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A Poincare-Birkhoff-Witt Theorem For Infinite Dimensional Lie Algebra

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A POINCARE-BIRKHOFF-WITT THEOREM FOR INFINITE DIMENSIONAL LIE ALGEBRAS

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§0. INTRODUCTION

Let $(1<)\ \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots$ be a series of positive real numbers such that

$$\sum_{n\geq 1} \lambda_n^{-s_0} < \infty \quad \text{for some integer } s_0.$$

For each $n \in \mathbf{N}$, formally consider e_n to be an eigenvector corresponding to the eigenvalue λ_n. Define for any $s \in \mathbf{Z}$

$$\mathfrak{g}^s = \{p = \sum_{n\in\mathrm{N}} a_n e_n;\ a_n \in \mathbf{C}, \sum_{n\in\mathrm{N}} |a_n|^2 \lambda_n^{2s} < \infty\}.$$

$\mathfrak{g}^s$ is a Hilbert space for every $s \in \mathbf{Z}$ with the norm $\|p\|_s^2 = \sum_{n\in\mathrm{N}} |a_n|^2\lambda_n^{2s}$. The inclusion mapping $\iota : \mathfrak{g}^s \to \mathfrak{g}^{s-1}$ is a compact operator for every $s \in \mathbf{Z}$. Set $\mathfrak{g} = \bigcap_s \mathfrak{g}^s$. $\{\mathfrak{g}, \mathfrak{g}^s; s \in \mathbf{Z}\}$ will be called a *Sobolev chain*. Set $\mathfrak{g}^* = \bigcup_s \mathfrak{g}^s$. As $\mathfrak{g}^{-s}$ is the dual space of $\mathfrak{g}^s$, $\mathfrak{g}^*$ is the dual space of $\mathfrak{g}$.

We denote by $C^\infty(\mathfrak{g}^s)$ the commutative algebra of all C^∞ functions on $\mathfrak{g}^s$. Since $C^\infty(\mathfrak{g}^{s-1}) \subset C^\infty(\mathfrak{g}^s)$, we set $C^\infty(\mathfrak{g}^*) = \bigcap_s C^\infty(\mathfrak{g}^s)$. Any $u \in \mathfrak{g}$, regarded as a linear function on $\mathfrak{g}^*$, is an element of $C^\infty(\mathfrak{g}^*)$. Let $(\hat{\circledast}\mathfrak{g}^s)^m$ be the Banach space of all continuous symmetric m-linear mappings of $\mathfrak{g}^{-s} \times \cdots \times \mathfrak{g}^{-s}$ into $\mathbf{C}$ with the natural operator norm, $\|\ \|_{-s}$ and set $(\circledast\mathfrak{g})^m = \bigcap_s (\hat{\circledast}\mathfrak{g}^s)^m$ with the projective limit topology. Hence, any element of $(\circledast\mathfrak{g})^m$ can be naturally viewed as an element of $C^\infty(\mathfrak{g}^*)$ as a homogeneous polynomial of degree m. Thus, we define a *polynomial* of degree m as an element of $\sum_{k=0}^m \oplus(\circledast\mathfrak{g})^k$, where we set $(\circledast\mathfrak{g})^0 = \mathbf{C}$. Denote by $\mathcal{P}(\mathfrak{g}^*)$ the space of all polynomials on $\mathfrak{g}^*$.

We define the C^∞-topology on $C^\infty(\mathfrak{g}^*)$, i.e. the C^∞ uniform topology on each compact subset: a basis of neighborhoods of 0 is given by the family $\{N(K, m, s, \epsilon)\}$ for compact subsets $K \subset \mathfrak{g}^*$, non-negative integers m, integers s and $\epsilon > 0$, where

$$N(K, m, s, \epsilon) = \{f \in C^\infty(\mathfrak{g}^*); \|(d^k f)(p)\|_{-s} < \epsilon, \text{ for} \forall p \in K\ \ , 0 \leq \forall k \leq m, \},$$

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where $(d^k f)(p)$ is the k-differential of f regarded as an element of $(\circledast\mathfrak{g})^k$.

In the following, we denote $C^\infty(\mathfrak{g}^*)$ with the C^∞ topology by $\mathfrak{a}$ for simplicity. $\mathfrak{a}$ is a topological algebra over $\mathbf{C}$.

We are now interested in "deforming" $\mathfrak{a}$ to a noncommutative but associative algebra.

Introducing a formal parameter ν, we consider the direct product

$$\mathfrak{a}[[\nu]] = \prod_{n=0}^{\infty} \nu^n \mathfrak{a}$$

with the direct product topology. We want to define a continuous product $*$ on $\mathfrak{a}[[\nu]]$ with the following properties:

(A.1) $* : \mathfrak{a}[[\nu]] \times \mathfrak{a}[[\nu]] \to \mathfrak{a}[[\nu]]$ is an associative product.
(A.2) ν commutes with any element of $\mathfrak{a}[[\nu]]$ and $1 * \tilde{f} = \tilde{f} * 1 = \tilde{f}$ for any $\tilde{f} \in \mathfrak{a}[[\nu]]$.

For a product $*$ on $\mathfrak{a}[[\nu]]$ with (A.1~2), we set for any $f, g \in \mathfrak{a}$,

$$f * g = \sum_{m=0}^{\infty} \nu^m \pi_m(f,g), \quad \pi_m(f,g) \in \mathfrak{a}.$$

By (A.1~2), we see for any $f, g, h \in \mathfrak{a}$,

$$(0.1) \quad \begin{cases} (\Box_m) \quad \sum_{k+l=m} \pi_k(\pi_l(f,g),h) = \sum_{k+l=m} \pi_k(f, \pi_l(g,h)), \ \forall m \geq 0, \\ \qquad\qquad \pi_0(f,1) = \pi_0(1,f) = f, \ \ \pi_m(f,1) = \pi_m(1,f) = 0, \ \forall m > 0. \end{cases}$$

A continuous m-linear mapping $\pi : \mathfrak{a} \times \cdots \times \mathfrak{a} \to \mathfrak{a}$ is called an *m-differential operator* of order k, if at any $p \in \mathfrak{g}^*$, $\pi(f_1, \cdots, f_m)(p) = 0$ holds whenever $(f_1, \cdots, f_m)$ satisfies $(d^{k+1}(f_1 f_2 \cdots f_m))(p) = 0$.

Now suppose $\mathfrak{g}$ is a topological Lie algebra with Lie bracket $[\ ,\ ]'$. For any $f, g \in \mathfrak{a}$, $df(p)$, $dg(p)$ are elements of $\mathfrak{g}^{**} = \mathfrak{g}$ for any $p \in \mathfrak{g}^*$, and $(df)_* : \mathfrak{g}^* \to \mathfrak{g}$ is a C^∞ mapping, i.e. $df : \mathfrak{g}^{-s} \to \mathfrak{g}^t$ is C^∞ for any s, t. Thus, we may define $\{f, g\} \in C^\infty(\mathfrak{g}^*)$ by

$$\{f,g\}(p) = [df(p), dg(p)]'(p).$$

It is obvious that $(\mathfrak{a}, \{\ ,\ \})$ is a Poisson algebra.

Definition 1. $(\mathfrak{a}[[\nu]], *)$ is called a *deformation quantization of* $\mathfrak{a}$ if $*$ satisfies (A.1~2) and the following (A.3~4):

(A.3) $\pi_0(f,g) = fg$ (the usual product) and $\pi_1(f,g) = -\frac{1}{2}\{f,g\}$ for any $f, g \in \mathfrak{a}$.
(A.4) π_m is a bidifferential operator of order $2m$ and $\pi_m(f,g) = (-1)^m \pi_m(g,f)$.

Our main theorem of this paper is as follows:

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Theorem A. *There exists a deformation quantization* $(\mathfrak{a}[[\nu]], *)$ *of* $\mathfrak{a}$ *such that* $\pi_m(\mathfrak{g}, \mathfrak{g}) = 0$ *for any* $m \geq 2$. *Moreover,* $\mathcal{P}(\mathfrak{g}^*)[[\nu]]$ *is a subalgebra of* $(\mathfrak{a}[[\nu]], *)$.

Thus, the quantized algebra $(\mathfrak{a}[[\nu]], *)$ naturally contains the universal enveloping algebra of the Lie algebra $\mathfrak{g}_\nu$ i.e. the Lie algebra generated by $\mathfrak{g}$ and ν with the relations $[X, Y] = \nu[X, Y]'$.

For any $k \in \mathbf{N}$, let x_k be the linear function on $\mathfrak{g}^*$ defined by $x_k(p) = \langle e_k, p\rangle_0$. $x_1, \cdots, x_k, \cdots$ are elements of $C^\infty(\mathfrak{g}^*)$.

In the quantized algebra $(\mathfrak{a}[[\nu]], *)$, we have

$$x_i * x_j = x_i x_j + \frac{1}{2}\nu[x_i, x_j]', \text{ so } x_i * x_j - x_j * x_i = \nu[x_i, x_j]'.$$

Hence, the above theorem extends the Poincare-Birkhof-Witt theorem for finite dimensional Lie algebras.

The method of proof of our main theorem is as follows : suppose we have $\{\pi_0, \pi_1, \cdots, \pi_{m-1}\}$ satisfying $(\Box_s)$ in (0.1) for $0 \leq s \leq m-1$. Our problem is to construct π_m such that $(\Box_s)$ is satisfied for $s = m$.

For multi-indices $\alpha = (\alpha_1, \cdots, \alpha_k, \cdots)$, we set $|\alpha| = \sum \alpha_k$. For α with $|\alpha| < \infty$, we set $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} \cdots$. We shall first construct $\pi_m(x^\alpha, x^\beta)$ for monomials x^α, x^β, and then applying Taylor's formula. To show key properties of π_m, we use the following polynomial approximation theorem :

Theorem B. *The space of all polynomials is dense in* $C^\infty(\mathfrak{g}^*)$ *in the* C^∞ *topology.*

The condition $\lim_{n\to\infty} \lambda_n = \infty$ is essentially used in this theorem.

Note that the assumption $\sum_{n\geq 1} \lambda_n^{-s_0} < \infty$, for some integer s_0, is crucial for Theorem A. In fact, for a separable Hilbert space E, let $H = E \oplus E \oplus \mathbf{C}$ be an infinite dimensional Heisenberg Lie algebra with the skew-symmetric continuous bilinear mapping $\theta : (E \oplus E) \times (E \oplus E) \to \mathbf{C}$ given by $\theta((u, v), (u', v')) = \langle u, v'\rangle - \langle v, u'\rangle$. Then, $f((u, v, c)) = \|u\|^2$, $g((u, v, c)) = \|v\|^2$ are polynomials of degree 2 on $H^* = H$, but the $*$-product $f * g$ diverges (cf. [OMY1] (2.9)). Thus, there is no deformation quantization of $C^\infty(H)$.

If $\mathfrak{g}$ is the Lie algebra of all C^∞ vector fields on a compact manifold, then Theorem A can be applied for $\mathfrak{g}$. Thus, there are several applications including quantizations on coadjoint orbits, which will be given in forthcoming papers.

§1. Smooth functions on $\mathfrak{g}^*$

1.1 Polynomial approximation theorem.

First, we note the following :

Lemma 1.1. *There exists an increasing series of compact subsets* $K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots$ *such that* $\bigcup K_n = \mathfrak{g}^*$. *For any compact subset* $K \subset \mathfrak{g}^*$, *there is* K_n *containing* K.

Proof. For any positive integer s, let D_{-s} be the open ball in $\mathfrak{g}^{-s}$ of radius s. It is easy to see that $D_{-s} \subset D_{-s-1} \subset \cdots$. Since the inclusion mapping ι is compact, D_{-s} is a relatively compact subset of $\mathfrak{g}^{-s-1}$, and hence of $\mathfrak{g}^*$. Set $K_s = \overline{D_{-s}}$ in $\mathfrak{g}^*$.

Let $p \in \mathfrak{g}^*$. By the definition of $\mathfrak{g}^*$, there exists s such that $p \in \mathfrak{g}^{-s}$. Suppose $\|p\|_{-s} < m$ for a positive integer m. Setting $n = \max\{s, m\}$, we have $p \in D_{-n}$.

Let $K \subset \mathfrak{g}^*$ be a compact subset. Suppose for each positive n, there exists $p_n \in K$ such that $p_n \in \mathfrak{g}^* - K_n$. By taking a subsequence if necessary, there exists $p_0 \in \mathfrak{g}^*$ such that $p_0 \in \mathfrak{g}^* - D_{-n}$ for any n. This contradicts the above fact. □

Proof of Theorem B. Consider now a C^∞ function f on $\mathfrak{g}^*$. Let K be an arbitrary fixed compact subset of $\mathfrak{g}^*$. By Lemma 1.1, one may assume that $K \subset D_{-n}$ for some n. Since D_{-n} is relatively compact in $\mathfrak{g}^{-l}$ for any $l > n$ and f is C^∞ on $\mathfrak{g}^{-l}$, for any ϵ and N, there exists $\delta > 0$ such that if $\|p - q\|_{-l} < \delta$, then $\|d^j f(p) - d^j f(q)\|_{-l} < \epsilon$ for any $0 \le j \le N$.

