Research Report

KSTS/RR-92/003
June 4, 1992

On A Constraction of Deformation Quantization of Poisson Algebra

by

H. Omori, Y. Maeda and A. Yoshioka

H. Omori and A. Yoshioka
Dept. of Math. Fac. of Sci. and Tech.
Science Univ. of Tokyo

Y. Maeda
Department of Mathematics
Faculty of Science and Technology
Keio University

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku, Yokohama, 223 Japan

KSTS/RR-92/003
June 4, 1992

ON A CONSTRUCTION OF DEFORMATION QUANTIZATION OF POISSON ALGEBRAS

HIDEKI OMORI*), YOSHIAKI MAEDA**), AKIRA YOSHIOKA*)

Dedicated to Professor Morio Obata on his 65th birthday

§0. INTRODUCTION

Let M be a C^∞ Poisson manifold with a Poisson structure $\{\ ,\ \}$, and $C^\infty(M)$ the commutative topological algebra over $\mathbf{C}$ with the C^∞ topology of all $\mathbf{C}$-valued C^∞ functions on M. In what follows, we put $C^\infty(M) = \mathfrak{a}$ for simplicity. $(\mathfrak{a}, \{\ ,\ \})$ is called a *Poisson algebra* (cf. [W]).

By introducing a formal parameter ν, we consider the direct product

$$\mathfrak{a}[[\nu]] = \prod_{n=0}^{\infty} \nu^n \mathfrak{a} \tag{0.1}$$

and consider a product $*:\mathfrak{a}[[\nu]] \times \mathfrak{a}[[\nu]] \to \mathfrak{a}[[\nu]]$. According to the decomposition (0.1), we can set, for any $f, g \in \mathfrak{a}$,

$$f * g = \sum_{n=0}^{\infty} \nu^n \pi_n(f, g). \tag{0.2}$$

Definition 1. For a Poisson algebra $(\mathfrak{a}, \{\ ,\ \})$, $(\mathfrak{a}[[\nu]], *)$ is called a *deformation quantization* of $(\mathfrak{a}, \{\ ,\ \})$ if $(\mathfrak{a}[[\nu]], *)$ satisfies

(A.1) $*$ is an associative product, and ν is in the center,
(A.2) $\pi_0(f, g) = fg$ and $\pi_1(f, g) = -\frac{1}{2}\{f, g\}$ for any $f, g \in \mathfrak{a}$,
(A.3) for each $m \geq 2$, $\pi_m : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ is a bidifferential operator of positive order and $\pi_m(f, g) = (-1)^m \pi_m(g, f)$ for any $f, g \in \mathfrak{a}$.

The Poisson algebra $(\mathfrak{a}, \{\ ,\ \})$ is called *deformation quantizable* if it has a deformation quantization.

Remark. (A.3) indicates 1 is the identity.

It is known in [OMY1],[DL] that if M is a symplectic manifold, then $(\mathfrak{a}, \{\ ,\ \})$ is deformation quantizable. However, as to Poisson algebras of nonconstant rank, there seems to be no general theory for the deformation quantizability.

*) Dept. of Math. Fac. of Science and Technology, Science Univ. of Tokyo, Noda, Chiba 278, Japan **) Dept. of Math. Fac. of Science and Technology, Keio Univ., Hiyoshi, Yokohama 223, Japan

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

KSTS/RR-92/003
June 4, 1992

The purpose of this paper is to propose an inductive method of constructing a deformation quantization of $(\mathfrak{a},\{\ ,\ \})$, and to give the necessary and sufficient conditions for this method to be continued.

To this end, we introduce k-truncated algebra: Set

$$\mathfrak{a}[[\nu]]_k = \mathfrak{a} \oplus \nu\mathfrak{a} \oplus \cdots \oplus \nu^k\mathfrak{a} \cong \mathfrak{a}[[\nu]]/\nu^{k+1}\mathfrak{a}[[\nu]] \tag{0.3}$$

and consider a product $*_k : \mathfrak{a}[[\nu]]_k \times \mathfrak{a}[[\nu]]_k \to \mathfrak{a}[[\nu]]_k$ such that

$$\nu^{k+1} = 0 \quad \text{and} \quad f *_k g = \prod_{n=0}^{k} \nu^n \pi_n(f,g) \quad \text{for any} \quad f, g \in \mathfrak{a}. \tag{0.4}$$

We denote this algebra by $(\mathfrak{a}[[\nu]]_k, *_k)$.

Definition 2. For $k \geq 2$, $(\mathfrak{a}[[\nu]]_k, *_k)$ is called a *deformation quantization of order* k of $(\mathfrak{a},\{\ ,\ \})$, if the following conditions are satisfied:

(B.1) $*_k$ is associative, and ν is in the center,
(B.2) $\pi_0(f,g) = fg$ and $\pi_1(f,g) = -\frac{1}{2}\{f,g\}$ for any $f,g \in \mathfrak{a}$,
(B.3) for each $m \geq 2$, π_m is a bidifferential operator of positive order and $\pi_m(f,g) = (-1)^m \pi_m(g,f)$ for any $f,g \in \mathfrak{a}$.

Now, suppose a deformation quantization $(\mathfrak{a}[[\nu]]_k, *_k)$ of order k of $(\mathfrak{a},\{\ ,\ \})$ is given. Our problem is to seek

$(\mathrm{P})_{k+1}$ a bidifferential operator $\pi_{k+1}\colon \mathfrak{a}\times\mathfrak{a}\to \mathfrak{a}$ such that $*_{k+1} = \sum_{n=0}^{k+1} \nu^n \pi_n$ yields a deformation quantization $(\mathfrak{a}[[\nu]]_{k+1}, *_{k+1})$ of order $k+1$ of $(\mathfrak{a},\{\ ,\ \})$.

We obtain the equation for the above π_{k+1} in §1. We introduce R_{k+1} in §§2-3, which comes from the Jacobi identity of $(\mathfrak{a}[[\nu]]_k, *_k)$. The necessary and sufficinet condition to solve $(\mathrm{P})_{k+1}$ will be described by $R_{k+1} = 0$ in §4. It is also studied that R_{k+1} determines a deRham-Chvalley 3-cocycle.

By means of our method, we give several concrete examples of deformation quantization of Poisson algebras in §5.

§1. Equations and Hochschild coboundary operators

Let $(\mathfrak{a}[[\nu]]_k, *_k)$ be a deformation quantization of order k of $(\mathfrak{a},\{\ ,\ \})$. The associativity of $(\mathfrak{a}[[\nu]]_{k+1}, *_{k+1})$ can be read as

$$\sum_{i+j=m} \pi_i(\pi_j(f,g),h) = \sum_{i+j=m} \pi_i(f,\pi_j(g,h)) \quad \text{for any} \quad f,g,h \in \mathfrak{a}, \tag{1.1$_m$}$$

for $0 \leq m \leq k+1$. Thus, in order to solve $(\mathrm{P})_{k+1}$, we construct π_{k+1} with (B.3) satisfying $(1.1)_{k+1}$.

