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by

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EXISTENCE OF A CLOSED STAR PRODUCT

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ABSTRACT. On any symplectic manifold, we show the existence of a closed star product by using a Weyl manifold. There is also an involutive anti-automorphism on a Weyl manifold. By using this, a complexification relating to quaternion ring is given.

Introduction. Let M be a symplectic $2n$-manifold and let $\mathcal{A}=C^{\infty}(M)$ be the space of smooth functions on M, on which one defines the Poisson structure. Denote by $\mathcal{A}[[\nu]]$ the algebra of formal power series with a formal parameter ν. According to Bayen et al. [B], we shall call an associative deformation $(\mathcal{A}[[\nu]],*)$ of the Poisson algebra $\mathcal{A}$ a *star product*. It is shown in [WL] and [OMY] that there is a star product on every symplectic manifold M.

Connes-Flato-Sternheimer [CFS] introduced a notion of a more refined star product, called a (*strongly*) *closed star product*. The closedness means that the integration of elements of $\mathcal{A}[[\nu]]$ is a trace of the deformed algebra:

DEFINITION. A star product is *strongly closed* iff

$$\int_M f*g\Omega^n=\int_M g*f\Omega^n$$

for all $f,g\in\mathcal{A}$, where Ω is the symplectic structure of M.

In the paper [CFS], they ask the existence of closed star products for general symplectic manifolds. For this question, we have an affirmative answer:

THEOREM. *On every symplectic manifold there exists a strongly closed star product.*

The Moyal bracket [M] on $\mathbf{R}^{2n}$ is an typical example of a closed star product. In the existence proof, [WL],[OMY] of star products, they coincides with the Moyal product on each local coordinates. However, it is not so clear from the construction whether they are closed star products. Our idea for obtaining star products in

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[OMY] was to construct a sort of non-commutative manifolds, called Weyl manifolds. To obtain Theorem, we will follow this construction more carefully.

As shown in [CFS], for the closed star products cyclic cohomology reflects the Hochschild cohomology in usual star products. So, the existence and the equivalence of star products can be explained in terms of the cyclic cohomology. They also defined the character of a closed star product as the cohomology class in the cyclic bicomplex. We expect this notion is enlargeable to quantum theory as asymptotic analysis.

1. Weyl algebra. A (real) Weyl algebra $\boldsymbol{W}^R$ (cf.[OMY]) is the associative algebra over $\mathbf{R}$ generated formally by $\nu, X_1, \cdots, X_n, Y_1, \cdots, Y_n$ with the commutation relations:

$$\begin{cases} [\nu, X_i] & = [\nu, Y_j] = [X_i, X_j] = [Y_i, Y_j] = 0, \\ [X_i, Y_j] & = -\nu\delta_{ij}, \quad (1 \le i, j \le n). \end{cases} \tag{1}$$

$\boldsymbol{W}^R$ can be regarded as the formal universal enveloping algebra of the Heisenberg Lie algebra. Namely, $\boldsymbol{W}^R$ is the space of all formal power series $\mathbf{R}[[\nu, X_1, \cdots, X_n, Y_1, \cdots, Y_n]]$ with the noncommutative product $*$. Here, the product $*$ is given by the following formula:

$$a * b = a e^{\frac{\nu}{2}\{\overleftarrow{\partial}_Y \dot{\wedge} \overrightarrow{\partial}_X\}} b, \tag{2}$$

where the right hand side is computed by the following conventions:

$$a\{\overleftarrow{\partial}_Y \dot{\wedge} \overrightarrow{\partial}_X\} b = \sum_{1 \le i \le n} \{\partial_{Y_i} a \cdot \partial_{X_i} b - \partial_{X_i} a \cdot \partial_{Y_i} b\},$$

$$e^{\frac{\nu}{2}\{\overleftarrow{\partial}_Y \dot{\wedge} \overrightarrow{\partial}_X\}} = \sum_k \frac{\nu^k}{2^k k!} \{\overleftarrow{\partial}_Y \dot{\wedge} \overrightarrow{\partial}_X\}^k.$$

$\boldsymbol{W}^R$ is a complete noncommutative topological algebra.

Let $\dagger$ be a continuous anti-automorphism on $\boldsymbol{W}^R$ defined by $X_i^\dagger = X_i$, $Y_j^\dagger = Y_j$ and $\nu^\dagger = -\nu$. Obviously, $\dagger^2 = 1$ and $(a * b)^\dagger = b^\dagger * a^\dagger$.

