{j-1}\log\Gamma\cdot\hat{u}_j + \Gamma^{j-1}{}^{y}\nabla\Gamma\cdot\hat{u}_j + \gamma^{i}\log\Gamma\cdot{}^{y}\nabla\hat{u}_j]
\end{aligned} \tag{4.25}
$$

Then we have

$$
{}^{y}\nabla G(x,y) \sim \sum_{j=0}^{\infty}\Gamma^{j-\tau-1}U_j(x,y) + \sum_{j=0}^{\infty}\Gamma^{j}\log\Gamma\cdot V_j(x,y)
$$

where for dim M = 4

$$
\begin{cases}
U_0 &= -{}^{y}\nabla\Gamma\cdot u_0, \quad U_1 = {}^{y}\nabla u_0 + {}^{y}\nabla\Gamma\cdot\hat{u}_0, \quad U_2 = {}^{y}\nabla\Gamma\cdot\hat{u}_1 \\
U_j &= {}^{y}\nabla\Gamma\cdot\hat{u}_{j-1}, \quad j\geq 3 \\
V_j &= {}^{y}\nabla\hat{u}_j + (j+1){}^{y}\nabla\Gamma\cdot\hat{u}_{j+1}, \quad j\geq 0
\end{cases} \tag{4.26}
$$

and for dim M ≥ 6

$$
\begin{cases}
U_0 &= -\tau{}^{y}\nabla\Gamma\cdot u_0, \quad U_j = (j-\tau){}^{y}\nabla\Gamma\cdot u_j + {}^{y}\nabla u_{j-1}, \quad j = 1,\ldots,\tau-1 \\
U_\tau &= {}^{y}\nabla u_{\tau-1} + {}^{y}\nabla\Gamma\cdot\hat{u}_0 \\
U_j &= {}^{y}\nabla\Gamma\cdot\hat{u}_{j-1}, \quad j\geq\tau+1 \\
V_j &= (j+1){}^{y}\nabla\Gamma\cdot\hat{u}_{j+1} + {}^{y}\nabla\hat{u}_j, \quad j\geq 0
\end{cases} \tag{4.27}
$$

4.3 Proof of Theorem 3.7

PROOF. We first show that

$$
\begin{array}{ll}
tr\ \tilde{U}_0(x,x) = tr\ \tilde{U}_1(x,x) = 0 & for\ dim\ M = 2 \\
tr\ \tilde{U}_0(x,x) = tr\ \tilde{U}_1(x,x) = tr\ \tilde{V}_1(x,x) = 0 & for\ dim\ M\ odd \\
tr\ \tilde{U}_0(x,x) = tr\ \tilde{U}_1(x,x) = 0 & for\ dim\ M\ even,\ dim\ M\geq 4
\end{array} \tag{4.28}
$$

The vanishing of the left hand column and of $tr\ \tilde{U}_1(x,x)$ for dim M $= 2$ in (4.28) follows from Proposition 4.11 and (3.5). For the vanishing of the remaining coefficients, for x close to y put

$$
u(x,y) = \theta^{-1/2}(x,y)P(x,y) \tag{4.29}
$$

where $P(x,y)$ is parallel translation from y to x along the unique minimal geodesic and θ is defined in §4.1. Then

$$ {}^y\nabla_i u = {}^y\nabla_i \theta^{-1/2} \cdot P(x,y) + \theta^{-1/2} \cdot {}^y\nabla_i P(x,y) \tag{4.30} $$

By [5, C.III.2], $\theta(x,y)$ can be expressed in normal coordinates $(x_1,\ldots,x_n)$ around x as $\theta(x,y) = \sqrt{\det g_{kl}(y)}$. Using ${}^y\nabla_i g_{kl} = 0$, we have

$$ {}^y\nabla_i \theta^{-1/2}(x,x) = 0 \tag{4.31} $$

and so by (4.30) and (A.14) we obtain ${}^y\nabla_i u = 0$. This together with Propositions 4.1 and 4.11 proves the vanishing of the remaining $tr\ \tilde{U}_1$ terms. The vanishing of $tr\ \tilde{V}_1$ is similar, since $\nabla\phi_\lambda = 0$.

Thus it suffices to show (3.10), (3.13), (3.16), (3.18). Set

$$ w(x,y) = \frac{u(x,y)}{r} \int_0^r u^{-1}(x(s),y)\ {}^x\Delta u(x(s),y)\ ds \tag{4.32} $$

with ${}^x\Delta$ denoting differentiation with respect to x. According to Proposition 4.11, we must compute ${}^y\nabla_i w(x,x)$. Note that

$$ \begin{aligned} w(x,y) &= \theta^{-1/2}(x,y)P(x,y)\textstyle\int_0^1 u^{-1}(\exp_y(\tau \exp_y^{-1}(x),y)) \\ &\quad \times {}^x\Delta u_0(\exp_y(\tau\exp_y^{-1}(x)),y)\ d\tau \end{aligned} \tag{4.33} $$

Using (4.31), (A.14) and $\theta(x,x) = 1$, we have

$$ \begin{aligned} {}^y\nabla_i w(x,x) &= \textstyle\int_0^1 {}^y\nabla_i[u^{-1}(\exp_y(\tau\exp_y^{-1}(x),y))] \cdot {}^x\Delta u(\exp_y(\tau\exp_y^{-1}(x)),y) \\ &\quad + {}^y\nabla_i {}^x\Delta u(\exp_y(\tau\exp_y^{-1}(x)),y)\ d\tau\Big|_{x=y}. \end{aligned} \tag{4.34} $$

Lemma 4.35 ${}^y\nabla_i\theta^{1/2}(\exp_y(\tau\exp_y^{-1}(x)),y)\Big|_{x=y} = {}^y\nabla P(\exp_y(\tau\exp_y^{-1}(x)),y)|_{x=y} = 0.$

PROOF. Putting $y(\tau,x,y) = \exp_y(\tau\exp_y^{-1}(x))$, we have

$$ \begin{aligned} {}^y\nabla_i\theta^{1/2}(y(\tau,x,y),y)\Big|_{x=y} &= \partial_{y^i}\theta^{1/2}(y(\tau,x,y),y)\Big|_{x=y} \\ &\quad + \textstyle\sum_j \partial_{x^j}\theta^{1/2}(y(\tau,x,y),y) \cdot \frac{\partial x^j}{\partial y^i}\Big|_{x=y} \end{aligned} \tag{4.36} $$

Note that $y(\tau,x,y)\Big|_{x=y} = y$. Using (4.31) gives the first half of Lemma 4.35. With the aid of (A.14), the vanishing of ${}^y\nabla P(\exp_y(\tau\exp_y^{-1}(x)),y)|_{x=y}$ is similar.

Thus

$$\int_0^1 {}^y\nabla_i[u^{-1}(\exp_y(\tau\exp_y^{-1}(x),y))]^x\Delta u_0(\exp_y(\tau\exp_y^{-1}(x),y))\,d\tau\Big|_{x=y}=0 \tag{4.37}$$

so to compute (4.34), we must study ${}^y\nabla_i\ {}^x\Delta u(\exp_y(\tau\exp_y^{-1}(x)),y)\Big|_{x=y}$. Using the notation $y(\tau,x,y)$ as above, we have

$$\begin{aligned}
&[{}^y\nabla_i\ {}^x\Delta u_{db}^{ca}(y(\tau,x,y),y)]_{x=y}= \\
&\quad[\partial_{y^i}[{}^x\Delta u_{db}^{ca}(y(\tau,x,y),y)]+A_{if}^{\ a}(y)^x\Delta u_{db}^{cf}(y(\tau,x,y),y) \\
&\qquad -A_{ib}^{\ f}(y)^x\Delta u_{df}^{ca}(y(\tau,x,y),y)]_{x=y}
\end{aligned}\tag{4.38}$$

where $A_{ib}^{\ a}$ are the components of the connection A. Since $u(x,y)=\theta^{1/2}(x,y)P_{db}^{ca}(x,y)$, we have

$$\begin{aligned}
&A_{if}^{\ a}(y)\ {}^x\Delta[\theta^{1/2}(y(\tau,x,y),y)P_{db}^{cf}(y(\tau,x,y),y)]\Big|_{x=y} \\
&= A_{if}^{\ a}(x)[{}^x\Delta\theta^{1/2}\cdot P_{db}^{cf}(x,y)+2\langle{}^x\nabla\theta^{1/2}(x,y),{}^x\nabla P_{db}^{cf}(y,\tau,x,y),y\rangle \\
&\qquad +{}^x\Delta P_{db}^{cf}(y(\tau,x,y),y)]\Big|_{x=y} \\
&= {}^x\Delta\theta^{1/2}\cdot A_{if}^{\ a}(x)P_{db}^{cf}(x,x)+A_{if}^{\ a}(x)^x\Delta P_{db}^{cf}(x,x)
\end{aligned}\tag{4.39}$$

Similarly, we have

$$A_{ib}^{\ f}\ {}^x\Delta u_{df}^{ca}(y(\tau,x,y),y)\Big|_{x=y}={}^x\Delta\theta^{1/2}\cdot A_{ib}^{\ f}(x)P_{df}^{ca}(x,x)+A_{ib}^{\ f}(x)^x\Delta P_{df}^{ca}(x,x) \tag{4.40}$$

On the other hand, we have

$$\begin{aligned}
&\partial_{y^i}[{}^x\Delta u_{db}^{ca}(y(\tau,x,y),y)]\Big|_{x=y} \\
&= \partial_{y^i}[{}^x\Delta\theta^{1/2}(y(\tau,x,y),y)P_{db}^{ca}(y(\tau,x,y),y) \\
&\qquad +2\langle{}^x\nabla\theta^{1/2}(y(\tau,x,y),y),{}^x\nabla P_{db}^{ca}(y(\tau,x,y),y)\rangle \\
&\qquad +\theta^{1/2}(y(\tau,x,y),y)^x\Delta P_{db}^{ca}(y(\tau,x,y),y)]\Big|_{x=y} \\
&= [\partial_{y^i}{}^x\Delta\theta^{1/2}(y(\tau,x,y),y)P_{db}^{ca}(x,x)+{}^x\Delta\theta^{1/2}(y(\tau,x,y),y)\partial_{y^i}P_{db}^{ca}(y(\tau,x,y),y) \\
&\qquad +\partial_{y^i}{}^x\Delta P_{db}^{ca}(y(\tau,x,y),y)]\Big|_{x=y}
\end{aligned}\tag{4.41}$$

where we have used Lemma 4.35. Combining (4.38), (4.39), (4.40) and (4.41) gives

$$\begin{aligned}[{}^y\nabla_i{}^x\Delta u^{ca}_{db}(y(\tau,x,y),y)]\Big|_{x=y} &= {}^x\Delta\theta^{1/2}{}^y\nabla_i P^{ca}_{db} + \partial_{y^i}{}^x\Delta P^{ca}_{db}(y(\tau,x,y),y)\Big|_{x=y} \\ &\quad + A_{if}^{\ a}(x)^x\Delta P^{cf}_{db}(x,x) - A_{ib}^{\ f}(x)^x\Delta P^{ca}_{df}(x,x) \\ &\quad + \partial_{y^i}{}^x\Delta\theta^{1/2}(y(\tau,x,y),y)\Big|_{x=y} P^{ca}_{db}(x,x) \\ &= \partial_{y^i}{}^x\Delta P^{ca}_{db}(y(\tau,x,y),y)\Big|_{x=y} \\ &\quad + \partial_{y^i}{}^x\Delta\theta^{1/2}(y(\tau,x,y),y)\Big|_{x=y} P^{ca}_{db}(x,x)\end{aligned} \tag{4.42}$$

where we have used (A.14) and (A.20). We also have

$$\begin{aligned}&\partial_{y^i}{}^x\Delta P^{ca}_{db}(y(\tau,x,y),y)\Big|_{x=y} \\ &= \partial_{x^j}{}^x\Delta P^{ca}_{db}(y(\tau,x,y),y)\cdot\frac{\partial x^j}{\partial y^i} + \partial_{y^i}{}^x\Delta P^{ca}_{db}(y(\tau,x,y),y)\Big|_{x=y} \\ &= \partial_{x^i}{}^x\Delta P^{ca}_{db}(x,x)(1-\tau) + \partial_{y^i}{}^x\Delta P^{ca}_{db}(x,x)\end{aligned} \tag{4.43}$$

So applying Corollary A.44 to (4.43) and plugging the result into (4.42), we finally get

$${}^y\nabla_i w(x,x) = \partial_{y^i}\ {}^x\Delta\theta^{1/2}(x,x)\delta^c_d\delta^a_b + \frac{1}{2}\partial_{y^i}\ {}^x\Delta P^{ca}_{db}(x,x) \tag{4.44}$$

since $P^{ca}_{db}(x,x) = \delta^c_d\delta^a_b$. Thus by (3.5), $tr\ \tilde{U}_1(x,x)$ is given by

$$\begin{aligned}&Tr\Big[-g^{ij}[{}^y\nabla_i w^{ce}_{da}N_{je}^{\ \ b} - {}^y\nabla_i w^{cb}_{de}N_{ja}^{\ \ e}]\Big] \\ &= \frac{1}{2}Tr\Big[-g^{ij}[\partial_{y^i}\ {}^x\Delta P^{ce}_{db}(x,x)N_{je}^{\ \ a}(x,x) - \partial_{y^i}\ {}^x\Delta P^{cb}_{de}(x,x)N_{ja}^{\ \ e}(x,x)]\Big]\end{aligned} \tag{4.45}$$

Notice that the terms involving ${}^x\Delta\theta^{1/2}$ in (4.44), which contain curvature information of the Riemannian metric, do not appear in (4.45), because $P^{ca}_{db}(x,x)$ is a diagonal matrix. By Proposition A.30, the right hand side of (4.45) is the local expression for $\delta(N,ym)(x)$, so Theorem 3.7 is proved.