Before treating this, we introduce some algebraic tools.

1.1. Hochschild coboundary operators. Denote by $\mathrm{C}^p(\mathfrak{a})$, $p \geq 1$, the space of all continuous p-linear mappings of $\mathfrak{a} \times \cdots \times \mathfrak{a}$ to $\mathfrak{a}$. We denote by $\mathrm{AC}^p(\mathfrak{a})$ and

$\mathrm{SC}^p(\mathfrak{a})$ $(p \geq 1)$ the set of the alternative and the symmetric elements of $\mathrm{C}^p(\mathfrak{a})$ respectively. If p=0, we set $\mathrm{C}^0(\mathfrak{a})$=$\mathrm{AC}^0(\mathfrak{a})$= $\mathrm{SC}^0(\mathfrak{a})$=$\mathfrak{a}$.

For any $\pi \in \mathrm{C}^2(\mathfrak{a})$, we define the *Hochschild coboundary operator* $\delta_\pi : \mathrm{C}^p(\mathfrak{a}) \to \mathrm{C}^{p+1}(\mathfrak{a})$, $p \geq 1$ by

$$(1.2) \qquad \begin{aligned}(\delta_\pi F)(v_1, \cdots, v_{p+1}) &= \pi(v_1, F(v_2, \cdots, v_{p+1})) \\ &+ \sum_{i=1}^{p}(-1)^i F(v_1, \cdots, \pi(v_i, v_{i+1}), \cdots, v_p) \\ &+ (-1)^{p+1}\pi(F(v_1, \cdots, v_p), v_{p+1})\end{aligned}$$

for $F \in \mathrm{C}^p(\mathfrak{a})$, and for p=0, we set for any $v \in \mathfrak{a}$,

$$(1.3) \qquad (\delta_\pi v)(v_1) = \pi(v_1, v).$$

By a direct computation using the linearization, we have the following:

Lemma 1.1. *For any $\pi, \pi', \pi'' \in \mathrm{C}^2(\mathfrak{a})$, we have*

$$(1) \qquad \delta_\pi \pi' = \delta_{\pi'} \pi, \qquad \delta_\pi I = \pi, \quad (I = \textit{identity})$$

$$(2) \qquad \delta_\pi \delta_\pi \pi = 0,$$

$$(3) \qquad \mathop{\mathfrak{S}}_{(\pi,\pi',\pi'')} \delta_\pi \delta_{\pi'} \pi'' = 0,$$

where $\mathop{\mathfrak{S}}_{(\pi,\pi',\pi'')}$ *means the cyclic summation with respect to* π, π', π''.

For any $\pi \in \mathrm{C}^2(\mathfrak{a})$, $-\frac{1}{2}\delta_\pi \pi$ is called the *associator* of $(\mathfrak{a},\pi)$. Namely, we have

$$(1.4) \qquad -\frac{1}{2}\delta_\pi \pi(u, v, w) = \pi(\pi(u, v), w) - \pi(u, \pi(v, w)).$$

Hence, $\delta_\pi \pi$=0, if and only if $(\mathfrak{a},\pi)$ is an *associative algebra.* If $(\mathfrak{a},\pi)$ is an associative algebra, then

$$(1.5) \qquad \delta_\pi^2 F = 0,$$

for any $F \in \mathrm{C}^p(\mathfrak{a})$ (cf. [Mc]). In particular, $\delta_\pi^2 I$=$\delta_\pi \pi$=0. Therefore, we have

Lemma 1.2. *δ_π^2=0 is equivalent to $\delta_\pi \pi$=0.*

1.2. Equation for (P)$_{k+1}$. For $\{\pi_i\}_{i=0}^{k+1}$, we denote $\delta_i = \delta_{\pi_i}$ for simplicity. By using these notations and owing to Lemma 1.1(1), $(1.1)_{k+1}$ is equivalent to the equation

$$(1.6)_{k+1} \qquad \delta_0 \pi_{k+1} = -\frac{1}{2} \sum_{i+j=k+1, i,j \geq 1} \delta_i \pi_j.$$

Thus, we shall solve $(1.6)_{k+1}$ for π_{k+1} satisfying (B.3), under the conditions $(1.1)_m$, (B.2) and (B.3) with $0 \leq m \leq k$.

Since the associativity of π_0 implies δ_0^2=0, the right hand side of $(1.6)_{k+1}$ must satisfy

$$(1.7) \qquad \sum_{i+j=k+1, i,j \geq 1} \delta_0 \delta_i \pi_j = 0.$$

Hence, (1.7) is a necessary condition for $(\mathfrak{a}[[\nu]]_k, *_k)$ to be extended to the deformation quantization of order $k+1$. However, (1.7) substantially gives no restriction on $(\mathfrak{a}[[\nu]]_k, *_k)$. Namely, we have:

Proposition 1.3. *If $(\mathfrak{a}[[\nu]]_k, *_k)$ is a deformation quantization of order k of $(\mathfrak{a}, \{\ ,\ \})$, then $\pi_0, \cdots, \pi_k$ satisfy (1.7).*

Proof. We put $\hat{\pi} = \pi_0 + \nu\pi_1 + \cdots + \nu^k\pi_k$. Since $(\mathfrak{a}[[\nu]]_k, *_k)$ is associative, i.e. $\delta_{\hat{\pi}}\hat{\pi} = 0$, we have

$$\sum_{i+j=m} \delta_i \pi_j = 0 \tag{1.8}$$

for $0 \leq m \leq k$. Applying δ_{k-m}, we get $\sum_{i+j=m} \delta_{k-m}\delta_i\pi_j = 0$ for $1 \leq m \leq k$. It follows

$$\sum_{i+j=k+1, i,j\geq 1} \delta_i\delta_0\pi_j + \sum_{i+j=k+1, i,j\geq 1} \delta_i\delta_j\pi_0 + \sum_{a+b+c=k+1, a,b,c\geq 1} \delta_a\delta_b\pi_c = 0. \tag{1.9}$$

Lemma 1.1 implies

$$\sum_{a+b+c=k+1, a,b,c\geq 1} \delta_a\delta_b\pi_c = 0. \tag{1.10}$$

Thus, using Lemma 1.1(1), we get

$$\sum_{i+j=k+1, i,j\geq 1} \delta_i\delta_0\pi_j = 0. \tag{1.11}$$

On the other hand, Lemma 1.2 gives

$$\sum_{i+j=m} \delta_i\delta_j = 0 \tag{1.12}$$

for $0 \leq m \leq k$. Hence, we have

$$\sum_{i+j=k+1, i,j\geq 1} \delta_0\delta_i\pi_j + \sum_{i+j=k+1, i,j\geq 1} \delta_i\delta_0\pi_j + \sum_{a+b+c=k+1, a,b,c\geq 1} \delta_a\delta_b\pi_c = 0. \tag{1.13}$$

(1.10) and (1.11) yields the desired result. □

§2. deRham-Chevalley cohomology

2.1. p-derivations. We will introduce the following notion:

Definition 2.1. For given $\pi \in \mathrm{C}^2(\mathfrak{a})$ and $p \geq 1$, we define $\partial_i^\pi : \mathrm{C}^p(\mathfrak{a}) \to \mathrm{C}^{p+1}(\mathfrak{a})$ $(1 \leq i \leq p)$ by

$$\begin{aligned}(\partial_i^\pi F)(v_1, \cdots, v_{p+1}) = &\ \pi(v_i, F(v_1, \cdots, \hat{v}_i, \cdots, v_{p+1})) \\ &- F(v_1, \cdots, \pi(v_i, v_{i+1}), \cdots, v_{p+1}) \\ &+ \pi(F(v_1, \cdots, \hat{v}_{i+1}, \cdots, v_{p+1}), v_{i+1})\end{aligned} \tag{2.1}$$

for any $F \in \mathrm{C}^p(\mathfrak{a})$.