2. Weyl functions. Let U be an open subset of $\mathbf{R}^{2n}$ with the coordinates $x_1, \cdots, x_n, y_1, \cdots, y_n$, and let $\boldsymbol{W}_U^R$ be the trivial algebra bundle $U \times \boldsymbol{W}^R$ over U. By $\Gamma(\boldsymbol{W}_U^R)$ we denote the space of all continuous sections of $\boldsymbol{W}_U^R$. By extending the product $*$, $\Gamma(\boldsymbol{W}_U^R)$ is a topological algebra over $\mathbf{R}$ via compact open topology.

For any $\mathbf{R}[[\nu]]$-valued C^∞ function f, we set

$$f^\sharp(p) = \sum_\alpha \frac{1}{\alpha!} \partial^\alpha f(p) (Z\cdot)^\alpha$$

where $(Z_1, \cdots, Z_{2n}) = (X_1, \cdots, X_n, Y_1 \cdots, Y_n)$ and $(Z\cdot)^\alpha = Z_1^{\alpha_1} \cdots Z_{2n}^{\alpha_{2n}}$. $f^\sharp$ is well-defined as a section of $\boldsymbol{W}_U^R$ and called a *Weyl function* on $\boldsymbol{W}_U^R$ (cf. [OMY]). The procedure for making $f^\sharp$ from f is called the *Weyl continuation*. Denote by $\mathcal{F}(\boldsymbol{W}_U^R)$ the set of all Weyl functions on $\boldsymbol{W}_U^R$. $\mathcal{F}(\boldsymbol{W}_U^R)$ is a closed subalgebra of $\Gamma(\boldsymbol{W}_U^R)$ (cf. [OMY]). The product $f^\sharp * g^\sharp$ is given by the same formula as (2), that is,

$$f^\sharp * g^\sharp(p) = (f e^{\frac{\nu}{2}\{\overleftarrow{\partial}_y \dot{\wedge} \overrightarrow{\partial}_x\}} g)^\sharp. \quad \text{(cf.[OMY])} \tag{3}$$

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The anti-automorphism † can be extended naturally on $\Gamma(\boldsymbol{W}_U^R)$ and

$$\mathcal{F}(\boldsymbol{W}_U^R)^\dagger = \mathcal{F}(\boldsymbol{W}_U^R),$$

i.e. for $f = \sum_l \nu^l f_l,\ f_l \in C^\infty(U)$, setting $f^\dagger = \sum_l (-\nu)^l f_l$, we have

$$(f^\sharp)^\dagger = (f^\dagger)^\sharp. \tag{4}$$

3. Integration. For any Weyl function $f^\sharp \in \mathcal{F}(\boldsymbol{W}_U^R)$ such that f is summable on U, we define the *integral* of $f^\#$ as follows:

$$\int_U f^\sharp = \int_U f dV \in \mathbf{R}[[\nu]] \tag{5}$$

where $dV = dx_1 \cdots dx_n dy_1 \cdots dy_n$, the canonical volume element on U.

Since the integration by parts shows

$$\int_U f\{\overleftarrow{\partial}_y \wedge \overrightarrow{\partial}_x\}^k g dV = 0$$

if one of f and g is support compact, we have

$$\int_U f^\sharp * g^\sharp = \int_U f \cdot g dV. \tag{6}$$

Thus, we get

$$\int_U [f^\sharp, g^\sharp] = 0. \tag{7}$$

Also, we easily have

$$\left(\int_U f^\sharp\right)^\dagger = \int_U (f^\sharp)^\dagger. \tag{8}$$

Denote by $\mathbf{R}[[\mu]]_+$ the set of all formal power series $\sum_k a_k \mu^k$ such that the first nonzero coefficient is positive, i.e. if $\sum_k a_k \mu^k = a_l \mu^l + a_{l+1}\mu^{l+1} + \cdots$, then $a_l > 0$. Then, we have

$$\mathbf{R}[[\mu]]_+ \bigcap -\mathbf{R}[[\mu]]_+ = \{0\}, \quad \mathbf{R}[[\mu]]_+ \bigcup -\mathbf{R}[[\mu]]_+ = \mathbf{R}[[\mu]].$$

Regarding $\mathbf{R}[[\mu]]_+$ as a positive convex cone, we can define a linear order on $\mathbf{R}[[\mu]]$. This is the lexicographic order with respect to degrees.