4.4 Proof of Theorem 3.19

Denote the right hand side of (4.2), with the infinite sum replaced by a sum to $N >> 0$, by G^{asy}_A. It is well known that ${}^y d_A G_A(x,y)$ and ${}^y d_A G^{asy}_A(x,y)$ differ by a term which is bounded as $y \to x$; cf. the remark after Theorem 5.10. Therefore, by (3.5) and the following invariant reformulation, $tr\ h_N(x,x)$ will be finite if $[*_y\ {}^y d_A G^{asy}_A(x,y), N(y)]|_{x=y}$ is finite.

We show the stronger result that the terms $\Gamma^{j-\tau-1}\tilde{U}_j(x,y)$, $\Gamma^{j-1}\tilde{V}_j(x,y)$ in (3.8), and the similar terms in (3.11), (3.14), all vanish as $y \rightarrow x$; note that it does not suffice to just prove the vanishing of $\tilde{U}_j(x,x), \tilde{V}_j(x,x)$. The expressions for the asymptotic coefficients u_j, $\hat{u}_j$, v_j for G_A in Appendix B involve $\theta^{-1/2}(x,y)$, $P(x,y)$, $\omega(x,y)$ and their derivatives in x, and the asymptotic coefficients for $d_A G_A$ involve further derivatives in y by Proposition 4.11. By a straightforward induction argument similar to the computations in §A.3, there exist smooth functions $k_j(x,y), l_j(x,y), m_j(x,y)$ defined near the diagonal in $M \times M$ such that

$$\begin{aligned} {}^y\nabla u_{j\,ad}^{\;cb}(x,y) &= k_j(x,y)\delta_a^c\delta_d^b + O(\Gamma^N) \\ {}^y\nabla \hat{u}_{j\,ad}^{\;cb}(x,y) &= l_j(x,y)\delta_a^c\delta_d^b + O(\Gamma^N) \\ {}^y\nabla v_{j\,ad}^{\;cb}(x,y) &= m_j(x,y)\delta_a^c\delta_d^b + O(\Gamma^N) \end{aligned}$$

for any N. In this expression, we have calculated the coefficients in a synchronous frame as in Appendix A, so that all derivatives of the components of the flat connection vanish at y, by virtue of [6, Prop. 3.7]. Thus, $d_A G_A^{asy}$ is a diagonal matrix to arbitrary order, as is $*d_A G_A^{asy}$, and so $[*d_A G_A^{asy}, N] = 0$ at $x = y$.

5 Regularization of the second fundamental form

In this section we propose a zeta function regularization for the trace of the operator H_N of Section 2 and discuss cases in which this regularization is finite. We show that the regularization scheme is consistent with the Faddeev-Popov formulation of the volume element for a gauge orbit in the sense that an infinite dimensional version of Hsiang's theorem holds. Combining Hsiang's theorem with Theorem 3.19, we show that there exist minimal gauge orbits of flat connections on certain 3-manifolds. Finally, we prove that the gauge orbits of reducible connections are minimal in the regularized sense for the class of 4-manifolds covered by Donaldson's theorem,

Before defining the regularization, we make a few remarks about the case when the structure group G is abelian. This gives a trivial gauge theory in the sense that all relevant equations reduce to linear Hodge theory. In fact, the operator d_A is independent of A. For if ∇ is a base point connection on P then $d_A = \nabla + [A, \cdot] = \nabla$, where A in the bracket denotes the connection form for A. Thus $\delta_N d_A^* = 0$, and $H_N(X)$ vanishes for all $N \in Ker\ d_A^*$. Thus no matter how we regularize the trace of H_N, in the abelian case all gauge orbits are totally geodesic.

Lemma 5.1 *Let P be a principal bundle with abelian structure group. Then every gauge orbit is a totally geodesic submanifold of $\mathcal{A}$.*

This observation holds in particular for $G = U(1)$, the gauge theory of electromagnetism. A more interesting case from our point of view occurs in quantum mechanics, for which $G = GL^+(1, R) \simeq R$, where the plus sign indicates positively oriented matrices, and P is the frame bundle of the trivial line bundle $\mathbf{R}$ over $\mathcal{M}$. (The reader can check that the following discussion does not require G to be compact.) Here $\mathcal{G} \simeq \{e^f : f \in C^\infty(M)\}$ is the space of positive functions on $\mathcal{M}$ and $\mathcal{A}$ is the space of derivations on $\mathcal{M}$. Note that the infinitesimal gauge action of $T_{Id}\mathcal{G}$ is the linear action $f \mapsto e^f \cdot A = A + df$, so it is not surprising that the orbits of $\mathcal{G}$, which are affine subspaces of $\mathcal{A}$, are totally geodesic.

In particular, for $A = d$, the exterior derivative on functions, the action of $\mathcal{G}$ is $d \mapsto d_f = e^{-f} d e^f$, the classical gauge transformation extended to forms by Witten in [7]. Recall that the Witten Laplacian on functions is $\Delta_f = d_f^* d_f$. More generally, for $A \in \mathcal{A}$ we have a Bochner Laplacian A^*A and Witten-Bochner Laplacians $\Delta_{A_f} = A_f^* A_f$ with $A_f = e^{-f} A e^f$.

Lemma 5.2 *The set of Witten-Bochner Laplacians $\{\Delta_{A_f} : f \in C^\infty(M)\}$ associated to a connection A is in 1-1 correspondence with the orbit $\mathcal{O}_A$ of A in $\mathcal{A}$.*

PROOF. We must show that the stabilizer of $\mathcal{G}$ on A equals the stabilizer of the action $f \mapsto \Delta_{A_f}$. For fixed A, the stabilizer of the action $(f, A) \mapsto A + df$ is clearly $\{f = constant\} \simeq R$. For the action on Witten-Bochner Laplacians, we first for simplicity assume $A = d$. If Δ_f equals $\Delta = d^*d$, then $Ker\ \Delta_f$ consist of constants c. But $Ker\ \Delta_f = Ker\ d_f$, and $d_f c = 0$ implies $e^f c$ is constant, so f is constant.

For general A, we first recall that a local slice for the gauge group action at $A = d$ is isomorphic to $Ker\ d_A^* = Ker\ \delta = Ker\ \Delta^1 \oplus Im\ \delta\Omega^2 \subset \Omega^1$. In fact this is a global slice: given $A \in \mathcal{A}$, we have $A = d + \omega$ for some $\omega \in \Omega^1$. ω has the Hodge decomposition $\omega = df + \omega'$, with $\omega' \in Ker\ \delta$, so $f \cdot A = d + \omega'$, which says that the slice $Ker\ \delta$ hits $\mathcal{O}_A$ at $-f \cdot A$.

For an arbitrary gauge orbit, we may now choose a representative connection $A = d + \omega$ with $\omega \in Ker\ \delta$. Note that $\Delta_A = \Delta + \delta\omega + \omega^* d + \omega^*\omega$, where ω^* is the adjoint of the map $\omega\wedge : \Omega^0 \to \Omega^1$. Under the action of f, $\Delta_A \mapsto \Delta_{A_f} = \Delta_f + e^{-f}\omega^* d e^f + \omega^*\omega + e^f \delta e^{-f}\omega$. On functions, the Witten Laplacian Δ_f equals $\Delta + |\nabla f|^2 - \Delta f$, so $\Delta_A = \Delta_{A,f}$ if and only if $\delta\omega + \omega^* d = |\nabla f|^2 - \Delta f + e^{-f}\omega^* d e^f + e^f \delta e^{-f}\omega$ as operators taking Ω^0 to Ω^1. Applying both sides to the function 1 yields $\delta\omega = |\nabla f|^2 - \Delta f + \omega^* df + e^f \delta e^{-f}\omega$, so

$$
\begin{aligned}
0 = \int_M \delta\omega &= \int_M |\nabla f|^2 + \int_M \omega^* df + \int_M e^f \delta e^{-f}\omega \quad (\text{as } \int_M \Delta f = 0) \\
&= \int_M |\nabla f|^2 + \int_M \omega^* df \wedge *1 + \int_M e^f \delta e^{-f}\omega \wedge *1 \\
&= \int_M |\nabla f|^2 + \int_M df \wedge *\omega + \int_M e^{-f}\omega \wedge *de^f
\end{aligned}
$$

$$
\begin{aligned}
&= \int_M |\nabla f|^2 + 2\int_M df \wedge *\omega \\
&= \int_M |\nabla f|^2 + 2\int_M f \wedge *\delta\omega \\
&= \int_M |\nabla f|^2
\end{aligned}
$$

which implies that f is constant.

Therefore the map $A \mapsto \Delta_A$ from derivations on M to second-order elliptic operators (with leading term the Laplacian on M) is injective on each gauge orbit in $\mathcal{A}$. Thus this family of operators $\mathcal{F}$ is parametrized by $\mathcal{A}$ and can be given the structure of an infinite dimensional Riemannian manifold.

Corollary 5.3 *The set of Witten Laplacians Δ_f is a totally geodesic submanifold in the set of Witten-Bochner Laplacians $\mathcal{F}$.*

For the nonabelian case, we now give a regularization for $Tr\ H_N$. Let $\Delta = \Delta_A$.

$$
Definition: \ Tr\ H_N = Tr(G_A(\delta_N d_A^*)d_A) = \int_0^\infty t^s Tr(e^{-t\Delta}(\delta_N d_A^*)d_A)dt\Big|_{s=0} \tag{5.4}
$$

If d_A were a finite-dimensional linear transformation, this definition would be an identity by directly setting $s = 0$ and using $G_A = \int_0^\infty (e^{-t\Delta} - P)\ dt$, where P is projection onto $Ker\ d_A$. Moreover, if $tr\ h_N(x,x)$ exists for all $x \in M$, this identity for G_A shows that the right hand side of (5.4) equals $\int_M tr\ h_N(x,x)$, and so (5.4) equals $Tr\ H_N$ in the ordinary sense. In general, the right hand side of (5.4) is to be understood as an analytic function of $s \in C$ for $Re(s) >> 0$ which admits a meromorphic continuation to all of **C**. The meromorphic continuation is standard but it is not clear at this point that the right hand side of (5.4) is finite at $s = 0$.

Before we discuss (5.4) for general structure groups, we consider a smooth family of positive self-adjoint elliptic differential operators $\{\Delta\} = \{\Delta_A\}_{A\in\mathcal{A}}$, where $\mathcal{A}$ is now just a smooth manifold, acting on sections of a bundle over a compact manifold. We assume that $dim\ Ker\ \Delta_A$ is independent of A. As the proof of Lemma 5.5 will show, for each $N \in T_A\mathcal{A}$ the zeta function

$$
\zeta_N(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}Tr((\delta_N\Delta_A)e^{-t\Delta_A})\ dt
$$

converges for $Re(s) >> 0$ and has a meromorphic continuation to **C** having at worst simple poles occurring at a finite set of integers (resp. half integers) for $dim\ M$ even (resp. odd).

Let P be the projection onto $Ker\ \Delta_A$ and denote the standard zeta function $\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}Tr\ (e^{-t\Delta_A} - P)\ dt$ by $\zeta_A(s)$.