We call $F \in C^p(\mathfrak{a})$ a *p-derivation with respect to* π, if for any j, $(1 \leq j \leq p)$

$$\partial_j^\pi F = 0. \tag{2.2}$$

By $Der^p(\mathfrak{a}, \pi)$, we denote the space of all p-derivations with respect to π. We also set

$$\mathcal{A}^p(\mathfrak{a}, \pi) = AC^p(\mathfrak{a}) \cap Der^p(\mathfrak{a}, \pi). \tag{2.3}$$

2.2. Chevalley coboundary operators. For any $\pi \in AC^2(\mathfrak{a})$, we define the *Chevalley coboundary operator*

$$d_\pi : AC^p(\mathfrak{a}) \to AC^{p+1}(\mathfrak{a})$$

by

$$\begin{aligned}(d_\pi F)&(v_1, \cdots, v_{p+1}) \\ &= \sum_{i=1}^{p+1} (-1)^{i+1} \pi(v_i, F(v_1, \cdots, \hat{v}_i, \cdots, v_{p+1})) \\ &\quad + \sum_{i<j} (-1)^{i+j} F(\pi(v_i, v_j), v_1, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_{p+1}).\end{aligned} \tag{2.4}$$

By a direct computation using the linearization, we have

Lemma 2.2. *For any* $\pi, \pi', \pi'' \in AC^2(\mathfrak{a})$,

(1) $$d_\pi \pi' = d_{\pi'} \pi, \quad d_\pi I = \pi,$$

(2) $$d_\pi d_\pi \pi = 0,$$

(3) $$\sum_{(\pi, \pi', \pi'')} d_\pi d_{\pi'} \pi'' = 0.$$

Since π is skew symmetric, we have

$$(d_\pi \pi)(u, v, w) = 2 \sum_{(u,v,w)} \pi(u, \pi(v, w)). \tag{2.5}$$

Thus, $d_\pi \pi = 0$ if and only if $(\mathfrak{a}, \pi)$ is a Lie algebra. If $(\mathfrak{a}, \pi)$ is a Lie algebra, then $d_\pi^2 F = 0$ for any $F \in AC^p(\mathfrak{a})$ (cf.[Ma]). Therefore,

Lemma 2.3. $d_\pi^2 = 0$ *is equivalent to* $d_\pi \pi = 0$.

In the following, we use the notations

$$\pi^\pm(u, v) = \frac{1}{2}\{\pi(u, v) \pm \pi(v, u)\}. \tag{2.6}$$

for $\pi \in C^2(\mathfrak{a})$.

We first remark the following:

Lemma 2.4. *$(\mathfrak{a},\pi)$, $\pi \in C^2(\mathfrak{a})$, is an associative algebra if and only if $\delta_\pi\pi \in AC^3(\mathfrak{a})$, and $(\mathfrak{a},\pi^-)$ is a Lie algebra.*

Proof. The necessity is obvious. Note at first that $\delta_\pi\pi \in \mathrm{AC}^3(\mathfrak{a})$ implies that $(\mathfrak{a},\pi)$ is an alternative algebra (cf. [S]). It is known in [S], p. 76, that

$$3\,\delta_\pi\pi(u,v,w) = 4 \sum_{(u,v,w)} \pi^-(u,\pi^-(v,w)). \tag{2.7}$$

Thus, if $(\mathfrak{a},\pi^-)$ is a Lie algebra, then $\delta_\pi\pi = 0$, hence $(\mathfrak{a},\pi)$ is associative. □

2.3. Cohomology. The following is not hard to prove:

Lemma 2.5.

$$d_{\pi_1}^2 = 0 \quad \text{and} \quad d_{\pi_1}\mathcal{A}^p(\mathfrak{a},\pi_0) \subset \mathcal{A}^{p+1}(\mathfrak{a},\pi_0),$$

where $\pi_0(f,g) = fg$ and $\pi_1(f,g) = -\frac{1}{2}\{f,g\}$ for any $f,g \in \mathfrak{a}$.

Definition 2.6. For Poisson algebra $(\mathfrak{a},\{\ ,\ \})$, we denote the p-th cohomology group of the cochain complex :

$$\cdots \to \mathcal{A}^p(\mathfrak{a},\pi_0) \xrightarrow{d_{\pi_1}} \mathcal{A}^{p+1}(\mathfrak{a},\pi_0) \longrightarrow \cdots \tag{2.8}$$

by $H^p(\mathfrak{a},\{\ ,\ \})$. $H^*(\mathfrak{a},\{\ ,\ \})$ will be called the *deRham-Chevalley cohomology group* of the Poisson algebra $(\mathfrak{a},\{\ ,\ \})$.

§3. Jacobi identity and the deRham-Chevalley 3-cocycle

Let $(\mathfrak{a}[[\nu]]_k, *_k)$, $*_k = \sum_{n=0}^k \nu^k\pi_n$, be a deformation quantization of order k of $(\mathfrak{a},\{\ ,\ \})$.

Definition 3.1. For simplicity, we set $d_i^- = d_{\pi_i^-}$. We define for $2 \le m \le k+1$

$$\begin{cases} Q_m &= \frac{1}{2}\sum_{i+j=m,i,j\ge 1} \delta_i\pi_j, \\ R_m &= \frac{1}{2}\sum_{i+j=m,i,j\ge 1} d_i^-\pi_j^-. \end{cases} \tag{3.1}$$

Remark if m is odd, then it is clear R_m=0 from (B.3). Moreover, the Jacobi identity of $(\mathfrak{a}[[\nu]]_k, *_k)$ yields

Proposition 3.2.

$$R_m = 0 \quad \text{for} \quad 2 \le m \le k.$$

By Proposition 1.3, we have $\delta_0 Q_{k+1}$=0. Similarly, we have the following:

Proposition 3.3.
$$d_1^- R_{k+1} = 0.$$

Proof. We put

(3.2) $$\hat{\pi} = \pi_0 + \nu\pi_1 + \cdots + \nu^k \pi_k.$$

Then, $(\mathfrak{a}[[\nu]]_k, \hat{\pi}^-)$ is a Lie algebra, so we see, by Lemma 2.3, $d_{\hat{\pi}^-}\hat{\pi}^- = 0$ and $d^2_{\hat{\pi}^-} = 0$. Hence, we have