From (8), we see

$$\int_U f^\sharp * f^{\sharp\dagger} \in \mathbf{R}[[-\nu^2]]_+. \tag{9}$$

So, the bilinear form

$$\langle f^\sharp, g^\sharp \rangle_0 = \int_U f^\sharp * (g^\sharp)^\dagger \quad \in \mathbf{R}[[\nu]] \tag{10}$$

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gives a hermitian inner product; $\langle f^{\#}, g^{\#}\rangle_0^{\dagger} = \langle g^{\#}, f^{\#}\rangle_0$, and $\langle f^{\sharp}, f^{\sharp}\rangle_0 = 0$ implies $f^{\sharp} = 0$.

4. Weyl diffeomorphisms. For open sets U and U' in $\mathbf{R}^{2n}$, consider trivial algebra bundles $\boldsymbol{W}_U^R$, $\boldsymbol{W}_{U'}^R$ and a bundle isomorphism

$$\begin{array}{ccc} \boldsymbol{W}_U^R & \xrightarrow{\Phi} & \boldsymbol{W}_{U'}^R \\ \downarrow & & \downarrow \\ U & \xrightarrow{\varphi} & U' \end{array} \tag{11}$$

The pullback $\Phi^* : \Gamma(\boldsymbol{W}_{U'}^R) \to \Gamma(\boldsymbol{W}_U^R)$ is defined by $(\Phi^* S)(p) = \Phi^{-1} S(\varphi(p))$. A bundle isomorphism Φ is called a *pre-Weyl diffeomorphism* if Φ^* induces a continuous algebra isomorphism $\Phi^* : \mathcal{F}(\boldsymbol{W}_{U'}^R) \to \mathcal{F}(\boldsymbol{W}_U^R)$ such that $\Phi(\nu) = \nu$. In particular, the induced diffeomorphism $\varphi : U \to U'$ is a symplectic diffeomorphism with respect to the standard symplectic 2-form $\Omega_0 = \sum dx_i \wedge dy_i$.

Conversely, we have the following:

LEMMA 1. ([OMY] LEMMA 3.2.). *For any continuous algebra isomorphism $\Psi : \mathcal{F}(\boldsymbol{W}_{U'}^R) \to \mathcal{F}(\boldsymbol{W}_U^R)$ such that $\Psi(\nu) = \nu$, there exists uniquely a pre-Weyl diffeomorphism Φ such that $\Psi = \Phi^*$.*

A pre-Weyl diffeomorphism $\Phi : \boldsymbol{W}_U^R \to \boldsymbol{W}_{U'}^R$ is called a *Weyl diffeomorphism*, if Φ satisfies

$$\Phi^*(f^{\#\dagger}) = \Phi^*(f^{\#})^{\dagger}, \qquad \int_U \Phi^*(f^{\#}) = \int_{U'} f^{\#}. \tag{12}$$

REMARK. In [OMY], a Weyl diffeomorphism has been defined as a pre-Weyl diffeomorphism $\Phi : \boldsymbol{W}_U^R \to \boldsymbol{W}_{U'}^R$ with

$$\Phi^*(f^{\sharp}) = (\varphi^* f)^{\sharp} \ mod\ \nu^2. \tag{13}$$

Any pre-Weyl diffeomorphism with (12) satisfies the condition (13). We replace the definition of "Weyl diffeomorphisms" by (12).

5. Lift of symplectic transformations. Now, suppose U, U' are diffeomorphic to the open unit disk D^n of $\mathbf{R}^n$. Let $\overline{U}$ and $\overline{U}'$ be the closures of U and U' respectively. A symplectic diffeomorphism $\varphi : \overline{U} \to \overline{U}'$ means that φ is the restriction of a symplectic diffeomorphism of a neighborhood of $\overline{U}$ onto a neighborhood of $\overline{U}'$.

In this section, we shall prove the following:

PROPOSITION 2. *With the above notations, for any symplectic diffeomorphism $\varphi : \overline{U} \to \overline{U}'$, there is a Weyl diffeomorphism Φ which induces φ on the base spaces. Φ is called a lift of φ, although this is not unique.*

To prove this, we need at first the following Lemma A:

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LEMMA A. *For the above $\varphi : \overline{U} \to \overline{U}'$, there is a 1-parameter family $\varphi_t : \overline{U}_t \to \overline{U}'$ of symplectic diffeomorphisms such that $\varphi_0 = id., U_0 = U', \varphi_1 = \varphi, U_1 = U$, and $\varphi_t(z)$ is C^∞ with respect to (t, z).*

Although this is known in [KN, Lemma 13], we shall give the short proof in the last section.