Lemma 5.5 *For all $s \in C$, $(s-1)\zeta_N(s) = -\delta_N \zeta_A(s-1)$.*

At poles of $\zeta_A(s)$, this equation is to be interpreted as saying that the poles of $\zeta_N(s)$ coincide with the poles of $\zeta_A(s)$ shifted by one.

PROOF. By the uniqueness of the meromorphic continuation of $\delta_N(s)$ and $\zeta_A(s)$, it suffices to prove the equation for $Re(s) >> 0$. Then

$$\begin{aligned}(s-1)\zeta_N(s) &= \tfrac{s-1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr((\delta_N \Delta_A) e^{-t\Delta_A})\, dt \\ &= -\tfrac{1}{\Gamma(s-1)} \int_0^\infty t^{s-2} Tr(-t(\delta_N \Delta_A) e^{-t\Delta_A})\, dt \\ &= -\tfrac{1}{\Gamma(s-1)} \int_0^\infty t^{s-2} \delta_N Tr(e^{-t\Delta_A} - P)\, dt \\ &= -\delta_N \zeta_A(s-1).\end{aligned}$$

Note that the proof shows that $\zeta_N(s)$ exists for $Re(s) >> 0$. The functional equation (5.5) and the standard mermorphic continuation of $\zeta_A(s)$ now give the meromorphic continuation of $\zeta_N(s)$.

Now consider $\Delta = \Delta_A = d_A^* d_A$. The assumption that $dim\ Ker\ \Delta$ is constant means that we consider variations only through orbits of the same orbit type as $\mathcal{O}_A$. Then

$$\begin{aligned}\zeta_N(s) &= \tfrac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr((\delta_N \Delta) e^{-t\Delta})\, dt \\ &= \tfrac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr((d_A^* \delta_N d_A + (\delta_N d_A^*) d_A) e^{-t\Delta})\, dt\end{aligned}$$

Let $\{\phi_i\}$ be an orthonormal basis of $\Omega^0(Ad\ P)$ consisting of eigensections for Δ, and let tr denote the pointwise trace in $Ad\ P$. Then

$$\begin{aligned}&Tr((d_A^* \delta_N d_A + (\delta_N d_A^*) d_A) e^{-t\Delta}) \\ &= \sum_i \int_M tr[(d_A^* \delta_N d_A + (\delta_n d_A^*) d_A) e^{-t\Delta} \phi_i \wedge *\phi_i] \qquad (5.6) \\ &= \sum_i e^{-\lambda_i t} \int_M tr[(d_A^* \delta_N d_A + (\delta_N d_A^*) d_A) \phi_i \wedge *\phi_i]\end{aligned}$$

For $\omega \in \Omega^1(Ad\ P)$, $\eta \in \Omega^0(Ad\ P)$, we have

$$\langle (\delta_N d_A^*)\omega, \eta\rangle = \delta_N \langle d_A^* \omega, \eta\rangle = \delta_N \langle \omega, d_A \eta\rangle = \langle \omega, (\delta_N d_A)\eta\rangle$$

so $\delta_N d_A^* = (\delta_N d_A)^*$, and (5.6) equals

$$\sum_i e^{-\lambda_i t} \int_M tr[(\delta_N d_A)\phi_i \wedge *d_A \phi_i + d_A \phi_i \wedge *\delta_N d_A \phi_i]$$

$$= 2\sum_i e^{-\lambda_i t}\int_M tr[(\delta_N d_A)^* d_A\phi_i \wedge *\phi_i] \qquad (5.7)$$

$$= 2\ Tr((\delta_N d_A)^* d_A e^{-t\Delta})$$

By [8, p. 152], (5.7) equals $2\ Tr(e^{-t\Delta}(\delta_N d_A)^* d_A)$, so

$$\frac{\Gamma(s)}{2}\zeta_N(s) = \int_0^\infty t^{s-1}Tr(e^{-t\Delta}(\delta_N d_A)^* d_A)\ dt \qquad (5.8)$$

Setting $s = 1$ in (5.8) gives:

Proposition 5.9 *Let $\zeta_N(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}Tr((\delta_N\Delta)e^{-t\Delta})\ dt$ with $\Delta = d_A^* d_A$. Then $Tr\ H_N = \frac{1}{2}\zeta_N(1)$ provided $\zeta_N(1)$ is finite.*

We can now give several cases in which the regularized definition of Tr H_N makes sense.

Theorem 5.10 *Assume A is an irreducible connection. Then the regularization (5.4) for $Tr\ H_N$ is finite if dim M is odd or dim $M = 2$. If dim $M = 4$, then the regularization is finite if and only if A is a Yang-Mills connection. In all dimensions, the regularization is finite within the space of flat connections.*

PROOF. By Lemma 5.5 and Proposition 5.9, $Tr\ H_N$ is finite if and only if $\delta_N\zeta_A(0) = 0$. In odd dimensions, it is well known that $\zeta_A(0) = -dim\ Ker\ \Delta_A$, so $\zeta_A(0) = 0$ on the set of irreducible connections, which form an open set in $\mathcal{A}$. (In fact, if P admits no flat connections, then the set of irreducible connections is dense [9].)

In even dimensions n, $\zeta_A(0) = c_n\int_M tr\ a_{n/2} - dim\ Ker\ \Delta_A$, where c_n is a dimension constant and the heat kernel $e(t,x,y)$ for $e^{-t\Delta}$ has asymptotics $e(t,x,x) \sim \sum_{k=0}^\infty a_k(x)t^{k-n/2}$. By [10, Ch. 4.8], $\int_M tr\ a_1(x) = c_1\int_M s(q) + (dim\ Ad\ P)(vol\ M)$, for c_1 a constant and $s(g)$ the scalar curvature of the metric g on M. Thus in dimension 2, $\delta_N\zeta_A(0) = 0$ at irreducible connections. Moreover, $\int_M tr\ a_2(x) = c_2(g) + c_3\int_M tr\ |F_A|^2$, where $c_2(g)$ depends only on the metric g, c_3 is a nonzero constant, and $\int_M tr\ |F_A|^2$ is the Yang-Mills functional. Thus for irreducible connections, $\delta_N\zeta_A(0) = 0$ precisely at Yang-Mills connections.

Finally, in even dimensions, $a_{n/2}$ is an expression in the curvatures of g and A and their covariant derivatives. For flat connections, this expression depends only on g, and hence is independent of the flat connection.

REMARK. Notice that Theorem 5.10 provides examples where the zeta function regularization of $Tr\ H_N$ is finite but the kernel h_N does not exist on the diagonal. More specifically, the direct method of trying to compute $Tr\ H_N$ by integrating the kernel over the diagonal has many obstructions, namely the nonvanishing of the coefficients $\tilde{U}_j(x,x)$, $\tilde{V}_j(x,x)$, at each $x \in M$, for a number of values of j (e.g. $j =$

$(n+1)/2$ if $dim\ M$ is odd.) In contrast, the regularization given by (5.4) has only one global obstruction, the nonvanishing of $\int_M \tilde{W}_{\frac{n}{2}-1}$, where $\tilde{W}_k$ are the coefficients in the asymptotic expansion of the kernel of $e^{-t\Delta_A}(\delta_N d_A^*)d_A$, as $(1/2)\int_M \tilde{W}_{\frac{n}{2}-1}$ is the residue at $s = 1$ of $\zeta_N(s)$ (cf. (5.8)). This can be better understood in the simpler case of G_A, whose trace does not exist due to the many pointwise obstructions $u_j(x,x)$, $\hat{u}_j(x,x)$ of Proposition 4.1. If we instead attempt to regularize the trace by setting $Tr\ G_A = Tr(\Delta_A^{-s}\Delta_A^{-1})|_{s=0}$, then formally $Tr\ G_A = \zeta_A(1)$. However, $\zeta_A(1)$ is a simple pole with residue $\int_M \tilde{u}_0(x,x)$ in the case $dim\ M$ even; there are similar formulas for the other cases. Thus there is only one global obstruction to the regularization of the trace.

To complete this analogy, we will show $(1/2)\tilde{W}_{\frac{n}{2}-1}(x,x)$ is precisely one of the asymptotic coefficients in Theorem 3.7 in each case. We have

$$\begin{aligned} H_N\phi(x) &= G_A(\delta_N d_A^*)d_A\phi(x) = \int_0^\infty e^{-t\Delta_A}(\delta_N d_A^*)d_A\phi(x)\ dt \\ &= \int_0^\infty \int_M e(t,x,y)\ {}^y(\delta_N d_A^*)\ {}^y d_A\phi(y)\ dy\ dt \\ &= \int_0^\infty \int_M {}^y(d_A^*\delta_N d_A)e(t,x,y)\phi(y)\ dy\ dt \end{aligned}$$

where $e(t,x,y)$ is the heat kernel for Δ_A, so the kernel of H_N has the expression

$$h_N(x,y) = \int_0^\infty {}^y(d_A^*\delta_N d_A)e(t,x,y)\ dt \tag{5.11}$$

We now assume that $dim\ M \geq 4$ is even and leave the other cases to the reader. For $L > (n/2)+2$, $e(t,x,y)$ has the parametrix

$$P_L(t,x,y) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{r}{4t}}\left(\sum_{k=0}^{\frac{n}{2}-2} u_k(x,y)t^k + \sum_{k=0}^{L-\frac{n}{2}} \hat{u}_k(x,y)t^k\right)$$

[11, §2], so

$$\begin{aligned} h_N(x,y) &= \int_0^{1/4} {}^y(d_A^*\delta_N d_A)(e(t,x,y) - P_L)\ dt + \int_{1/4}^\infty {}^y(d_A^*\delta_N d_A)e(t,x,y)\ dt \\ &\quad + \int_0^{1/4} {}^y(d_A^*\delta_N d_A)P_L\ dt \end{aligned} \tag{5.12}$$

The first two terms on the right hand side of (5.12) are bounded; the role of $1/4$ is unessential, as replacing it by another positive real changes the right hand side by a bounded amount. Thus the last term must give the asymptotics of $h_N(x,y)$, at least up to bounded terms. On the other hand, a similar argument shows that the kernel of $e^{-t\Delta_A}(\delta_N d_A^*)d_A$ is ${}^y(d_A^*\delta_N d_A)e(t,x,y)$, which equals ${}^y(d_A^*\delta_N d_A)P_L$ plus a bounded remainder. We can now explicitly differentiate P_L as in [12], plug the results into (5.12), and do the t integration. As in [11], the terms with exponent t^{-1} integrate to give the first nonzero log term in Proposition 4.1, while the other terms give the coefficients of $\Gamma^{j-\tau}$. In particular, we see that $(1/2)\tilde{W}_{\frac{n}{2}-1}(x,x)$ equals $\tilde{V}_1(x,y)$.

Thus Theorem 5.10 gives a relatively easy proof that certain integrals of asymptotic coefficients of h_N vanish. However, this approach does not yield the more refined pointwise results of Theorem 3.7.

It is tempting to regularize Tr H_N so that orbits of Yang-Mills connections are minimal. By Theorem 5.10, this could be done by setting Tr H_N equal to one-half the residue of $\zeta_N(s)$ at $s = 1$, as this residue equals $\delta_N \zeta_A(0)$ by Lemma 5.5. This in turn could be accomplished by defining $Tr\ H_N$ to be

$$\frac{1}{\Gamma(s)} \int_0^\infty t^s\ Tr(e^{-t\Delta}(\delta_N d_A)^* d_A)\ dt\Big|_{s=0}$$

–note the extra factor of $\Gamma(s)^{-1}$. However, only the regularization given in (5.4) is compatible with the Faddeev-Popov approach to quantizing Yang-Mills theory, as we shall now explain.

In finite dimensions, a submanifold is minimal if the variation of the volume element at each point vanishes under deformations of the submanifold. For gauge orbits, the volume element is associated to $T_A\mathcal{O}_A \simeq Im\ d_A \subset \Omega^1(Ad\ P)$. Given an orthonormal basis $\{\phi_i\}$ of $\Omega^0(Ad\ P)$ consisting of eigensections of Δ_A with eigenvalues $\{\lambda_i\}$, the volume element is formally given by $\phi_1 \wedge \phi_2 \wedge \ldots$ times the "Jacobian" of d_A, the Faddeev-Popov ghost determinant:

$$(det'(\langle d_A\phi_i, d_A\phi_j\rangle))^{1/2} = (det'(\langle \Delta_A\phi_i, \phi_j\rangle))^{1/2} = (\textstyle\prod' \lambda_i)^{1/2}$$

where the prime indicates exclusion of the λ_i equal to zero; for irreducible connections, we may drop the prime. Thus the volume element is formally $(det^{1/2}\ \Delta_A)\phi_1 \wedge \phi_2 \wedge \ldots$ (strictly speaking, the volume element should involve the wedge product of cotangent vectors ϕ_i^* dual to ϕ).

