Research Report

KSTS/RR-92/005
Sept. 30, 1992

On the Existence of a Global Solution for the Yang-Mills Graident Flow on 4-Manifolds

by

H. Kozono, Y. Maeda and H. Naito

H. Kozono
Department of Mathematics
Kyushu University

Y. Maeda
Department of Mathematics
Keio University

H. Naito
Department of Mathematics
Nagoya University

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku, Yokohama, 223 Japan

KSTS/RR-92/005
September 30, 1992

August 1992

On the Existence of a Global Solution for the Yang-Mills Gradient Flow on 4-Manifolds

Hideo KOZONO, Yoshiaki MAEDA and Hisashi NAITO†

1. Introduction.

Let M be a closed Riemannian 4-manifold and P a G-bundle over M, where G is a compact Lie group embedded as a subgroup of $SO(l)$ (or $SU(l)$). We denote by $\mathfrak{g}$ the Lie algebra of G. Let $\mathfrak{g}_P$ and $\mathfrak{G}_P$ be the adjoint and automorphism bundles of P, respectively. Using the metric on G induced by the Killing form, we fix a metric on P compatible with the action of G. Let $\Omega^k(\mathfrak{g}_P)$ be the space of smooth $\mathfrak{g}$-valued k-forms, i.e. $\Omega^k(\mathfrak{g}_P) = C^\infty(M; \mathfrak{g}_P \otimes \wedge^k T^*M)$.

Let $\mathfrak{U}$ be the affine space of smooth connections on P compatible with the metric on P. Picking a base connection $D_0 \in \mathfrak{U}$, we have

$$\mathfrak{U} = \{D = D_0 + A : A \in \Omega^1(\mathfrak{g}_P)\}.$$

Define the $W^{m,p}$-Sobolev space $\mathfrak{U}^{m,p}$ of connections as follows:

$$\mathfrak{U}^{m,p} = \{D = D_0 + A : A \in W^{m,p}(\Omega^1(\mathfrak{g}_P))\},$$

where $W^{m,p}(\Omega^k(\mathfrak{g}_P))$ is the Sobolev space of $\mathfrak{g}$-valued k-forms with m derivatives in L^p. Since $\mathfrak{U}$ is an affine space, we can identify $\mathfrak{U}$ (resp. $\mathfrak{U}^{m,p}$) with $\Omega^1(\mathfrak{g}_P)$ (resp. $W^{m,p}(\Omega^1(\mathfrak{g}_P))$). For a connection $D = D_0 + A$, we denote by d_A and d_A^* the covariant exterior derivative and its formal adjoint, respectively. Moreover, we write the covariant derivatives on tensors as $\tilde{\nabla}_A$ and ∇ for the connections D and D_0. If $D = D_0 + A \in \mathfrak{U}$, then its curvature is given by $R_A = {d_A}^2 \in \Omega^2(\mathfrak{g}_P)$.

The Yang-Mills gradient flow is the steepest descent flow of the Yang-Mills functional $E(A) = \dfrac{1}{2}\displaystyle\int_M |R_A|^2\, dV$:

$$(1.1) \qquad \begin{cases} \partial_t A = -d_A^* R_A & \text{on} \quad M \times [0, \infty), \\ A(0) = A_0 & \text{on} \quad M \times \{0\}. \end{cases}$$

In this paper, we will construct a global solution of (1.1) and show that its singularities consist of only finitely many points in space-time. Indeed, we shall solve (1.1) in the class

†Partially supported by The Ishida Foundation.

$V(M,[0,\infty])$;

$$V(M,[0,\infty]) := \Big\{ A \in \Omega^1(M\times[0,\infty],\mathfrak{g}_P);$$
$$\sup_{0\le t\le\infty}\int_M \left(|R_A|^2 + |\tilde\nabla_A R_A|^2 + |\tilde\nabla_A^2 R_A|^2 + |A|^2 + |\nabla A|^2 + |\partial_t A|^2\right) dV < \infty \Big\}.$$

Theorem. *Let M be a closed 4-manifold. For any initial value $A_0 \in W^{1,2}(\Omega^1(\mathfrak{g}_P))$, there exists a global weak solution of the Yang-Mills gradient flow equation (1.1) in the class $V(M,[0,\infty])$. Moreover there exists a finite set $\mathcal{S}$ in $M\times(0,\infty]$ such that the solution is regular and unique on $(M\times(0,\infty))\backslash\mathcal{S}$.*

The gauge transformation $s \in \mathfrak{G} = C^\infty(\mathfrak{G}_P)$ acts on connections by conjugation: $A \mapsto s^*A = s^{-1}ds + s^{-1}As$. The curvature is actually a section of the bundle $P\otimes T^*M\wedge T^*M$, and so a gauge transformation $s\in\mathfrak{G}$ also acts on curvature tensors by $R_A \mapsto s^*R_A = R_{s^*A} = s^{-1}R_A s$. The fact that gauge transformations leave the Yang-Mills functional invariant, i.e. $E(s^*A) = E(A)$, creates a major difficulty for treating the regularity of the solution.

If a connection $D = D_0 + A$ transforms to $s^*D = \tilde D = D_0 + \tilde A$ under a gauge transformation s, then the Yang-Mills gradient flow (1.1) transforms to

$$\begin{cases} \partial_t A = -d_A^* R_A + d_A\alpha & \text{on} \quad M\times[0,\infty), \\ A(0) = A_0 & \text{on} \quad M\times\{0\}, \end{cases} \tag{1.2}$$

where $\alpha = s^{-1}\partial_t s \in \Omega^0(\mathfrak{g}_P)$ (cf. Jost [**6**]). We call (1.2) a modified Yang-Mills gradient flow. Conversely, a solution A, α or s of (1.2) yields a solution $(s^{-1})^*A$ of (1.1).

For the proof of the Theorem, we first construct a solution of (1.2) in a finite time interval $(0,T]$. Then, we return to (1.1) and show that the energy functional $E(A(t))$ is monotone non-increasing with respect to t, which enables us to extend the life span of the above solution beyond the time T. The singular set $\mathcal{S}$ can be characterized in terms of the local concentration of the L^2-norm of the curvature R_A.

2. Fundamental inequalities.

Let M be a closed 4-manifold. As preparation, we prove some fundamental inequalities.

Proposition 2.1. *Let $T>0$. There exist constants C, $R_0>0$ such that for any u, $v\in L^2(0,T;W^{1,2}(M))$, and any $r\in(0,R_0]$, we have*

$$\int_0^T\int_M |u|\,|v|^2\,dVdt \le C \sup_{(x,t)\in M\times[0,T]} \left(\int_{B_r(x)} |u|^2\,dV\right)^{1/2} \cdot\left(\int_0^T\int_M |\nabla v|^2\,dVdt + r^{-2}\int_0^T\int_M |v|^2\,dVdt\right).$$

This proposition depends on the following local result:

Lemma 2.2. *There exist constants C, $R_0 > 0$ such that for any u, $v \in L^2(0,T;W^{1,2}(M))$, $r \in (0, R_0]$, $x \in M$ and a monotone decreasing radial function $\varphi = \varphi(d(x,\cdot)) \in L^\infty(B_r(x))$, the following inequality holds:*

$$\int_0^T \int_M |u|\,|v|^2 \varphi\, dVdt \le C \sup_{0\le t\le T} \left(\int_{B_r(x)} |u|^2\, dV\right)^{1/2} \cdot \left(\int_0^T \int_M |\nabla v|^2 \varphi\, dVdt + r^{-2} \int_0^T \int_M |v|^2 \varphi\, dVdt\right).$$

Proof. First we assume $\varphi = 1$ and let $\bar{v}_t = \mathrm{vol}(B_r(x))^{-1} \int_{B_r(x)} v(\cdot,t)\, dV$ be the mean value of v. By Hölder's inequality, we have