(3.3) $$\sum_{i+j=m} d_i^- \pi_j^- = 0,$$

(3.4) $$\sum_{i+j=m} d_i^- d_j^- = 0,$$

for $2 \le m \le k$. Computing similarly as in the proof of Proposition 1.3 and using (3.3), we have

(3.5)
$$\sum_{i+j=k+1, i,j\ge 2} d_i^- d_1^- \pi_j^- + \sum_{i+j=k+1, i,j\ge 2} d_i^- d_j^- \pi_1^- + \sum_{a+b+c=k+1, a,b,c\ge 2} d_a^- d_b^- \pi_c^- = 0.$$

Lemma 2.2 gives

(3.6) $$\sum_{i+j=k+1, i,j\ge 2} d_i^- d_1^- \pi_j^- = 0.$$

Thus, (3.4) with (3.6) gives the desired result. □

Note that R_{k+1} is computed only by using $\{\pi_1^-(=\pi_1), \cdots, \pi_k^-\}$.
In the rest of this section, we shall prove the following theorem.

Theorem 3.4. *R_{k+1} is a deRham-Chevalley 3-cocycle.*

It is sufficient to show $\partial_j R_{k+1} = 0$ $(j = 1, 2, 3)$, where $\partial_j = \partial_j^{\pi_0}$.
We define two mappings $\sigma_p, \mathfrak{c}_p : C^p(\mathfrak{a}) \to C^p(\mathfrak{a})$ by

(3.7) $$(\sigma_p F)(v_1, v_2, \cdots, v_p) = F(v_p, v_{p-1}, \cdots, v_1),$$

(3.8) $$(\mathfrak{c}_p F)(v_1, v_2, \cdots, v_p) = F(v_p, v_1, v_2, \cdots, v_{p-1}).$$

Obviously $\sigma_2 = \mathfrak{c}_2$. Since $\mathfrak{c}_3^3 = 1$, we have

(3.9) $$(1 + \mathfrak{c}_3 + \mathfrak{c}_3^2)(1 - \mathfrak{c}_3) = 0,$$

(3.10) $$(1 - \mathfrak{c}_3 + \mathfrak{c}_3^2)(1 + \mathfrak{c}_3) = 2.$$

We have the following formulas.

Lemma 3.5. *(i) For any $\pi \in C^2(\mathfrak{a})$ and $F \in C^2(\mathfrak{a})$, we have*

$$(1) \qquad \delta_\pi \sigma_p F = (-1)^{p+1} \sigma_{p+1} \delta_{\sigma_2 \pi} F,$$

$$(2) \qquad \partial_j^\pi \mathfrak{c}_p F = \mathfrak{c}_{p+1} \partial_{j+1}^\pi F \quad (1 \le j \le p-1), \quad \partial_p^\pi \mathfrak{c}_p F = \mathfrak{c}_{p+1}^2 \partial_1^\pi F.$$

(ii) In particular, if $\pi \in SC^2(\mathfrak{a})$, we have

$$\partial_j^\pi \sigma_p F = \sigma_{p+1} \partial_{p-j+1}^\pi F \quad (1 \le j \le p).$$

Using the above, we get the following relation.

Lemma 3.6.

$$(1 + \mathfrak{c}_3 + \mathfrak{c}_3^2)(1 - \sigma_3) Q_{k+1} = 4 R_{k+1}.$$

Proof. Since $\delta_i = \delta_i^+ + \delta_i^-$, where $\delta_i^\pm = \delta_{\pi_i^\pm}$, we see by the definition of Q_{k+1},

$$(3.11) \qquad Q_{k+1} = \frac{1}{2} \sum_{i+j=k+1, i,j \ge 1} (\delta_i^+ \pi_j^+ + \delta_i^- \pi_j^-) + \sum_{i+j=k+1,i,j\ge 1} \delta_i^+ \pi_j^-.$$

Lemma 3.5(i)(1) gives $\sigma_3 \delta_i^+ \pi_j^- = \delta_i^+ \pi_j^-$, $\sigma_3 \delta_i^+ \pi_j^+ = -\delta_i^+ \pi_j^+$, $\sigma_3 \delta_i^- \pi_j^- = -\delta_i^- \pi_j^-$. Then, we have

$$(3.12) \qquad \begin{cases} Q_{k+1} - \sigma_3 Q_{k+1} &= \sum_{i+j=k+1,i,j\ge 1} (\delta_i^+ \pi_j^+ + \delta_i^- \pi_j^-), \\ Q_{k+1} + \sigma_3 Q_{k+1} &= 2 \sum_{i+j=k+1,i,j \ge 1} \delta_i^+ \pi_j^-. \end{cases}$$

By the direct calculation, we get

$$\begin{aligned} (1 + \mathfrak{c}_3 + \mathfrak{c}_3^2)(1 - \sigma_3) Q_{k+1}(f,g,h) &= 4 \sum_{i+j=k+1,i,j\ge 1} \mathop{\mathfrak{S}}_{(f,g,h)} \pi_i^-(f, \pi_j^-(g,h)) \\ &= 4 R_{k+1}(f,g,h) \end{aligned}$$

which gives Lemma 3.6. □

To simplify the notations, we shall use the following:

$$(3.13) \qquad \begin{cases} f \cdot g &= \pi_0(f,g), \quad \langle f, g \rangle_m^\pm = \pi_m^\pm(f,g), \quad (m \ge 1), \\ \langle f, g \cdot \langle h,t \rangle^\pm \rangle_m^\pm &= \sum_{i+j=m,i,j\ge 1} \pi_i^\pm(f, g \cdot \pi_j^\pm(h,t)) \qquad (m \ge 2), \\ \langle \langle f, \langle g,h \rangle^\pm \rangle^\pm, t \rangle_m^\pm &= \sum_{a+b+c=m,a,b,c\ge 1} \pi_a^\pm(\pi_b^\pm(f, \pi_c^\pm(g,h)), t) \quad (m \ge 3), \\ \langle \langle f,g \rangle^\pm, \langle h,t \rangle^\pm \rangle_m^\pm &= \sum_{a+b+c=m,a,b,c,\ge 1} \pi_a^\pm(\pi_b^\pm(f,g), \pi_c^\pm(h,t)) \quad (m \ge 3). \end{cases}$$

Proof of Theorem 3.4. By using notations (3.13), R_{k+1} is written as

(3.14)
$$\begin{aligned} R_{k+1}(f,g,h) &= \langle f, \langle g,h\rangle^-\rangle^-_{k+1} + \langle g, \langle h,f\rangle^-\rangle^-_{k+1} + \langle h, \langle f,g\rangle^-\rangle^-_{k+1} \\ &= \frac{1}{4} \mathop{\mathfrak{S}}_{(f,g,h)} \sum_{i+j=k+1} \delta_i^- \pi_j^-(f,g,h). \end{aligned}$$