Proof of Proposition 2: For φ_t in Lemma A, we define the Hamiltonian vector field H_t on $\overline{U}_t$ by

$$\frac{d}{dt}\varphi_t^{-1}(z) = H_t(\varphi_t^{-1}(z)).$$

There exists a $\mathbf{R}$-valued C^∞ function h_t on $\overline{U}_t$ such that $i(H_t)\Omega = -dh_t$. Taking the Weyl continuation $h_t^\sharp$ of h_t, we consider the differential equation

$$\frac{d}{dt}g_t^\sharp = \frac{1}{\nu}[h_t^\sharp, g_t^\sharp], \quad g_0^\sharp = g^\sharp, \quad g \in C^\infty(U', \mathbf{R}[[\nu]]). \tag{14}$$

By setting $g_t^\sharp = \sum_l g_{t,l}^\sharp \nu^l$, the equation (14) can be solved uniquely in $\mathcal{F}(\boldsymbol{W}^R(U_t))$ (cf. [OMY] Theorem 3.6). We denote the solution by $\Psi_t(g^\sharp)$.

We note the following properties which are easy to see by the uniqueness of the solution of (14):

(a) Since $\frac{1}{\nu}ad(h_t^\sharp) = \frac{1}{\nu}[h_t^\sharp, *]$ is a continuous derivation of $\mathcal{F}(\boldsymbol{W}^R(U_t))$ such that $\frac{1}{\nu}ad(h_t^\sharp)\nu = 0$, Ψ_t is a continuous isomorphism of $(\mathcal{F}(\boldsymbol{W}^R(U_0)), *)$ onto $(\mathcal{F}(\boldsymbol{W}^R(U_t)), *)$ such that $\Psi_t(\nu) = \nu$ (cf.[OMY] Theorem 3.7).

(b) Since $h_t^{\sharp\dagger} = h_t^\sharp$, and $(\frac{1}{\nu}[h_t^\sharp, g_t^\sharp])^\dagger = \frac{1}{\nu}[h_t^\sharp, g_t^{\sharp\dagger}]$, Ψ_t satisfies $\Psi_t(g^\sharp)^\dagger = \Psi_t(g^{\sharp\dagger})$.

(c) Since $\int_{U_t} \frac{d}{dt}g_t^\sharp = \int_{U_t} \frac{1}{\nu}[h_t^\sharp, g_t^\sharp] = 0$, Ψ_t must satisfy $\int_{U_t} \Psi_t(g^\sharp) = \int_{U_0} g^\sharp$.

Using Lemma 1, we have a Weyl diffeomorphism Φ such that $\Phi^* = \Psi_1$ by setting $t = 1$. □

REMARK. In the above proof, we used only that $h_t^{\sharp\dagger} = h_t^\sharp$, hence we can replace $h_t^\sharp$ by $h_t^\sharp + \nu^2 f_t^\sharp + \nu^4 g_t^\sharp + \cdots$ using $f_t, g_t, \cdots \in C^\infty(\overline{U}_t)$.

6. Real Weyl manifolds. Let M be a C^∞ paracompact $2n$-manifold. Consider a locally trivial algebra bundle $\boldsymbol{W}_M^R$ with the fiber isomorphic to $\mathbf{W}^R$. By definition of $\boldsymbol{W}_M^R$, there is an open covering $\{V_\alpha\}_\alpha$ of M with local trivializations $\Phi_\alpha : \boldsymbol{W}_{V_\alpha}^R \to \boldsymbol{W}_{U_\alpha}^R$ where $\boldsymbol{W}_{V_\alpha}^R$ is the restriction of $\boldsymbol{W}_M^R$ and $\boldsymbol{W}_{U_\alpha}^R$ is the trivial algebra bundle over $U_\alpha \subset \mathbf{R}^{2n}$. Let $\varphi_\alpha : V_\alpha \to U_\alpha$ be the induced homeomorphism such that the following diagram commutes:

$$\begin{array}{ccc} \boldsymbol{W}_{V_\alpha}^R & \xrightarrow{\Phi_\alpha} & \boldsymbol{W}_{U_\alpha}^R \\ \downarrow & & \downarrow \\ V_\alpha & \xrightarrow{\varphi_\alpha} & U_\alpha \end{array}$$