(2.1)
$$\begin{aligned}\int_0^T \int_{B_r} |u|\,|v|^2\, dVdt \le& C \int_0^T \left(\int_{B_r} |u|^2\, dV\right)^{1/2} \left(\int_M |v|^4\, dV\right)^{1/2} dt \\ \le& C \sup_{0\le t\le T} \left(\int_{B_r} |v-\bar{v}_t|^2\, dV\right)^{1/2} \cdot \int_0^T \left(\int_{B_r} |v-\bar{v}_t|^4 + |\bar{v}_t|^4\, dV\right)^{1/2} dt.\end{aligned}$$

By the Sobolev embedding theorem, we have

$$\int_{B_r} |v-\bar{v}_t|^4\, dV \le C \left(\int_{B_r} |\nabla v|^2\, dV\right)^2. \tag{2.2}$$

On the other hand, by Hölder's inequality,

$$\begin{aligned}\int_{B_r} |\bar{v}_t|^4\, dV &\le C \int_{B_r} \left|\frac{1}{\mathrm{vol}(B_r)} \int_{B_r} v\, dV\right|^4 dV \\ &\le C\,\mathrm{vol}(B_r)^{-3} \left|\int_{B_r} v\, dV\right|^4 \\ &\le C\,\mathrm{vol}(B_r)^{-1} \left(\int_{B_r} |v|^2\, dV\right)^2 \\ &\le C r^{-2} \left(\int_{B_r} |v|^2\, dV\right)^2.\end{aligned} \tag{2.3}$$

By (2.1), (2.2) and (2.3), we have Lemma 2.2 for $\varphi = 1$.

By linearity, Lemma 2.2 holds also for step functions. For general φ, we can show the assertion by approximating φ by step functions. ∎

Proposition 2.1 is derived from Lemma 2.2 via the following argument. For the proof, see Struwe [**15**].

Lemma 2.3. *There exist constants K, $R_0 > 0$ depending only on M such that for any $r \in (0, R_0]$ there exists a covering of M by balls $B_{r/2}(x_i)$ with the property that at any point $x \in M$ at most K of the balls $B_r(x_i)$ meet.*

We next give some identities for the curvature form R_A. For the connection $D = D_0+A$, we note that R_A has the following expression.

$$R_A = R_{D_0} + dA + [A, A].$$

In what follows, we shall abbreviate d_{D_0} by d.

Lemma 2.4. *If A is a smooth solution of (1.1), then*

(2.4) $\quad \partial_t R_A = -\Delta_A^H R_A,$

(2.5) $\quad \partial_t R_A = -\Delta_A^r R_A + [R_A, R_A],$

(2.6) $\quad \partial_t |R_A| \leq \Delta |R_A| + C|R_A|^2,$

(2.7)
$$\partial_t |\tilde{\nabla}_A^{(n)} R_A| \leq \Delta |\tilde{\nabla}_A^{(n)} R_A| + C \sum_{i=0}^{n} |\tilde{\nabla}_A^{(i)} R_A| \, |\tilde{\nabla}_A^{(n-i)} R_A|, \qquad \text{for} \quad n = 1, 2, \cdots$$

where Δ_A^H and Δ_A^r are the Hodge and the rough Laplacian, respectively, i.e., $\Delta_A^H = d_A^* d_A + d_A d_A^*$ and $\Delta_A^r = D^* D$.

Proof. Note that $d_A \partial_t A = \partial_t R_A$. Applying d_A to (1.1), we have

$$\partial_t R_A = -d_A d_A^* R_A = -\Delta_A^H R_A.$$

The Bochner-Weitzenböck formula gives

$$(\Delta_A^r - \Delta_A^H) R_A = [R_A, R_A],$$

hence we obtain (2.4) and (2.5).

Moreover, for $\psi \in \Omega^2(\mathfrak{g}_P)$ we have

$$|\psi| \, \Delta |\psi| \geq \langle \psi, \Delta_A^r \psi \rangle,$$

which implies (2.6).

For the inequality (2.7), we will establish first that the derivatives of R_A evolve according to the equation

$$\partial_t \tilde{\nabla}_A^{(n)} R_A = \tilde{\nabla}_A^2 \tilde{\nabla}_A^{(n)} R_A + \sum_{i=0}^{n} \tilde{\nabla}_A^{(i)} R_A * \tilde{\nabla}_A^{(n-i)} R_A, \tag{2.8}$$

where $A * B$ denotes some linear combination of tensor products of components of A and B. Indeed, the case $n = 0$ is just (2.5). Assuming (2.8) for n and using (1.1), we have

$$\begin{aligned}\partial_t \tilde{\nabla}_{A_k} \tilde{\nabla}_A^{(n)} R_A &= \tilde{\nabla}_{A_k} \partial_t \tilde{\nabla}_A^{(n)} R_A + [d_A^* R_{A_k}, \tilde{\nabla}_A^{(n)} R_A] \\ &= \tilde{\nabla}_{A_k} \left(\tilde{\nabla}_A^2 \tilde{\nabla}_A^{(n)} R_A + \sum_{i=0}^{n} \tilde{\nabla}_A^{(i)} R_A * \tilde{\nabla}_A^{(n-i)} R_A \right) + [d_A^* R_{A_k}, \tilde{\nabla}_A^{(n)} R_A] \\ &= \tilde{\nabla}_A^2 (\tilde{\nabla}_{A_k} \tilde{\nabla}_A^{(n)} R_A) + \sum_{i=0}^{n+1} \tilde{\nabla}_A^{(i)} R_A * \tilde{\nabla}_A^{(n+1-i)} R_A,\end{aligned}$$

where $A = \sum A_k dx^k$, this implies that (2.8) is true for $n + 1$. The inequality (2.7) follow from (2.8). ∎

3. Construction of the local strong solution.

In this section, we show the existence of a time-local smooth solution for (1.2). First we rewrite (1.2) as an equation for the connection A. To make (1.2) a parabolic system for A, we take $\alpha = -d_A^* A$, (cf. Kono-Nagasawa [8]). Then, (1.2) is equivalent to the following for $\nabla = D_0$ a connection on $\mathfrak{g}_P$,

$$(3.1) \qquad \begin{cases} \dfrac{\partial A_i(t)}{\partial t} = \nabla^j \nabla_j A_i - [R_i^j, A_j] \\ \qquad + [A^j, \nabla_j A_i + [A_j, A_i]] + [\nabla_i A^j - \nabla^j A_i + [A_i, A^j], A_j] \\ \qquad + \nabla^j [A_j, A_i] + [A^j, [A_j, A_i]], \\ A_i(t)|_{t=0} = A_i^{(0)}, \end{cases}$$

where $A(t) = A_i(t) dx^i \in \Omega^1(\mathfrak{g}_P)$ is the unknown function, $A^{(0)} = A_i^{(0)} dx^i \in \Omega^1(\mathfrak{g}_P)$ is the given initial data, and $R = R_{ij} dx^i \wedge dx^j$ is the curvature 2-form of ∇.