Let $\mathbf{R}^m$ be the subspace of $\mathfrak{g}$ spanned by $e_1, \cdots, e_m$ and π_m the projection of $\mathfrak{g}^*$ onto $\mathbf{R}^m$. We regard π_m as a linear mapping of $\mathfrak{g}^*$ into itself. For any point $p = \sum a_i e_i$ of D_{-n}, set $p_m = \pi_m(p)$ ($= \sum_{i=1}^m a_i e_i$). Then

$$\|p - p_m\|_{-l} < n\lambda_m^{-l+n}$$

for any $p \in D_{-n}$. Since $\lim \lambda_m = \infty$, taking m so large that $n\lambda_m^{-l+n} < \delta$, we find that f is approximated on K by $\pi_m^* f$.

By the polynomial approximation theorem on $\mathbf{R}^m$, we see that on K, $\pi_m^* f$ is approximated by a series of polynomials on $\mathfrak{g}^*$. Thus, the space of all polynomials is dense in $C^\infty(\mathfrak{g}^*)$ in the C^∞ topology. □

1.2 Tensor products and differential operators.

For a Sobolev chain $\{\mathfrak{g}, \mathfrak{g}^s; s \in \mathbf{Z}\}$, we introduced the tensor products $(\hat{\circledast}\mathfrak{g}^s)^m$ as the Banach space of all continuous symmetric m-linear mappings of $\mathfrak{g}^{-s} \times \cdots \times \mathfrak{g}^{-s}$ into $\mathbf{C}$ with the natural operator norm, and set $(\circledast\mathfrak{g})^m = \bigcap_s (\hat{\circledast}\mathfrak{g}^s)^m$ with the projective limit topology. For $L \in (\hat{\circledast}\mathfrak{g}^s)^m$, setting $\|L\|_{-s} = \sup_{\|x\|_{-s}=1} |L(x, \cdots, x)|$ defines a Banach norm on $(\hat{\circledast}\mathfrak{g}^s)^m$.

On the other hand, let $(\circledast\mathfrak{g}^s)^m$ be the usual symmetric tensor product of $\mathfrak{g}^s$ as a Hilbert space, that is, any element $a \in (\circledast\mathfrak{g}^s)^m$ can be written as $a = \sum a_{i_1 \cdots i_m} e_{i_1} \circledast \cdots \circledast e_{i_m}$ with the Hilbert norm $|a|_s$ defined by

$$|a|_s^2 = \sum |a_{i_1 \cdots i_m}|^2 \lambda_{i_1}^{2s} \cdots \lambda_{i_m}^{2s}. \tag{1.1}$$

Obviously, the dual space of $(\circledast\mathfrak{g}^s)^m$ is $(\circledast\mathfrak{g}^{-s})^m$.

There is a natural continuous inclusion of $(\circledast\mathfrak{g}^s)^m$ into $(\hat{\circledast}\mathfrak{g}^s)^m$. Moreover, by the assumption that $\sum_{n \ge 1} \lambda_n^{-s_0} < \infty$, we see also that there is a continuous inclusion of $(\hat{\circledast}\mathfrak{g}^s)^m$ into $(\circledast\mathfrak{g}^{s-s_0/2})^m$. Hence $(\circledast\mathfrak{g})^m$ coincides with the inverse limit of $(\circledast\mathfrak{g}^s)^m$. Taking its dual, we see that the dual space of $(\circledast\mathfrak{g})^m$ is $\bigcup_s (\circledast\mathfrak{g}^{-s})^m$ with the inductive limit topology, which will be denoted by $(\circledast\mathfrak{g}^*)^m$.

For multi-indices $\alpha = (\alpha_1, \cdots, \alpha_k, \cdots)$, we set $|\alpha| = \sum \alpha_k$. For α such that $|\alpha| < \infty$, we set $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_k! \cdots$, and

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} \cdots . \quad \partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_k}^{\alpha_k} \cdots .$$

For any $t \in \mathbf{Z}$, $\lambda^{t\alpha} = \lambda_1^{t\alpha_1} \lambda_2^{t\alpha_2} \cdots \lambda_k^{t\alpha_k} \cdots$.

$\sum_{|\alpha|=m} \frac{1}{\alpha!} a_\alpha x^\alpha$ is a homogeneous polynomial of degree m on $\mathfrak{g}^*$ if and only if

$$\sum_{|\alpha|=m} |a_\alpha|^2 \lambda^{2s\alpha} < \infty$$

for any $s > 0$. For any $f \in \mathfrak{a}$, $d^l f(p)$ is a continuous symmetric l-linear mapping of $\mathfrak{g}^{-s} \times \cdots \times \mathfrak{g}^{-s}$ into $\mathbf{C}$ for any s, hence $d^l f(p) \in (\circledast \mathfrak{g}^s)^l$ for any s. It follows that $d^l f(p) \in (\circledast \mathfrak{g})^l$. We define the norm $|d^l f(p)|_s$ by

$$|d^l f(p)|_s^2 = \sum_{|\gamma|=l} |\partial^\gamma f|^2(p) \lambda^{2s\gamma}. \tag{1.2}$$

The following is easy to see by the converse of Taylor's theorem:

Lemma 1.2. *$f \in \mathfrak{a}$, if and only if $|d^l f(p)|_s < \infty$ for any non-negative integer l and any integer s, and $d^l f(p)$ is continuous with respect to $p \in \mathfrak{g}^*$.*

It is easy to see that any l-differential operator π of order d has the expression

$$\pi = \sum_{|\alpha+\cdots+\delta| \le d} \pi_{\alpha,\cdots,\delta} \underbrace{\partial^\alpha \otimes \cdots \otimes \partial^\delta}_{l}.$$

For any linear differential operator $L = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha$ of order m mapping $\mathfrak{a}$ to itself, by evaluation at each $p \in \mathfrak{g}^*$, L defines a continuous linear mapping

$$L_p = \sum_{|\alpha| \le m} a_\alpha(p) \partial^\alpha : \sum_{k=0}^{m} \oplus (\circledast \mathfrak{g})^k \to \mathbf{C}.$$

Thus, $L_p \in \sum_{k=0}^m \oplus (\circledast \mathfrak{g}^*)^k$. This implies that

$$L_p \in \sum_{k=0}^{m} \oplus (\circledast \mathfrak{g}^{-s})^k \quad \text{for some } s = s(p).$$

Since L is a differential operator of order m, $p \mapsto L_p$ is a C^∞ mapping of $\mathfrak{g}^*$ into $\sum_{k=0}^m \oplus (\circledast \mathfrak{g}^*)^k$. In particular, for any N, $(d^N L_*)_p \in (\circledast \mathfrak{g})^N \otimes \sum_{k=0}^m \oplus (\circledast \mathfrak{g}^*)^k$. This implies that for any t, there exists $s = s(t)$ such that $(d^N L_*)_p \in (\circledast \mathfrak{g}^t)^N \otimes \sum_{k=0}^m \oplus (\circledast \mathfrak{g}^{-s})^k$.

The continuity of $(d^N L_*)_p$ implies that for any $p \in \mathfrak{g}^*$ and for any integers $t, N \ge 0$, there exist $s = s(t, N, p)$ and a neighborhood V_p of p in $\mathfrak{g}^{-s}$ such that $p \mapsto (d^N L_*)_p$ is a continuous mapping of V_p into $(\circledast \mathfrak{g}^t)^N \otimes \sum_{k=0}^m \oplus (\circledast \mathfrak{g}^{-s})^k$.

Similarly, we have the following criterion:

Lemma 1.3. $\pi = \sum_{|\alpha+\beta|\leq m} \frac{1}{\alpha!\beta!}\pi_{\alpha,\beta}\partial^{\alpha}\otimes\partial^{\beta}$, $\pi_{\alpha,\beta}\in\mathfrak{a}$, *is a bidifferential operator of order* m, *if and only if* $\pi_{\alpha,\beta}$ *satisfies for any non-negative integers* t, N *and for any* $p\in\mathfrak{g}^*$,
(i) *there is an integer* $s = s(t,N,p) > 0$ *such that*

$$\sum_{|\gamma|=N}\sum_{\alpha,\beta}|\partial^{\gamma}\pi_{\alpha,\beta}(p)|^2\lambda^{2t\gamma}\lambda^{-2s(\alpha+\beta)} < \infty,$$

(ii) *for any* $\epsilon > 0$, *there exist* $s = s(N,t,p,\epsilon)$ *and a neighborhood* V_p *of* p *in* $\mathfrak{g}^{-s}$ *such that*

$$\sum_{|\gamma|=N}\sum_{\alpha,\beta}|\partial^{\gamma}\pi_{\alpha,\beta}(p)-\partial^{\gamma}\pi_{\alpha,\beta}(q)|^2\lambda^{2t\gamma}\lambda^{-2s(\alpha+\beta)} < \epsilon. \quad \text{for any } q\in V_p.$$

Proof. Suppose π is a bidifferential operator of order m. Then, we have

$$\pi_{\alpha,\beta}(p) = \pi((x-x(p))^{\alpha},(x-x(p))^{\beta})(p).$$

At every $p\in\mathfrak{g}^*$, by the same argument as above π induces

$$\pi_p = \sum_{|\alpha+\beta|\leq m}\frac{1}{\alpha!\beta!}\pi_{\alpha,\beta}(p)\partial^{\alpha}\otimes\partial^{\beta}\in(\sum_{k=0}^{m}\oplus(\circledast\mathfrak{g}^*)^k)\otimes(\sum_{k=0}^{m}\oplus(\circledast\mathfrak{g}^*)^k). \tag{1.3}$$

The differentiability of π_p gives the first inequality. The continuity of $p\mapsto(d^N\pi_*)_p$ yields the second one.

Conversely, given $\pi(\alpha,\beta)\in\mathfrak{a}$, $|\alpha+\beta|\leq m$, satisfying (i) and (ii), we define π_p by (1.3). Then, by (i), we have

$$\pi_p\in(\sum_{k=0}^{m}\oplus(\circledast\mathfrak{g}^*)^k)\otimes(\sum_{k=0}^{m}\oplus(\circledast\mathfrak{g}^*)^k) \tag{1.4}$$

for any $p\in\mathfrak{g}^*$. The second inequality (ii) gives the smoothness of $p\mapsto\pi_p$. Note that $\pi(f,g)(p) = \pi_p(f,g)$ for any f, $g\in\mathfrak{a}$ and $\pi(f,g)(p)$ depends only on $\partial^{\alpha}f(p), \partial^{\beta}g(p)$ for $|\alpha+\beta|\leq m$. Thus, $\pi(f,g)\in\mathfrak{a}$ by (i) and (ii). It is easy to see that π gives a continuous bilinear mapping of $\mathfrak{a}\times\mathfrak{a}$ into $\mathfrak{a}$. □

For any $f\in\mathfrak{a}$ and $p\in\mathfrak{g}^*$, we see that $f = f(p)+\sum_{1\leq i<\infty}F_i(x,p)(x_i-x_i(p))$, where $F_i(x,p) = \int_0^1\frac{\partial f}{\partial x_i}(x(p)+t(x-x(p))dt$. By Lemma 1.3, we have the following:

Lemma 1.4. *Let* π *be a bidifferntial operator of order* m. *Then, the operator* L *defined by*

$$L(f)(p) = \sum_{i=1}^{\infty}\pi(F_i,\, x_i-x_i(p))(p)$$

is a linear differential operator of order m.

Note that a similar criterion is available for 3-differential operators. If π, π' are bidifferential operators of order m, m' respectively, then $\pi(f,\pi'(g,h))$ is a 3-differential operator of order $m+m'$. If $E(f,g,h)$ is a 3-differential operator of order m, then $E(x_i,f,x_l)$ is a linear differential operator of order $m-2$ with respect to f.

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§2. Algebraic preliminaries

To introduce the obstructions R_m given in §3, we prepare some algebraic tools, called Hochschild and deRham-Chevalley coboundary operators. This notion is given in a purely algebraic manner. So, in this section, we do not specify $\mathfrak{a}$ and take it only as an abstract topological vector space.