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DEFINITION. $\boldsymbol{W}_M^R$ is called a *real Weyl manifold*, if for each V_α, V_β such that $V_\alpha \cap V_\beta \neq \emptyset$,

$$\Phi_{\alpha\beta} = \Phi_\beta \Phi_\alpha^{-1} : \varphi_\alpha(V_\alpha \cap V_\beta) \times \boldsymbol{W}^R \to \varphi_\beta(V_\beta \cap V_\alpha) \times \boldsymbol{W}^R$$

is a Weyl diffeomorphism, that is, $\Phi_{\alpha\beta}^* \mathcal{F}(\boldsymbol{W}_{U_{\beta\alpha}}^R) = \mathcal{F}(\boldsymbol{W}_{U_{\alpha\beta}}^R)$ with (12) where $U_{\alpha\beta} = \varphi_\alpha(V_\alpha \cap V_\beta)$. $\Phi_\alpha : \boldsymbol{W}_{V_\alpha}^R \to \boldsymbol{W}_{U_\alpha}^R$ will be called a *local Weyl chart* on $\boldsymbol{W}_M^R$.

It is known that the base manifold M of a Weyl manifold $\boldsymbol{W}_M^R$ is a C^∞ symplectic manifold.

On a real Weyl manifold $\boldsymbol{W}_M^R$, the concept of Weyl functions make sense by using local Weyl charts as sections of $\boldsymbol{W}_M^R$, which form a closed subalgebra of $\Gamma(\boldsymbol{W}_M^R)$ (cf.[OMY]). Let $\mathcal{F}(\boldsymbol{W}_M^R)$ be the ring of all Weyl functions on $\boldsymbol{W}_M^R$. By the above definition of Weyl diffeomorphisms, $\dagger$ is well-defined on $\mathcal{F}(\boldsymbol{W}_M^R)$ as an anti-automorphism such that $\dagger^2 = 1$.

THEOREM 3. *On any paracompact C^∞ symplectic manifold M, there exists a real Weyl manifold $\boldsymbol{W}_M^R$.*

PROOF. Let M be a paracompact C^∞ symplectic manifold and let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a locally finite, simple open covering of M. For each $\alpha \in \Lambda$ there is a coordinate mapping $\varphi_\alpha : V_\alpha \to U_\alpha \subset \mathbf{R}^{2n}$, which is a symplectic diffeomorphism. For each $\alpha, \beta \in \Lambda$ such that $V_\alpha \cap V_\beta \neq \emptyset$, we set $U_{\alpha\beta} = \varphi_\alpha(V_\alpha \cap V_\beta)$. One may assume that the coordinate transformation $\varphi_{\alpha\beta} = \varphi_\beta\varphi_\alpha^{-1}$ can be extended to a symplectic diffeomorphism of $\overline{U}_{\alpha\beta}$ onto $\overline{U}_{\beta\alpha}$.

For above $\alpha, \beta \in \Lambda$, let

$$\tilde{\Phi}_{\alpha\beta} : \boldsymbol{W}_{U_{\alpha\beta}}^R \to \boldsymbol{W}_{U_{\beta\alpha}}^R$$

be a Weyl diffeomorphism given as a lift of $\varphi_{\alpha\beta}$ by Proposition 2.

It is easy to see that for any $\alpha, \beta, \gamma \in \Lambda$ such that $V_\alpha \cap V_\beta \cap V_\gamma \neq \emptyset$,

(i) $\tilde{\Phi}_{\beta\alpha}\tilde{\Phi}_{\alpha\beta}$ is a Weyl diffeomorphism which induces the identity on the base space $\varphi_\alpha(V_\alpha \cap V_\beta)$.

(ii) $\tilde{\Phi}_{\gamma\alpha}\tilde{\Phi}_{\beta\gamma}\tilde{\Phi}_{\alpha\beta}$ is a Weyl diffeomorphism on $\boldsymbol{W}_{\varphi_\alpha(V_\alpha\cap V_\beta\cap V_\gamma)}^R$ which induces the identity on the base space $\varphi_\alpha(V_\alpha \cap V_\beta \cap V_\gamma)$.