Note that (1.8) holds for $1 \le m \le k$. By lemma 1.1 and (3.1), (1.8) can be written

$$\delta_0 \pi_m = -Q_m. \tag{3.15}$$

Set $\pi_m^\pm = \frac{1}{2}(1 \pm \sigma_2)\pi_m$. Remarking $\sigma_2 = \mathfrak{c}_2$, and using Lemma 3.5, we have

$$\delta_0 \pi_m^+ = \frac{1}{2}(1-\sigma_2)\delta_0\pi_m = -(1-\mathfrak{c}_3)\partial_2 \pi_m^+, \tag{3.16}$$

$$\delta_0 \pi_m^- = \frac{1}{2}(1+\sigma_2)\delta_0\pi_m = -(1+\mathfrak{c}_3)\partial_2 \pi_m^- \tag{3.17}$$

By (3.10), the equation (3.15) splits into two equations:

(3.18)
$$\begin{aligned} \partial_2 \pi_m^- &= \frac{1}{4}(1 - \mathfrak{c}_3 + \mathfrak{c}_3^2)(1+\sigma_3)Q_m, \\ (1-\mathfrak{c}_3)\partial_2 \pi_m^+ &= \frac{1}{2}(1-\sigma_3)Q_m. \end{aligned}$$

By using (3.12), (3.18) is equivalent to the following:

(3.19)
$$\begin{aligned} &\pi_m^-(f, g\cdot h) \\ =&g\cdot \pi_m^-(f,h) + \pi_m^-(f,g)\cdot h + \langle\langle f,g\rangle^-, h\rangle^+_m + \langle\langle f,h\rangle^-, g\rangle^+_m - \langle f, \langle g,h\rangle^+\rangle^-_m. \end{aligned}$$

We now compute the following quantity:

$$R_{k+1}(f\cdot g,h,t) = \langle f\cdot g, \langle h,t\rangle^-\rangle^-_{k+1} + \langle h, \langle t, f\cdot g\rangle^-\rangle^-_{k+1} + \langle t, \langle f\cdot g, h\rangle^-\rangle^-_{k+1}. \tag{3.20}$$

By using (3.19), (3.20) can be rewritten as

(3.21)
$$\begin{aligned} & f\cdot \langle g, \langle h,t\rangle^-\rangle^-_k + g\cdot \langle f, \langle h,t\rangle^-\rangle^-_{k+1} + \langle\langle f, \langle h,t\rangle^-\rangle^-, g\rangle^+_{k+1} \\ &\quad + \langle\langle g, \langle h,t\rangle^-\rangle^-, f\rangle^+_{k+1} + \langle\langle h,t\rangle^-, \underset{\blacktriangle}{\langle f,g\rangle^+}\rangle^-_{k+1} \\ & + \underline{\langle h, \langle t,f\rangle^-\cdot g\rangle^-_{k+1}} + \underline{\langle h, \langle t,g\rangle^-\cdot f\rangle^-_{k+1}} + \langle h, \langle\langle t,f\rangle^-, g\rangle^+\rangle^-_{k+1} \\ &\quad + \langle h, \langle\langle t,g\rangle^-, f\rangle^+\rangle^-_{k+1} + \langle h, \langle\underset{\blacktriangle}{\langle f,g\rangle^+}, t\rangle^-\rangle^-_{k+1} \\ & - \underline{\langle t, \langle h,f\rangle^-\cdot g\rangle^-_{k+1}} - \underline{\langle t, \langle h,g\rangle^-\cdot f\rangle^-_{k+1}} - \langle t, \langle\langle h,f\rangle^-, g\rangle^+\rangle^-_{k+1} \\ &\quad - \langle t, \langle\langle h,g\rangle^-, f\rangle^+\rangle^-_{k+1} - \langle t, \langle\underset{\blacktriangle}{\langle f,g\rangle^+}, h\rangle^-\rangle^-_{k+1}. \end{aligned}$$

The three terms marked by ▲ vanish by virtue of Proposition 3.2, for setting $A_l = \langle f, g\rangle_l^+$ we see that these terms are

(3.22)

$$\sum_{l=1}^{k}\{\langle\langle h,t\rangle^-, A_l\rangle_{k+1-l}^- + \langle\langle t, A_l\rangle^-, h\rangle_{k+1-l}^- + \langle\langle A_l, h\rangle^-, t\rangle_{k+1-l}^-\}$$

$$= -\sum_{l=1}^{k+1-1} R_{k+1-l}(A_l, h, t) = 0.$$

Computing the underlined 4 terms in (3.21) by using (3.19), we have

$$R_{k+1}(f\cdot g, h, t)$$

$$= f\cdot\{\langle g, \langle h,t\rangle^-\rangle_{k+1}^- + \langle h, \langle t,g\rangle^-\rangle_{k+1}^- + \langle t, \langle g,h\rangle^-\rangle_{k+1}^-\}$$

$$+ g\cdot\{\langle f, \langle h,t\rangle^-\rangle_{k+1}^- + \langle h, \langle t,f\rangle^-\rangle_{k+1}^- + \langle t, \langle f,h\rangle^-\rangle_{k+1}^-\}$$

$$+ \langle h,g\rangle^-\cdot\langle t,f\rangle^- + \langle h,f\rangle^-\cdot\langle t,g\rangle^- - \langle t,g\rangle^-\cdot\langle h,f\rangle^- - \langle t,f\rangle^-\cdot\langle h,g\rangle^-$$

(3.23)

$$+\langle\langle\underset{\blacktriangle}{\langle t,f\rangle^-}, g\rangle^+, h\rangle_{k+1}^- \quad +\langle\langle h, \langle t,f\rangle^-\rangle^-, g\rangle_{k+1}^+ \quad +\langle\langle h,g\rangle^-, \underset{\bigstar}{\langle t,f\rangle^-}\rangle_{k+1}^+$$

$$+\langle\langle\underset{\blacktriangle}{\langle t,g\rangle^-}, f\rangle^+, h\rangle_{k+1}^- \quad +\langle\langle h, \langle t,g\rangle^-\rangle^-, f\rangle_{k+1}^+ \quad +\langle\langle h,f\rangle^-, \underset{\bigstar}{\langle t,g\rangle^-}\rangle_{k+1}^+$$

$$-\langle\langle\underset{\blacktriangle}{\langle h,f\rangle^-}, g\rangle^+, t\rangle_{k+1}^- \quad -\langle\langle t, \langle h,f\rangle^-\rangle^-, g\rangle_{k+1}^+ \quad -\langle\langle t,g\rangle^-, \underset{\bigstar}{\langle h,f\rangle^-}\rangle_{k+1}^+$$

$$-\langle\langle\underset{\blacktriangle}{\langle h,g\rangle^-}, f\rangle^+, t\rangle_{k+1}^- \quad -\langle\langle t, \langle h,g\rangle^-\rangle^-, f\rangle_{k+1}^+ \quad -\langle\langle t,f\rangle^-, \underset{\bigstar}{\langle h,g\rangle^-}\rangle_{k+1}^+$$

$$+\langle h, \langle\underset{\blacktriangle}{\langle t,f\rangle^-}, g\rangle^+\rangle_{k+1}^-$$

$$+\langle h, \langle\underset{\blacktriangle}{\langle t,g\rangle^-}, f\rangle^+\rangle_{k+1}^-$$

$$-\langle t, \langle\underset{\blacktriangle}{\langle h,f\rangle^-}, g\rangle^+\rangle_{k+1}^- \quad +\langle\langle f, \langle h,t\rangle^-\rangle^-, g\rangle_{k+1}^+ \quad +\langle\langle g, \langle h,t\rangle^-\rangle^-, f\rangle_{k+1}^+$$

$$-\langle t, \langle\underset{\blacktriangle}{\langle h,g\rangle^-}, f\rangle^+\rangle_{k+1}^-.$$