2.1. Hochschild coboundary operators.

Let $\mathfrak{a}$ be a topological vector space over $\mathbf{C}$. Denote by $\mathrm{C}^p(\mathfrak{a})$, $p \geq 1$, the space of all continuous p-linear mappings of $\mathfrak{a}\times\cdots\times\mathfrak{a}$ to $\mathfrak{a}$. We denote by $\mathrm{AC}^p(\mathfrak{a})$ and $\mathrm{SC}^p(\mathfrak{a})$ ($p \geq 1$) the set of the alternative and the symmetric p-linear mappings, respectively. If p=0, we set $\mathrm{C}^0(\mathfrak{a})= \mathrm{AC}^0(\mathfrak{a})= \mathrm{SC}^0(\mathfrak{a})=\mathfrak{a}$.

For any $\pi \in\mathrm{C}^2(\mathfrak{a})$, we define the *Hochschild coboundary operator* $\delta_\pi : \mathrm{C}^p(\mathfrak{a}) \to \mathrm{C}^{p+1}(\mathfrak{a})$, $p \geq 1$, by

$$(2.1)\qquad \begin{aligned}(\delta_\pi F)(v_1,\cdots,v_{p+1}) &= \pi(v_1, F(v_2,\cdots,v_{p+1}))\\ &+\sum_{i=1}^{p}(-1)^i F(v_1,\cdots,\pi(v_i,v_{i+1}),\cdots,v_{p+1})\\ &+(-1)^{p+1}\pi(F(v_1,\cdots,v_p),v_{p+1})\end{aligned}$$

for $F \in \mathrm{C}^p(\mathfrak{a})$, and for p=0, we set $(\delta_\pi v)(v_1) = \pi(v_1, v)$ for any $v \in \mathfrak{a}$.

By a direct computation using the linearization, we have the following:

Lemma 2.1. *For any $\pi, \pi', \pi'' \in \mathrm{C}^2(\mathfrak{a})$, we have*

$$\delta_\pi\pi' = \delta_{\pi'}\pi, \quad \delta_\pi I = \pi, \ (I = \mathit{identity}) \text{ and } \delta_\pi\delta_\pi\pi = 0,$$
$$\mathop{\mathfrak{S}}_{(\pi,\pi',\pi'')} \delta_\pi\delta_{\pi'}\pi'' = 0,$$

where $\mathop{\mathfrak{S}}_{(\pi,\pi',\pi'')}$ *means the cyclic summation with respect to* π, π', π''.

$\delta_\pi\pi$=0, if and only if ($\mathfrak{a}$,π) is an associative algebra. If ($\mathfrak{a}$,π) is an associative algebra, then $\delta_\pi^2 F = 0$, for any $F \in \mathrm{C}^p(\mathfrak{a})$ (cf. [Mc]). In particular, $\delta_\pi^2 I=\delta_\pi\pi=0$. Therefore, $\delta_\pi^2=0$ is equivalent to $\delta_\pi\pi=0$.

Let $(\mathfrak{a}, \pi_0)$ be any associative algebra. Suppose $\pi_0, \pi_1,\cdots, \pi_{k-1} \in \mathrm{C}^2(\mathfrak{a})$ satisfy $(\Box_l)$ in (0.1) for any integer l such that $0 \leq l \leq k-1$. We denote $\delta_i = \delta_{\pi_i}$ for simplicity. We consider the equation $(\Box_k)$, which is equivalent to

$$(2.2)\qquad \delta_0\pi_k = -Q_k, \ \text{ where } \ Q_k = \frac{1}{2}\sum_{i+j=k,i,j\geq 1}\delta_i\pi_j.$$

Since δ_0^2=0 by the associativity of π_0, if (2.2) can be solved, then the right hand side must satisfy $\delta_0 Q_k = 0$. At the first glance, this looks like a necessary condition for $(\mathfrak{a}, \pi_0)$ to be deformed associatively, but in fact this is fulfilled automatically. Namely, we have

Proposition 2.2. *Let* $(\mathfrak{a}, \pi_0)$ *be any associative algebra. If* $\pi_0, \pi_1, \cdots, \pi_{k-1} \in \mathrm{C}^2(\mathfrak{a})$ *satisfy* $(\Box_l)$ *for any integer* l *such that* $0 \leq l \leq k-1$, *then* $\pi_0, \cdots, \pi_{k-1}$ *satisfy also* $\delta_0 Q_k = 0$.

Proof. is seen in [OMY2], Proposition 1.3.

2.2. p-derivations.

For $\pi \in \mathrm{C}^2(\mathfrak{a})$, we define $\partial_i^\pi : \mathrm{C}^p(\mathfrak{a}) \to \mathrm{C}^{p+1}(\mathfrak{a}) \quad (1 \leq i \leq p),\ p \geq 1$, by

$$
\begin{aligned}
(\partial_i^\pi F)(v_1, \cdots, v_{p+1}) = {} & \pi(v_i, F(v_1, \cdots, \hat{v}_i, \cdots, v_{p+1})) \\
& - F(v_1, \cdots, \pi(v_i, v_{i+1}), \cdots, v_{p+1}) \\
& + \pi(F(v_1, \cdots, \hat{v}_{i+1}, \cdots, v_{p+1}), v_{i+1})
\end{aligned}
\tag{2.3}
$$

for any $F \in \mathrm{C}^p(\mathfrak{a})$.

We call $F \in \mathrm{C}^p(\mathfrak{a})$ a *p-derivation with respect to* π, if $\partial_j^\pi F = 0$ for any j, $(1 \leq j \leq p)$. By $Der^p(\mathfrak{a}, \pi)$, we denote the space of all p-derivations with respect to π. Set also

$$\mathcal{A}^p(\mathfrak{a}, \pi) = \mathrm{AC}^p(\mathfrak{a}) \cap Der^p(\mathfrak{a}, \pi).$$

We define mappings σ_p, $\mathfrak{c}_p : \mathrm{C}^p(\mathfrak{a}) \to \mathrm{C}^p(\mathfrak{a})$ by

$$(\sigma_p F)(v_1, v_2, \cdots, v_{p-1}, v_p) = F(v_p, v_{p-1}, \cdots, v_2, v_1), \tag{2.4}$$

$$(\mathfrak{c}_p F)(v_1, v_2, \cdots, v_{p-1}, v_p) = F(v_p, v_1, v_2, \cdots, v_{p-1}). \tag{2.5}$$

Since $\mathfrak{c}_3^3 = 1$, we have

$$(1 + \mathfrak{c}_3 + \mathfrak{c}_3^2)(1 - \mathfrak{c}_3) = 0, \tag{2.6}$$

$$(1 - \mathfrak{c}_3 + \mathfrak{c}_3^2)(1 + \mathfrak{c}_3) = 2. \tag{2.7}$$

The following formulas are useful for later computations:

Lemma 2.3. (i) *For any* $\pi \in \mathrm{C}^2(\mathfrak{a})$ *and* $F \in \mathrm{C}^p(\mathfrak{a})$, *we have*

$$
\begin{aligned}
& \delta_\pi \sigma_p F = (-1)^{p+1} \sigma_{p+1} \delta_{\sigma_2 \pi} F, \\
& \partial_j^\pi \mathfrak{c}_p F = \mathfrak{c}_{p+1} \partial_{j+1}^\pi F \quad (1 \leq j \leq p-1), \quad \partial_p^\pi \mathfrak{c}_p F = \mathfrak{c}_{p+1}^2 \partial_1^\pi F.
\end{aligned}
$$

(ii) *In particular, if* $\pi \in \mathrm{SC}^2(\mathfrak{a})$, *we have*

$$\delta_\pi F = \sum_{1 \leq i \leq p} (-1)^{i-1} \partial_i^\pi F, \quad \partial_j^\pi \sigma_p F = \sigma_{p+1} \partial_{p+1-j}^\pi F \quad (1 \leq j \leq p).$$

(iii) *If* $\pi \in \mathrm{SC}^2(\mathfrak{a})$ *and* $\delta_\pi \pi = 0$, *we have*

$$(\partial_j^\pi - \partial_{j+1}^\pi) \partial_j^\pi = 0 \quad \textit{for } 1 \leq j \leq p.$$

2.3. deRham-Chevalley coboundary operators.

For any $\pi \in \mathrm{AC}^2(\mathfrak{a})$, we define the *Chevalley coboundary operator* $d_\pi : \mathrm{AC}^p(\mathfrak{a}) \to \mathrm{AC}^{p+1}(\mathfrak{a})$ by

$$(2.8)\qquad \begin{aligned}&(d_\pi F)(v_1, \cdots, v_{p+1})\\ &= \sum_{i=1}^{p+1}(-1)^{i+1}\pi(v_i, F(v_1, \cdots, \hat{v}_i, \cdots, v_{p+1}))\\ &\quad + \sum_{i<j}(-1)^{i+j}F(\pi(v_i, v_j), v_1, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_{p+1}).\end{aligned}$$

By a direct computation using the linearization, we have

Lemma 2.4. *For any $\pi, \pi', \pi'' \in \mathrm{AC}^2(\mathfrak{a})$,*

$$d_\pi \pi' = d_{\pi'}\pi,\ \ d_\pi I = \pi,\ \ (I = identity),\ \text{and}\ d_\pi d_\pi \pi = 0,$$
$$\mathop{\mathfrak{S}}_{(\pi,\pi',\pi'')} d_\pi d_{\pi'}\pi'' = 0, \quad (d_\pi \pi)(u, v, w) = 2 \mathop{\mathfrak{S}}_{(u,v,w)} \pi(u, \pi(v, w)).$$

By the last identity in Lemma 2.4, $d_\pi \pi = 0$ if and only if $(\mathfrak{a}, \pi)$ is a Lie algebra. If $(\mathfrak{a}, \pi)$ is a Lie algebra, then $d_\pi^2 F = 0$ for any $F \in \mathrm{AC}^p(\mathfrak{a})$ (cf.[Ma]). Therefore, $d_\pi^2 = 0$ is equivalent to $d_\pi \pi = 0$.

In the following, we use the notations

$$(2.9)\qquad \pi^{\pm}(u, v) = \frac{1}{2}\{\pi(u, v) \pm \pi(v, u)\}.$$

for $\pi \in \mathrm{C}^2(\mathfrak{a})$.

Definition 2.5. For $\pi_0, \cdots, \pi_{m-1} \in \mathrm{C}^2(\mathfrak{a})$, we set

$$(2.10)\qquad \begin{cases} Q_m &= \frac{1}{2}\sum_{i+j=m, i,j\geq 1} \delta_i \pi_j, (\text{c.f}\quad (2.2))\\ R_m &= \frac{1}{2}\sum_{i+j=m, i,j\geq 1} d_i^- \pi_j^-, \end{cases}$$

where $d_i^- = d_{\pi_i^-}$.

By Proposition 2.2, we have $\delta_0 Q_k$=0, if π_0, π_1, $\cdots$, π_{k-1} satisfy $(\square_l)\ 0 \leq l \leq k-1$.

Assume that $(\mathfrak{a}, \pi_0, \pi_1)$ is a Poisson algebra, i.e. $\pi_0 \in \mathrm{SC}^2(\mathfrak{a})$, $\pi_1 \in \mathrm{AC}^2(\mathfrak{a})$ such that $\delta_0 \pi_0 = 0$, $\delta_0 \pi_1 = 0$, $d_1 \pi_1 = 0$.

We easily have

$$d_{\pi_1}\mathcal{A}^p(\mathfrak{a}, \pi_0) \subset d_{\pi_1}\mathcal{A}^{p+1}(\mathfrak{a}, \pi_0), \quad d_{\pi_1}^2 = 0.$$

Thus, we can give the following p-th cohomology group $H^p(\mathfrak{a}, \pi_0, \pi_1)$ of the cochain complex

$$\cdots \to \mathcal{A}^p(\mathfrak{a}, \pi_0) \xrightarrow{d_{\pi_1}} \mathcal{A}^{p+1}(\mathfrak{a}, \pi_0) \to \cdots,$$

which is called the *deRham-Chevalley cohomology group* of the Poisson algebra. By a similar manner as in Proposition 2.2, we have the following:

Proposition 2.6. *Suppose* $(\mathfrak{a}, \pi_0, \pi_1)$ *is a Poisson algebra. If* $\pi_0, \cdots, \pi_{k-1} \in C^2(\mathfrak{a})$ *satisfy* $(\Box_l)$ *for* $0 \le l \le k-1$, *then* $R_l = 0$ *for* $2 \le l \le k-1$ *and* $d_1^- R_k = 0$.

Proof. is seen in [OMY2], Propositions 3.2 - 3.3.