Here $det\ \Delta_A$ is defined by the Ray-Singer regularization $det\ \Delta_A = \exp(-\frac{d}{ds}\Big|_{s=0} \zeta_A(s))$, as this formula generalizes a standard identity for finite dimensional transformations. Therefore an irreducible orbit $\mathcal{O}_A$ will be minimal among all nearby orbits of the same type if $\delta_N\ det\ \Delta_A = 0$ for all $N \in Ker\ d_A^*$. Note that by the gauge invariance of $det\ \Delta_A$, we need only check the variation of the determinant at one point of the gauge orbit.

Recall the following theorem of W.-Y. Hsiang [13]:

Theorem 5.13 *Let G be a compact connected Lie group acting via isometries on a Riemannian manifold M. Then any orbit of G whose volume is extremal among nearby orbits of the same type is a minimal submanifold of M.*

For noncompact manifolds, we say a possibly noncompact submanifold W has extremal volume if the variation of the volume element in normal directions vanishes at every point of W. The compatibility of our regularization of Tr H_N with the Faddeev-Popov approach is given by the following infinite dimensional analogue of Hsiang's Theorem:

Theorem 5.14 *A gauge orbit of irreducible connections whose volume is extremal among nearby orbits is a minimal submanifold of the space of connections.*

PROOF. Here a gauge orbit has extremal volume if $\delta_N\ det\ \Delta_A = 0$ in the notation above. We must show that at A, $Tr\ H_N = 0$ iff $\delta_N\ det\ \Delta_A = 0$. This follows by taking $\frac{d}{ds}\Big|_{s=1}$ of the equation in Lemma 5.5 and using Proposition 5.9.

In summary, by the first variation formula, we measure minimality in finite dimensions either by the infinitesimal variation of the volume element or by $Tr\ II$. In Yang-Mills theory, we cannot define minimality via the volume element, as we only have volume elements along gauge orbits, and the first variation formula cannot be shown directly. Nevertheless, for the gauge orbits given in Theorem 5.10, we have a regularized notion of $Tr\ II$ which we can use to define minimality, and only this regularization is compatible with the Faddeev-Popov volume element to the extent that a version of Hsiang's theorem holds.

Hsiang's theorem implies the existence of minimal gauge orbits for a class of 3-manifolds.

Corollary 5.15 *Let Σ be a Seifert fibered homology sphere with at least four exceptional orbits. Then the space of flat $SU(2)$ connections over Σ contains at least two minimal gauge orbits.*

PROOF. By [14, Prop. 2.7], the moduli space of irreducible flat $SU(2)$ connections over such a Σ has at least one component which is a closed manifold of dimension $2m-6$, where M is the number of exceptional orbits. On this component, the function $det\ \Delta_A$ has at least two critical points. By Theorem 5.14, the gauge orbit over any critical point is a minimal submanifold of the space of all connections lying over this component. (Note that this space is a manifold, as $dim\ Ker\ d_A = 0$ on this space.)

Of course, this proof applies to any manifold and gauge group such that the moduli space of connections contains a closed manifold.

The proof of our version of Hsiang's theorem easily carries over to any family of connections with $dim\ Ker\ \Delta_A$ constant. However, the case where the dimension of the kernel is nonconstant is also important. For in finite dimensions, Hsiang's theorem immediately implies that orbits of isolated type are minimal submanifolds, which indicates that we should look for minimal submanifolds among the orbits of reducible connections. This involves a reworking of the argument leading up to Theorem 5.10, since the Ray-Singer regularization is not continuous near a reducible connection.

To begin, note that the right hand side of Definition 5.4 remains well defined for $Re(s) >> 0$, since the trace still dies exponentially as $t \to \infty$. The argument in (5.6)-(5.8) remains valid, so we still have

$$Tr\ H_N = \frac{1}{2}\zeta_N(1) \tag{5.16}$$

provided $\zeta_N(1)$ is finite.

On the other hand, the proof of Lemma 5.5 no longer holds, since $dim\ Ker\ d_A$ is not constant near a reducible connection. To modify this argument, we pick $\tilde{\lambda}$ not in the spectrum of Δ_A; by the minimax characterization of the spectrum, $\tilde{\lambda}$ is not in the spectrum of Δ_B for all B in some neighborhood of A. Let $\tilde{P} = \tilde{P}_A$ be the projection of the L^2 zero-forms into the eigenspaces lying below $\tilde{\lambda}$. Finally, set

$$\hat{\zeta}(s) = \hat{\zeta}_A(s) = \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}Tr(e^{-t\Delta})\ dt + \frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}Tr(e^{-t(\Delta-\tilde{P})})\ dt \tag{5.17}$$

Notice that both terms on the right hand side of (5.17) are smooth in A. Since $\delta_N Tr(e^{-t\Delta})$ dies exponentially as $t \to \infty$ as in (5.6)-(5.8), we have

$$(s-1)\zeta_N(s) = -\frac{1}{\Gamma(s-1)}\int_0^\infty t^{s-2}\delta_N Tr(e^{-t\Delta})\ dt$$

as in the proof of Lemma 5.5. Thus

$$\begin{aligned}\delta_N\hat{\zeta}(s-1) &= -(s-1)\zeta_N(s) - \frac{1}{\Gamma(s-1)}\int_1^\infty t^{s-2}\delta_N Tr(e^{-t\Delta})\ dt \\ &\quad + \frac{1}{\Gamma(s-1)}\int_1^\infty t^{s-2}\delta_N Tr(e^{-t(\Delta-\tilde{P})})\ dt\end{aligned}$$

which yields:

Lemma 5.18

$$\delta_N\hat{\zeta}_A(s-1) = -(s-1)\zeta_N(s) - \frac{1}{\Gamma(s-1)}\int_1^\infty t^{s-2}\delta_N Tr(e^{-t\tilde{P}})\ dt$$

We need to know that the integral in the lemma has uniform exponential decay for $Re(s)$ less than a constant. For a curve of connections $A(q)$ with $A(0) = A$, $\dot{A}(0) = N$, we may write

$$Tr(e^{-t\tilde{P}_q}) = \sum_i e^{-t\lambda_i(q)}\int_M \omega_i(q) \wedge *\omega_i(q)$$

for some smoothly varying orthonormal bases $\{\omega_i(q)\}$ of $Im\ \tilde{P}_q$. Since $1 = \int_M \omega_i \wedge *\omega_i$, we have $0 = \int_M \omega_i \wedge *\dot{\omega}_i$, and so

$$\delta_N Tr(e^{-t\tilde{P}_q}) = \sum_i e^{-t\lambda_i(q)}\dot{\lambda}_i(q)$$

Since the minimum of the λ_i is zero, we must have $\sum_{\lambda_i(0)=0} \dot{\lambda}_i(0) = 0$. Thus

$$\delta_N Tr(e^{-t\tilde{P}}) = O(e^{-t\lambda_1})$$

where λ_1 is the first nonzero eigenvalue of Δ_A. In particular,

$$\frac{1}{\Gamma(s-1)}\int_1^\infty t^{s-2}Tr(e^{-t\tilde{P}})\ dt\Big|_{s=1} = 0 \tag{5.19}$$

Just as for the ordinary zeta function, $\hat{\zeta}(0)$ is computed by plugging the heat kernel asymptotics into the first integral in (5.17) and integrating, since the second term vanishes at $s = 0$. The result is

$$\hat{\zeta}(0) = \begin{cases} \int a_{n/2} & dim\ M\ even \\ 0 & dim\ M\ odd \end{cases} \tag{5.20}$$

in the notation of Theorem 5.10. Note that we have adjusted $\hat{\zeta}(s)$ so that $dim\ Ker\ d_A$ does not appear in $\hat{\zeta}(0)$. Thus $\delta_N\hat{\zeta}(0) = 0$ at reducible connections for all the cases considered in Theorem 5.10. Setting $s = 1$ in Lemma 5.18 and using (5.19), we see that $\zeta_N(1)$ is finite. By (5.16), we obtain:

Proposition 5.21 *Theorem 5.10 is valid for reducible connections.*

We now specialize to the setup of Donaldson's theorem. Assume that the base manifold is a simply connected 4-manifold with positive definite intersection form and that P is an $SU(2)$ bundle with topological charge one. For a generic metric on M, the moduli space $\mathcal{M}$ of selfdual connections $\mathcal{SD}$ is a smooth 5-manifold in the quotient topology except at the finite number of reducible connections [15, 16]. The proofs of [16, Thms. 4.9, 4.11] show that at a reducible connection A, we can take a six dimensional orthonormal basis $\{N_i\}$ of the normal bundle of $T_A\mathcal{O}_A$ in $T\mathcal{SD}$. By the last Lemma, we may set

$$Tr\ II = Tr\ II_A = \sum_i (Tr\ H_{N_i})N_i \tag{5.22}$$

This vector field along $T_A\mathcal{O}_A$ is well defined, as the linearity of H_N in N shows that there is a unique vector in the normal bundle at A given by the right hand side of (5.22) for any orthonormal basis.

We now claim that $Tr\ II$ is gauge invariant:

$$g \cdot Tr\ II_A = Tr\ II_{g \cdot A} \tag{5.23}$$

The left hand side of this equation is $\sum(Tr\ H_{N_i})g_*N_i$, while the right hand side equals $\sum(Tr\ H_{g_*N_i})g_*N_i$, since the gauge group acts via isometries. Thus it suffices to show that $Tr\ H_{N_i} = Tr\ H_{g_*N_i}$. Dropping the asterisk, we note that $e^{-t\Delta_{gA}} = ge^{-t\Delta_A}g^{-1}$ and

$$(\delta_{gN}d^*_{gA})d_{gA} = g\delta_N d_A g^{-1} g d_A g^{-1} = g(\delta_N d^*_A)d_A g^{-1}$$

Using Definition 5.4, we get (5.23).

In finite dimensions, an orbit of isolated type for an isometric action of a compact Lie group on a manifold admits no nonzero invariant vector field, as the exponential map applied to this vector field would give a family of nearby orbits of the same type. The exponential map has difficulties in infinite dimensions, but we can reach the same conclusion in our case by appealing to the structure of $\mathcal{M}$. By [16, Thm. 4.4], $Ker\ d^*_A$ gives a local slice for the action of $\mathcal{G}$ on $\mathcal{A}$ near A, so the normal space at A within $\mathcal{SD}$ is $Ker\ d^*_A \cap Ker\ P_-$, where P_-N is the projection of d_AN into the anti-selfdual 2-forms $\Omega^2_-(Ad\ P)$. Thus the normal space is the first cohomology group H^1_A of the elliptic complex

$$\Omega^0(Ad\ P) \xrightarrow{d_A} \Omega^1(Ad\ P) \xrightarrow{P_-} \Omega^2_-(Ad\ P)$$

The stabilizer $\mathcal{G}_A$ of A is isomorphic to $U(1)$, and by [16, Prop. 4.9], $\mathcal{G}_A$ acts on $H^1_A \simeq C^3$ by the usual action of $U(1)$ on C^3. Thus there are no nonzero $\mathcal{G}$ invariant vectors in the normal space, which proves

Theorem 5.24 *Let M be a simply connected 4-manifold with positive definite intersection form and let P be an $SU(2)$ bundle over M of topological charge one. For generic metrics on M, the reducible selfdual connections form minimal gauge orbits within the space of all selfdual connections.*

Thus examples of minimal nonabelian gauge orbits exist over e.g. CP^2 and $S^2 \times S^2$.