Here six terms below the line of (3.23) come directly from (3.21). $A^-\cdot B^-$ means $\sum_{i+j=k+1, i,j\geq 1} A_i^-\cdot B_j^-$. Note that the terms marked by ★ and ▲ vanish by themselves, and the third line of the right hand side of (3.23) also vanish by itself. Hence, we have

(3.24)

$$
\begin{aligned}
&R_{k+1}(f\cdot g,h,t)\\
&\quad= f\cdot R_{k+1}(g,h,t)+g\cdot R_{k+1}(f,h,t)\\
&\quad+\langle\langle h,\langle t,g\rangle^-\rangle^-,f\rangle^+_{k+1}+\langle\langle t,\langle g,h\rangle^-\rangle^-,f\rangle^+_{k+1}+\langle\langle g,\langle h,t\rangle^-\rangle^-,f\rangle^+_{k+1}\\
&\quad+\langle\langle h,\langle t,f\rangle^-\rangle^-,g\rangle^+_{k+1}+\langle\langle t,\langle f,h\rangle^-\rangle^-,g\rangle^+_{k+1}+\langle\langle f,\langle h,t\rangle^-\rangle^-,g\rangle^+_{k+1}.
\end{aligned}
$$

The last six terms in (3.24) vanish by virtue of Proposition 3.2, and (3.14). Hence, we have $\partial_1 R_{k+1}=0$. As R_{k+1} is alternative, we have $\partial_j R_{k+1}=0$ $(j=1,2,3)$. Then, Theorem 3.4 is obtained. □

§4. Main results

We state the following theorem ([OMY2]).

Main Theorem. *Suppose $(\mathfrak{a}[[\nu]]_k,*_k)$ is a deformation quantization of order k of $(\mathfrak{a},\{\ ,\ \})$. Then, there exists a bidifferential operator $\pi_{k+1}:\mathfrak{a}\times\mathfrak{a}\to\mathfrak{a}$ satisfying* $(\mathrm{P})_{k+1}$ *if and only if R_{k+1}=0.*

We will give a brief sketch of the proof of Main theorem. The necessity is obvious by Proposition 3.2. For the sufficiency, we work on each coordinates $(x_1,\cdots,x_n)$, and seek $\pi_{k+1}(x^\alpha,x^\beta)$ for every multi-index (α,β). Then, the equation (1.6) for $\pi_{k+1}(x^\alpha,x^\beta)$ gives a huge linear system and the assumption $R_{k+1}=0$ plays to the role of the solvability condition. Lastly, using polynomial approximation theorem and the partition of unity, we obatin π_{k+1} globally.

The construction is given by long direct calculations. Here, cohomological theories do not seem to be useful for our construction.

Our method of the proof is very primitive, so that this can be applied to infinite dimensional manifolds. Hence several application to field theory can be expected.

If R_{k+1} is s non-trivial deRham-Chevalley 3-coboundary, we can replace π_k to $\pi_k+\theta_k$ by the deRham-Chevalley 2-cochain θ_k so that R_{k+1} for $\pi_k+\theta_k$ vanishes. Since the above replacement gives another solution of (1.6), Main theorem and the above property give a step up for the construction of deformation quantization.

Now, suppose $(\mathfrak{a}[[\nu]]_{2k-1},*_{2k-1})$ is a deformation quantization of odd order $2k-1$ of $(\mathfrak{a},\{\ ,\ \})$. We consider the cohomology class $[R_{2k}]$. Note the solution of (1.6) has the ambiguity by Hochschild 2-cocycles. Let θ be a symmetric bidifferentail operator of positive order satisfying $\delta_0\theta=0$. By a replacement $\pi'_{2k-2}=\pi_{2k-2}+\theta$, $\{\pi_0,\pi_1,\cdots,\pi_{2k-3},\pi'_{2k-2}\}$ gives a deformation quantization $(\mathfrak{a}[[\nu]]_{2k-2},*'_{2k-2})$ of order $2k-2$. Since $R'_{2k-1}=0$ for $(\mathfrak{a}[[\nu]]_{2k-2},*'_{2k-2})$, there exists π'_{2k-1} such that $\{\pi_0,\pi_1,\cdots,\pi_{2k-3},\pi'_{2k-2},\pi'_{2k-1}\}$ defines a deformation quantization $(\mathfrak{a}[[\nu]]_{2k-1},*'_{2k-1})$ of order $2k-1$ by Main theorem. We have

Proposition 4.1. $[R_{2k}]=[R'_{2k}]$*, where R_{2k} and R'_{2k} are determined by* (3.1) *for* $(\mathfrak{a}[[\nu]]_{2k-1},*_{2k-1})$ *and* $(\mathfrak{a}[[\nu]]_{2k-1},*'_{2k-1})$*, respectively.*

Before proving Proposition 4.1, we prepare the follwing Lemma.

Lemma 4.2. *Let θ be a symmetric bidifferential operator of positive order such that $\delta_0\theta = 0$. Then, there exists a linear differential operator ξ such that $\theta = \delta_0\xi$.*

Proof. Suppose $(U_\alpha, x_1, \cdots, x_n)$ is a local coordinate system on M. If $\theta=\delta_0\xi_\alpha$ on each U_α, then using a partition of unity, $\{\phi_\alpha\}$, we see that $\theta= \delta_0 \sum_\alpha \phi_\alpha\xi_\alpha$. Thus, we have only to show that $\theta=\delta_0\xi_\alpha$ on U_α. For any point $\mathbf{a}=(a_1, \cdots, a_n)$, and $f \in C^\infty(U_\alpha)$, we set $f_i^{\mathbf{a}}= f(a_1, \cdots, a_{i-1}, x_i, \cdots, x_n)$. Obviously,

$$f(\mathbf{x}) - f(\mathbf{a}) = \sum_{i=1}^{n} \frac{f_i^{\mathbf{a}} - f_{i+1}^{\mathbf{a}}}{x_i - a_i}(x_i - a_i).$$