Since we construct only the local solution for (3.1), we do not have to restrict the dimension of M. Making use of fractional powers of the Laplacian, we shall prove the existence of a strong solution $A(t)$ of (3.1) on a finite time interval $(0, T)$. To this end, let us introduce some notation. The space $L^r(\Omega^1(\mathfrak{g}_P))$ denotes the usual L^r-space with the norm denoted by $\| \cdot \|_r$. We define an operator $\mathcal{L}_r$ on $L^r(\Omega^1(\mathfrak{g}_P))$ by

$$\mathcal{L}_r A_i := -\nabla^j \nabla_j A_i - [R_i^j, A_j], \quad \text{for} \quad A \in D(\mathcal{L}_r)$$

with domain $D(\mathcal{L}_r) = W^{2,r}(\Omega^1(\mathfrak{g}_P))$. Then (3.1) may be rewritten as the following equation on $L^r(\Omega^1(\mathfrak{g}_P))$:

$$(3.2) \qquad \begin{cases} \dfrac{\partial A}{\partial t} + \mathcal{L}_r A + Q(A) = 0, \\ A(0) = A^{(0)}, \end{cases}$$

where $Q(A) = Q_1(A) + Q_2(A)$;

$$Q_1(A)_i = 2[A^j, \nabla_j A_i] + [\nabla_i A^j, A_j] + \nabla^j [A_j, A_i],$$
$$Q_2(A)_i = 3[A^j, [A_j, A_i]].$$

Our result now reads:

Theorem 3.1. *Let* $\dim M = n$ *and let* $A^{(0)} \in L^n(\Omega^1(\mathfrak{g}_P))$. *Then there exist* $T > 0$ *and a function* $A(t)$ *on* $[0, T)$ *with the following properties:*

(1) $A \in C([0,T); L^n(\Omega^1(\mathfrak{g}_P))) \cap C^1((0,T); L^n(\Omega^1(\mathfrak{g}_P)))$;
(2) $A(t) \in D(\mathcal{L}_n)$ *for* $t > 0$, $\mathcal{L}_n A \in C((0,T); L^n(\Omega^1(\mathfrak{g}_P)))$;
(3) A *is a solution of (3.2).*

To prove the Theorem, we need some preliminaries. Since we are interested only in the local solution, we may assume that $\mathcal{L}_r$ has a bounded inverse $\mathcal{L}_r^{-1}$ on $L^r(\Omega^1(\mathfrak{g}_P))$. Hence by the wellknown theory of elliptic differential equations,

$$\|A\|_{H^{2,r}} \leq C_r \|\mathcal{L}_r A\|_r, \quad \text{for} \quad A \in D(\mathcal{L}_r) \quad (1 < r < \infty) \tag{3.3}$$

with a constant C_r independent of A. Moreover, $-\mathcal{L}_r$ generates a contractive holomorphic semi-group $\{e^{-t\mathcal{L}_r}\}_{t \geq 0}$ of class C^0 in $L^r(\Omega^1(\mathfrak{g}_P))$. Therefore we can define the fractional power $\mathcal{L}_r^\alpha$ $(0 < \alpha < 1)$ of $\mathcal{L}_r$ and get a continuous embedding

$$D(\mathcal{L}_r^\alpha) \hookrightarrow H^{2\alpha, r}(\Omega^1(\mathfrak{g}_P)), \quad 0 \leq \alpha \leq 1, \tag{3.4}$$

where $H^{m,r}$ denotes the space of the Bessel potentials. (see, e.g., Fujiwara [**3**]).

In the following, we shall work mainly with $r = n$ and write $\mathcal{L}_n = \mathcal{L}$ for simplicity.

Lemma 3.2. *If* $A \in D(\mathcal{L}^\alpha)$ *for* $\frac{1}{2} < \alpha < 1$, *then* $Q_1(A)$, $Q_2(A) \in L^n(\Omega^1(\mathfrak{g}_P))$. *In fact,*

$$\begin{aligned} \|Q_1(A)\|_n &\leq C \|\mathcal{L}^\alpha A\|_n \|\mathcal{L}^{1/2} A\|_n, \\ \|Q_2(A)\|_n &\leq C \|\mathcal{L}^\alpha A\|_n \|\mathcal{L}^{1/4} A\|_n^2. \end{aligned} \tag{3.5}$$

If A, $B \in D(\mathcal{L}^\alpha)$ *for* $\frac{1}{2} < \alpha < 1$, *then*

$$\begin{aligned} \|Q_1(A) - Q_1(B)\|_n &\leq C(\|\mathcal{L}^\alpha (A - B)\|_n \|\mathcal{L}^{1/2} A\|_n \\ &\qquad + \|\mathcal{L}^\alpha A\|_n \|\mathcal{L}^{1/2}(A - B)\|_n), \\ \|Q_2(A) - Q_2(B)\|_n &\leq C(\|\mathcal{L}^{1/4} A\|_n^2 + \|\mathcal{L}^{1/4} B\|_n) \|\mathcal{L}^\alpha (A - B)\|_n, \end{aligned} \tag{3.6}$$

where the constant C *depends only on* α.

Proof. By (3.4) and the Sobolev embedding theorem, we have $D(\mathcal{L}^\alpha) \hookrightarrow L^\infty(\Omega^1(\mathfrak{g}_P))$, $D(\mathcal{L}^{1/2}) \hookrightarrow H^{1,n}(\Omega^1(\mathfrak{g}_P))$, $D(\mathcal{L}^{1/4}) \hookrightarrow L^{2n}(\Omega^1(\mathfrak{g}_P))$, where $\hookrightarrow$ means a continuous inclusion. Hence it follows from Hölder's inequality that

$$\begin{aligned}\|Q_1(A)\|_n \leq& C\|A\|_\infty\|\nabla A\|_n \leq C\|\mathcal{L}^\alpha A\|_n\|\mathcal{L}^{1/2}A\|_n,\\ \|Q_2(A)\|_n \leq& C\|A\|_{2n}^2\|A\|_\infty \leq C\|\mathcal{L}^\alpha A\|_n\|\mathcal{L}^{1/4}A\|_n^2,\end{aligned}$$

which shows (3.5). The inequality (3.6) is an immediate consequence of (3.5). ∎

Lemma 3.3.

(1) *If $A \in D(\mathcal{L}^\alpha)$ for $\frac{1}{2} < \alpha < 1$, then*

$$\begin{aligned}\|\mathcal{L}^{-1/4}Q_1(A)\|_n &\leq M\|\mathcal{L}^{1/4}A\|_n\|\mathcal{L}^{1/2}A\|_n\\ \|\mathcal{L}^{-1/4}Q_2(A)\|_n &\leq M\|\mathcal{L}^{1/4}A\|_n^3.\end{aligned} \tag{3.7}$$

(2) *If $A,\ B \in D(\mathcal{L}^\alpha)$ for $\frac{1}{2} < \alpha < 1$, then*

$$\begin{aligned}\|\mathcal{L}^{-1/4}(Q_1(A)-Q_1(B))\|_n \leq& M(\|\mathcal{L}^{1/4}(A-B)\|_n\|\mathcal{L}^{1/2}B\|_n\\ &+\|\mathcal{L}^{1/4}A\|_n\|\mathcal{L}^{1/2}(A-B)\|_n)\\ \|\mathcal{L}^{-1/4}(Q_2(A)-Q_2(B))\|_n \leq& M(\|\mathcal{L}^{1/4}A\|_n^2+\|\mathcal{L}^{1/4}B\|_n^2)\|\mathcal{L}^{1/4}(A-B)\|_n,\end{aligned} \tag{3.8}$$

where the constant M is independent of A and B.

Proof. It is easy to see that $\mathcal{L}_r^*$, the adjoint operator of $\mathcal{L}_r$ in $L^r(\Omega^1(\mathfrak{g}_P))$, satisfies

$$\mathcal{L}_r^* = \mathcal{L}_{r'},$$

where $1/r + 1/r' = 1$.