§3. Jacobi identities

3.1. The obstruction R_m.

Let $\mathfrak{a} = C^\infty(\mathfrak{g}^*)$ and assume the following:

(H.1) Set $\pi_0(f,g) = fg$, $\pi_1(f,g) = -\frac{1}{2}\{f,g\}$. Furthermore, $\pi_2, \cdots, \pi_{m-1} \in C^2(\mathfrak{a})$ are given so that $(\Box_l)$: $\sum_{i+j=l} \delta_i \pi_j = 0$ for any $l, 0 \le l \le m-1$.
(H.2) $\pi^+_{\text{odd}} = \pi^-_{\text{even}} = 0$ and $\pi_s(x_i, x_j) = 0$ for $2 \le s \le m-1$.
(H.3) π_s is a bidifferential operator of order $2s$ for any $0 \le s \le m-1$.

Remark that if m is odd, then $R_m = 0$. $R_m(f,g,h)$ is a 3-differential operator of order $2m$.

Let Q_m be given in (2.2). Under the assumptions (H.1)~(H.3), we want to solve the equation $\delta_0 \pi_m = -Q_m$ (cf.(2.2)). By remarking $\sigma_2 = \mathfrak{c}_2$, and using Lemma 2.3, the above equation is rewritten as

$$(3.1) \qquad \begin{cases} (1-\mathfrak{c}_3)\partial_2^0 \pi_m^+ = -\delta_0 \pi_m^+ = -\frac{1}{2}(1-\sigma_3)\delta_0 \pi_m = \frac{1}{2}(1-\sigma_3) Q_m, \\ (1+\mathfrak{c}_3)\partial_2^0 \pi_m^- = -\delta_0 \pi_m^- = -\frac{1}{2}(1+\sigma_3)\delta_0 \pi_m = \frac{1}{2}(1+\sigma_3) Q_m, \end{cases}$$

where $\partial_i^{\pi_0} = \partial_i^0$. By (2.7), the equation (3.1) splits into two equations:

$$(3.2) \qquad \partial_2^0 \pi_m^- = \frac{1}{4}(1 - \mathfrak{c}_3 + \mathfrak{c}_3^2)(1+\sigma_3) Q_m,$$

$$(3.3) \qquad (1-\mathfrak{c}_3)\partial_2^0 \pi_m^+ = \frac{1}{2}(1-\sigma_3) Q_m.$$

Assume (3.1) has a solution π_m. By applying Lemma 2.3, and (2.6), (2.7), in addition to $\delta_0 Q_m = 0$, Q_m must satisfy the following consistency conditions for (3.2–3):

$$(3.4) \qquad (\partial_2^0 - \partial_3^0)(1 - \mathfrak{c}_3 + \mathfrak{c}_3^2)(1+\sigma_3) Q_m = 0,$$

$$(3.5) \qquad (1 + \mathfrak{c}_3 + \mathfrak{c}_3^2)(1-\sigma_3) Q_m = 0.$$

However, (3.4) is not a new condition. Namely, we have the following;

Lemma 3.1. *If* $\delta_0 Q = 0$ *for* $Q \in C^3(\mathfrak{a})$, *then* (3.4) *is satisfied.*

Proof. is seen in Appendix 6.1.

Next, we consider (3.5), the consistency condition for (3.3).

Lemma 3.2. $(1+\mathfrak{c}_3+\mathfrak{c}_3^2)(1-\sigma_3)Q_m = 4R_m$. *Thus, the consistency condition of* (3.3) *is* $R_m = 0$.

Proof. Since $\delta_i = \delta_i^+ + \delta_i^-$, where $\delta_i^{\pm} = \delta_{\pi_i^{\pm}}$, we see by the definition of Q_m, that

$$Q_m = \frac{1}{2}\sum_{i+j=m,i,j\geq 1}(\delta_i^+\pi_j^+ + \delta_i^-\pi_j^-) + \sum_{i+j=m,i,j\geq 1}\delta_i^+\pi_j^-. \tag{3.6}$$

Note $\sigma_3\delta_i^+\pi_j^- = \delta_i^+\pi_j^-$, $\sigma_3\delta_i^+\pi_j^+ = -\delta_i^+\pi_j^+$, $\sigma_3\delta_i^-\pi_j^- = -\delta_i^-\pi_j^-$ by Lemma 2.3. Then, we have

$$\begin{cases} Q_m - \sigma_3 Q_m &= \sum_{i+j=m,i,j\geq 1}(\delta_i^+\pi_j^+ + \delta_i^-\pi_j^-), \\ Q_m + \sigma_3 Q_m &= 2\sum_{i+j=m,i,j\geq 1}\delta_i^+\pi_j^-. \end{cases} \tag{3.7}$$

By (2.2), (3.7) and Lemma 2.4, we have

$$\begin{aligned}(1+\mathfrak{c}_3+\mathfrak{c}_3^2)(1-\sigma_3)Q_m(f,g,h) &= 4\sum_{i+j=m,i,j\geq 1}\mathop{\mathfrak{S}}_{(f,g,h)}\pi_i^-(f,\pi_j^-(g,h)) \\ &= 4R_m(f,g,h). \qquad \square\end{aligned} \tag{3.8}$$

3.2. Cohomological property for R_m.

By Lemma 3.2, $R_m = 0$ must hold for π_m to exist. First,recall the following fact whose proof is seen in [OMY2], Theorem 3.4.

Theorem 3.3. *Suppose* $\pi_2, \cdots, \pi_{m-1} \in \mathrm{C}^2(\mathfrak{a})$ *satisfy* (H.1) $\sim$ (H.3). *Then,*

$$\partial_j^0 R_m = 0, \quad \textit{for} \quad j = 1,2,3 \textit{ i.e. } R_m \in \mathcal{A}_3(\mathfrak{a}, \pi_0).$$

Hence, by Proposition 2.6 R_m *is a deRham-Chevalley 3-cocycle.*

Using Theorem 3.3, we have

Corollary 3.4. *Assume that (H.1)~(H.3) hold for* $\mathfrak{a} = C^\infty(\mathfrak{g}^*)$. *Then,* $R_m = 0$.

Proof. $\pi_l(x_i, x_j) = 0$ for $l \geq 2$. By the 3-derivation property and by the polynomial approximation theorem, we have only to check the quantities

$$R_m(x_i, x_j, x_k) = \mathop{\mathfrak{S}}_{(i,j,k)}\pi_{m-1}^-(x_i, \pi_1^-(x_j, x_k)).$$

R_2 always vanishes because $d_{\pi_1}\pi_1 = 0$. Hence, if $\pi_1(x_i, x_j) = c_{ij} + \sum_k c_{ij}^k x_k$, then $R_m = 0$. $\square$

Remark. We shall call $R_m = 0$ *the Jacobi identities.*

For the convenience sake, in what follows, we use the notation:

$$(3.9)\quad \begin{cases} f\cdot g &= \pi_0(f,g), \quad \langle f,g\rangle^{\pm}_m = \pi^{\pm}_m(f,g), \quad (m\geq 1),\\ \langle f, g\cdot\langle h,t\rangle^{\pm}\rangle^{\pm}_m &= \sum_{i+j=m,i,j\geq 1} \pi^{\pm}_i(f, g\cdot \pi^{\pm}_j(h,t)) \qquad (m\geq 2),\\ \langle\langle f,\langle g,h\rangle^{\pm}\rangle^{\pm},t\rangle^{\pm}_m &= \sum_{a+b+c=m,a,b,c\geq 1} \pi^{\pm}_a(\pi^{\pm}_b(f,\pi^{\pm}_c(g,h)),t) \quad (m\geq 3),\\ \langle\langle f,g\rangle^{\pm},\langle h,t\rangle^{\pm}\rangle^{\pm}_m &= \sum_{a+b+c=m,a,b,c,\geq 1} \pi^{\pm}_a(\pi^{\pm}_b(f,g),\pi^{\pm}_c(h,t)) \quad (m\geq 3). \end{cases}$$

Now, we shall discuss the cases m = even and m = odd separately.
(E) Case $m=2k$: The equations (3.2-3) for $\pi_{2k}=\pi^+_{2k}+\pi^-_{2k}$ are rewritten as follows:

$$(3.10)\qquad \begin{cases} (a)\quad (1-\mathfrak{c}_3)\partial^0_2\pi^+_{2k} = \frac{1}{2}\sum_{i+j=2k,i,j\geq 1}(\delta^+_i\pi^+_j+\delta^-_i\pi^-_j)\\ (b)\quad \partial^0_2\pi^-_{2k}=0, \end{cases}$$

where we used (3.7). One may set $\pi^-_{2k}=0$, for this is the trivial solution of (3.10,(b)). By a little careful computation together with the definition of $\delta^+_i\pi^+_j$, $\delta^-_i\pi^-_j$, we see that (3.10,(a)) is equivalent to the following:

$$(3.11)\qquad \pi^+_{2k}(f,gh)-\pi^+_{2k}(h,gf)=E_{2k}(f,g,h),$$

where

$$(3.12)\qquad \begin{aligned} E_{2k}(f,g,h) = \pi^+_{2k}(f,g)h &- \pi^+_{2k}(h,g)f\\ &+\langle\langle f,g\rangle^+,h\rangle^+_{2k}-\langle\langle h,g\rangle^+,f\rangle^+_{2k}\\ &-\langle\langle h,f\rangle^-,g\rangle^-_{2k}. \end{aligned}$$

$E_{2k}(f,g,h)$ is a 3-differential operator of order $4k$.

(O) Case $m=2l+1$: The equations (3.2-3) are changed into

$$(3.13)\qquad \begin{cases} (a)\quad \partial^0_2\pi^-_{2l+1}=\frac{1}{4}(1-\mathfrak{c}_3+\mathfrak{c}^2_3)(1+\sigma_3)Q_{2l+1}\\ (b)\quad (1-\mathfrak{c}_3)\partial^0_2\pi^+_{2l+1}=\frac{1}{2}\sum_{i+j=2l+1,i,j\geq 1}(\delta^+_i\pi^+_j+\delta^-_i\pi^-_j). \end{cases}$$

By (H.2), the right hand side of (3.13,(b)) vanishes. In what follows we set $\pi^+_{2l+1}=0$.

§4. Construction of π_{odd}

In this section, we prove the following:

Theorem 4.1. *Let $l \geq 1$. Under the assumptions* (H.1-3), *there exists* $\pi_{2l+1} \in AC^2(\mathfrak{a})$ *such that* $\sum_{i+j=2l+1, i,j\geq 0} \delta_i \pi_j = 0$ *and* π_{2l+1} *is a bidifferential operator of order* $2(2l+1)$ *satisfying* $\pi_{2l+1}(x_i, x_j) = 0$.

Let x_k be the linear functional on $\mathfrak{g}^*$ defined by $x_k(p) = < e_k, p >_0$ and set

$$\pi^-_{2l+1}(x_i, x_j) = 0. \tag{4.1}$$

4.1. Construction of π^-_{odd}.

First, we show how to construct π_{2l+1}. By (3.7), we see that (3.13,(a)) is equivalent to

$$\begin{aligned}\pi^-_{2l+1}(f, gh) &= g\pi^-_{2l+1}(f,h) + \pi^-_{2l+1}(f,g)h \\ &+ \langle\langle f,g\rangle^-, h\rangle^+_{2l+1} + \langle\langle f,h\rangle^-, g\rangle^+_{2l+1} - \langle f, \langle g,h\rangle^+\rangle^-_{2l+1}.\end{aligned} \tag{4.2}$$

Setting $\zeta_j = x_j - x_j(p)$, we have

$$g(x) = g(p) + \sum_{j\geq 1} G_j(x,p)\zeta_j,$$

where $G_j(x,p) = \int_0^1 \frac{\partial g}{\partial x_j}(p + t(x-p))dt$. Putting $f = x_i$ in (4.2), we get

$$\begin{aligned}\pi^-_{2l+1}(x_i, g)(p) &= \sum_{j\geq 1}\{\langle\langle x_i, G_j\rangle^-, x_j\rangle^+_{2l+1}(p) \\ &+ \langle\langle x_i, x_j\rangle^-, G_j\rangle^+_{2l+1}(p) - \langle x_i, \langle G_j, x_j\rangle^+\rangle^-_{2l+1}(p)\}.\end{aligned} \tag{4.3}$$

Remark that $\partial_x^\alpha G_j(x,p)|_{x=p} = \frac{1}{|\alpha|+1}(\partial_x^\alpha \partial_{x_j} g)(p)$. By the assumptions (H.1-3) and Lemma 1.4, the right hand side (4.3) is a linear differential operator of order $4l+1$ with respect to g.

Define $\pi^-_{2l+1}(h, x_i)$ by

$$\pi^-_{2l+1}(h, x_i) = -\pi^-_{2l+1}(x_i, h). \tag{4.4}$$

By (4.2), we have

$$\begin{aligned}\pi^-_{2l+1}(f,g)(p) &= \sum_{j\geq 1}\{\frac{\partial g}{\partial x_j}(p)\pi^-_{2l+1}(f, x_j)(p) \\ &+ \langle\langle f, G_j\rangle^-, x_j\rangle^+_{2l+1}(p) + \langle\langle f, x_j\rangle^-, G_j\rangle^+_{2l+1}(p) - \langle f, \langle G_j, x_j\rangle^+\rangle^-_{2l+1}(p)\}.\end{aligned} \tag{4.5}$$

By a similar proof as in Lemma 1.4, the right hand side of (4.5) is a bidifferential operator of order $2(2l+1)$ with respect to f, g.