Notice that $\frac{f_i^{\mathbf{a}} - f_{i+1}^{\mathbf{a}}}{x_i - a_i}$ is C^∞ with respect to $(\mathbf{a}, \mathbf{x}) \in U_\alpha \times U_\alpha$. Define $\xi_\alpha(f)$ by

$$\xi_\alpha(f)(\mathbf{a}) = -\sum_{i=1}^{n} \theta(\frac{f_i^{\mathbf{a}} - f_{i+1}^{\mathbf{a}}}{x_i - a_i}, x_i - a_i)(\mathbf{a}).$$

Note that $\theta(*, 1)$=0. If θ is a bidifferential operator of order k, $\xi_\alpha(f)$ is a linear differential operator of order k. Thus, we have

$$\begin{aligned}
&(\delta_0\xi_\alpha)(f, g)(\mathbf{a}) \\
&\quad = -(\xi_\alpha(fg) - f\xi_\alpha(g) - \xi_\alpha(f)g)(\mathbf{a}) \\
&\quad = \sum_{i=1}^{n}(\theta(f_i^{\mathbf{a}} \cdot \frac{g_i^{\mathbf{a}} - g_{i+1}^{\mathbf{a}}}{x_i - a_i}, x_i - a_i)(\mathbf{a}) - f(\mathbf{a})\theta(\frac{g_i^{\mathbf{a}} - g_{i+1}^{\mathbf{a}}}{x_i - a_i}, x_i - a_i)(\mathbf{a})) \\
&\quad + \sum_{i=1}^{n}(\theta(g_{i+1}^{\mathbf{a}} \cdot \frac{f_i^{\mathbf{a}} - f_{i+1}^{\mathbf{a}}}{x_i - a_i}, x_i - a_i)(\mathbf{a}) - g(\mathbf{a})\theta(\frac{f_i^{\mathbf{a}} - f_{i+1}^{\mathbf{a}}}{x_i - a_i}, x_i - a_i)(\mathbf{a})).
\end{aligned}$$

Since $\delta_0\theta$=0 implies

$$\theta(fg, h) - f\theta(g, h) = \theta(f, gh) - \theta(f, g)h,$$

we have by using $\theta(f, g)$=$\theta(g, f)$ and $\theta(*, 1)$=0 that

$$\begin{aligned}
(\delta_0\xi_\alpha)(f, g)(\mathbf{a}) &= \sum_{i=1}^{n}(\theta(f_i^{\mathbf{a}}, g_i^{\mathbf{a}} - g_{i+1}^{\mathbf{a}}) + \theta(g_{i+1}^{\mathbf{a}}, f_i^{\mathbf{a}} - f_{i+1}^{\mathbf{a}}))(\mathbf{a}) \\
&= \sum_{i=1}^{n}(\theta(f_i^{\mathbf{a}}, g_i^{\mathbf{a}}) - \theta(f_{i+1}^{\mathbf{a}}, g_{i+1}^{\mathbf{a}}))(\mathbf{a}) \\
&= \theta(f, g)(\mathbf{a}). \qquad \square
\end{aligned}$$

Proof of Proposition 4.1. Due to skew symmetricity of solutions, $(1.6)_{2k-1}$ is reduced to the first equality of (3.18) with $m = 2k - 1$, for the second being always true. Therefore, if we write $\pi'_{2k-1} = \pi_{2k-1} + w$, then w is given by

$$\partial_2 w(f, g, h) = -(\delta_0\xi)(\pi_1(f, g), h) - (\delta_0\xi)(\pi_1(f, h), g) + \pi_1(f, (\delta_0\xi)(g, h)) \tag{4.1}$$

for any $f, g, h \in \mathfrak{a}$. The right hand side of (4.1) is equal to

$$gA(f,h) - A(f,gh) + A(f,g)h = -(\partial_2 A)(f,g,h),$$

where $A(f,g) = \xi(\pi_1(f,g)) - \pi_1(f,\xi(g))$.

Put

$$w(f,g) = \pi_1(g,\xi(f)) - \pi_1(f,\xi(g)) + \xi(\pi_1(f,g)). \tag{4.2}$$

Then, it is easy to see

$$w(f,g) = -w(g,f) \quad \text{and} \quad \partial_2 w = -\partial_2 A,$$

so (4.2) gives a solution of (4.1). Hence, we have $\pi'_{2k-1} = \pi_{2k-1} - (d_1\xi)(f,g)$ for any $f, g \in \mathfrak{a}$. By (3.1), we get $R'_{2k} = R_{2k} + d_1 d_1 \xi = R_{2k}$.

Note that the ambiguity of π'_{2k-1} is $\mathcal{A}^2(\mathfrak{a}, \pi_0)$, i.e, another solution of $(1.6)_{2k-1}$ is given by $\pi''_{2k-1} = \pi_{2k-1} + \theta$, $\theta \in \mathcal{A}^2(\mathfrak{a}, \pi_0)$. It is easy to check R''_{2k} for this π''_{2k-1} is cohomologous to R_{2k-1}. Thus, we get the desired result. □

Therefore, if a Poisson algebra $(\mathfrak{a}, \{\ ,\ \})$ is given, then the cohomology class of the first obstruction cocycle R_4 is determined only by $(\mathfrak{a}, \{\ ,\ \})$. If there exists $(\mathfrak{a}, \{\ ,\ \})$ such that $[R_4] \neq 0$, then such a Poisson algebra has no deformation quantization. However, such examples are not known yet.

§5. Examples

We give several examples of deformation quantization of Poisson algebras.

Ex.1. (cf. [B]) Let $\mathcal{G}^*$ be the dual space of a finite dimensional Lie algebra $\mathcal{G}$. Regarding $X \in \mathcal{G}$ as a linear function on $\mathcal{G}^*$, we define $\{X, Y\} = [X, Y]$, i.e, for a linear basis $X_1, \cdots, X_n$ of $\mathcal{G}$, we set

$$\{X_i, X_j\} = \sum_{k=1}^{n} c_{ij}^k X_k$$

using the structure constants c_{ij}^k of $\mathcal{G}$. By the polynomial approximation theorem, the above procedure makes $C^\infty(\mathcal{G}^*)$ a Poisson algebra whose rank is not constant. $(C^\infty(\mathcal{G}^*), \{\ ,\ \})$ is deformation quantizable and the deformation quantization is given by the closure of the universal enveloping algebra $\mathcal{U}_\nu(\mathcal{G})$ of $\mathcal{G}$ with the parameter ν, i.e. the algebra generated by $X_1, \cdots, X_n$ with the relations

$$[X_i, X_j] = -\nu \sum_{k=1}^{n} c_{ij}^k X_k.$$

In the construction of π_{k+1}, we can set always $\pi_m(X_i, X_j) = 0$ for $m \geq 2$. If this is the case, we have only to check the quantities

$$R_{k+1}(X_i, X_j, X_l) = \sum_{(i,j,l)} \pi_k^-(X_i, \pi_1^-(X_j, X_l)). \tag{5.1}$$

R_2 always vanishes because $d_{\pi_1}\pi_1 = 0$. Hence, if $\pi_1(X_i, X_j) = c_{ij} + \sum_l c_{ij}^l X_l$, then $R_s = 0$ for any $s \geq 2$. This will be also applicable to Poisson algebras of constant rank, and linearizable Poisson algebras (cf. [W]) for making deformation quantization.