Take $r \in (1,\infty)$ so that $1/r = 1/n + 1/2n$. Then by (3.4) we have $\|A\|_{r'} \leq C\|\mathcal{L}_{n'}^{1/4}A\|_{n'}$ for all $A \in D(\mathcal{L}_{n'}^{1/4})$ with C independent of A ($n' = \frac{n}{n-1}$). Hence Hölder's inequality yields

$$\begin{aligned}\left|\langle\mathcal{L}^{-1/4}Q_1(A),\varphi\rangle\right| &= \left|\langle Q_1(A),\mathcal{L}_{n'}^{-1/4}\varphi\rangle\right|\\ &\leq \|Q_1(A)\|_r\|\mathcal{L}_{n'}^{-1/4}\varphi\|_{r'}\\ &\leq C\|A\|_{2n}\|\nabla A\|_n\|\mathcal{L}_{n'}^{1/4}\mathcal{L}_{n'}^{-1/4}\varphi\|_{n'}\\ &\leq M\|\mathcal{L}^{1/4}A\|_n\|\mathcal{L}^{1/2}A\|_n\|\varphi\|_{n'}\end{aligned}$$

for all $\varphi \in \Omega^1(\mathfrak{g}_P)$. By duality we obtain

$$\|\mathcal{L}^{-1/4}Q_1(A)\|_n \leq M\|\mathcal{L}^{1/4}A\|_n\|\mathcal{L}^{1/2}A\|_n.$$

Similarly, we have for $r = \frac{2n}{3}$.

$$\begin{aligned}\left|\langle \mathcal{L}^{-1/4}Q_2(A), \varphi\rangle\right| &\le \|Q_2(A)\|_r\|\varphi\|_{n'} \\ &\le C\|A\|_{3r}^3\|\varphi\|_{n'} = C\|A\|_{2n}^3\|\varphi\|_{n'} \\ &\le M\|\mathcal{L}^{1/4}A\|_n^3\|\varphi\|_{n'},\end{aligned}$$

for all $\varphi \in \Omega^1(\mathfrak{g}_P)$, from which it follows that

$$\|\mathcal{L}^{-1/4}Q_2(A)\|_n \le M\|\mathcal{L}^{1/4}A\|_n^3.$$

Using (3.7), we easily get (3.8). ∎

Lemma 3.4. *Let $A^{(0)} \in L^n(\Omega^1(\mathfrak{g}_P))$. Then there exist $T > 0$ and a function $A(t)$ on $[0, T)$ such that*

(1) $A \in C([0,T); L^n(\Omega^1(\mathfrak{g}_P))) \cap C((0,T); D(\mathcal{L}^\alpha))$ with

$$\sup_{0<t<T} t^\alpha\|\mathcal{L}^\alpha A(t)\|_n < \infty \quad \text{for} \quad 0 \le \alpha < \frac{3}{4}; \tag{3.9}$$

(2) A is a solution of the integral equation

$$A(t) = e^{-t\mathcal{L}}A^{(0)} - \int_0^t e^{-(t-s)\mathcal{L}}Q(A)(s)\,ds, \quad 0 \le t < T. \tag{3.10}$$

Proof. We solve (3.10) by the following successive approximation:

$$\begin{cases} A_1(t) = e^{-t\mathcal{L}}A^{(0)}, \\ A_{j+1}(t) = A_1(t) - \int_0^t e^{-(t-s)\mathcal{L}}Q(A_j)(s)\,ds, \quad j = 1, 2, \cdots. \end{cases} \tag{3.11}$$

Let us first show that

$$\sup_{0<t<T} t^\alpha\|\mathcal{L}^\alpha A_j(t)\|_n \le K_{\alpha,j}, \quad 0 \le \alpha < \frac{3}{4}, \quad j = 1, 2, \cdots. \tag{3.12}$$

Indeed for $j = 1$, we have

$$t^\alpha\|\mathcal{L}^\alpha A_1(t)\|_n = t^\alpha\|\mathcal{L}^\alpha e^{-t\mathcal{L}}A^{(0)}\|_n \le \|A^{(0)}\|_n \quad \text{for all } t > 0$$

and hence we may set $K_{\alpha,1} := \sup_{0<t<T} t^\alpha\|\mathcal{L}^\alpha e^{-t\mathcal{L}}A^{(0)}\|_n$.

Suppose that (3.12) is true for j. Then it follows from Lemma 3.3 that

$$\begin{aligned}\|\mathcal{L}^{\alpha}A_{j+1}(t)\|_n \leq&\|\mathcal{L}^{\alpha}A_1(t)\|_n + \int_0^t \|\mathcal{L}^{\alpha+1/4}e^{-(t-s)\mathcal{L}}\mathcal{L}^{-1/4}Q(A_j)(s)\|_n\,ds\\ \leq&K_{\alpha,1}t^{-\alpha} + \int_0^t (t-s)^{-\alpha-1/4}\|\mathcal{L}^{-1/4}Q(A_j)(s)\|_n\,ds\\ \leq&K_{\alpha,1}t^{-\alpha}\\ &+ M\int_0^t (t-s)^{-\alpha-1/4}(\|\mathcal{L}^{1/4}A_j(s)\|_n\|\mathcal{L}^{1/2}A_j(s)\|_n + \|\mathcal{L}^{1/4}A_j(s)\|_n^3)\,ds\\ \leq&K_{\alpha,1}t^{-\alpha} + M(K_{1/4,j}K_{1/2,j} + K_{1/4,j}^3)\int_0^t (t-s)^{-\alpha-1/4}s^{-3/4}\,ds\\ \leq&K_{\alpha,1}t^{-\alpha} + MB(3/4-\alpha,1/4)(K_{1/4,j}K_{1/2,j} + K_{1/4,j}^3)t^{-\alpha}\end{aligned}$$

for $0 \leq \alpha < 3/4$ and $0 < t < T$, where $B(\cdot,\cdot)$ denotes the beta function. Hence (3.12) is satisfied with j replaced by $j+1$ and

$$K_{\alpha,j+1} := K_{\alpha,1} + MB(3/4-\alpha,1/4)(K_{1/4,j}K_{1/2,j} + K_{1/4,j}^3). \tag{3.13}$$

(3.13) shows that $\{K_{\alpha,j}\}_{j=1}^{\infty}$ is a *closed* recurrence for $\alpha = \frac{1}{4}$ and $\alpha = \frac{1}{2}$. Now let $k_j := \max\{K_{1/4,j}, K_{1/2,j}\}$ $(j = 1,2,\cdots)$. Then by (3.13), we have

$$k_{j+1} \leq k_1 + 2M\beta(k_j^2 + k_j^3), \quad \beta = B(1/4,1/4). \tag{3.14}$$

for $j = 1,2,\cdots$. By (3.14), we see that there exist positive constants m_* and k such that if

$$k_1 < m_*, \tag{3.15}$$

then

$$k_j \leq k \quad \text{for all } j = 1,2,\cdots. \tag{3.16}$$

In fact, m_* is determined by the local maximum of the function $f(x) = x - 2M\beta(x^2 + x^3)$ and k is the positive root of the equation $f(x) = k_1$.

Assume (3.15) for a moment and set

$$B_j(t) := A_j(t) - A_{j-1}(t), \quad j = 1,2,\cdots, \quad (A_0(t) = 0).$$

Then it follows from Lemma 3.3, (3.7) and (3.15) that
(3.17)

$$\begin{aligned}\|\mathcal{L}^\alpha B_{j+1}(t)\|_n &\le \int_0^t \|\mathcal{L}^{\alpha+1/4}e^{-(t-s)\mathcal{L}}\mathcal{L}^{-1/4}(Q(A_j)(s)-Q(A_{j-1})(s))\|_n\,ds\\ &\le \int_0^t \|\mathcal{L}^{\alpha+1/4}e^{-(t-s)\mathcal{L}}\|_{B(L^n)}\|\mathcal{L}^{-1/4}(Q(A_j)(s)-Q(A_{j-1})(s))\|_n\,ds\\ &\le M\int_0^t (t-s)^{-\alpha-1/4}\left\{\|\mathcal{L}^{1/2}B_j(s)\|_n\|\mathcal{L}^{1/4}A_j(s)\|_n\right.\\ &+\|\mathcal{L}^{1/2}A_{j-1}(s)\|_n\|\mathcal{L}^{1/4}B_j(s)\|_n\\ &\left.+(\|\mathcal{L}^{1/4}A_j(s)\|_n^2+\|\mathcal{L}^{1/4}A_{j-1}\|_n^2)\|\mathcal{L}^{1/4}B_j(s)\|_n\right\}\,ds\\ &\le Mk\int_0^t (t-s)^{-\alpha-1/4}(\|\mathcal{L}^{1/2}B_j(s)\|_n s^{-1/4}+\|\mathcal{L}^{1/4}B_j(s)\|_n s^{-1/2})\,ds\\ &+2Mk^2\int_0^t (t-s)^{-\alpha-1/4}\|\mathcal{L}^{1/4}B_j(s)\|_n s^{-1/2}\,ds\end{aligned}$$

for $0 \le \alpha < \frac{3}{4}$.