It seems difficult to find minimal gauge orbits in the interior of $\mathcal{SD}$. According to Theorem 5.14, we must calculate $\delta_N\zeta'_A(0)$ for an irreducible selfdual connection A. We have

$$\begin{aligned}
\delta_N\zeta'_A(0) &= \frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\delta_N Tr(e^{-t\Delta})\ dt \\
&= \frac{d}{ds}\Big|_{s=0} -\frac{1}{\Gamma(s)}\int_0^\infty t^s Tr((\delta_N\Delta)e^{-t\Delta})\ dt \\
&= \frac{d}{ds}\Big|_{s=0} -\frac{2}{\Gamma(s)}\int_0^\infty Tr(d^*_A(\delta_N d_A)e^{-t\Delta})\ dt
\end{aligned}$$

$$= \left.\frac{d}{ds}\right|_{s=0} -\frac{2}{\Gamma(s)}\int_0^\infty t^s Tr(d_A^*[N, e^{-t\Delta}])\ dt$$

$$= \left.\frac{d}{ds}\right|_{s=0} -\frac{4}{\Gamma(s)}\int_0^\infty t^s Tr(d_A^* N e^{-t\Delta})\ dt$$

where in the last line we use $Tr(A(\overline{BC})^t) = -Tr(A(\overline{CB})^t)$ for $A, B, C \in U(N)$. However, we know of no nonflat case where the last line vanishes; proceeding directly by differentiating in s, we see that the last line is still nonlocal and hence not computable (cf. [2], where the ratio of the Faddeev-Popov determinant and the determinant of $P_-^* P_-$ is shown to be locally computable for $M = S^4$).

Note that $\delta_N \zeta_A'(0)$ would vanish if $\delta_N\Delta$ had terms $d_A^*(\delta_N d_A)$ and $(-\delta_N d_A^*)d_A$. In other words, if we introduce a Z_2 grading on δ_N by replacing δ_N by $(-1)^k\delta_N$ whenever δ_N acts on an operator with domain $\Omega^k(Ad\ P)$, then $\delta_N\Delta = d_A^*\delta_n d_A - (\delta_n d_A^*)d_A$, and we would have $\delta_N\zeta_A'(0) = 0$. Is there a supersymmetric theory that justifies this *ad hoc* grading? In other words, are gauge orbits minimal in some supersymmetric sense?

A Properties of parallel translation

In this Appendix we prove some fundamental properties of parallel translation which are used in the computation of the kernel $h_N(x, y)$ in §3. We use the notation of that section.

A.1 Parallel translation

Let $x, y \in M$ be contained in a neighborhood V whose diameter is less than the injectivity radius. Let $P(x, y)$ denote parallel translation from y to x along the unique minimal geodesic $x(t) = (x^i(t))$, parametrized so that $x = x(1)$. Let p be the origin of normal coordinates on V. Then

$$\ddot{x}^i + \Gamma^i_{jk}(x(t))\dot{x}^j\dot{x}^k = 0, \quad x^i(0) = y \tag{A.1}$$

where Γ^i_{jk} are the Christroffel symbols and $\dot{x}^i = \frac{d}{dt}x^i(t)$. Note that $\dot{x}^i(0) = x^i$ if $y = p$, as $\Gamma^i_{jk}(p) = 0$, while in general by (A.1)

$$\dot{x}^j(0) = x^j - y^j + \frac{1}{2}\Gamma^j_{sr}(x^s - y^s)(x^r - y^r) + O(|x - y|^3) \tag{A.2}$$

Let A be a fixed connection on $Ad\ P$ with connection one-form A_{ib}^a taking values in $\mathfrak{u}(N)$; the manifold indices will be denoted by $i, j, k, \ldots$ and the Lie algebra indices by $a, b, c, \ldots$. Parallel translation of a vector $\xi(y) = (\xi_d^c(y)) \in \Omega_y^0(Ad\ P)$ satisfies the following differential equation:

$$\begin{cases} \frac{d\xi}{dt}(t)_d^c + A_{jr}^{\ c}(x(t))\dot{x}^j\xi_d^r - A_{jd}^{\ r}(x(t))\dot{x}^j\xi_r^c = 0 \\ \xi_d^c(0) = \xi_d^c(y) \end{cases} \tag{A.3}$$

and $\xi_d^c(x) = P_{da}^{cb}(x,y)\xi_b^a(y)$.

The Taylor expansion of $\xi_d^c(t)$ is given by:

$$\begin{aligned} \xi_d^c(t) &= \xi_d^c(0) + \xi_{d,(1)}^c(0)t + \tfrac{1}{2!}\xi_{d,(2)}^c(0)t^2 + \tfrac{1}{3!}\xi_{d,(3)}^c(0)t^3 + \ldots \\ &\quad + \tfrac{1}{k!}\xi_{d,(k)}^c t^k + \cdots \end{aligned} \tag{A.4}$$

where

$$\xi_{d,(i)}^c(0) = \frac{d^i}{dt^i}\xi_d^c(0) \qquad (i = 1, 2, \ldots) \tag{A.5}$$

By (A.3) we have

$$\xi_{d,(1)}^c(0) = \dot{\xi}_d^c(0) = -A_{jr}^{\ c}(y)\dot{x}^j(0)\xi_d^r(0) + A_{jd}^{\ r}(y)\dot{x}^j(0)\xi_r^c(0) \tag{A.6}$$

We put

$$P_{da}^{cb} = {}^{(0)}P_{da}^{cb} + {}^{(1)}P_{da}^{cb} + {}^{(2)}P_{da}^{cb} + {}^{(3)}P_{da}^{cb} + {}^{(4)}P_{da}^{cb} + 0(|x-y|^5) \tag{A.7}$$

where

$${}^{(l)}P_{da}^{cb}\xi_b^a(y) = \frac{1}{l!}\frac{d^l}{dt^l}\xi_d^c(0) \qquad (l = 0, 1, 2, \ldots) \tag{A.8}$$

It is easily seen that ${}^{(l)}P_{da}^{cb} = 0(|x-y|^l)$.

Lemma A.9

$${}^{(0)}P_{da}^{cb} = \delta_a^c\delta_d^b \tag{A.10}$$

and

$${}^{(1)}P_{da}^{cb} = -A_{ja}^{\ c}(y)\dot{x}^j(0)\delta_d^b + A_{jd}^{\ b}(y)\dot{x}^j(0)\delta_a^c \tag{A.11}$$

PROOF. (A.10) and (A.11) follow from the initial condition in (A.3) and from (A.6).

A.2 First and second derivatives of $P(x,y)$

We first show:

Lemma A.12

$$P(p,p) = Id \tag{A.13}$$

$$^x\nabla P(p,p) = {^y\nabla} P(p,p) = 0 \tag{A.14}$$

where $^x\nabla$, $^y\nabla$ *denote covariant derivatives with respect to* x, y.

PROOF. Lemma A.9 implies (A.13). Note that

$$^x\nabla_j P^{cb}_{da} = \partial_{x_j} P^{cb}_{da} + A_{j\,r}^{\;c}(x) P^{rb}_{da} - A_{j\,d}^{\;r}(x) P^{cb}_{ra}. \tag{A.15}$$

Substituting (A.10) and (A.11) into (A.15), we obtain $^x\nabla P(p,p) = 0$. Using

$$P(x,y)P(y,x) = Id, \tag{A.16}$$

we get $^y\nabla P(p,p) = 0$.

From now on, we work in a synchronous frame for $Ad\ P$ over V, which by definition is given by parallel translating a chosen orthonormal frame at p out the radial geodesics. In such a frame, we have

$$A_{i\,j}^{\;k}(p) = 0, \quad \forall i,j,k \tag{A.17}$$

[6, Prop. 3.7]. (In some formulas below, we keep terms involving $A_{i\,j}^{\;k}(p)$ as we will need to differentiate them later.) For example, from Lemma A.12 we have

$$\partial_{x^i} P(p,p) = \partial_{y^j} P(p,p) = 0 \tag{A.18}$$

Lemma A.19

$$^x\Delta P(p,p) = {^y\Delta} P(p,p) = 0 \tag{A.20}$$

where $^x\Delta = -g^{ij}\ {^x\nabla_i}{^x\nabla_j}$.

PROOF. Since

$$^x\Delta P^{cb}_{da} = -g^{ij}\{\partial_{x^i}{^x\nabla_j} P^{cb}_{da} - \Gamma^m_{ij}(x){^x\nabla_m} P^{cb}_{da} + A_{i\,r}^{\;c}(x){^x\nabla_j} P^{rb}_{da} - A_{j\,d}^{\;r}(x){^x\nabla_j} P^{cb}_{ra}\}, \tag{A.21}$$

by (A.14) we have

$$^x\Delta P^{cb}_{da}|_{x=y=p} = -g^{ij}\partial_{x^i}{^x\nabla_j} P^{cb}_{da}|_{x=y=p} \tag{A.22}$$

This gives

$$
\begin{aligned}
{}^{x}\Delta P^{cb}_{da}|_{x=y=p} &= -\,\delta^{ij}[\partial_{x^i}\partial_{x^j}P^{cb}_{da} + \partial_{x^i}A_{jr}^{c}\cdot P^{cb}_{da} + A_{jr}^{c}\partial_{x^i}P^{rb}_{da} \\
&\qquad -\partial_{x^i}A_{jd}^{r}\cdot P^{cb}_{ra} - A_{jd}^{r}\partial_{x^i}P^{cb}_{ra}]|_{x=y=p}
\end{aligned}
\tag{A.23}
$$

To obtain (A.20), we compute $\ddot{\xi}^c_d(0)$. Differentiating (A.3) in t gives

$$
\begin{aligned}
\ddot{\xi}^c_d(0) &= -\partial_l A_{jr}^{c}(y)\dot{x}^l\dot{x}^j\xi^r_d(0) - A_{jr}^{c}(y)\ddot{x}^j\xi^r_d(0) - A_{jr}^{c}(y)\dot{x}^j(0)\dot{\xi}^r_d(0) \\
&\quad +\partial_l A_{jd}^{r}(y)\dot{x}^l\dot{x}^j\xi^c_r(0) + A_{jd}^{r}(y)\ddot{x}^j\xi^c_r(0) + A_{jd}^{r}(y)\dot{x}^j(0)\dot{\xi}^c_r(0) \\
&= -\partial_l A_{jr}^{c}(y)\dot{x}^l\dot{x}^j\xi^r_d(0) + A_{jr}^{c}(y)\Gamma^j_{mn}(y)\dot{x}^m\dot{x}^n\xi^r_d(0) \\
&\quad -A_{jr}^{c}(y)(-A_{ls}^{r}(y)\dot{x}^l\xi^s_d(0) + A_{ld}^{s}(y)\dot{x}^l\xi^r_s(0))\dot{x}^j \\
&\quad +\partial_l A_{jd}^{r}(y)\dot{x}^l\dot{x}^i\xi^c_r(0) - A_{jd}^{r}\Gamma^j_{mn}(y)\dot{x}^m\dot{x}^n\xi^c_r(0) \\
&\quad +A_{jd}^{r}(y)\dot{x}^j(-A_{ls}^{c}(y)\dot{x}^l\xi^s_r(0) + A_{lr}^{s}(y)\dot{x}^l\xi^c_s(0))
\end{aligned}
\tag{A.24}
$$

This implies

$$
\begin{aligned}
{}^{(2)}P^{cb}_{da} &= \tfrac{1}{2}[-\partial_l A_{ja}^{c}(y)\dot{x}^l\dot{x}^j\delta^b_d + A_{ja}^{c}(y)\Gamma^j_{mn}(y)\dot{x}^m\dot{x}^n\delta^b_d + A_{jr}^{c}(y)A_{la}^{r}(y)\dot{x}^l\dot{x}^j\delta^b_d \\
&\quad -A_{ja}^{c}(y)A_{ld}^{b}(y)\dot{x}^l\dot{x}^j + \partial_l A_{jd}^{b}(y)\dot{x}^l\dot{x}^j\delta^c_a - A_{jd}^{b}(y)\Gamma^j_{mn}(y)\dot{x}^m\dot{x}^n\delta^c_a] \\
&\quad A_{jd}^{b}(y)A_{la}^{c}\dot{x}^j\dot{x}^l + A_{jd}^{r}(y)A_{lr}^{b}(y)\dot{x}^l\dot{x}^j\delta^c_a
\end{aligned}
\tag{A.25}
$$

Thus,

$$
\begin{aligned}
P^{cb}_{da} &= \delta^c_a\delta^b_a + (-A_{ja}^{c}(y)(S^j + \tfrac{1}{2}\Gamma^i_{mn}(y)S^mS^n)\delta^b_d) + A_{jd}^{b}(y)(S^j + \tfrac{1}{2}\Gamma^i_{mn}(y)S^mS^n)\delta^c_a \\
&\quad +\tfrac{1}{2!}[-\partial_l A_{ja}^{c}(y)S^lS^j\delta^b_d + A_{ja}^{c}(y)\Gamma^j_{mn}(y)S^mS^n\delta^b_d \\
&\quad -A_{jr}^{c}(y)(-A_{la}^{r}(y)\delta^b_d + A_{ld}^{b}(y)\delta^r_a)S^lS^j + \partial_l A_{jd}^{b}(y)S^lS^j\delta^c_a \\
&\quad -A_{jd}^{b}(y)\Gamma^j_{mn}(y)S^mS^n\delta^c_a + A_{jd}^{r}(y)(-A_{la}^{c}(y)\delta^b_r + A_{lr}^{b}(y)\delta^r_a)S^lS^j] + 0(S^3)
\end{aligned}
\tag{A.26}
$$

where $S^j = x^j - y^j$. Applying $\partial_{x^i}\partial_{x^j}$ to (A.26), taking $x = y = p$, using (A.17), and substituting into (A.23), we get ${}^{x}\Delta P(p,p) = 0$. Differentiating (A.16) and using Lemma A.12 gives ${}^{y}\Delta P(p,p) = 0$.