Thus, we obtain $\pi^-_{2l+1}(f,g)$ for any $f, g \in \mathfrak{a}$. However, we only see that $\pi^-_{2l+1}(x_i, x_j) = 0$ for $l \geq 1$ and $\pi^-_{2l+1}(x_i, h) = -\pi^-_{2l+1}(h, x_i)$.

4.2. Skewness of π_{2l+1}.

To prove Theorem 4.1, we only show the following:

Proposition 4.2. *$\pi^-_{2l+1}(f,h)$ given by (4.5) is skew-symmetric.*

Proof. By the polynomial approximation theorem, we have only to show the skewness for polynomials. Thus in what follows, we assume the following:

$$(\mathrm{S})_s \qquad \pi^-_{2l+1}(x^\alpha, x^\beta) = -\pi^-_{2l+1}(x^\beta, x^\alpha) \quad \text{for any} \quad \alpha, \beta \quad \text{such that} \quad |\alpha+\beta| \le s.$$

Consider $\pi^-_{2l+1}(x^\alpha, x^\beta)$ such that $|\alpha+\beta| = s+1$. If either of $|\alpha|, |\beta|$ is 1, then (4.4) shows the skew-symmetricity. We now show $(\mathrm{S})_{s+1}$ for $|\alpha|, |\beta| \ge 2$. Since π^-_{2l+1} is a continuous bilinear mapping, it is enough to show that

$$\pi^-_{2l+1}(x^\alpha x^{\alpha'}, x^\beta x^{\beta'}) = -\pi^-_{2l+1}(x^\beta x^{\beta'}, x^\alpha x^{\alpha'}) \quad \text{for} \quad |\alpha|, |\alpha'|, |\beta|, |\beta'| \ge 1.$$

For simplicity, set $f = x^\alpha$, $g = x^{\alpha'}$, $h = x^\beta$, $t = x^{\beta'}$. By the assumption $(\mathrm{S})_s$, one obtains

$$(4.6) \qquad \pi^-_{2l+1}(fg, h) = -\pi^-_{2l+1}(h, fg), \quad \pi^-_{2l+1}(f, gh) = -\pi^-_{2l+1}(gh, f), \quad \text{etc.}$$

By (4.2), we have

$$\begin{aligned}\pi^-_{2l+1}(fg, ht) = \pi^-_{2l+1}(fg,h)t + \pi^-_{2l+1}(fg,t)h + \langle\langle fg,h\rangle^-, t\rangle^+_{2l+1} \\ + \langle\langle fg,t\rangle^-, h\rangle^+_{2l+1} - \langle fg, \langle h,t\rangle^+\rangle^-_{2l+1}.\end{aligned}$$

Using (4.2), and the assumption $(\mathrm{S})_s$, we have

$$(4.7)$$
$$\begin{aligned}\pi^-_{2l+1}(fg, ht) = {}& \pi^-_{2l+1}(f,h)gt + \pi^-_{2l+1}(g,h)ft + \pi^-_{2l+1}(f,t)gh + \pi^-_{2l+1}(g,t)fh \\ & - t\langle\langle h,f\rangle^-, g\rangle^+_{2l+1} - t\langle\langle h,g\rangle^-, f\rangle^+_{2l+1} + t\langle h, \langle f,g\rangle^+\rangle^-_{2l+1} \\ & - h\langle\langle t,f\rangle^-, g\rangle^+_{2l+1} - h\langle\langle t,g\rangle^-, f\rangle^+_{2l+1} + h\langle t, \langle f,g\rangle^+\rangle^-_{2l+1} \\ & + \langle\langle fg,h\rangle^-, t\rangle^+_{2l+1} + \langle\langle fg,t\rangle^-, h\rangle^+_{2l+1} - \langle fg, \langle h,t\rangle^+\rangle^-_{2l+1}.\end{aligned}$$

The first line of the right hand side of (4.7) is skew-symmetric under the permutation of $(f,g,h,t) \to (h,t,f,g)$, which we shall denote by σ. Let $\mathfrak{S}$ denote $1+\sigma$. Then, using (4.2) and applying the assumption to the last line of (4.7), we have the following:

$$
\begin{array}{lll}
\multicolumn{3}{l}{\mathfrak{S}\pi^-_{2l+1}(fg,ht) =} \\
-\mathfrak{S}t\langle\langle h,f\rangle^-,g\rangle^+_{2l+1} & -\mathfrak{S}t\langle\langle h,g\rangle^-,f\rangle^+_{2l+1} & +\underset{\blacktriangle}{\mathfrak{S}t\langle h,\langle f,g\rangle^+\rangle^-_{2l+1}} \\
-\mathfrak{S}h\langle\langle t,f\rangle^-,g\rangle^+_{2l+1} & -\mathfrak{S}h\langle\langle t,g\rangle^-,f\rangle^+_{2l+1} & +\underset{\blacktriangle}{\mathfrak{S}h\langle t,\langle f,g\rangle^+\rangle^-_{2l+1}} \\
 & -\underset{\blacktriangle}{\mathfrak{S}f\langle g,\langle h,t\rangle^+\rangle^-_{2l+1}} & -\underset{\blacktriangle}{\mathfrak{S}g\langle f,\langle h,t\rangle^+\rangle^-_{2l+1}} \\
+\underset{\blacklozenge}{\mathfrak{S}\langle\langle f,g\rangle^+,\langle h,t\rangle^+\rangle^-_{2l+1}} & & \\
 & +\underset{\blacktriangledown}{\mathfrak{S}\langle\langle\langle h,t\rangle^+,f\rangle^-,g\rangle^+_{2l+1}} & +\underset{\blacktriangledown}{\mathfrak{S}\langle\langle\langle h,t\rangle^+,g\rangle^-,f\rangle^+_{2l+1}} \\
-\mathfrak{S}\langle\langle\langle h,f\rangle^-,g\rangle^+,t\rangle^+_{2l+1} & -\mathfrak{S}\langle\langle\langle h,g\rangle^-,f\rangle^+,t\rangle^+_{2l+1} & -\underset{\blacktriangledown}{\mathfrak{S}\langle\langle\langle f,g\rangle^+,h\rangle^-,t\rangle^+_{2l+1}} \\
-\mathfrak{S}\langle\langle\langle t,f\rangle^-,g\rangle^+,h\rangle^+_{2l+1} & -\mathfrak{S}\langle\langle\langle t,g\rangle^-,f\rangle^+,h\rangle^+_{2l+1} & -\underset{\blacktriangledown}{\mathfrak{S}\langle\langle\langle f,g\rangle^+,t\rangle^-,h\rangle^+_{2l+1}} \\
+\mathfrak{S}\langle f\langle g,h\rangle^-,t\rangle^+_{2l+1} & +\mathfrak{S}\langle g\langle f,h\rangle^-,t\rangle^+_{2l+1} & \\
+\mathfrak{S}\langle f\langle g,t\rangle^-,h\rangle^+_{2l+1} & +\mathfrak{S}\langle g\langle f,t\rangle^-,h\rangle^+_{2l+1} &
\end{array}
$$

The terms marked by $\blacktriangle$, $\blacktriangledown$, $\blacklozenge$ are cancelled out. Denoting by σ_{12},σ_{34} the permutations $(f,g,h,t) \to (g,f,h,t)$, $(f,g,h,t) \to (f,g,t,h)$ respectively, we have:

$$
\begin{aligned}
\text{(4.8)} \qquad & \mathfrak{S}\pi^-_{2l+1}(fg,ht) \\
&= -\mathfrak{S}(1+\sigma_{34})(1+\sigma_{12})\{t\langle\langle h,f\rangle^-,g\rangle^+_{2l+1} \\
&\qquad + \langle\langle\langle h,f\rangle^-,g\rangle^+,t\rangle^+_{2l+1} - \langle f\langle g,h\rangle^-,t\rangle^+_{2l+1}\}.
\end{aligned}
$$

Substitute the equality (ϵ_{2l}) given in Appendix 6.2 to the last term of (4.8), where we remark that (ϵ_{2l}) is valid for any π^+_m such that $m \le 2l$. Note that

$$
\text{(4.9)} \qquad \mathfrak{S}(1+\sigma_{34})(1+\sigma_{12})S_a(f,\pi^-_b(g,h),t) = 0.
$$

By a little complicated calculation, we have

(4.10)

$$\begin{aligned}\mathfrak{S}\pi^-_{2l+1}(fg,ht) &= -\frac{1}{3}\mathfrak{S}(1+\sigma_{34})(1+\sigma_{12})\langle f,\langle t,\langle g,h\rangle^-\rangle^-\rangle^-_{2l+1}\\ &=\frac{1}{3}\langle t,\langle f,\langle g,h\rangle^-\rangle^-\rangle^-_{2l+1} - \frac{1}{3}\langle f,\langle t,\langle g,h\rangle^-\rangle^-\rangle^-_{2l+1}\\ &+\frac{1}{3}\langle t,\langle g,\langle f,h\rangle^-\rangle^-\rangle^-_{2l+1} - \frac{1}{3}\langle g,\langle t,\langle f,h\rangle^-\rangle^-\rangle^-_{2l+1}\\ &+\frac{1}{3}\langle h,\langle f,\langle g,t\rangle^-\rangle^-\rangle^-_{2l+1} - \frac{1}{3}\langle f,\langle h,\langle g,t\rangle^-\rangle^-\rangle^-_{2l+1}\\ &+\frac{1}{3}\langle h,\langle g,\langle f,t\rangle^-\rangle^-\rangle^-_{2l+1} - \frac{1}{3}\langle g,\langle h,\langle f,t\rangle^-\rangle^-\rangle^-_{2l+1}.\end{aligned}$$

We see by (3.8) that

$$\begin{aligned}&\langle t,\langle f,\langle g,h\rangle^-\rangle^-\rangle^-_{2l+1} - \langle f,\langle t,\langle g,h\rangle^-\rangle^-\rangle^-_{2l+1}\\ &\qquad= -\langle\langle g,h\rangle^-,\langle t,f\rangle^-\rangle^-_{2l+1} + R_{2l}(t,f,\pi^-_1(g,h)).\end{aligned}$$

Substituting these to (4.10), we have

$$\begin{aligned}\mathfrak{S}\pi^-_{2l+1}(fg,ht) &= \frac{1}{3}R_{2l}(t,f,\pi^-_1(g,h)) + \frac{1}{3}R_{2l}(\pi^-_1(t,f),g,h)\\ &+\frac{1}{3}R_{2l}(t,g,\pi^-_1(f,h)) + \frac{1}{3}R_{2l}(\pi^-_1(t,g),f,h)\\ &=0,\end{aligned} \tag{4.11}$$

because $R_m = 0$ by Corollary 3.4. Proposition 4.2 is thereby proved. □

§5. THE CONSTRUCTION OF π_{even}

The goal of this section is as follows

Theorem 5.1. *Assume* (H.1)∼(H.3) *for* $m = 2k$. *There exists* $\pi_{2k} \in \mathrm{SC}^2(\mathfrak{a})$ *such that* $\sum_{i+j=2k}\delta_i\pi_j = 0$, *and* π_{2k} *is a bidifferential operator of order* $4k$.

Notice at first that several existence theorems which will be given in what follows for monomials x^α, x^β etc. are evenly valid for monomials $(x - x(p))^\alpha$, $(x - x(p))^\beta$ etc. for any $p \in \mathfrak{g}^*$ by usual parallel displacements.

5.1. Induction for constructing π_{ev}.

To construct π^+_{2k}, we work at first on monomials of $x_1, \cdots, x_n \cdots$. We set

$$\pi^+_{2k}(x_i,x_j) = 0, \quad (k \geq 1). \tag{5.1}$$

For multi-indices α,β, we construct $\pi^+_{2k}(x^\alpha,x^\beta)$ inductively.

Assume the following:

$(\mathrm{B})_s$ $\pi^+_{2k}(x^\alpha, x^\beta)$ are obtained for any x^α, x^β such that $|\alpha+\beta| \le s$, and these satisfy (3.10), and $\pi^+_{2k}(x^\alpha, x^\beta) = \pi^+_{2k}(x^\beta, x^\alpha)$.

In what follows, we put unknown quantities $\pi^+_{2k}(x^\alpha, x^\beta)$ by $\varpi^+_{2k}(x^\alpha, x^\beta)$ for $|\alpha+\beta| = s+1$. Under $(\mathrm{B})_s$, we want at first to obtain $\varpi^+_{2k}(x_i, x^\gamma)$ for $|\gamma|+1 = s+1$.