Taking $\alpha = 1/4$ and $\alpha = 1/2$ in the above estimate, we get by induction

$$(3.18)\qquad \begin{cases}\|\mathcal{L}^{1/4}B_j(t)\|_n \le k\{2M\beta(k+k^2)\}^{j-1}t^{-1/4}\\ \|\mathcal{L}^{1/2}B_j(t)\|_n \le k\{2M\beta(k+k^2)\}^{j-1}t^{-1/2}, \quad j=1,2,\cdots.\end{cases}$$

By (3.17) and (3.18),

$$(3.19)\qquad \|\mathcal{L}^\alpha B_{j+1}(t)\|_n \le k\{2M\beta(k+k^2)\}^{j-1}\{2MB(\frac{3}{4}-\alpha,\frac{1}{4})(k+k^2)\}t^{-\alpha},$$

$(0 < t < T)$.

Since k satisfies $k = k_1 - 2M\beta(k^2+k^3)$, under the assumption (3.15) we have $2M\beta(k+k^2) = 1 - k_1/k \in (0,1)$ and hence by (3.19) the sequence $A_j(t) = \sum_{r=1}^{j} B_r(t)$ converges absolutely and uniformly in $L^n(\Omega^1(\mathfrak{g}_P))$ with respect to $t \in [0,T]$: $A_j(t) \to A(t)$, where $A \in BC([0,T);L^n(\Omega^1(\mathfrak{g}_P)))$.

Moreover, again by (3.19), for each $0 < \alpha < \frac{3}{4}$ there exists $A^{(\alpha)} \in C((0,T);L^n(\Omega^1(\mathfrak{g}_P)))$ with $t^\alpha A^{(\alpha)}(t) \in BC([0,T];L^n(\Omega^1(\mathfrak{g}_P)))$ such that $\sup_{0<t<T} t^\alpha\|\mathcal{L}^\alpha A_j(t) - A^{(\alpha)}(t)\|_n \to 0$ as $j \to \infty$. Since $\mathcal{L}$ is a closed operator on $L^n(\Omega^1(\mathfrak{g}_P))$, we can conclude that $A \in C((0,T);D(\mathcal{L}^\alpha))$ with $\mathcal{L}^\alpha A(t) = A^{(\alpha)}(t)$ for all $0 < t < T$, and hence

$$(3.20)\qquad \sup_{0<t<T} t^\alpha\|\mathcal{L}^\alpha(A_j(t)-A(t))\|_n \to 0 \quad (0 \le \alpha < \frac{3}{4}) \text{ as } j \to \infty.$$

Now again by Lemma 3.3, (3.8), (3.16) and (3.20),

$$\begin{aligned}
&\|\int_0^t e^{-(t-s)\mathcal{L}}(Q(A_j)(s)-Q(A)(s))\,ds\|_n \\
\le&\int_0^t \|\mathcal{L}^{1/4}e^{-(t-s)\mathcal{L}}\|_{B(L^n)}\|\mathcal{L}^{-1/4}(Q(A_j)(s)-Q(A)(s))\|_n\,ds \\
\le& M\int_0^t (t-s)^{-1/4}\Big\{\|\mathcal{L}^{1/2}(A_j(s)-A(s))\|_n\|\mathcal{L}^{1/4}A_j(s)\|_n \\
&+\|\mathcal{L}^{1/2}A(s)\|_n\|\mathcal{L}^{1/4}(A_j(s)-A(s))\|_n \\
&+(\|\mathcal{L}^{1/4}A(s)\|_n^2+\|\mathcal{L}^{1/4}A_j(s)\|_n^2)\|\mathcal{L}^{1/4}(A_j(s)-A(s))\|_n\Big\}\,ds \\
\le& Mk\beta\sup_{0<s<t} s^{1/2}\|\mathcal{L}^{1/2}(A_j(s)-A(s))\|_n \\
&+(Mk\beta+2Mk^2\beta)\sup_{0<s<t} s^{1/4}\|\mathcal{L}^{1/4}(A_j(s)-A(s))\|_n \\
&\to 0 \quad \text{as } j\to\infty
\end{aligned}$$

Hence under the assumption (3.15), we see by (3.11) and the above convergence that A is the desired solution of (3.10).

It remains to show that we can take T so small that (3.15) is satisfied. Since $D(\mathcal{L}^{1/2})$ is dense in $L^n(\Omega^1(\mathfrak{g}_P))$, there exists a function $\tilde{A}^{(0)}\in D(\mathcal{L}^{1/2})$ such that $\|A^{(0)}-\tilde{A}^{(0)}\|_n<\frac{m_*}{2}$. Hence it follows that

$$\begin{aligned}
t^\alpha\|\mathcal{L}^\alpha e^{-t\mathcal{L}}A^{(0)}\|_n &\le t^\alpha\|\mathcal{L}^\alpha e^{-t\mathcal{L}}(A^{(0)}-\tilde{A}^{(0)})\|_n+t^\alpha\|\mathcal{L}^\alpha e^{-t\mathcal{L}}\tilde{A}^{(0)}\|_n \\
&\le \|A^{(0)}-\tilde{A}^{(0)}\|_n+t^\alpha\|\mathcal{L}^\alpha\tilde{A}^{(0)}\|_n \\
&< \frac{m_*}{2}+t^\alpha\|\mathcal{L}^\alpha\tilde{A}^{(0)}\|_n, \quad t>0,
\end{aligned}$$

for $\alpha=1/4$ and $\alpha=1/2$. Now T may be taken to be $T=\left(\frac{m_*}{2\|\mathcal{L}^{1/2}\tilde{A}^{(0)}\|_n}\right)^4$. This completes the proof of Lemma 3.4. ∎

To show that A in Lemma 3.4 also satisfies (1), (2) and (3) in Theorem 3.1, we need:

Lemma 3.5. *Let A be the solution of (3.10) given by Lemma 3.4. Then for $0\le\alpha<\frac{3}{4}$, $\mathcal{L}^\alpha A(t)$ is a Hölder continuous function on $(0,T)$ with values in $L^n(\Omega^1(\mathfrak{g}_P))$. More precisely, for $0\le\alpha<\frac{3}{4}$ there exists $0<\eta<\frac{3}{4}-\alpha$ such that*

$$\|\mathcal{L}^\alpha A(t+h)-\mathcal{L}^\alpha A(t)\|_n\le C(h^\eta t^{-\alpha-\eta}+h^{3/4-\alpha}t^{-3/4}) \tag{3.21}$$

holds for all $h>0$ and $0<t\le T$, where $C=C(\alpha,\eta,M,k)$ is independent of h and t.