Ex.2. Consider the symplectic form $\frac{1}{y^2}dx \wedge dy$ on the upper half plane H_+. This gives a Poisson algebra structure $\{,\}$ on $C^\infty(H_+)$ such that

$$\{f,g\} = y^2(\partial_x f \partial_y g - \partial_y f \partial_x g)$$

which can be extended to $C^\infty(\mathbf{R}^2)$. $(C^\infty(\mathbf{R}^2),\{,\})$ has a deformation quantization. Since all π_m are bidifferential operators, the restriction $f * g|H_+$ depends only on $f|H_+, g|H_+$. Hence, any deformation quantization $(C^\infty(\mathbf{R}^2)[[\nu]],*)$ defines a $*$-product on $C^\infty(H_+)[[\nu]]$. Taking the cartesian coordinates $(x,y) \in \mathbf{R}^2$, we can construct the quantized algebra $(C^\infty(\mathbf{R}^2)[[\nu]],*)$ with $\pi_m(x,y) = 0$ for $m \geqq 2$. So we have the relation $[x,y] = -\nu y^2$ where $y^2 = yy = y * y$. This is equivalent to

$$y * x = (x + \nu y) * y$$

and the algebra $C^\infty(\mathbf{R}^2)[[\nu]]$ can be characterized only by this relation. Its restriction onto H_+ is isomorphic to the algebra of covariant symbol calculus given in [Be] [Mo].

Ex.3. Let x,y,z be the natural coordinate functions on $\mathbf{R}^3$. For any positive integers k,l,m, the relations

$$\{x,y\} = z^k, \quad \{y,z\} = x^l, \quad \{z,x\} = y^m$$

define a Poisson algebra structure on $C^\infty(\mathbf{R}^3)$, in which the function

$$f_0(x,y,z) = \frac{1}{l+1}x^{l+1} + \frac{1}{m+1}y^{m+1} + \frac{1}{k+1}z^{k+1}$$

Poisson-commutes with all elements of $C^\infty(\mathbf{R}^3)$ (i.e, f_0 is in the center). The Poisson algebra $(C^\infty(\mathbf{R}^3),\{,\})$ has a deformation quantization such that

$$\pi_j(x,y) = \pi_j(y,z) = \pi_j(z,x) = 0$$

for $j \geq 2$. The obtained quantized algebra is characterized by the relations

$$[x,y] = -\nu z^k, \quad [y,z] = -\nu x^l, \quad [z,x] = -\nu y^m$$

where $z^k = (z\cdot)^k = (z*)^k$ etc.

Ex.4. Let $x_1, x_2, \cdots, x_n$ be the coordinates on $\mathbf{R}^n$. For any skew symmetric matrix $(a_{ij})_{1\leq i,j\leq n}$ and for any positive integers $p_1,\cdots,p_n$, the relations

$$\{x_i,x_j\} = a_{ij}x_i^{p_i}x_j^{p_j}, \quad (1 \leq i,j \leq n)$$

define a Poisson algebra structure on $C^\infty(\mathbf{R}^n)$. If $p_1 = \cdots = p_n = 1$, then $(C^\infty(\mathbf{R}^n),\{,\})$ has a deformation quantization such that

$$\pi_k(x_i,x_j) = 0 \quad (1 \leq i,j \leq n) \quad \text{for } k \geq 2$$

or such that

$$\sum_{n=0}^{\infty} \nu^n \pi_n(x_i,x_j) = \sqrt{\frac{1-\frac{\nu}{2}a_{ij}}{1+\frac{\nu}{2}a_{ij}}}x_i x_j.$$

The latter relates to a noncommutative torus, for

$$x_i * x_j = \sqrt{\frac{1-\frac{\nu}{2}a_{ij}}{1+\frac{\nu}{2}a_{ij}}}x_i x_j, \quad x_j * x_i = \sqrt{\frac{1+\frac{\nu}{2}a_{ij}}{1-\frac{\nu}{2}a_{ij}}}x_i x_j,$$

hence $x_j * x_i = \frac{1+\frac{\nu}{2}a_{ij}}{1-\frac{\nu}{2}a_{ij}} x_i * x_j$. Thus, a noncommutative torus can be understood as a deformation quantization of Poisson algebra of this type.

Ex.5. Let $\mathfrak{g}$ be the algebra of the so called quantum group $Gl_q(2, R)$ (cf.[Wo],[D]). This is the algebra generated by x, y, u, v with the relations

$$\begin{cases} x * u = e^{\nu} u * x, \quad x * v = e^{\nu} v * x \\ u * y = e^{\nu} y * u, \quad v * y = e^{\nu} y * v \\ u * v = v * u \\ x * y - e^{\nu} u * v = y * x - e^{-\nu} u * v. \end{cases}$$

$\mathfrak{g}$ defines the structure of Poisson algebra on $C^{\infty}(M(2))$, where $M(2)$ is the space of 2×2 matrices, as follows:

$$\{x, u\} = xu, \quad \{x, v\} = xv, \quad \{x, y\} = 2uv,$$
$$\{u, v\} = 0, \quad \{u, y\} = uy, \quad \{v, y\} = vy.$$

This Poisson algebra $(C^{\infty}(M(2)), \{\ ,\ \})$ has a deformation quantization such that

$$\pi_m(linear function\,,\, linear function) = 0 \quad \text{for} \quad m \geq 2.$$

REFERENCES

[B] F. Bayen et al, *Deformation theory and quantization I*, Annals of Physics **111** (1978), 61-110.
[Be] F.A. Berezin, *Quantization*, Math. USSR Izvestija **8** (1974), 1109-1165.
[DL] M. De Wilde and P.B. Lecomte, *Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds*, Lett. Math. Phys. **7** (1983), 487-496.
[D] V.G. Drinfeld, *Quantum groups*, Proc. of ICM '86, Berkeley Calf. **1** (1987), 798-820.
[Ma] Y. Matsushima, *The theory of Lie algebra*, (in Japanese) Gendai Suugaku-kouza 15, Kyouritsu Press.
[Mc] S. MacLane, *Homology*, Springer, 1963.
[Mo] S. Moreno, **-products on some Kaehler manifolds*, Lett. Math. Phys. **11** (1986), 361-372.
[S] R. Schafer, *An introduction to non associative algebras*, Academic Press, 1976.
[OMY1] H. Omori, Y. Maeda and A. Yoshioka, *Weyl manifolds and deformation quantization*, Advances in Mathematics **85** (1991), 224-255.
[OMY2] H. Omori, Y. Maeda and A. Yoshioka, *Deformation quantization of Poisson algebras (preprint)*.
[W] A. Weinstein, *The local structure of Poisson manifolds*, J. Diff. Geom. **18** (1983), 523-557.
[Wo] S. L. Woronowicz, *Twisted SU(2) group. An example of non-commutative differential calculus*, Publ. RIMS, Kyoto Univ. **23** (1987), 117-181.