Use the following notation:

$$(x^\alpha) \in x^\mu, \quad (x^\alpha, x^\beta, x^\gamma) \in x^\mu \quad \text{etc},$$

if there exist $x^\delta, x^{\delta'}$ such that $x^\alpha x^\delta = x^\mu$, $x^\alpha x^\beta x^\gamma x^{\delta'} = x^\mu$ etc.

Now, for any (x_i, x^β, x_j) such that $x_i x_j x^\beta = x^\mu$, (3.10,(a)) is read as follows:

$$\varpi^+_{2k}(x_i, x^\beta x_j) - \varpi^+_{2k}(x_j, x^\beta x_i) = E_{2k}(x_i, x^\beta, x_j), \tag{5.2}$$

where E_{2k} is defined by (3.12). Set the right hand side of (5.2) by $A_{ij}(= -A_{ji})$. Under the assumption $(\mathrm{B})_s$, A_{ij}'s are known quantities.

5.2. Left extremals. We now assume that x^μ is fixed as $|\mu| = s+1$. $\varpi^+_{2k}(x_i, x^\beta x_j)$ depends only on i such that $(x_i) \in x^\mu$. Set

$$T_i = \varpi^+_{2k}(x_i, x^\beta x_j). \tag{5.3}$$

Then, (5.2) is nothing but an over determined linear system

$$T_i - T_j = A_{ij} \quad \text{for} \quad (x_i, x_j) \in x^\mu.$$

This can be solved if and only if A_{ij} satisfy

$$A_{ij} + A_{jh} + A_{hi} = 0 \quad \text{for any} \quad (x_i, x_j, x_h) \in x^\mu. \tag{5.4}$$

First of all, we remark the following:

Proposition 5.2. *For any fixed x^μ such that $|\mu| = s+1$, the solubility condition (5.4) is satisfied.*

Proof. is seen in Appendix 6.2. .

By Proposition 5.2, T_i is given by

$$T_i = \frac{1}{n(\mu)} \sum_l A_{il} + K_{2k}(x^\mu), \tag{5.5}$$

where $n(\mu)$ is the number of (l) such that $(x_l) \in x^\mu$, and

$$K_{2k}(x^\mu) = \text{arbitrary element of} \quad C^\infty(\mathfrak{g}^*) \quad \text{depending only on} \quad x^\mu.$$

We choose simply $K_{2k} = 0$ in what follows.

For a fixed μ such that $|\mu| = s + 1$, we define a set of pairs of multi-indices by

$$S_\mu = \{(\alpha, \beta); \alpha + \beta = \mu, |\alpha| \geq 1, |\beta| \geq 1\}.$$

For any i, $i \geq 1$, we denote $< i >= (0, \cdots, 0, 1, 0, \cdots)$. An element $(< i >, \mu- < i >)$ (resp. $(\mu- < i >, < i >)$) will be called a *left extremal point* (resp. a *right extremal point*) of S_μ.

For a fixed x^μ, set $\mu(i) = \mu- < i >$, $\mu(i,j) = \mu- < i > - < j >$ for any (x_i), $(x_i, x_j) \in x^\mu$. Then, we have by (5.5)

$$\begin{aligned} &\varpi^+_{2k}(x_i - x_i(p), (x - x(p))^{\mu(i)}) \\ (5.6) \qquad &= \frac{1}{n(\mu)} \sum_j E_{2k}(x_i - x_i(p), (x - x(p))^{\mu(i,j)}, x_j - x_j(p)) \quad \forall p \in \mathfrak{g}^*. \end{aligned}$$

Lemma 5.3. *Let $L_i(f)(p) = \sum_\alpha \varpi^+_{2k}(x_i - x(p), (x - x(p))^\alpha)(p)\partial^\alpha f(p)$ by using $\varpi^+_{2k}(x_i - x_i(p), (x - x(p))^\alpha)$ obtained by (5.6) for any $(x_i - x(p), (x - x(p))^\alpha)$. Then, L_i is a linear differential operator of order $4k - 1$ for any i.*

Proof. Replace $\varpi^+_{2k}(x_i - x(p), (x - x(p))^\alpha)(p)$ in $L_i(f)(p)$ by the right hand side of (5.6) and remark that $E_{2k}(x_i - x_i(p), (x - x(p))^{\alpha - <j>}, x_j - x_j(p))(p)$ involves only the terms $\langle\langle\ ,\ \rangle^\pm,\ \rangle^\pm_{2k}$. Since $\langle\langle\ ,\ \rangle^\pm,\ \rangle^\pm_{2k}$ is a 3-differential operator of order $4k$ by the assumptions (H,1)-(H,3), L_i satisfies that at every $p \in \mathfrak{g}^*$ that

$$\varpi^+_{2k}(x_i, (x - x(p))^\alpha)(p) = 0 \quad \text{for} \quad |\alpha| > 4k - 1.$$

By using the similar criterion of Lemma 1.3 for 3-differential operators $\langle\langle\ ,\ \rangle^\pm,\ \rangle^\pm_{2k}$, we have that there is an integer s such that

$$\sum_{|\mu| < 4k} |\varpi^+_{2k}(x_i, (x - x(p))^\mu)(p)|^2 \lambda^{-2s\mu} < \infty.$$

Similarly, for any $\epsilon > 0$, and for any $p \in \mathfrak{g}^*$, there is a neighborhood V_p of p and an integer $s > 0$ such that for any $q \in V_p$,

$$\sum_\mu |\varpi^+_{2k}(x_i, (x - x(q))^\mu)(q) - \varpi^+_{2k}(x_i, (x - x(p))^\mu)(p)|^2 \lambda^{-2s\mu} < \epsilon.$$

Now, assume that

(1) For a fixed integer $l - 1$ and an arbitrary t, there is $s = s(l - 1, t)$ such that

$$\sum_{|\gamma| = l-1} \sum_\mu |\partial^\gamma \varpi^+_{2k}(x_i - x(p), (x - x(p))^\mu)(p)|^2 \lambda^{2t\gamma} \lambda^{-2s\mu} < \infty.$$

(2) For any $\epsilon > 0$ t, and for any $p \in \mathfrak{g}^*$, there is a neighborhood V_p of p and an integer $s = s(l-1, t, V_p)$ such that for any $q \in V_p$,

$$\sum_{|\gamma|=l-1} \sum_{\mu} |\partial^\gamma \varpi_{2k}^+(x_i, (x - x(q))^\mu)(q) - \partial^\gamma \varpi_{2k}^+(x_i, (x - x(p))^\mu)(p)|^2 \lambda^{2t\gamma} \lambda^{-2s\mu} < \epsilon.$$

We shall show that same inequalities as (1),(2) hold for l. Recall (3.11), and we see that $(\partial^\gamma E_{2k}(x_i - x_i(p), (x - x(p))^\alpha, x_j - x_j(p))(p)$ involves the partial derivatives $\partial^\beta \varpi_{2k}^+$ up to only $|\beta| \leq l - 1$. Hence, the assumptions (1),(2) can be applied. Other terms are written as $\langle\langle\ ,\ \rangle^\pm,\ \rangle_{2k}^\pm$. By using the similar criterion as in Lemma 1.3 for 3-differential operators $\langle\langle\ ,\ \rangle^\pm,\ \rangle_{2k}^\pm$, we obtain the lemma. □

5.3. Bridges.

Using the left extremal points, we shall construct $\varpi_{2k}^+(x^\alpha, x^\beta)$ for the pair of multi-indices (α, β) with $\alpha + \beta = \mu$,

Definition 5.4. For pairs of multi-indices (α, β) and (α', β') such that there is γ with
$\alpha' = \alpha + \gamma$, $\beta' = \beta - \gamma$, and $\alpha + \beta = \alpha' + \beta' = \mu$. the *bridge relation* $(Br)_\gamma$ from (α, β) to (α', β') is the following:

$$(Br)_\gamma \qquad \varpi_{2k}^+(x^{\alpha'}, x^{\beta'}) - \varpi_{2k}^+(x^\alpha, x^\beta) = -E_{2k}(x^\alpha, x^\gamma, x^{\beta'}),$$

where

$$\begin{aligned} E_{2k}(x^\alpha, x^\gamma, x^{\beta'}) = {} & \pi_{2k}^+(x^\alpha, x^\gamma) x^{\beta'} - x^\alpha \pi_{2k}^+(x^\gamma, x^{\beta'}) \\ & + \langle\langle x^\alpha, x^\gamma\rangle^+, x^{\beta'}\rangle_{2k}^+ - \langle x^\alpha, \langle x^\gamma, x^{\beta'}\rangle^+\rangle_{2k}^+ \\ & - \langle x^\gamma, \langle x^\alpha, x^{\beta'}\rangle^-\rangle_{2k}^- \quad \text{(cf. (3.12))}. \end{aligned}$$

If $(\alpha, \beta), (\alpha', \beta') \in S_\mu$ have the bridge relation $(Br)_\gamma$, we denote by $(\alpha, \beta) \overset{\gamma}{\leadsto} (\alpha', \beta')$ (or $(x^\alpha, x^\beta) \overset{\gamma}{\leadsto} (x^{\alpha'}, x^{\beta'})$).

Note that if $(\alpha, \beta) \overset{\gamma}{\leadsto} (\alpha', \beta')$, then $(\beta', \alpha') \overset{\gamma}{\leadsto} (\beta, \alpha)$, which is called the *dual bridge relation* to $(\alpha, \beta) \overset{\gamma}{\leadsto} (\alpha', \beta')$. The following lemma shows that any chain of bridges from a point of S_μ to another can be replaced by a direct bridge:

Lemma 5.5. *For $(\alpha, \beta + \gamma + \gamma')$, $(\alpha + \gamma, \beta + \gamma')$, $(\alpha + \gamma + \gamma', \beta) \in S_\mu$, the relations $(\alpha, \beta + \gamma + \gamma') \overset{\gamma}{\leadsto} (\alpha + \gamma, \beta + \gamma')$ and $(\alpha + \gamma, \beta + \gamma') \overset{\gamma'}{\leadsto} (\alpha + \gamma + \gamma', \beta)$ generate the relation $(\alpha, \beta + \gamma + \gamma') \overset{\gamma+\gamma'}{\leadsto} (\alpha + \gamma + \gamma', \beta)$.*

Proof. Let $f = x^\alpha, g = x^\gamma, h = x^{\gamma'}, k = x^\beta$ for the simplicity. By Proposition 2.2, we see that $\delta_0 Q_{2k} = 0$. Using (3.6) and Corollary 3.4, we have

$$(5.7) \qquad Q_{2k}(a, b, c) = \langle a, \langle b, c\rangle^+\rangle_{2k}^+ - \langle\langle a, b\rangle^+, c\rangle_{2k}^+ + \langle b, \langle a, c\rangle^-\rangle_{2k}^-.$$

The bridge relations $(Br)_\gamma$, $(Br)_{\gamma'}$, $(Br)_{\gamma+\gamma'}$ are written as follows:

$$\begin{cases} -f\pi_{2k}^+(g,ht) + \varpi_{2k}^+(fg,ht) - \varpi_{2k}^+(f,ght) + \pi_{2k}^+(f,g)ht & = Q_{2k}(f,g,ht), \\ -fg\pi_{2k}^+(h,t) + \varpi_{2k}^+(fgh,t) - \varpi_{2k}^+(fg,ht) + \pi_{2k}^+(fg,h)t & = Q_{2k}(fg,h,t), \\ -f\pi_{2k}^+(gh,t) + \varpi_{2k}^+(fgh,t) - \varpi_{2k}^+(f,ght) + \pi_{2k}^+(f,gh)t & = Q_{2k}(f,gh,t). \end{cases}$$

Computing $-(Br)_\gamma - (Br)_{\gamma'} + (Br)_{\gamma+\gamma'}$, we get

$$\begin{aligned} f(\delta_0\pi_{2k}^+)(g,h,t)+&(\delta_0\pi_{2k}^+)(f,g,h)t \\ &= -Q_{2k}(f,g,ht) - Q_{2k}(fg,h,t) + Q_{2k}(f,gh,t). \end{aligned} \tag{5.8}$$

By the assumption $(\mathrm{B})_s$, we have

$$(\delta_0\pi_{2k}^+)(g,h,t) = -Q_{2k}(g,h,t), \quad (\delta_0\pi_{2k}^+)(f,g,h) = -Q_{2k}(f,g,h).$$

Hence, (5.8) is

$$-fQ_{2k}(g,h,t) - Q_{2k}(f,g,h)t = -Q_{2k}(fg,h,t) + Q_{2k}(f,gh,t) - Q_{2k}(f,g,ht).$$

This holds because of $\delta_0 Q_{2k} = 0$. □

Note that by (5.7), we see easily that

$$\sum_{(f,g,h)} Q_{2k}(f,g,h) = 0. \tag{5.9}$$

By a similar manner, we have

Lemma 5.6. *If there are relations*

$$(<i>,\mu-<i>) \overset{\gamma}{\rightsquigarrow} (\alpha,\beta), \quad (<j>,\mu-<j>) \overset{\gamma'}{\rightsquigarrow} (\alpha,\beta),$$

then the computation of $\varpi_{2k}^+(x^\alpha,x^\beta)$ *does not depend on* $(Br)_\gamma$ *and* $(Br)_{\gamma'}$*, where the initial conditions for the bridges are given by* (5.3), (5.5).