Proof. An elementary calculation shows

$$\|(e^{-h\mathcal{L}}-1)A\|_n\le h^\gamma\|\mathcal{L}^\gamma A\|_n,\quad A\in D(\mathcal{L}^\gamma),\quad 0<\gamma<1,$$

for all $h > 0$. Hence by Lemma 3.3 (2), (3.12) and (3.16),

$$\begin{aligned}
&\|\mathcal{L}^\alpha A(t+h) - \mathcal{L}^\alpha A(t)\|_n \\
\leq &\|(e^{-h\mathcal{L}} - 1)\mathcal{L}^\alpha e^{-t\mathcal{L}} A^{(0)}\|_n \\
&+ \int_t^{t+h} \|\mathcal{L}^{\alpha+1/4} e^{-(t+h-s)\mathcal{L}} \mathcal{L}^{-1/4} Q(A)(s)\|_n \, ds \\
&+ \int_t^{t+h} \|(e^{-h\mathcal{L}} - 1)\mathcal{L}^{\alpha+1/4} e^{-(t-s)\mathcal{L}} \mathcal{L}^{-1/4} Q(A)(s)\|_n \, ds \\
\leq &h^\eta \|\mathcal{L}^{\alpha+\eta} e^{-t\mathcal{L}} A^{(0)}\|_n \\
&+ \int_t^{t+h} (t+h-s)^{-\alpha-1/4} (\|\mathcal{L}^{1/2} A(s)\|_n \|\mathcal{L}^{1/4} A(s)\|_n + \|\mathcal{L}^{1/4} A(s)\|_n^3) \, ds \\
&+ h^\eta \int_t^{t+h} (t-s)^{-\alpha-1/4-\eta} (\|\mathcal{L}^{1/2} A(s)\|_n \|\mathcal{L}^{1/4} A(s)\|_n + \|\mathcal{L}^{1/4} A(s)\|_n^3) \, ds \\
\leq &h^\eta t^{-\alpha-\eta} \|A^{(0)}\|_n + M(k^2 + k^3) \int_t^{t+h} (t+h-s)^{-\alpha-1/4} s^{-3/4} \, ds \\
&+ M(k^2+k^3) h^\eta \int_t^{t+h} (t-s)^{-\alpha-1/4-\eta} s^{-3/4} \, ds \\
\leq &h^\eta t^{-\alpha-\eta} \|A^{(0)}\|_n + \frac{M(k^2+k^3)}{3/4-\alpha} h^{3/4-\alpha} t^{-3/4} \\
&+ M(k^2+k^3) B(3/4-\alpha-\eta, 1/4) h^\eta t^{-\alpha-\eta}, \quad 0 \leq \alpha < \frac{3}{4},
\end{aligned}$$

for all $t > 0$, $h > 0$, where $0 \leq \eta < \alpha - \frac{3}{4}$, from which (3.21) follows. ∎

Proof of Theorem 3.1. Let $A(t)$ be the solution of (3.11) given in Lemma 3.4. Then by Lemmas 3.2 and 3.5, we see that the function $Q(A)(t)$ is Hölder continuous on $(0, T)$ with values in $L^n(\Omega^1(\mathfrak{g}_P))$. By the general theory of holomorphic semi-groups (see e.g. Tanabe [**16**, Theorem 3.3.2], A is also a solution of (3.2) in the class of (1) and (2) in Theorem 3.1. This completes the proof. ∎

4. Estimates.

Now we return to the case when M is a closed 4-manifold. In this section, we give various estimates for the curvature tensor, which will be used to characterize the singular set $\mathcal{S}$. Recall that, for $0 \leq t_0 < t_1$,

$$\begin{aligned}
V(M, [t_0, t_1]) := \Big\{ &A \in \Omega^1(M \times [t_0, t_1], \mathfrak{g}_P); \\
&\sup_{t_0 \leq t \leq t_1} \int_M \left(|R_A|^2 + |\tilde{\nabla}_A R_A|^2 + |\tilde{\nabla}_A^2 R_A|^2 + |A|^2 + |\nabla A|^2 + |\partial_t A|^2 \right) dV < \infty \Big\}.
\end{aligned}$$

Lemma 4.1. *Let A be a solution of (1.1) in the class of $V(M,[0,T])$. Then the function*

$$E(t) = \frac{1}{2}\int_M |R_A(\cdot,t)|^2\,dV,$$

is non-increasing.

Proof. Taking the L^2-inner product of R_A in (2.4), we have

$$\frac{d}{dt}\frac{1}{2}\int_M |R_A|^2\,dV = -\int_M |d_A^* R_A|^2\,dV \le 0,$$

for any $t \in [0,T]$, which gives Lemma 4.1. ∎

By Lemma 4.1, if A is a solution of (1.1), then for any $t \in [0,T]$,

$$E(t) = \int_M |R_A(\cdot,t)|^2\,dV,$$

is bounded from above. For the solution $A \in V(M,[0,T])$, put

$$\varepsilon(r) = \varepsilon(r,A,T) = \sup_{(x,t)\in M\times[0,T]} \left(\int_{B_r(x)} |R_A(\cdot,t)|^2\,dV\right)^{1/2}. \tag{4.1}$$

In the sequel we give a priori bounds for the norm of A in terms of the initial energy $E(A_0) = E_0$, T and ε_1. Here $\varepsilon_1 > 0$ is a parameter depending only on M and P which will be determined in Lemmas 4.2-4.4. To obtain these, we use Sobolev embedding theorem for 4-dimensional case, which plays an important role for our purpose. We will set ε_1 to be the smallest of the numbers ε_1 occurring in these Lemmas.

Lemma 4.2. *There exist constants $\varepsilon_1 > 0$ and $C > 0$ such that for any solution $A \in V(M;[0,T))$ and any number $r \in (0,R_0]$, the following inequalities hold:*

$$\sup_{0<t<T_0} \int_M |\tilde\nabla_A R_A|^2\,dV \le C, \tag{4.2}$$

$$\sup_{0<t<T_0} \int_M |\tilde\nabla_A^2 R_A|^2\,dV \le C, \tag{4.3}$$

provided $\varepsilon(r) < \varepsilon_1$.

Proof. Multiplying (2.7) by $|\tilde\nabla_A R_A|$ for $n=1$, we have

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_0^T\int_M |\tilde\nabla_A R_A|^2\,dV dt + \int_0^T\int_M |\nabla|\tilde\nabla_A R_A||^2\,dV dt \\ \le C\int_0^T\int_M |\tilde\nabla_A R_A|^2 |R_A|\,dV dt.\end{aligned} \tag{4.4}$$

The right hand side of (4.4) is estimated by
(4.5)

$$\begin{aligned}&\int_0^T\int_M |\tilde{\nabla}_A R_A|^2 |R_A|\, dV dt\\ \le & C \sup_{0<t<T}\left(\int_{B_r(x)} |R_A|^2\, dV\right)^{1/2}\left(\int_0^T\int_M |\nabla|\tilde{\nabla}_A R_A||^2\, dV dt + r^{-2}\int_0^T\int_M |R_A|^2\, dV dt\right),\end{aligned}$$

with the help of Lemma 2.1. If $\varepsilon(r) < \varepsilon_1$, by (4.4) and (4.5), for the function $u(t) = \int_M |\tilde{\nabla}_A R_A|^2(t)\, dV$, we have

$$\frac{d}{dt}u(t) \le CE_0 r^{-2} u(t), \quad E_0 = E(0) = E(A_0),$$

which implies the result (4.2) by Gronwall's inequality.

On the other hand, (4.4) and (4.5) yield

$$\int_0^T\int_M |\nabla|\tilde{\nabla}_A R_A||^2\, dV dt \le C, \tag{4.6}$$

where the constant C depends on E_0, M, P and T.