Using (i), (ii) and (13), we have

$$\tilde{\Phi}_{\alpha\beta}^*\tilde{\Phi}_{\beta\alpha}^* = 1 \quad mod\ \nu^2, \quad \tilde{\Phi}_{\alpha\beta}^*\tilde{\Phi}_{\beta\gamma}^*\tilde{\Phi}_{\gamma\alpha}^* = 1 \quad mod\ \nu^2. \tag{15}$$

As it is proved in [OMY], §5, for any α, β with $V_\alpha \cap V_\beta \neq \emptyset$, there is a C^∞ real function $f_{\alpha\beta}$ such that a Weyl diffeomorphism $\hat{\Phi}_{\alpha\beta}$ given by

$$\hat{\Phi}_{\alpha\beta}^* = \tilde{\Phi}_{\alpha\beta}^* e^{-\frac{1}{\nu}ad(\nu^2 f_{\alpha\beta}^\#)}$$

satisfies

$$\hat{\Phi}_{\alpha\beta}^*\hat{\Phi}_{\beta\alpha}^* = 1 \quad mod\ \nu^3, \quad \hat{\Phi}_{\alpha\beta}^*\hat{\Phi}_{\beta\gamma}^*\hat{\Phi}_{\gamma\alpha}^* = 1 \quad mod\ \nu^3. \tag{16}$$

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Note that the left hand sides of (16) are the pull back of Weyl diffeomorphisms and hence these commute with †. It follows that the above equalities must hold up to mod ν^4.

In this repair, note that $e^{-\frac{1}{\nu}ad(\nu^2 f^{\#}_{\alpha\beta})}$ is a pull back of Weyl diffeomorphism which induces the identity on the base space. Repeating this procedure, we have a family of Weyl diffeomorphisms $\Phi_{\alpha\beta}$ for any α, β such that $V_\alpha \cap V_\beta \neq \emptyset$, and

$$\Phi^*_{\alpha\beta}\Phi^*_{\beta\alpha} = 1, \quad \Phi^*_{\alpha\beta}\Phi^*_{\beta\gamma}\Phi^*_{\gamma\alpha} = 1. \tag{17}$$

where $\Phi_{\alpha\beta}$ is written in the shape

$$\Phi^*_{\alpha\beta} = \tilde{\Phi}^*_{\alpha\beta} e^{-\frac{1}{\nu}ad(\nu^2 f^{\#}_{\alpha\beta})} e^{-\frac{1}{\nu}ad(\nu^4 g^{\#}_{\alpha\beta})} \cdots .$$

The desired algebra bundle $\boldsymbol{W}^R_M$ is obtained by patching $\{\boldsymbol{W}^R_{U_\alpha}\}_{\alpha\in\Lambda}$ via the above Weyl diffeomorphisms. □

7. Proof of Theorem. Let M be a symplectic 2-manifold and $\boldsymbol{W}^R_M$ a real Weyl manifold. It is known in [OMY] that $\mathcal{F}(\boldsymbol{W}^R_M)/\nu\mathcal{F}(\boldsymbol{W}^R_M)$ is isomorphic to $C^\infty(M)$. Let π be the natural projection of $\mathcal{F}(\boldsymbol{W}^R_M)$ onto $C^\infty(M)$. Suppose $W^R_M = \bigcup_\alpha W^R_{V_\alpha}$, and $W^R_{V_\alpha}$ is isomorphic to $W^R_{U_\alpha}$, $U_\alpha \subset \mathbf{R}^{2n}$, via a Weyl diffeomorphism Φ_α. Let $\varphi_\alpha : V_\alpha \to U_\alpha$ be the symplectic diffeomorphism induced by Φ_α. Let $\{\phi_\alpha\}_\alpha$ be a partition of unity subordinate to the covering $\bigcup_\alpha V_\alpha$. Define a linear mapping $\sharp_M$ of $C^\infty(M, \mathbf{R}[[\nu]])$ into $\mathcal{F}(\boldsymbol{W}^R_M)$ by $f^{\sharp_M} = \sum_\alpha \Phi^*_\alpha(\varphi^{*-1}\phi_\alpha f)^{\sharp_\alpha}$, where $\sharp_\alpha$ is the Weyl continuation on U_α. It is known in [OMY] that $\sharp_M$ gives a linear isomorphism of $C^\infty(M, \mathbf{R}[[\nu]])$ onto $\mathcal{F}(\boldsymbol{W}^R_M)$ such that $\pi f^{\sharp_M} = f \ mod\ \nu^2$, hence $\mathcal{F}(\boldsymbol{W}^R_M)$ is understood as $C^\infty(M, \mathbf{R}[[\nu]])$ which has a star product.