Proof. One may assume that $i \neq j$. Since there are bridges, (x^α,x^β) must be given in the shape (x_ix_jh,x^β). We set $t = x^\beta$ for simplicity. Then, $(Br)_\gamma$, $(Br)_{\gamma'}$ are written as follows:

$$\varpi_{2k}^+(x_ix_jh,t) = \varpi_{2k}^+(x_i,x_jht) + x_i\pi_{2k}^+(x_jh,t) - \pi_{2k}^+(x_i,x_jh)t + Q_{2k}(x_i,x_jh,t), \tag{5.10}$$

$$\varpi_{2k}^+(x_jx_ih,t) = \varpi_{2k}^+(x_j,x_iht) + x_j\pi_{2k}^+(x_ih,t) - \pi_{2k}^+(x_j,x_ih)t + Q_{2k}(x_j,x_ih,t). \tag{5.11}$$

We have only to show the right hand side of (5.10) − (5.11) vanishes. Note that $\varpi^+_{2k}(x_i, x^\alpha)$ satisfies (5.2). By (5.2), we have

$$\begin{aligned}(5.12)\qquad & \varpi^+_{2k}(x_i, htx_j) - \varpi^+_{2k}(x_j, htx_i) \\ &= -x_i\pi^+_{2k}(ht, x_j) + \pi^+_{2k}(x_i, ht)x_j - Q_{2k}(x_i, ht, x_j).\end{aligned}$$

Using (5.12), we compute the right hand side of (5.11). So, the right hand side of (5.10) − (5.11) is

$$\begin{aligned}(5.13)\qquad & x_i(\pi^+_{2k}(x_jh, t) - \pi^+_{2k}(ht, x_j)) \\ &+ x_j(\pi^+_{2k}(x_i, ht) - \pi^+_{2k}(x_ih, t)) \\ &+ t(\pi^+_{2k}(x_j, x_ih) - \pi^+_{2k}(x_i, x_jh)) \\ &+ Q_{2k}(x_i, x_jh, t) - Q_{2k}(x_j, x_ih, t) - Q_{2k}(x_i, ht, x_j).\end{aligned}$$

By the assumption $(\mathrm{B})_s$, (5.13) is

$$\begin{aligned} & x_iQ_{2k}(x_j, h, t) - x_jQ_{2k}(x_i, h, t) - tQ_{2k}(x_j, h, x_i) \\ &+ Q_{2k}(x_i, x_jh, t) + Q_{2k}(t, x_ih, x_j) + Q_{2k}(x_j, ht, x_i).\end{aligned}$$

Recalling the definition of $\delta_0 Q_{2k}$ and using (5.9), we see that the above quantity is

$$(5.14)\qquad (\delta_0Q_{2k})(x_i, x_j, h, t) - (\delta_0Q_{2k})(x_j, x_i, h, t) = 0. \qquad \square$$

5.4. Right extremals.

As we have shown in 5.2, we have obtained $\varpi^+_{2k}(x_i, x^\alpha)$ for $\alpha + \langle i \rangle = \mu$, $|\mu| = s+1$. Next, we shall determine $\varpi^+_{2k}(x^\alpha, x_i)$ for $\alpha + \langle i \rangle = \mu$, $|\mu| = s + 1$. Given (x^α, x_i), there are a pair (x_j, x^β) and a multi-index γ such that $(x_j, x^\beta) \overset{\gamma}{\rightsquigarrow} (x^\alpha, x_i)$. Thus, we can get $\varpi^+_{2k}(x^\alpha, x_i)$ by $(Br)_\gamma$. By Lemma 5.6, $\varpi^+_{2k}(x^\alpha, x_i)$ is independent of the choice of γ and (x_j, x^β). We now show that $\varpi^+_{2k}(x_i, x^\alpha) = \varpi^+_{2k}(x^\alpha, x_i)$.

First of all, we easily have

Lemma 5.7. *For any i, j and a multi-index α, we have*

$$(5.15)\qquad \varpi^+_{2k}(x^\alpha x_i, x_j) = \varpi^+_{2k}(x_j, x^\alpha x_i).$$

Proof. Consider a bridge relation $(\langle i \rangle, \alpha + \langle j \rangle) \overset{\alpha}{\rightsquigarrow} (\alpha + \langle i \rangle, \langle j \rangle)$ and we have

$$(5.16)\qquad \varpi^+_{2k}(x^\alpha x_i, x_j) = \varpi^+_{2k}(x_i, x^\alpha x_j) - E_{2k}(x_i, x^\alpha, x_j)$$

by $(Br)_\alpha$. On the other hand, we write down (5.2) for $(x_j, x^\alpha x_i)$:

$$(5.17)\qquad \varpi^+_{2k}(x_j, x^\alpha x_i) = \varpi^+_{2k}(x_i, x^\alpha x_j) + A_{ji}.$$

Combining (5.16) with (5.17), we have (5.15). $\square$

Using Lemma 5.7, we have:

Lemma 5.8. *$\varpi^+_{2k}(x_i, x^\alpha) = \varpi^+_{2k}(x^\alpha, x_i)$ for any i and α.*

5.5. Determination for $\varpi^+_{2k}(x^\alpha, x^\beta)$.

To determine $\varpi^+_{2k}(x^\alpha, x^\beta)$, we choose an left extremal point (x_i, x^δ) such that $(x_i, x^\delta) \overset{\gamma}{\rightsquigarrow} (x^\alpha, x^\beta)$. Thus, we put $\varpi^+_{2k}(x^\alpha, x^\beta)$ by $(Br)_\gamma$, which also does not depend on the choice of γ and (x_i, x^δ).

We now prove

Proposition 5.9. *Under the assumptions* (HE.1-3), *$\varpi^+_{2k}(x^\alpha, x^\beta)$ can be constructed so that they satisfy $(Br)_\gamma$, $\varpi^+_{2k}(x^\alpha, x^\beta) = \varpi^+_{2k}(x^\beta, x^\alpha)$, and ϖ^+_{2k} is a bidifferential operator of order $4k$.*

Proof. Using the bridge relation

$$
(5.18) \qquad \begin{cases} \varpi^+_{2k}(x^{\gamma+<i>}, x^\beta) - \varpi^+_{2k}(x_i, x^{\gamma+\beta}) &= -E_{2k}(x_i, x^\gamma, x^\beta), \\ \varpi^+_{2k}(x^{\gamma+\beta}, x_i) - \varpi^+_{2k}(x^\beta, x^{\gamma+<i>}) &= -E_{2k}(x^\beta, x^\gamma, x_i). \end{cases}
$$

Hence, we have $\varpi^+_{2k}(x^\alpha, x^\beta) = \varpi^+_{2k}(x^\beta, x^\alpha)$ for $|\alpha + \beta| = s + 1$. This implies that for any α, β, γ with $\alpha+\beta+\gamma = \mu$, the equation $(Br)_\gamma$ is equal to that of (3.11) substituted by $f = x^\alpha, g = x^\gamma, h = x^\beta$. Then, we get the first and the second part of Proposition 5.9. This construction can be applied for monomials $(x - x(p))^\alpha, (x - x(q))^\beta$ etc..

To prove the last part, remark that

$$
\begin{aligned}
&\varpi^+_{2k}((x - x(p))^\alpha, (x - x(p))^\beta) \\
&\quad = \varpi^+_{2k}(x_i, (x - x(p))^{\alpha+\beta-<i>}) - E_{2k}(x_i, (x - x(p))^{\alpha-<i>}, (x - x(p))^\beta),
\end{aligned}
$$

for an $(x_i) \in x^\alpha$. By a similar proof as in Lemma 5.3, we have the desired result. Namely, we obtain by induction that ϖ^+_{2k} satisfies that for any l, t, there is an integer $s = s(l, t)$ such that

$$
\sum_{|\gamma|=l} \sum_{\alpha,\beta} |\partial^\gamma \varpi_{2k}((x - x(p))^\alpha, (x - x(p))^\beta)(p)|^2 \lambda^{2t\gamma} \lambda^{-2s(\alpha+\beta)} < \infty,
$$

and that for any $\epsilon > 0$ and l, t, there is a neighborhood V_p of p in $\mathfrak{g}^*$ and s such that for any $q \in V_p$,

$$
\begin{aligned}
\sum_{|\gamma|=k} \sum_{\alpha,\beta} |\partial^\gamma \varpi_{2k}((x - x(p))^\alpha, (x - x(p))^\beta)(p) - \partial^\gamma \varpi_{2k}((x - x(q))^\alpha, (x - x(q))^\beta)(q)|^2 \\
\times \lambda^{2t\gamma} \lambda^{-2s(\alpha+\beta)} < \epsilon.
\end{aligned}
$$

□

We now put $\pi^+_{2k}(x^\alpha, x^\beta) = \varpi^+_{2k}(x^\alpha, x^\beta)$. The symmetricity of π_{2k} is obtained by the polynomial approximation theorem and Proposition 5.9. Theorem 5.1 is thereby proved, and we obtain Theorem A.

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§6. Appendix

6.1 Proof of Lemma 3.1.

If $\delta_0 Q = 0$, then $\delta_0(1+\sigma_3)Q = 0$ by Lemma 2.3. Set $Q^+ = \frac{1}{2}(1+\sigma_3)Q$. Note that $\delta_0 = \partial_1^0 - \partial_2^0 + \partial_3^0$ by Lemma 2.3, (ii). Thus, we have $(\partial_2^0 - \partial_3^0)Q^+ = \partial_1^0 Q^+$. Using Lemma 2.3, we have $(\partial_2^0 - \partial_3^0)\mathfrak{c}_3^2 = \mathfrak{c}_4^3(\partial_1^0 - \partial_2^0)$. So, we get

$$(\partial_2^0 - \partial_3^0)\mathfrak{c}_3^2 Q^+ = -\mathfrak{c}_4^3 \partial_3^0 Q^+.$$

Hence,

$$(\partial_2^0 - \partial_3^0)(1 - \mathfrak{c}_3 + \mathfrak{c}_3^2)Q^+ = \partial_1^0 Q^+ - (\partial_2^0 - \partial_3^0)\mathfrak{c}_3 Q^+ - \mathfrak{c}_4^3 \partial_3^0 Q^+. \tag{6.1}$$

Evaluating the right hand side of (6.1) at (f,g,h,t), we have

$$\begin{aligned} & f\cdot \underset{\blacktriangle}{Q^+(g,h,t)} - Q^+(f\cdot g,h,t) + \underline{Q^+(f,h,t)\cdot g} \\ & - g\cdot Q^+(t,f,h) + Q^+(t,f,g\cdot h) - \underline{Q^+(h\cdot t,f,g) + Q^+(h,f,g)\cdot t} \\ & - t\cdot Q^+(f,h,g) + Q^+(g,h,t\cdot f) - \underset{\blacktriangle}{Q^+(g,h,t)}\cdot f, \end{aligned} \tag{6.2}$$

where $f\cdot g = \pi_0(f,g)$. The terms marked by ▲ are trivially cancelled. Use $\sigma_3 Q^+ = Q^+$, $\delta_0 Q = 0$, to the underlined terms of (6.2). Then, these terms are changed into $Q^+(g\cdot f,h,t) - Q^+(g,f\cdot h,t)$. Hence (6.2) is

$$-Q^+(g,f\cdot h,t) - g\cdot Q^+(t,f,h) + Q^+(t,f,g\cdot h) - t\cdot Q^+(f,h,g) + Q^+(g,h,t\cdot f).$$

Using $\sigma_3 Q^+ = Q^+$ to $Q^+(g,h,t\cdot f)$, we see that (6.2) is $-(\delta_0 Q^+)(t,f,h,g) = 0$. □

6.2. Proof of Proposition 5.2.

We shall show that (5.4) is satisfied under the assumptions (H.1-2). For that purpose, we shall investigate (3.11) more precisely. For any fixed (f,g,h), (3.11) can be regarded as a linear system with unknowns $\pi_{2k}^+(f,gh)$, $\pi_{2k}^+(g,hf)$, $\pi_{2k}^+(h,fg)$:

$\pi_{2k}^+(f,gh)$	$\pi_{2k}^+(g,hf)$	$\pi_{2k}^+(h,fg)$	
1	0	-1	$: E_{2k}(f,g,h)$
-1	1	0	$: E_{2k}(g,h,f)$
0	-1	1	$: E_{2k}(h,f,g)$

The solubility condition of the above linear system is satisfied by virtue of $R_{2k} = 0$. Set

$$S_{2k}(f,g,h) = \mathop{\mathfrak{S}}_{(f,g,h)} \pi_{2k}^{+}(f, gh). \tag{6.3}$$

Then, $S_{2k} \in \mathrm{SC}^3(\mathfrak{a})$. By using (3.12), the solution of the linear system is written as follows:

(ϵ_{2k})

$$\begin{aligned}\pi_{2k}^{+}(f,gh) &= \frac{1}{3}S_{2k}(f,g,h) + \frac{1}{3}\pi_{2k}^{+}(f,g)h + \frac{1}{3}\pi_{2k}^{+}(f,h)g - \frac{2}{3}f\pi_{2k}^{+}(g,h) \\ &+ \frac{1}{3}\langle\langle f,g\rangle^{+},h\rangle_{2k}^{+} + \frac{1}{3}\langle\langle f,h\rangle^{+},g\rangle_{2k}^{+} - \frac{2}{3}\langle\langle g,h\rangle^{+},f\rangle_{2k}^{+} \\ &+ \frac{1}{3}\langle\langle f,g\rangle^{-},h\rangle_{2k}^{-} + \frac{1}{3}\langle\langle f,h\rangle^{-},g\rangle_{2k}^{-}.\end{aligned}$$

All others are obtained by the cyclic permutation of (f,g,h). Note also that the above formula can be applied for π_m^{+} such that $m \le 2k-1$.