For the statement (4.3) multiplying (2.7) by $|\tilde{\nabla}_A^2 R_A|$ for $n = 2$, we obtain

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_0^T\int_M |\tilde{\nabla}_A^2 R_A|^2\, dV dt + \int_0^T\int_M |\nabla|\tilde{\nabla}_A^2 R_A||^2\, dV dt\\ \le & C\int_0^T\int_M |\tilde{\nabla}_A^2 R_A|^2 |R_A|\, dV dt + C\int_0^T\int_M |\tilde{\nabla}_A R_A|^2 |\tilde{\nabla}_A^2 R_A|\, dV dt.\end{aligned} \tag{4.7}$$

Using Lemma 2.1 and (4.6), we obtain
(4.8)

$$\begin{aligned}&\int_0^T\int_M |\nabla|\tilde{\nabla}_A^2 R_A||^2 |R_A|\, dV dt + \int_0^T\int_M |\tilde{\nabla}_A R_A|^2 |\tilde{\nabla}_A^2 R_A|\, dV dt\\ \le & C \sup_{0<t<T}\left(\int_{B_r(x)} |R_A|^2\, dV\right)^{1/2}\left(\int_0^T\int_M |\nabla|\tilde{\nabla}_A R_A||^2\, dV dt + r^{-2}\int_0^T\int_M |R_A|^2\, dV dt\right)\\ & + C\int_0^T\int_M |\tilde{\nabla}_A^2 R_A|^2\, dV dt + C\int_0^T\int_M |\nabla|\tilde{\nabla}_A R_A||^2\, dV dt + r^{-2}\int_0^T\int_M |\tilde{\nabla}_A R_A|^2\, dV dt.\end{aligned}$$

If $\varepsilon(r) < \varepsilon_1$, by (4.4) and (4.5), for the function $v(t) = \int_M |\tilde{\nabla}_A^2 R_A|^2(t)\, dV$, we have

$$\frac{d}{dt}v(t) \le CE_0 r^{-2} v(t) + C(t),$$

which implies the result (4.3), by Gronwall's inequality. ∎

Lemma 4.3. *There exist constants $\varepsilon_1 > 0$ and $C > 0$ such that for any solution $A \in V(M;[0,T))$ and any number $r \in (0, R_0]$, the following inequality holds:*

$$\sup_{0<t<T_0} \int_M |A|^2\, dV \leq C, \tag{4.9}$$

provided $\varepsilon(r) < \varepsilon_1$.

Proof. Multiplying (1.1) by A and then integrating, we have

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int_0^T \int_M |A|^2\, dV dt &\leq \int_0^T \int_M -\langle d_A^* R_A, A\rangle\, dV dt \\ &\leq C \int_0^T \int_M |\tilde{\nabla}_A R_A|^2\, dV dt + \int_0^T \int_M |A|^2\, dV dt. \end{aligned} \tag{4.10}$$

For the function $u(t) = \int_M |A|^2(t)\, dV$, we obtain

$$\frac{d}{dt} u(t) \leq C u(t) + C(t),$$

by using (4.10). The claim follows from Gronwall's inequality. ∎

Lemma 4.4. *There exist constants $\varepsilon_1 > 0$ and $C > 0$ such that for any solution $A \in V(M;[0,T))$ and any number $r \in (0, R_0]$, the following inequality holds:*

$$\sup_{0<t<T_0} \int_M |\nabla A|^2\, dV \leq C, \tag{4.11}$$

provided $\varepsilon(r) < \varepsilon_1$.

Proof. Applying $\tilde{\nabla}_A$ to (1.1), multiplying by $|\tilde{\nabla}_A A|$ and then integrating, we have

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int_0^T \int_M |\tilde{\nabla}_A A|^2\, dV dt &\leq \int_0^T \int_M -\langle \tilde{\nabla}_A d_A^* R_A, \tilde{\nabla}_A A\rangle\, dV dt \\ &\leq C \int_0^T \int_M |\tilde{\nabla}_A^2 R_A|^2\, dV dt + \int_0^T \int_M |\tilde{\nabla}_A A|^2\, dV dt. \end{aligned} \tag{4.12}$$

For the function $u(t) = \int_M |\tilde{\nabla}_A A|^2(t)\, dV$, using (4.12) we obtain

$$\frac{d}{dt} u(t) \leq C u(t) + C(t).$$

By Gronwall's inequality, we obtain

$$\sup_{0<t<T} \int_M |\tilde{\nabla}_A A|^2\, dV dt \leq C. \tag{4.13}$$

On the other hand, since $R_A = d_A A - \frac{1}{2}[A, A]$, we have

$$\int_M |[A, A]|^2 \, dV \le \int_M |R_A|^2 \, dV + \int_M |\tilde{\nabla}_A A|^2 \, dV \le C. \tag{4.14}$$

Using $\tilde{\nabla}_A A = \nabla A + [A, A]$, (4.13) and (4.14), we get (4.11). ∎

Lemma 4.5. *For any solution $A \in V(M, [0, T])$ of (1.1), we have*

$$\int_0^T \int_M |\partial_t A|^2 \, dV dt \le E_0 = E(0).$$

Proof. Multiplying (1.1) by $\partial_t A$ and integrating over $M \times [0, T]$, we have

$$\int_0^T \int_M |\partial_t A|^2 \, dV dt = \frac{1}{2} \int_0^T \int_M \partial_t |R_A|^2 \, dV dt \le \frac{1}{2} \int_M |R_A|^2(0) \, dV.$$

∎

As the consequence of Lemmas 4.2-4.4, we obtain:

Proposition 4.6. *There exist constants $\varepsilon_1 > 0$ and $C > 0$ such that for any solution $A \in V(M; [0, T))$ and any number $r \in (0, R_0]$, the following inequality holds:*

$$\sup_{0<t<T_0} \int_M \left(|R_A|^2 + |\tilde{\nabla}_A R_A|^2 + |\tilde{\nabla}_A^2 R_A|^2 + |A|^2 + |\nabla A|^2 + |\partial_t A|^2 \right) dV \le C,$$

provided $\varepsilon(r) < \varepsilon_1$.

As a corollary of Proposition 4.6, we have:

Proposition 4.7. *Let $T_0 > 0$ be the maximal existence time for the local smooth solution constructed in Theorem 3.1, with the initial value $A_0 \in W^{1,2}(\Omega^2(\mathfrak{g}_P))$. Then there exists a weak limit A_1 of the solution $A(t)$ in $W^{1,2}(\Omega^2(\mathfrak{g}_P))$ as $t \to T_0$.*

5. Proof of Theorem.

Let $T = T(A_0)$ be the maximal existence time for the smooth solution of (1.1).

Theorem 5.1. *Let $A_0 \in \mathfrak{U}^{1,2}$, and let $A \in V(M, [0, T))$ be a solution of (1.1). The existence time $T(A_0)$ is characterized by the condition*

$$\limsup_{T' \to T} \left(\int_{B_r(x)} |R_A|^2 \, dV \right)^{1/2} \ge \varepsilon_1 \quad \textit{for all} \quad r \in (0, R_0].$$

The solution A is regular on $M \times (0,T]$ except for finitely many points $\{(x^l,t^l) : 1 \le l \le L\}$, characterized by

$$\limsup_{T' \to t^l} \left(\int_{B_r(x^l)} |R_A|^2 \, dV \right)^{1/2} \ge \varepsilon_1, \quad \text{for all} \quad r \in (0, R_0].$$

Moreover, the energy $E(A(\cdot,t))$ is non-increasing.

Proof. Let $A_0 \in \mathfrak{U}^{1,2}$. By Theorem 3.1, there exist a local regular solutions $A(t)$ of (1.1). By Proposition 4.6, we see that the maximal existence time $T(A_0)$ is characterized by

$$\limsup_{T' \to T(A_0)} \left(\int_{B_r(x)} |R_A|^2 \, dV \right)^{1/2} \ge \varepsilon_1, \tag{5.1}$$

for some $r > 0$ and all $x \in M$. Theorem 5.1 follows immediately from:

Lemma 5.2. *Put*

$$\mathcal{S}_{T^*} := \left\{ x \in M : \int_{B_r(x)} |R_A(\cdot,T^*)|^2 \, dV \ge \varepsilon_1 \quad \text{for all} \quad r \in (0,R_0] \right\}.$$

Then $\mathcal{S}_{T^*}$ consists of finitely many points.

