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Hölder estimates of solutions for difference-differential equations of elliptic-parabolic type

by

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1. Introduction

Let Ω be a bounded open set in the m-dimensional Euclidean space $R^m, m \geq 2$ and let T be a given positive number. For a positive integer N, we put

$$h = T/N \quad \text{and} \quad t_n = nh \ (n = 0,1,\cdots,N). \tag{1.1}$$

In this paper we shall consider a family of linear elliptic partial differential scalar-valued equations of the divergence form:

$$(w_n - w_{n-1})/h = \sum_{i,j=1}^{m} D_i(a_{ij}(x)D_j w_n) \ (1 \leq n \leq N) \ \text{in} \ \Omega, \tag{1.2}$$

where $D_i = \partial/\partial x_i (1 \leq i \leq m)$. We assume that the coefficients $a_{ij}(x)(1 \leq i,j \leq m)$ are measurable functions defined in Ω and fulfill the uniform ellipticity and boundedness condition with positive numbers λ and μ, $\lambda \leq \mu$:

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \eta|\xi|^2 \quad \text{for any} \quad \xi \in R^m \quad \text{and} \quad x \in \Omega. \tag{1.3}$$

Here and hereafter summation convention is used.

Around 1957, epoch-making papers [1] and [7] were published by E.DeGiorgi and J.Nash, each of which succeeded in achieving Hölder estimates of solutions for scalar-valued elliptic or parabolic partial differential equations of the divergence form with bounded and measurable coefficients. Since then, O.A.Ladyzenskaya and N.N.Ural'tseva have made it capable to deal with non-linear elliptic and parabolic ones([3],[4]),whose works have become classical results of great importance. On the other hand, J.Moser has proved in celebrated papers [5] and [6] that a Harnack inequality is valid for solutions of elliptic and parabolic differential equations with bounded and measurable coefficients.

The purpose of this paper is to extend some results of Hölder estimates obtained in [3] and [4] to the equations of the type (1.2), which we call difference- partial differential equations of the elliptic-parabolic type having the divergence form. In deriving Hölder estimates for solutions of such equations, we are essentially to rely on two theorems demonstrated by Ladyzhenskaya-Ural'tzeva. The reason is that the equations (1.2) represent the feature of elliptic or parabolic type, depending on whether a time discrete mesh h is relatively large or small compared with the size of considered local domain. From this standpoint we introduce later two function classes, variances of classical function classes stated in [3] and [4].

Standard notations

$$Q = Q_T = (0,T) \times \Omega, \tag{1.4}$$

$$\Gamma = \Gamma_T = \{(t,x): 0 < t < T, x \in \partial\Omega\} \cup \{(t,x): t = 0, x \in \Omega\},$$

where the set $\partial\Omega$ is the boundaray of Ω.

For a point $(t_0,x_0) \in Q$,we put

$$\begin{aligned} B_\rho(x_0) &= \{x \in R^m : |x-x_0| < \rho\}, \\ \Omega_\rho(x_0) &= \Omega \cap B_\rho(x_0) \end{aligned} \tag{1.5}$$

and

$$Q(\rho,\tau) = \{(t,x) \in Q : t_0 - \tau < t < t_0, x \in B_\rho(x_0)\}, \tag{1.6}$$

the set being called a local parabolic cylinder.

Throughout the paper we suppose the domain Ω is of the type A ([3]), that is, there exist two positive numbers ρ_0 and A, $0 < A < 1$, such that $|\Omega_\rho{}^i| \le A|B_\rho|$ for any ball B_ρ with center on $\partial\Omega$ of radius $\rho \le \rho_0$ and any connected component Ω_ρ^i of Ω_ρ. Here and hereafter, the notation $|\cdot|$ denotes an m-dimensional Lebesgue measure.

We adopt usual Sobolev spaces $W_2^1(\Omega)$ and $\overset{0}{W}{}_2^1(\Omega)$ from [3] and [4].

Let $\psi(t,x)$ be a Hölder continuous function on Γ with some positive numbers C and α_0, $0 < \alpha_0 < 1$, such that

$$|\psi(t,x)-\psi(s,y)| \le C(|t-s|^{\alpha_0/2}+ |x-y|^{\alpha_0}) \tag{1.7}$$

for any $(t,x),(s,y) \in \Gamma$.