A.3 Third derivatives of $P(x,y)$

A computation similar to (A.24) gives

$$
\begin{aligned}
\frac{d^3\xi^c_d}{dt^3} &= \dot{x}^j\dot{x}^l\dot{x}^q\Big[-\partial_q\partial_l A^{c}_{jr}\xi^r_d + 2\partial_l A^{c}_{kr}\Gamma^k_{jq}\xi^r_d + \partial_k A^{c}_{jr}\Gamma^k_{ql}\xi^r_d - 2\partial_l A^{c}_{jr}(-A^{r}_{qs}\xi^s_d + A^{s}_{qd}\xi^r_s) \\
&\quad -A^{c}_{kr}(-\partial_l\Gamma^k_{qj} + 2\Gamma^k_{sl}\Gamma^s_{jq})\xi^r_d + 2A^{c}_{kr}\Gamma^k_{ql}(-A^{r}_{js}\xi^s_d + A^{s}_{jd}\xi^r_s) - A^{c}_{jr}[-\partial_l A^{r}_{qs}\xi^s_d \\
&\quad +A^{r}_{ks}\Gamma^k_{ql}\xi^s_d - A^{r}_{ls}(-A^{s}_{qk}\xi^k_d + A^{k}_{qd}\xi^c_k) + \partial_l A^{s}_{qd}\xi^r_s - A^{s}_{kd}\Gamma^k_{lq}\xi^r_s \\
&\quad +A^{s}_{ld}(-A^{r}_{qk}\xi^k_s + A^{k}_{qs}\xi^r_k)] + \partial_q\partial_l A^{r}_{jd}\xi^c_r - 2\partial_l A^{r}_{kd}\Gamma^k_{jq}\xi^c_r - \partial_k A^{r}_{jd}\Gamma^k_{ql}\xi^c_r \\
&\quad +2\partial_l A^{r}_{jd}(-A^{c}_{qs}\xi^s_r + A^{s}_{qr}\xi^c_s) + A^{r}_{kd}(-\partial_l\Gamma^k_{qj} + 2\Gamma^k_{sl}\Gamma^s_{jq})\xi^c_r \\
&\quad -2A^{r}_{kd}\Gamma^k_{ql}(-A^{c}_{js}\xi^s_r + A^{s}_{jr}\xi^c_s) + A^{r}_{jd}[-\partial_l A^{c}_{qs}\xi^s_r + A^{c}_{ks}\Gamma^k_{ql}\xi^s_r \\
&\quad -A^{c}_{ls}(-A^{s}_{qk}\xi^k_r + A^{k}_{qr}\xi^s_k) + \partial_l A^{s}_{qr}\xi^c_s - A^{s}_{kr}\Gamma^k_{lq}\xi^c_s + A^{s}_{kr}(-A^{c}_{qk}\xi^k_s + A^{k}_{qs}\xi^c_k)]\Big]
\end{aligned}
\tag{A.27}
$$

This yields

$$
\begin{aligned}
{}^{(3)}P^{cb}_{da} &= \tfrac{1}{3!}\dot{x}^j\dot{x}^l\dot{x}^q\Big[-\partial_l\partial_q A^{c}_{ja}\delta^b_d + 2\partial_l A^{c}_{ka}\Gamma^k_{jq}\delta^b_d + \partial_k A^{c}_{ja}\Gamma^k_{ql}\delta^b_d + 2\partial_l A^{c}_{jr}A^{r}_{qa}\delta^b_d \\
&\quad -2\partial_l A^{c}_{ja}A^{b}_{qd} - A^{c}_{ka}(-\partial_l\Gamma^k_{qj} + 2\Gamma^k_{sl}\Gamma^s_{jq})\delta^b_d - 2A^{c}_{kr}\Gamma^k_{ql}A^{r}_{ja}\delta^b_d \\
&\quad +2A^{c}_{ka}A^{b}_{jd}\Gamma^k_{ql} + A^{c}_{jr}\partial_l A^{r}_{qa}\delta^b_d - A^{c}_{jr}A^{r}_{sa}\Gamma^s_{ql}\delta^b_d - A^{c}_{jr}A^{r}_{ls}A^{s}_{qa}\delta^b_d + A^{c}_{jr}A^{r}_{la}A^{b}_{qd} \\
&\quad -A^{c}_{ja}\partial_l A^{b}_{qd} + A^{c}_{ja}A^{b}_{sd}\Gamma^s_{lq} + A^{c}_{jr}A^{b}_{ld}A^{r}_{qa} - A^{c}_{ja}A^{s}_{ld}A^{b}_{qs} + \partial_l\partial_q A^{b}_{jd}\delta^c_a \\
&\quad -2\partial_l A^{b}_{kd}\Gamma^k_{jq}\delta^c_a - \partial_k A^{b}_{jd}\Gamma^k_{ql}\delta^c_a - 2\partial_l A^{b}_{jd}A^{c}_{qa} + 2\partial_l A^{r}_{jd}A^{b}_{qr}\delta^c_a \\
&\quad +A^{b}_{kd}(-\partial_l\Gamma^k_{qj} + 2\Gamma^k_{sl}\Gamma^s_{jq})\delta^c_a + 2A^{b}_{kd}\Gamma^k_{ql}A^{c}_{ja} - 2A^{r}_{kd}A^{b}_{jr}\Gamma^k_{ql}\delta^c_a - A^{b}_{jd}\partial_l A^{c}_{qa} \\
&\quad +A^{b}_{jd}A^{c}_{ka}\Gamma^k_{ql} + A^{b}_{jd}A^{c}_{ls}A^{s}_{qa} + A^{r}_{jd}A^{c}_{la}A^{b}_{qr} + A^{r}_{jd}\partial_l A^{b}_{qr}\delta^c_a - A^{r}_{jd}A^{b}_{kr}\Gamma^k_{lq} \\
&\quad -A^{r}_{jd}A^{b}_{kr}A^{c}_{qa} + A^{r}_{jd}A^{s}_{kr}A^{b}_{qs}\delta^c_a\Big]
\end{aligned}
\tag{A.28}
$$

We now state the first of our two main third derivative calculations. Recall that the curvature two-form F of the connection A is given in local coordinates by

$$F_{ija}{}^{b} = \partial_i A_{ja}^{\ b} - \partial_j A_{ia}^{\ b} + A_{ie}^{\ b} A_{ja}^{\ e} - A_{je}^{\ b} A_{ia}^{\ e}.$$

and that

$$\nabla^i F_{ija}{}^{b} = g^{ik}[\partial_k F_{ija}{}^{b} - \Gamma^l_{ki} F_{lja}{}^{b} - \Gamma^l_{kj} F_{ila}{}^{b} + A_{kc}^{\ b} F_{ija}{}^{c} - A_{ka}^{\ c} F_{ijc}{}^{b}]$$

Thus at the origin p in a normal coordinate system and with respect to a synchronous frame, we have

$$\begin{aligned} F_{ija}{}^{b}\Big|_{x=p} &= \partial_i A_{ja}^{\ b} - \partial_j A_{ia}^{\ b} \\ \nabla^i F_{ija}{}^{b}\Big|_{x=p} &= \sum_i (\partial_i \partial_i A_{ja}^{\ b} - \partial_i \partial_j A_{ia}^{\ b}) \end{aligned} \qquad \text{(A.29)}$$

Proposition A.30

$$\begin{aligned} {}^y\nabla_s\ {}^x\Delta P^{cb}_{da}(0,0) &= \tfrac{1}{3}(\nabla_i F^{i}{}_{sa}{}^{c}\delta^b_d - \nabla_i F^{i}{}_{sa}{}^{b}\delta^c_a) \\ &= -{}^x\nabla_s\ {}^y\Delta P^{cb}_{da}(0,0) \end{aligned} \qquad \textit{(A.31)}$$

PROOF. As before, differentiating (A.16) gives

$${}^y\nabla_s\ {}^x\Delta P^{cb}_{da}(0,0) = -{}^x\nabla_s\ {}^y\Delta P^{cb}_{da}(0,0)$$

As in Lemma A.19, we have

$${}^y\nabla_s\ {}^x\Delta P^{cb}_{da}\Big|_{x=y=p} = g^{ij}\partial_{y^s}\nabla_{x_i}\nabla_{x_j} P^{cb}_{da}\Big|_{x=y=p} \qquad \text{(A.32)}$$

where we use $A_{ij}^{\ k}(p) = 0, \partial_i P(p,p) = 0$ and $g^{ij}(p) = \delta^{ij}$. Set

$$(P^{cb}_{da})_{\leq 2} = \sum_{l=0}^{2} {}^{(l)}P^{cb}_{da}. \qquad \text{(A.33)}$$

Using (A.26), we have

$$\sum_i \partial_{y^s}\partial_{x^i}\partial_{x^i}(P^{cb}_{da})_{\leq 2}\Big|_{x=y=p} =$$

$$\sum_i [-A_{qa}^{\ c}\partial_{y^s}\Gamma^q_{ii}\delta^b_d + A_{qd}^{\ b}\partial_{y^s}\Gamma^q_{ii}\delta^c_a + \frac{1}{3}\{(A_{qa}^{\ c}\partial_{y^s}\Gamma^q_{ii} + A_{qa}^{\ c}\partial_{y^i}\Gamma^q_{si} + A_{qa}^{\ c}\partial_{y^i}\Gamma^q_{si})\delta^b_d$$

$$-(A_{qd}^{\;b}\partial_{y^s}\Gamma_{ii}^q + A_{qd}^{\;b}\partial_{y^i}\Gamma_{si}^q + A_{qd}^{\;b}\partial_{y^i}\Gamma_{si}^q)\delta_a^c\} + \{-\partial_{y^s}\partial_{y^i}A_{ja}^{\;c}\delta_d^b + A_{qa}^{\;c}\partial_{y^s}\Gamma_{ii}^q\delta_d^b$$

$$+\partial_{y^s}A_{jv}^{\;c}A_{ia}^{\;v}\delta_d^b + A_{jv}^{\;c}\partial_{y^s}A_{la}^{\;v}\delta_d^b - 2\partial_{y^s}A_{ja}^{\;c}A_{ld}^{\;b} - 2A_{ja}^{\;c}\partial_{y^s}A_{id}^{\;b} + \partial_{y^s}\partial_{y^i}A_{id}^{\;b}\delta_a^c$$

$$-A_{qd}^{\;b}\partial_{y^s}\Gamma_{ii}^q\delta_a^c + \partial_{y^s}A_{id}^{\;r}A_{ir}^{\;b}\delta_a^c + A_{id}^{\;r}\partial_{y^s}A_{ir}^{\;b}\delta_a^c\}]\Big|_{x=y=p}$$

$$= -\partial_{y^s}\partial_{y^i}A_{ia}^{\;c}\delta_d^b + -\partial_{y^s}\partial_{y^i}A_{id}^{\;b}\delta_a^c \tag{A.34}$$