The lower semi-continuity of the energy yields

$$\begin{aligned} \int_{M' \times T^*} |R_A|^2 \, dV &\le \liminf_{T \to T^*, T < T^*} \int_{M' \times T} |R_A|^2 \, dV \\ &\le \liminf_{T \to T^*, T < T^*} \int_{M \times T} |R_A|^2 \, dV - \sum_{l=1}^{L_1} \int_{B_r(x^l) \times T^*} |R_A|^2 \, dV \\ &\le E_0 - L_1 \varepsilon_1, \end{aligned} \tag{5.2}$$

for any $r \in (0, R_0]$ and any $M' \subset M \setminus \cup_{l=1}^{L_1} B_r(x^l)$. Passing to the limit $r \to 0$ and $M' \to M$, we have

$$E(\cdot, T^*) \le E_0 - L_1 \varepsilon_1,$$

which means that L_1 is finite by (5.2). This gives Lemma 5.2. ∎

To obtain the main Theorem in the introduction, it is enough to show:

Theorem 5.3. *For any initial value $A_0 \in \mathfrak{U}^{1,2}$, there exists a weak solution $A \in V(M,[0,\infty))$ of (1.1). The solution A is regular on $M \times (0,\infty)$ with the exception of finitely many points $\{(x^l,t^l) : 1 \le l \le L\}$ characterized by*

$$\limsup_{T' \to T} \int_{B_r(x)} |R_A|^2 \, dV \ge \varepsilon_1, \quad \text{for all} \quad r \in (0, R_0].$$

Moreover, the energy $E(A(\cdot,t))$ is non-increasing.

Proof. It is sufficient to prove that the solution can be extended to the time ∞.

Assume the solution is at most extendable to time $T_1^* > 0$. Since the singular set $\mathcal{S}$ is finite, $A(\cdot, T_1^*)$ is regular on $M\backslash\mathcal{S} \cap \{t = T_1^*\}$. Moreover, since the solution A belongs to $C([0,T); W^{1,2}(\Omega^1(\mathfrak{g}_P)))$, Proposition 4.6 yields $A(\cdot, T_1^*) \in \mathfrak{U}^{1,2}$. By Theorem 5.1, there exists a number $T_2^* > 0$ such that the solution A may be extended to $[0, T_1^* + T_2^*]$ and T_2^* is characterized by the property in Theorem 5.1. Iterating this procedure, we see that the solution A can be extended to time infinity. ∎

As an easy consequence of Theorem 5.3, we have:

Corollary 5.4. *If $E(A_0) \le \varepsilon_1$, then there exists a smooth global solution for (1.1).*

Finally, we characterize of the singular points for $A(t)$.

Theorem 5.5. *Let A be a solution of (1.1) constructed in Theorem 5.2, and suppose that (x_0, T), $T \le \infty$, is a singular point. Then there exist sequences $x_m \to x_0$, $t_m \uparrow T$, $r_m \in (0, R_0]$, $r_0 \to 0$ and a regular Yang-Mills connection A_0 on $\mathbb{R}^4$ such that*

$$A_{r_m,(x_m,t_m)}(x,t) := A(r_m \cdot x + x_0, r_m^2 \cdot t_m + T)$$

tends to A_0 locally in $\mathfrak{U}^{2,2}$. Moreover the Yang-Mills connection A_0 extends to a regular Yang-Mills connection on S^4.

Proof. Let x^0 be a singular point of A at time T characterized by the condition

$$\limsup_{T' \to T} \int_{B_r(x_0)} |R_A|^2 \, dV \ge \varepsilon_1.$$

Therefore there exist sequences $x_m \to x^l$, $t_m \uparrow T$, $r_m \in (0, R_0)$ with $r_m \to 0$ such that

$$\varepsilon_1 = \int_{B_{r_m}(x_m)\times t_m} |R_A|^2 \, dV.$$

For any $t \in [t_m - \varepsilon, t_m]$, we have

$$\int_{B_{2r_m}(x_m)\times t} |R_A|^2 \, dV \ge \frac{\varepsilon_1}{2}, \quad \int_{t_m-\varepsilon}^{t_m} \int_M |\nabla R_A|^2 \, dV \le CE_0.$$

Hence the sequence $A_m := A_{r_m,(x_m,t_m)}$ satisfies the estimates on $\mathcal{D}_m := \{(x,t) : r_m \cdot x + x_m \in B_\rho(x_0), r_m^2 \cdot t + t_m \ge 0\}$:

$$\int_{\mathcal{D}_m \times [-\varepsilon,0]} |R_{A_m}|^2 \, dVdt \le C,$$
$$\int_{\mathcal{D}_m \times [-\varepsilon,0]} |\partial_t A_m|^2 \, dVdt \longrightarrow 0 \quad (m \to \infty).$$

Therefore there exists a subsequence $\{A_m\}$ such that $A_m(\cdot, 0)$ converges to A_0 weakly in $\mathfrak{U}^{2,2}(\mathbb{R}^4)$ and strongly in $\mathfrak{U}^{1,2}_{\rm loc}(\mathbb{R}^4)$. Passing to the limit $m \to \infty$, it follows that A_0 is a Yang-Mills connection with finite energy on $\mathbb{R}^4$. By Uhlenbeck's result [**17**], A_0 extends to a Yang-Mills connection on a bundle P' over S^4. ∎

References.

[**1**] R. Courant and D. Hilbert, "Method of Mathematical Physics I, II," Interscience, New York, 1953, 1965.

[**2**] S. Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. **50** (1985), 1–26.

[**3**] D. Fujiwara, *On the asymptotic behaviour of the Green operators for elliptic boundary problems and the pure imaginary powers of the second order operators*, J. Math. Soc. Japan **21** (1969), 481–521.

[**4**] M. Giaquinta, "Multiple Integrals in the Calculus of Variations and Nonlinear Analysis," Princeton Univ. Press, Princeton, New Jersey, 1983.

[**5**] D. Gilberg and N. S. Trudinger, "Elliptic Partial Differential Equations of Second Order, Second edition," Springer-Verlag, Berlin-Heiderberg-New York, 1983.

[**6**] J. Jost, "Nonlinear Methods in Riemannian and Kählerian Geometry," Birkhäuser, Basel, 1988.

[**7**] H. Kozono and Y.Maeda, *On asymptotic stability for the Yang-Mills gradient flow*, Tokyo J. Math. **12** (1989), 387–413.

[**8**] K. Kono and T. Nagasawa, *Weak asymptotic stability of Yang-Mills gradient flow*, Tokyo J. Math. **11** (1988), 339–357.

[**9**] H. B. Lawson, "The Theory of Gauge Fields in Four Dimensions," CBMS Regional Conf. Series Vol. 58, AMS, Providence, 1987.

[**10**] C. B. Morrey, "Multiple Integrals in the Calculus of Variations," Springer-Verlag, Berlin-Herderberg-New York, 1966.

[**11**] H. Naito, H. Kozono and Y. Maeda, *A stable manifold theorem for the Yang-Mills gradient flow*, Tôhoku Math. J. **42** (1990), 45–66.

[**12**] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math. **113** (1981), 1–24.

[**13**] R. Schoen and K. Uhlenbeck, *A regularity theory for harmonic maps*, J. Differential Geom. **18** (1982), 307–335.

[**14**] S. Sedlacek, *A direct method for minimizing the Yang-Mills functional over 4-manifolds*, Comm. Math. Phys. **86** (1982), 515–527.

[**15**] M. Struwe, *On the evolution of harmonic mappings of Riemann surfaces*, Comm. Math. Helv. **4** (1985), 558–581.

[**16**] H. Tanabe, "Equation of Evolution," Pitmann, London, 1979.

[**17**] K. Uhlenbeck, *Removable singularities in Yang-Mills fields*, Comm. Math. Phys. **83** (1982), 11-29.

[18] K. Uhlenbeck, *Connections with L^p-bounds on curvature*, Comm. Math. Phys. **83** (1982), 31–42.

Hideo KOZONO

Department of Mathematics, Colleage of General Education, Kyusyu Univ., Fukuoka, 810, JAPAN
`f77177a@kyu-cc.cc.kyushu-u.ac.jp`

Yoshiaki MAEDA

Department of Mathematics, Faculty of Science and Technology, Keio Univ., Yokohama, 223, JAPAN
`maeda@math.keio.ac.jp`

Hisashi NAITO

Department of Mathematics, School of Sciences, Nagoya Univ., Nagoya, 464-01, JAPAN
`naito@math.nagoya-u.ac.jp`