For any $f^{\#_M} \in \mathcal{F}(\boldsymbol{W}^R_M)$ with the compact support, we put the integral of $f^{\#_M}$ as

$$\int_M f^{\#_M} = \sum_\alpha \int_{U_\alpha} \Phi^{*-1}_\alpha (\phi_\alpha f)^{\#_M}, \tag{18}$$

which is well-defined because $\int_{U_{\alpha\beta}} \Phi^*_{\alpha\beta} f^\sharp = \int_{U_{\beta\alpha}} f^\sharp$. Moreover, it is easily seen that

$$\int_M f^{\sharp_M} = \int_M f dV \tag{19}$$

and

$$\int_M [f^{\sharp_M}, g^{\sharp_M}] = 0, \tag{20}$$

which gives the Theorem.

For the integral of Weyl functions $\tilde{f}, \tilde{g}$, we have the following:

$$\int_M \tilde{f} * (\tilde{g})^\dagger = \left(\int_M \tilde{g} * (\tilde{f})^\dagger\right)^\dagger,$$
$$\int_M \tilde{f} * (\tilde{f})^\dagger \in \mathbf{R}_+[[-\nu^2]], \ = 0 \text{ implies } \tilde{f} = 0.$$

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8. Complexifications Using the anti-automorphism $\dagger$ on $\mathcal{F}(\boldsymbol{W}_M^R)$, we have the following decomposition:

$$\mathcal{F}(\boldsymbol{W}_M^R) = \mathcal{F}_R(\boldsymbol{W}_M^R) \oplus \nu\mathcal{F}_R(\boldsymbol{W}_M^R),$$

where $\mathcal{F}_R(\boldsymbol{W}_M^R)$ is the space of all $\dagger$-invariant (i.e. hermitian) Weyl functions.

By taking the tensor product $\boldsymbol{W}_M^R \otimes \mathbf{C}$, the usual complexification $\boldsymbol{W}_M^C$ is obtained as a complex Weyl algebra bundle, where $i^\dagger = -i$. $\boldsymbol{W}_M^C$ is a Weyl manifold constructed in [OMY].

Here, we shall give another complexification by requesting $i\nu = -\nu i$ and $(i\nu)^\dagger = -i\nu$ which relates to the quaternion structure.

Now, let

$$\mathcal{F}^Q(\boldsymbol{W}_M^R) = \mathcal{F}_R(\boldsymbol{W}_M^R) \oplus i\mathcal{F}_R(\boldsymbol{W}_M^R) \oplus \nu\mathcal{F}_R(\boldsymbol{W}_M^R) \oplus i\nu\mathcal{F}_R(\boldsymbol{W}_M^R).$$

For any $\tilde{f} = \tilde{f}_0 + i\tilde{f}_1 + \nu\tilde{f}_2 + i\nu\tilde{f}_3$, we set

$$\int_M \tilde{f} = \int_M \tilde{f}_0 + i\int_M \tilde{f}_1 + \nu\int_M \tilde{f}_2 + i\int_M \nu\tilde{f}_3,$$
$$\tilde{f}^\dagger = \tilde{f}_0 - i\tilde{f}_1 - \nu\tilde{f}_2 - i\nu\tilde{f}_3.$$

9. Proof of Lemma A. Let $\Omega_0 = d\theta_0$, $\theta_0 = \frac{1}{2}\sum_{1\leq i\leq n}(x_i dy_i - y_i dx_i)$ be the symplectic 2-form on $\mathbf{R}^{2n}$. For open subsets U, $U' \subset \mathbf{R}^{2n}$, let $\varphi : \overline{U} \to \overline{U}'$ be a symplectic diffeomorphism.

One may assume that U, U' contain the origin 0, and there are closed neighborhoods K_U, $K_{U'}$ of $\overline{U}$, $\overline{U}'$ such that K_U, $K_{U'}$ are C^∞ diffeomorphic to the unit closed ball, and φ is a C^∞ symplectic diffeomorphism of K_U onto $K_{U'}$.

Choose a C^∞ function h_0 on $\mathbf{R}^{2n}$ which has only one non degenerate critical point at the origin with the following properties:

(1) $h_0 = \sum(x_i^2 + y_i^2)$ on a neighborhood V of 0 and $h_0(p) > 0$, if $p \neq 0$.

(2) The boundary ∂K_U is the level surface $h_0 = 1$.

LEMMA 4. *There is a C^∞ vector field X on K_U such that $\mathcal{L}_X\Omega_0(= d(i(X)\Omega_0)) = \Omega_0$, and $\mathcal{L}_X h_0(= Xh_0) > 0$. Such a vector field will be called a conformally expansive vector field.*

PROOF. The gradient flow of h_0 gives a diffeomorphism Ψ of $K_U - \{0\}$ onto $(-\infty, 0] \times S^{2n-1}$ and we have $h_0 = e^t$, where t is the coordinate on $(-\infty, 0]$.