Suppose $(x_i, x_j, x_h) \in x^{\mu}$, i.e. there is a monomial g such that $x_i x_j x_h g = x^{\mu}$. By (3.12), we have

$$\begin{aligned}&A_{ij} + A_{jh} + A_{hi} \\ &= \mathop{\mathfrak{S}}_{(i,j,h)} [\pi_{2k}^{+}(x_i, gx_h)x_j - \pi_{2k}^{+}(x_j, gx_h)x_i \\ &+ \langle\langle x_i, gx_h\rangle^{+}, x_j\rangle_{2k}^{+} - \langle\langle x_j, gx_h\rangle^{+}, x_i\rangle_{2k}^{+} \\ &+ \langle\langle x_i, x_j\rangle^{-}, gx_h\rangle_{2k}^{-}] \\ &= (1) + (2) + (3),\end{aligned} \tag{6.4}$$

where

$$\begin{aligned}(1) &= \mathop{\mathfrak{S}}_{(i,j,h)} x_i\{\pi_{2k}^{+}(x_h, gx_j) - \pi_{2k}^{+}(x_j, gx_h)\} = \mathop{\mathfrak{S}}_{(i,j,h)} x_i E_{2k}(x_h, g, x_j) \\ (2) &= \mathop{\mathfrak{S}}_{(i,j,h)} \langle x_i, \langle x_h, gx_j\rangle^{+} - \langle x_j, gx_h\rangle^{+}\rangle_{2k}^{+} \\ (3) &= \mathop{\mathfrak{S}}_{(i,j,h)} \langle\langle x_i, x_j\rangle^{-}, gx_h\rangle_{2k}^{-}.\end{aligned}$$

Recalling (3.8) and using (4.2) for the term (3), we have

$$\begin{aligned}(3) &= \mathfrak{S} x_h\langle\langle x_i, x_j\rangle^{-}, g\rangle_{2k}^{-} \\ &+ \mathfrak{S}\langle\langle\langle x_i, x_j\rangle^{-}, g\rangle^{-}, x_h\rangle_{2k}^{+} - \mathfrak{S}\langle\langle x_i, x_j\rangle^{-}, \langle g, x_h\rangle^{+}\rangle_{2k}^{-},\end{aligned} \tag{6.5}$$

where we used

$$\mathfrak{S}\langle\langle\langle x_i, x_j\rangle^{-}, x_h\rangle^{-}, g\rangle_{2k}^{+} = \sum_{a+b=2k, a,b\ge 1} \pi_a^{+}(R_b(x_i, x_j, x_h), g) = 0.$$

From (3.12), we have

(6.6)
$$(1) = \mathfrak{S}\ x_i\{\langle\langle x_h, g\rangle^+, x_j\rangle_{2k}^+ - \langle\langle x_j, g\rangle^+, x_h\rangle_{2k}^+\} + \mathfrak{S}\ x_i\langle\langle x_h, x_j\rangle^-, g\rangle_{2k}^-.$$

Note that in (1) + (3) the last term of (6.6) and the first term of (6.5) are cancelled out. Use (3.11-12) to (2), and remark that $R_m = 0$. Then, we see

(6.7)
$$\begin{aligned}
&A_{ij} + A_{jh} + A_{hi}\\
&\quad= \mathfrak{S}\ \langle\langle g, x_h\rangle^+, \langle x_i, x_j\rangle^-\rangle_{2k}^- + \mathfrak{S}\ \langle\langle\langle x_i, x_j\rangle^-, g\rangle^-, x_h\rangle_{2k}^+\\
&\quad+ \mathfrak{S}\ x_i\{\langle\langle x_h, g\rangle^+, x_j\rangle_{2k}^+ - \langle\langle x_j, g\rangle^+, x_h\rangle_{2k}^+\} + \mathfrak{S}\ \langle x_i, \langle x_h, g\rangle^+ x_j - \langle x_j, g\rangle^+ x_h\rangle_{2k}^+\\
&\quad+ \mathfrak{S}\ \langle x_i, \langle\langle x_h, g\rangle^+, x_j\rangle^+ - \langle\langle x_j, g\rangle^+, x_h\rangle^+\rangle_{2k}^+ + \mathfrak{S}\ \langle x_i, \langle\langle x_h, x_j\rangle^-, g\rangle^-\rangle_{2k}^+.
\end{aligned}$$

Note that the second term and the last term of the right hand side of (6.7) are cancelled out. We now use (ϵ_{2k}) to the second term of the second line in (6.7). After a little complicated rearrangement of the terms, we have

(6.8)
$$\begin{aligned}
&A_{ij} + A_{jh} + A_{hi}\\
&= \mathfrak{S}\ x_i \cdot \langle\langle x_h, g\rangle^+, x_j\rangle_{2k}^+ - \mathfrak{S}\ x_i \cdot \langle\langle x_j, g\rangle^+, x_h\rangle_{2k}^+ + \mathfrak{S}\ \langle\langle g, x_h\rangle^+, \langle x_i, x_j\rangle^-\rangle_{2k}^-\\
&+ \mathfrak{S}\ \langle x_i, \langle x_j, \langle x_h, g\rangle^+\rangle^+\rangle_{2k}^+ - \mathfrak{S}\ \langle x_i, \langle x_h, \langle x_j, g\rangle^+\rangle^+\rangle_{2k}^+\\
&+ \underset{\bigstar}{\frac{1}{3}\mathfrak{S}\sum_{a+b=2k} S_a(x_i, \pi_b^+(x_h, g), x_j)} - \underset{\bigstar}{\frac{1}{3}\mathfrak{S}\sum_{a+b=2k} S_a(x_i, \pi_b^+(x_j, g), x_h)}\\
&+ \underset{\blacktriangle}{\frac{1}{3}\mathfrak{S}\langle x_i, x_j\rangle^+ \cdot \langle x_h, g\rangle^+} + \frac{1}{3}\mathfrak{S}x_j \cdot \langle x_i, \langle x_h, g\rangle^+\rangle_{2k}^+ - \frac{2}{3}\mathfrak{S}x_i \cdot \langle x_j, \langle x_h, g\rangle^+\rangle_{2k}^+\\
&- \underset{\blacktriangle}{\frac{1}{3}\mathfrak{S}\langle x_i, x_h\rangle^+ \cdot \langle x_j, g\rangle^+} - \frac{1}{3}\mathfrak{S}x_h \cdot \langle x_i, \langle x_j, g\rangle^+\rangle_{2k}^+ + \frac{2}{3}\mathfrak{S}x_i \cdot \langle x_h, \langle x_j, g\rangle^+\rangle_{2k}^+\\
&+ \underset{\blacklozenge}{\frac{1}{3}\mathfrak{S}\langle\langle x_i, x_j\rangle^+, \langle x_h, g\rangle^+\rangle_{2k}^+} + \frac{1}{3}\mathfrak{S}\langle\langle x_i, \langle x_h, g\rangle^+\rangle^+, x_j\rangle_{2k}^+ - \frac{2}{3}\mathfrak{S}\langle x_i, \langle x_j, \langle x_h, g\rangle^+\rangle^+\rangle_{2k}^+\\
&- \underset{\blacklozenge}{\frac{1}{3}\mathfrak{S}\langle\langle x_i, x_h\rangle^+, \langle x_j, g\rangle^+\rangle_{2k}^+} - \frac{1}{3}\mathfrak{S}\langle\langle x_i, \langle x_j, g\rangle^+\rangle^+, x_h\rangle_{2k}^+ + \frac{2}{3}\mathfrak{S}\langle x_i, \langle x_h, \langle x_j, g\rangle^+\rangle^+\rangle_{2k}^+\\
&+ \frac{1}{3}\mathfrak{S}\langle\langle x_i, x_j\rangle^-, \langle x_h, g\rangle^+\rangle_{2k}^- + \frac{1}{3}\mathfrak{S}\langle\langle x_i, \langle x_h, g\rangle^+\rangle^-, x_j\rangle_{2k}^-\\
&- \frac{1}{3}\mathfrak{S}\langle\langle x_i, x_h\rangle^-, \langle x_j, g\rangle^+\rangle_{2k}^- - \frac{1}{3}\mathfrak{S}\langle\langle x_i, \langle x_j, g\rangle^+\rangle^-, x_h\rangle_{2k}^-,
\end{aligned}$$

where $A^+ \cdot B^+$ means $\sum_{a+b=2k, a,b\geq 1} A_a^+ B_b^+$. The terms marked by $\blacktriangle$, $\bigstar$, $\blacklozenge$ are cancelled out respectively. Since

$$\mathfrak{S}\ x_i \cdot \langle\langle x_h, g\rangle^+, x_j\rangle_{2k}^+ = \mathfrak{S}\ x_i \cdot \langle x_j, \langle x_h, g\rangle^+\rangle_{2k}^+ = \mathfrak{S}\ x_h \cdot \langle x_i, \langle x_j, g\rangle^+\rangle_{2k}^+,$$

the six terms involving $\cdot$ of (6.8) are cancelled out. Note also that

$$(6.9)\qquad \mathfrak{S}\,\langle\langle x_i,\langle x_j,g\rangle^+\rangle^+,x_h\rangle^+_{2k}=\mathfrak{S}\,\langle x_i,\langle x_j,\langle x_h,g\rangle^+\rangle^+\rangle^+_{2k}$$

$$\mathfrak{S}\,\langle\langle x_i,x_h\rangle^-,\langle x_j,g\rangle^+\rangle^-_{2k}=-\mathfrak{S}\,\langle\langle x_i,x_j\rangle^-,\langle x_h,g\rangle^+\rangle^-_{2k}.$$

Finally, (6.8) is reduced to the following:

(6.10)

$$-\frac{1}{3}\mathfrak{S}\,\langle\langle x_i,x_j\rangle^-,\langle x_h,g\rangle^+\rangle^-_{2k}+\frac{1}{3}\mathfrak{S}\langle\langle x_i,\langle x_h,g\rangle^+\rangle^-,x_j\rangle^-_{2k}$$
$$-\frac{1}{3}\mathfrak{S}\langle\langle x_i,\langle x_j,g\rangle^+\rangle^-,x_h\rangle^-_{2k}$$

$$=-\frac{1}{3}\underset{(i,j,h)}{\mathfrak{S}}\,\{\langle\langle x_i,x_j\rangle^-,\langle x_h,g\rangle^+\rangle^-_{2k}+\langle\langle\langle x_h,g\rangle^+,x_i\rangle^-,x_j\rangle^-_{2k}+\langle\langle x_i,\langle x_j,g\rangle^+\rangle^-,x_h\rangle^-_{2k}\}$$
$$=-\frac{1}{3}\sum_{a+b=2k,a,b\geq 1}\underset{(i,j,h)}{\mathfrak{S}}R_a(x_i,x_j,\pi^+_b(x_h,g))=0.$$

So, $\varpi^+_{2k}(x_i,x^\alpha)$ is obtained by (5.5) for any (x_i,x^α) such that $x_ix^\alpha=x^\mu$. Thus, Proposition 5.2 is proved. □

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