Now we compute $\sum_i \partial_{y^s}\partial_{x^i}\partial_{x^i}{}^{(3)}P_{da}^{cb}\Big|_{x=y=p}$. Note that $\dfrac{d^3\xi_d^c}{dt^3} = 3!^{(3)}P_{da}^{cb}$. Differentiating (A.27) in x and y, we obtain

$$\frac{1}{3!}\sum_i \partial_{y^s}\partial_{x^i}\partial_{x^i}{}^{(3)}P_{da}^{cb}\Big|_{x=y=p}$$

$$= \frac{1}{3}\sum_i\Big[2\partial_{y^i}\partial_{y^s}A_{ia}^{\;c}\delta_d^b + \partial_{y^i}\partial_{y^i}A_{sa}^{\;c}\delta_d^b - 2\partial_{y^s}A_{ir}^{\;c}A_{ia}^{\;r}\delta_d^b - 2\partial_{y^i}A_{sr}^{\;c}A_{ia}^{\;r}\delta_d^b$$

$$-2\partial_{y^i}A_{ir}^{\;c}A_{sa}^{\;r}\delta_d^b - A_{qa}^{\;c}\partial_{y^s}\Gamma_{ii}^q\delta_d^b - 2A_{qa}^{\;c}\partial_{y^i}\Gamma_{si}^q\delta_d^b - A_{sr}^{\;c}\partial_{y^i}A_{ia}^{\;r}\delta_d^b - A_{ir}^{\;c}\partial_{y^s}A_{ia}^{\;r}\delta_d^b$$

$$-A_{ir}^{\;c}\partial_{y^i}A_{sa}^{\;r}\delta_d^b + A_{sr}^{\;c}A_{iv}^{\;r}A_{ia}^{\;v}\delta_d^b + A_{ir}^{\;c}A_{sv}^{\;r}A_{ia}^{\;v}\delta_d^b + A_{ir}^{\;c}A_{iv}^{\;r}A_{sa}^{\;v}\delta_d^b$$

$$-2\partial_{y^i}\partial_{y^s}A_{id}^{\;b}\delta_a^c - \partial_{y^i}\partial_{y^i}A_{sd}^{\;b}\delta_a^c - 2\partial_{y^s}A_{id}^{\;r}A_{ir}^{\;b}\delta_a^c - 2\partial_{y^i}A_{sd}^{\;r}A_{ir}^{\;b}\delta_a^c$$

$$-2\partial_{y^i}A_{id}^{\;r}A_{sr}^{\;b}\delta_a^c + A_{qd}^{\;b}\partial_{y^s}\Gamma_{ii}^q\delta_a^c + 2A_{qd}^{\;b}\partial_{y^i}\Gamma_{si}^q\delta_a^c - A_{sd}^{\;r}\partial_{y^i}A_{ir}^{\;b}\delta_a^c - A_{id}^{\;r}\partial_{y^s}A_{ir}^{\;b}\delta_a^c$$

$$-A_{id}^{\;r}\partial_{y^i}A_{sr}^{\;b}\delta_a^c - A_{sd}^{\;r}A_{ir}^{\;v}A_{iv}^{\;b}\delta_a^c - A_{id}^{\;r}A_{sr}^{\;v}A_{iv}^{\;b}\delta_a^c - A_{id}^{\;r}A_{ir}^{\;v}A_{sv}^{\;b}\delta_a^c \tag{A.35}$$

$$+3\partial_{y^s}A_{ia}^{\;c}A_{id}^{\;b} + 3\partial_{y^i}A_{sa}^{\;c}A_{id}^{\;b} + 3\partial_{y^i}A_{ia}^{\;c}A_{sd}^{\;b} + 3A_{sa}^{\;c}\partial_{y^i}A_{id}^{\;b} + 3A_{ia}^{\;c}\partial_{y^s}A_{id}^{\;b}$$

$$+3A_{ia}^{\;c}\partial_{y^i}A_{sd}^{\;b} - 3A_{sr}^{\;c}A_{ia}^{\;r}A_{id}^{\;b} - 3A_{ir}^{\;c}A_{sa}^{\;r}A_{id}^{\;b} - 3A_{ir}^{\;c}A_{ia}^{\;r}A_{sd}^{\;b} + 3A_{sa}^{\;c}A_{id}^{\;v}A_{iv}^{\;b}$$

$$+3A_{ia}^{\;c}A_{sd}^{\;v}A_{iv}^{\;b} + 3A_{ia}^{\;c}A_{id}^{\;v}A_{sv}^{\;b}\Big]$$

$$= \frac{1}{3}\sum_i(-\partial_{y^i}\partial_{y^s}A_{ia}^{\;c}\delta_d^b + \partial_{y^i}\partial_{y^i}A_{sa}^{\;c}\delta_d^b + \partial_{y^i}\partial_{y^s}A_{id}^{\;b}\delta_a^c - \partial_{y^i}\partial_{y^i}A_{sd}^{\;b}\delta_a^c)$$

Combining (A.32), (A.34) and (A.35), we have

$$
\begin{aligned}
&{}^y\nabla_s\ {}^x\Delta P^{cb}_{da}\Big|_{x=y=0} \\
&= \frac{1}{3}\sum_i\Big[[\partial_i\partial_i A_{sa}{}^c - \partial_i\partial_s A_{ia}{}^c]\delta^b_d - [\partial_i\partial_i A_{sd}{}^b - \partial_i\partial_s A_{id}{}^b]\delta^c_a\Big] \\
&= \frac{1}{3}\sum_i[\nabla_i F_{isa}{}^c\delta^d_b - \nabla_i F_{isb}{}^d\delta^c_a]
\end{aligned}
$$

which by (A.29) gives Proposition A.30.

We finish with one more third derivative calculation.

Proposition A.36

$$
{}^x\nabla_i\ {}^x\Delta P^{cb}_{da}\Big|_{x=y=p} = \frac{1}{3}[-\nabla_j F^j{}_{ia}{}^c\delta^b_d + \nabla_j F^j{}_{id}{}^b\delta^c_a] \qquad (A.37)
$$

PROOF. Since the result only involves x derivatives, we may assume that the geodesic coordinates are centered at y. The distance function satisfies $r^2 = \sum g_{ij}x^i x^j$, where $g_{ij} = \delta_{ij} - \frac{1}{3}R_{ikj\ell}x^k x^\ell + \cdots$, so

$$
\nabla r^2 = O(r) \ \text{ and } \ \nabla_i\nabla^j r^2 = 2\delta^j_i + O(r^2). \qquad (A.38)
$$

For any bundle, parallel translation is characterized by

$$
P(x,x) = Id \ \text{ and } \ \nabla_{\partial/\partial r}P = 0.
$$

Rewriting this last equation as $\nabla^i r^2\cdot\nabla_i P = 0$ and differentiating gives

$$
\nabla_j\nabla^i r^2\cdot\nabla_i P + \nabla^i r^2\cdot\nabla_j\nabla_i P = 0, \qquad (A.39)
$$

$$
\nabla^k\nabla_j\nabla^i r^2\cdot\nabla_i P + 2\nabla^k\nabla^i r^2\cdot\nabla_j\nabla_i P + \nabla^i r^2\cdot\nabla^k\nabla_j\nabla_i P = 0. \qquad (A.40)
$$

Evaluating (A.39) at $r=0$ and using (A.38) gives (A.14):

$$
(\nabla P)(x,x) = 0 \qquad (A.41)
$$

and after setting $k=j$, (A.40) gives (A.20):

$$
(\Delta P)(x,x) = 0. \qquad (A.42)
$$

We now differentiate (A.40) again by applying ∇_k and evaluating at $r=0$. Using (A.38), (A.41), we get

$$
\nabla_k\nabla^k\nabla_j P + 2\nabla^k\nabla_j\nabla_k P = 0.
$$

Using $\nabla^k\nabla_j = \nabla_j\nabla^k + F_j^k$ this becomes

$$\nabla_k(F_j^kP) + 3\nabla_j\nabla_k\nabla^kP + 3F_{kj}\nabla^kP = 0$$

which by (A.41) reduces to

$$\nabla_j(\Delta P) = -\frac{1}{3}(\nabla_kF_j^k)P \tag{A.43}$$

(intrinsically, this is $\nabla(\Delta P) = \frac{1}{3}d^*F\cdot P$). For the adjoint bundle, the curvature acts by the adjoint representation $F\cdot\xi = [F,\xi]$, so in coordinates (A.43) becomes

$$\nabla_j(\Delta P)_{cd}^{ab} = -\frac{1}{3}\left[\nabla_kF_{jc}^{ka}\delta_d^b - \delta_b^a\nabla_kF_{jd}^{kc}\right].$$

Corollary A.44

$$^y\nabla\ ^x\Delta P(p,p) = -^x\nabla\ ^x\Delta P(p,p)$$

This follows immediately from Propositions A.30 and A.36.

B The proof of Proposition 4.1

Fix y and a neighborhood V having geodesic polar coordinates centered at y. Put $r = dist(x,y)$ and $\Gamma = r^2$. Let $\exp_y\colon T_yM \to M$ be the exponential map at y, and set $\theta(x,y) = \det(d\exp_y(\exp_y^{-1}x))$ for $x\in V$. The following lemma can be found in [5, G.V.3]:

Lemma B.1 *For any smooth function $f = f(r)$, we have*

$$\Delta f(r) = -\frac{d^2}{dr^2}f(r) - \frac{d}{dr}f(r)\Big(\frac{\theta'}{\theta} + \frac{n-1}{r}\Big)$$

where θ' denotes the derivative in r.

The projection P_A onto $Ker\ \Delta_A = Ker\ d_A^*d_A = Ker\ d_A$ has a smooth kernel $\omega(x,y) = \sum_\lambda \phi_\lambda(x)\otimes\phi_\lambda(y)$, where $\{\phi_\lambda\}$ is a finite dimensional orthonormal basis of $Ker\ d_A$. To obtain the coefficients $u_j, \hat{u}_j$ in Proposition 4.1, we formally solve the equation

$$^x\Delta_AG_A(x,y) = -\omega(x,y). \tag{B.2}$$

for $x \neq y$ in V to iteratively determine $u_j, \hat{u}_j$. Note that the infinite sum obtained differs from $G_A(x,y)$ by a bounded term, as in the proof in §4.4, so it is an asymptotic expansion in the usual sense only up to bounded terms.

From now on we just write Δ for ${}^x\Delta_A$ and G for G_A. Let $dim\ M = n$.

(i) **Case n odd.**

We assume that G has the asymptotic expansion

$$G(x,y) \sim \sum_{j=0}^{\infty} \Gamma^{j-\tau} u_j(x,y) + \sum_{j=0}^{\infty} \Gamma^j v_j(x,y) \tag{B.3}$$

with $\tau = (n-2)/2$. By Lemma B.1, we have

$$\Delta(\Gamma^j) = -j(4j - 4 + 2n + 2r\frac{\theta'}{\theta})\Gamma^{j-1} \tag{B.4}$$

and so

$$\begin{aligned} \Delta(\Gamma^{j-\tau} u_j) &= \Delta u_j \cdot \Gamma^{j-\tau} - 4(j-\tau)\Gamma^{j-\tau} r \cdot {}^r\nabla u_j + u_j \Delta \Gamma^{j-\tau} \\ &= -\Gamma^{j-\tau}(j-\tau)\Big[4r \cdot {}^r\nabla u_j + (4j + 2r\tfrac{\theta'}{\theta})u_j\Big] + \Gamma^{j-\tau}\Delta u_j \end{aligned} \tag{B.5}$$

where ${}^r\nabla$ is the covariant derivative in the direction $\partial/\partial r$. Plugging in the right hand side of (B.3) for G gives

$$\begin{aligned} \Delta G(x,y) &= \sum_{j=0}^{\infty} -(j-\tau)\Gamma^{j-\tau-1}\Big[4r \cdot {}^r\nabla u_j + (4j + 2r\frac{\theta'}{\theta})u_j - \frac{1}{j-\tau}\Delta u_{j-1}\Big] \\ &\quad + \sum_{j=0}^{\infty}(-j)\Gamma^{j-1}\Big[(4j + 2n - 4 + 2r\frac{\theta'}{\theta})v_j - 4r \cdot {}^r\nabla v_j + \Delta v_{j-1}\Big] \\ &\equiv -\omega(x,y) \end{aligned} \tag{B.6}$$

where we put $u_{-1} \equiv v_{-1} \equiv 0$. Setting the right hand side of (B.6) equal to zero gives

$$\begin{cases} {}^r\nabla u_0 + 2\frac{\theta'}{\theta} u_0 = 0 & \\ {}^r\nabla(r^j u_0^{-1} u_j) = \frac{r^{j-1}u_0^{-1}}{4(j-\tau)}\Delta u_{j-1} & j \geq 1 \\ v_0 = 0 & \\ {}^r\nabla(r^{2n} u_0^{-1} v_1) = \frac{r^{2n-1}u_0^{-1}}{4r^{2n}}\omega(x,y) & \\ {}^r\nabla(r^{2n+4j-4} u_0^{-1} v_j) = -\frac{r^{2n+4j-5}u_0^{-1}}{4j}\Delta v_{j-1} & j \geq 2 \end{cases} \tag{B.7}$$