Remark $\Omega_0 = d(\theta_0 + dg)$ for any g, and $\psi^{-1*}(\theta_0 + dg) = fdt + \beta_t + d(\psi^{-1*}g)$, where β_t is a 1-from on S^{2n-1}. Set $\tilde{g} = \psi^{-1*}g$, and $d\tilde{g} = \partial_t\tilde{g}dt + d'\tilde{g}$, where d' implies the exterior derivative on S^{2n-1}. Thus, by a suitable choice of g, one may assume that $\psi^{-1*}(\theta_0 + dg)$ does not involve dt component, and hence this is regarded as a 1-form on each level manifold S^{2n-1} parameterized by t. Set $\omega_t = \psi^{-1*}(\theta_0 + dg)$. If $t \ll 0$, then $\omega_t = e^t\tilde{\omega}$, where $\tilde{\omega}$ is the standard contact form on S^{2n-1}. Since $d\omega_t = dt \wedge \frac{\partial\omega_t}{\partial t} + d'\omega$ is a symplectic 2-form, we have

$$\frac{\partial\omega_t}{\partial t} \wedge (d'\omega_t)^{n-1} = f_t\tilde{\omega} \wedge (d'\tilde{\omega})^{2n-1}, \quad f_t > 0. \tag{21}$$

Remark that $\omega_t \wedge (d'\omega_t)^{n-1} = e^{(2n-1)t}\tilde{\omega} \wedge (d'\tilde{\omega})^{2n-1}$ for $t \ll 0$. Thus, if $\omega_t(p) = 0$ at some point $(t, p), p \in S^{2n-1}$, then

$$\frac{d}{dt}(\omega_t \wedge (d'\omega_t)^{n-1}(p)) = \frac{\partial \omega_t}{\partial t} \wedge (d'\omega_t)^{n-1}(p). \tag{22}$$

If one considers a point (t, p) such that $\omega_t(p) = 0$, but $\omega_s(p) \neq 0$ for any s, $s < t$, then the right hand side of (22) has negative sign. This is a contradiction by (21). Therefore, ω_t gives a family of contact 1-forms on S^{2n-1}.

It is known in [O] that there exists a one parameter family F_t of diffeomorphisms on S^{2n-1} such that $\omega_t = e^{\phi_t} F_t^* \tilde{\omega}$, where for $t << 0$, $\phi_t = t$, and $F_t = 1$. This is proved by using implicit function theorem to the natural action of the group of diffeomorphisms onto the space of 1-forms.

Thus $K_U - \{0\}$ is identified symplectically with $((-\infty, 0] \times S^{2n-1},\ d(e^{f_t}\omega_0))$. Since $d(e^{f_t}\omega_0) = e^{(2n-1)t}\tilde{\omega}$ for $t \ll 0$, we have for any $t < 0$. $\frac{\partial f_t}{\partial t} > 0$. Thus, setting $X = (\frac{\partial f_t}{\partial t})^{-1}\frac{\partial}{\partial t}$, we see that $\mathcal{L}_X d(e^{f_t}\omega_0) = d(e^{f_t}\omega_0)$, and $Xe^t > 0$. □

Now, since $Sp(2n, \mathbf{R})$ is connected, one can take a suitable one parameter family A_t of linear symplectic transformations such that $A_0 = id.$ and $A_1 = (d\varphi)_0$. Thus, for the proof of Lemma A, one may assume that $(d\varphi)_0 = id$.

Take conformally expansive vector fields X, X' on $K_U, K_{U'}$ respectively. Let $\mathcal{E}_t$, $\mathcal{E}'_t$ $-\infty < t \leq 0$, be the flows generated by X, X' with $\mathcal{E}_0 = 1$, $\mathcal{E}'_0 = 1$. Remark that $\mathcal{E}_t^*\Omega_0 = e^t\Omega_0$, and the same equality holds for $\mathcal{E}'_t$. Since $Xh_0 > 0$, we see that $\mathcal{E}_t K_U$ shrinks to the origin as t tends to $-\infty$.

Thus, $\varphi_t = \mathcal{E}_t'^{-1}\varphi\mathcal{E}_t$ is a symplectic diffeomorphism of K_U into $\mathbf{R}^{2n}$ such that $\varphi_0 = \varphi$ and $\varphi_t = 1$ for $t << 0$. Then, $\varphi_{t-\infty}^{-1}\varphi$ is a desired one. This completes the proof of Lemma A.

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