This has a solution given by

$$\begin{cases} u_0(x,y) = \theta^{-\frac{1}{2}}(x,y)P(x,y) & \\ u_j(x,y) = \frac{u_0(x,y)}{4(j-\tau)r^j}\int_0^r s^{j-1}u_0^{-1}(x(s),y)\Delta u_{j-1}(x(s),y)\, ds & j \geq 1 \\ v_0(x,y) = 0 & \\ v_1(x,y) = -\frac{u_0(x,y)}{4r^{2n}}\int_0^r s^{2n-1}u_0^{-1}(x(s),y)\omega(x(s),y)\, ds & \\ v_j(x,y) = -\frac{u_0(x,y)}{jr^{2n}}\int_0^r s^{2n+4j-5}u_0^{-1}(x(s),y)\Delta v_{j-1}(x(s),y)\, ds & j \geq 2 \end{cases} \tag{B.8}$$

In particular, we have

$$u_1(x,y) = \frac{u_0(x,y)}{4(j-\tau)r^j} \int_0^r u_0^{-1}(x(s),y)\Delta u_0(x(s),y)\, ds \tag{B.9}$$

(ii) **Case n even,** $n \geq 4$.
Here we assume the asymptotic expansion

$$G(x,y) \sim \sum_{j=0}^{\infty} \Gamma^{j-\tau} u_j + \sum_{j=0}^{\infty} \Gamma^j \log \Gamma \cdot \hat{u}_j \tag{B.10}$$

By Lemma B.1 we have

$$\Delta(\Gamma^j \log \Gamma) = -\frac{d^2}{dr^2}(\Gamma^j \log \Gamma) - {}^r\nabla(\Gamma^j \log \Gamma)\Big(\frac{\theta'}{\theta} + \frac{n-1}{r}\Big) \tag{B.11}$$

and

$$\begin{cases} {}^r\nabla(\Gamma^j \log \Gamma) = 2jr\Gamma^{j-1} \log \Gamma + 2r\Gamma^{j-1} \\ \frac{d^2}{dr^2}(\Gamma^j \log \Gamma) = j(4j-2)\Gamma^{j-1} \log \Gamma + (8j-2)\Gamma^{j-1} \end{cases} \tag{B.12}$$

By a direct calculation, we get for $j \geq 1$,

$$\begin{aligned} \Delta(\hat{u}_j\Gamma^j \log \Gamma) &= \Delta\hat{u}_j \cdot \Gamma^j \log \Gamma - 2\langle d_A\hat{u}_j, \nabla\Gamma^j \log \Gamma\rangle + \hat{u}_j\Delta(\Gamma^j \log \Gamma) \\ &= \Delta\hat{u}_j \cdot \Gamma^j \log \Gamma - 4j\Gamma^{j-1} \log \Gamma \cdot r \cdot {}^r\nabla\hat{u}_j - 4\Gamma^{j-1} r \cdot {}^r\nabla\hat{u}_j \\ &\quad + \hat{u}_j\Delta(\Gamma^j \log \Gamma) \\ &= \Delta\hat{u}_j \cdot \Gamma^j \log \Gamma - 4j\Gamma^{j-1} \log \Gamma \cdot r \cdot {}^r\nabla\hat{u}_j - 4\Gamma^{j-1} r \cdot {}^r\nabla\hat{u}_j \\ &\quad + \hat{u}_j[-j(4j-2)\Gamma^{j-1} \log \Gamma - (8j-2)\Gamma^{j-1} \\ &\quad - (2jr\Gamma^{j-1} \log \Gamma + 2\Gamma^{j-1} r)\Big(\tfrac{\theta'}{\theta} + \tfrac{n-1}{r}\Big)] \end{aligned} \tag{B.13}$$

(where the connection ∇ acts in the x variable), and for $j = 0$

$$\begin{aligned} \Delta(\hat{u}_0 \log \Gamma) &= \Delta\hat{u}_0 \cdot \log \Gamma - 4\Gamma^{-1} r \cdot {}^r\nabla\hat{u}_0 + \hat{u}_0\Delta \log \Gamma \\ &= \Delta\hat{u}_0 \cdot \log \Gamma - \Gamma^{-1}[4r \cdot {}^r\nabla\hat{u}_0 + (2n - 4 + 2r\tfrac{\theta'}{\theta})\hat{u}_0] \end{aligned} \tag{B.14}$$

Computing as in Case (i), we get

$$\sum_{j=0}^{\tau-1} \Delta(\Gamma^{j-\tau} u_j) = -\sum_{j=0}^{\tau-1} \Gamma^{j-\tau-1}(j-\tau)[r \cdot {}^r\nabla u_j + (4j - 2r\frac{\theta'}{\theta})u_j - \frac{1}{j-\tau}\Delta u_{j-1}]$$

where we put $u_{-1} \equiv 0$. Plugging in (B.10) gives

$$\begin{aligned} \Delta G(x,y) &= -\textstyle\sum_{j=1}^{\infty} \Gamma^{j-1}[4r \cdot {}^r\nabla\hat{u}_j + (8j - 4 + M)\hat{u}_j] \\ &\quad -\tfrac{1}{\Gamma}[4r \cdot {}^r\nabla\hat{u}_0 + (M-4)\hat{u}_0 - \Delta u_{\tau-1}] \\ &\quad -\textstyle\sum_{j=1}^{\infty} j\Gamma^{j-1} \log \Gamma[4r \cdot {}^r\nabla\hat{u}_j + (4j - 4 + M)\hat{u}_j - \tfrac{1}{j}\Delta\hat{u}_{j-1}] \\ &\quad -\textstyle\sum_{j=0}^{\infty}(j-\tau)\Gamma^{j-\tau-1}[4r \cdot {}^r\nabla u_j + (4j - 2n + M)u_j - \tfrac{1}{j-\tau}\Delta u_{j-1}] \\ &\equiv -\omega(x,y) \end{aligned} \tag{B.15}$$

where $M = 2n + 2r(\theta'/\theta)$. To make each term in (B.15) vanish, we set

$$\begin{cases} u_0(x,y) = \theta^{-\frac{1}{2}}(x,y)P(x,y) \\ u_j(x,y) = -\frac{u_0(x,y)}{4(j-\tau)r^j}\int_0^r s^{j-1}u_0^{-1}(x(s),y)\Delta u_{j-1}(x(s),y)\,ds, \quad 1 \le j \le \tau - 1 \\ u_\tau(x,y) = 0 \\ u_{\tau+1}(x,y) = -\frac{u_0(x,y)}{4r^{\tau+1}}\int_0^r s^{\tau-1}u_0^{-1}(x(s),y)[-r\hat{u}_1(x(s),y) + \Delta\hat{u}_0(x(s),y) \\ \qquad -\omega(x(s),y)]\,ds \\ u_j(x,y) = \frac{u_0(x,y)}{4r^j}\int_0^r s^{j-1}u_0^{-1}(x(s),y)[\frac{1}{j-\tau}\Delta u_{j-1}(x(s),y) - 4(j-\tau)\hat{u}_{j-\tau}(x(s),y) \\ \qquad +\frac{1}{j-\tau}\Delta\hat{u}_{j-\tau-1}(x(s),y)]\,ds, \quad j \ge \tau + 1 \end{cases} \tag{B.16}$$

and

$$\begin{cases} \hat{u}_0(x,y) = -\frac{u_0(x,y)}{4r^2}\int_0^r s^{\tau-1}u_0^{-1}(x(s),y)\Delta u_{\tau-1}(x(s),y)\,ds \\ \hat{u}_j(x,y) = \frac{u_0(x,y)}{4jr^{j+\tau}}\int_0^r s^{j+\tau-1}u_0^{-1}(x(s),y)\Delta\hat{u}_{j-1}(x(s),y)\,ds \end{cases} \tag{B.17}$$

(iii) Case $n = 2$.

We assume the asymptotic expansion

$$G(x,y) \sim \sum_{j=0}^{\infty}\Gamma^j \log\Gamma \cdot \hat{u}_j(x,y) + \sum_{j=0}^{\infty}\Gamma^j u_j(x,y) \tag{B.18}$$

As before, we apply Δ to both sides of (B.18). By a computation similar to Case (ii), we must solve

$$\begin{aligned} \Delta G \;&\sim -\textstyle\sum_{j=1}^{\infty}\Gamma^{j-1}[4r\cdot{}^r\nabla\hat{u}_j + (8j-4+M)\hat{u}_j] - \Gamma^{-1}[4r\cdot{}^r\nabla\hat{u}_0 + (M-4)\hat{u}_0] \\ &\quad -\Gamma^{j-1}\log\Gamma[4r\cdot{}^r\nabla\hat{u}_j + (4j-4+M)\hat{u}_j - \tfrac{1}{j}\Delta\hat{u}_{j-1}] \\ &\quad -\textstyle\sum_{j=1}^{\infty} j\Gamma^{j-1}[4r\cdot{}^r\nabla u_j + (4j+2r\frac{\theta'}{\theta})u_j - \frac{1}{j}\Delta u_{j-1}] \\ &\equiv -\omega(x,y) \end{aligned} \tag{B.19}$$

with $M = 4 + 2r(\theta'/\theta)$. To make each term in (B.19) vanish, we set

$$\begin{cases} \hat{u}_0(x,y) = \theta^{-\frac{1}{2}}(x,y)P(x,y) \\ \hat{u}_j(x,y) = \frac{\hat{u}_0(x,y)}{4jr^j}\int_0^r s^{j-1}\hat{u}_0^{-1}(x(s),y)\Delta\hat{u}_{j-1}(x(s),y)\,ds \end{cases} \tag{B.20}$$

and

$$\begin{cases} u_0(x,y) = 0 \\ u_1(x,y) = -\frac{\hat{u}_0(x,y)}{4r}\int_0^r s^{-1}\hat{u}_0^{-1}(x(s),y)[-\omega(x(s),y) - 4\hat{u}_1(x(s),y) \\ \qquad -\Delta\hat{u}_0(x(s),y)]\,ds \\ u_j(x,y) = -\frac{\hat{u}_0(x,y)}{4r^j}\int_0^r s^{j-1}\hat{u}_0^{-1}(x(s),y)[-4j\hat{u}_j(x(s),y) - \frac{1}{j}\Delta\hat{u}_{j-1}(x(s),y)\,ds \end{cases} \tag{B.21}$$

In particular, we have

$$\hat{u}_1(x,y) = \frac{\hat{u}_0(x,y)}{4r}\int_0^r \hat{u}_0^{-1}(x(s),y)\Delta\hat{u}_0(x(s),y)\,ds \tag{B.22}$$

References

[1] M. F. Atiyah, R. Bott, and V. K. Patodi, *On the heat equation and the index theorem*, Inventiones Math. **19** (1973), 279–330.

[2] M. Berger, P. Gauduchon, and E. Mazet, *Le Spectre d'une Variété Riemannienne*, Lecture Notes in Mathematics, vol. 194, Springer-Verlag, Berlin, 1971.

[3] S. Donaldson, *An application of gauge theory to the topology of 4-manifolds*, J. Differential Geometry **18** (1983), 269–316.

[4] R. Fintushel and R. Stern, *Instanton homology of Seifert fibered homology three spheres*, Proc. London Math. Soc. **(3) 61** (1990), 109–137.

[5] D. Freed and K. Uhlenbeck, *Instantons and Four-Manifolds*, Springer-Verlag, New York, 1984.

[6] P. B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*, Publish or Perish, Wilmington, 1984.

[7] D. Groisser, *The geometry of the moduli space of CP^2 instantons*, Inventiones Math. **99** (1990), 393–409.

[8] D. Groisser and T. Parker, *The Riemannian geometry of the Yang-Mills moduli space*, Commun. Math. Phys. **112** (1987), 663–689.

[9] W.-Y. Hsiang, *On compact homogeneous minimal submanifolds*, Proc. Nat. Acad. Sci. USA **56** (1966), 5–6.

[10] K. Kodaira, *Harmonic fields in Riemannian manifolds (Generalized potential theory)*, Annals of Math. **50** (1949), 587–665.

[11] H. B. Lawson, Jr., *Lectures on Minimal Submanifolds*, Publish or Perish, Berkeley, 1980.

[12] T. Parker and S. Rosenberg, *Invariants of conformal Laplacians*, J. Differential Geometry **25** (1987), 535–557.

[13] D. B. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Adv. in Math. **7** (1971), 145–210.

[14] S. Rosenberg, *The variation of the de Rham zeta function*, Trans. AMS **299** (1987), 535–557.

[15] I. M. Singer, *Some remarks on the Gribov ambiguity*, Commun. Math. Phys **60** (1978), 5–12.

[16] E. Witten, *Supersymmetry and Morse theory*, J. Differential Geometry **17** (1982), 661–692.