We shall now define "a family of weak solutions for equation (1.2) with the prescribed boundary and initial date ψ" by a family of functions $w_n \in W_2^1(\Omega)(1 \le n \le N)$ which satisfy

$$\int_\Omega\{\frac{1}{h}(w_n - w_{n-1})\varphi + a_{ij}(x)D_jw_nD_i\varphi\}\, dx = 0(1 \le n \le N) \tag{1.8}$$

for any function $\varphi \in C_0^\infty(\Omega)$ and

$$\begin{aligned} &w_n = \psi_{t_n} \quad \text{on} \quad \partial\Omega(1 \le n \le N) \quad \text{in the trace sense,}\\ &w_0(x) = \psi(0,x) \quad \text{for} \quad x\in\Omega, \end{aligned} \tag{1.9}$$

where ψ_{t_n} $(1 \le n \le N)$ are sections such that

$$\psi_{t_n}(x) = \psi(t_n,x) \quad \text{for} \quad x \in \partial\Omega. \tag{1.10}$$

We set the function $w(t,x)$ by means of the equalities

$$\begin{aligned} &w(t,x) = w_n(x) \quad \text{for} \quad t_{n-1}< t \le t_n(1 \le n \le N),\\ &w(0,x) = \psi(x). \end{aligned} \tag{1.11}$$

Let θ, $0 < \theta < 1$, be a number to be determined later in Theorem P, not depending on h, while we take h in (1.1) sufficiently small satisfying

$$h < \min\{\theta^2, \theta^{-2/3}, 1/36\}. \tag{1.12}$$

We are now in a position to state our main result.

Theorem. Let $\{w_n\}(1 \le n \le N)$ be a family of weak solutions of (1.2) with the initial and boundary data ψ. Suppose that $\{w_n\}(1 \le n \le N)$ is equi-bounded:

$$\max_{1\le n\le N} \sup_{x\in\Omega}|w_n(x)| \le M \tag{1.13}$$

for some positive constant M.

Then there exist positive numbers C and $\alpha, 0<\alpha<1$, depending only on λ,μ,m and M, such that

$$\operatorname{osc}\{w_n:\Omega_\rho\} \le C\rho^{\alpha}(1 \le n \le N) \tag{1.14}$$

for any positive ρ and

$$|w_n(x)-w_{n'}(x)| \le C\{(n-n')h\}^{\alpha/4} \tag{1.15}$$

for any positive integer n and $n',(n - n')h < 1, 1 \le n' < n \le N$ and any $x \in \Omega$.

To make our proof more clear, we have confined ourselves to treating only linear equations of type (1.2). However, the similar assertion might hold for solutions of the equations (1.2) with a quadratic nonlinear term of the gradients in the right hand side. For linear equations with the initial and boundary data (1.9), the estimate (1.13) shall be derived in the usual way of maximum principle (see [4]).

We should here notice that (1.14) and (1.15) are uniform Hölder estimates with respect to h.

2. Preliminaries

In this section we desplay only a variation of results obtained by O.A.Ladydzenskaya and N.N.Ural'tseva(refer to [3],[4]), which is indispensable to our reasoning. First we recall two function spaces from [3] and [4].

Let f be an integrable function defined on Ω or Q. For measurable sets $\tilde{\Omega} \subset \Omega$ and $\tilde{Q} \subset Q$, we use the usual notations :

$$\|f\|_{2,\tilde{\Omega}} = (\int_{\tilde{\Omega}} |f|^2 dx)^{1/2}, \qquad \|\nabla f\|_{2,\tilde{Q}} = \iint_{\tilde{Q}} |\nabla f|^2 dxdt)^{1/2} \tag{2.1}$$

and

$$\mathrm{osc}\{f:\tilde{\Omega}\} = \sup\{f(x):x \in \tilde{\Omega}\} - \inf\{f(x):x \in \tilde{\Omega}\}, \qquad \mathrm{osc}\{f:\tilde{Q}\} = \sup\{f(t,x):(t,x) \in \tilde{Q}\}-\inf\{f(t,x):(t,x) \in \tilde{Q}\}, \tag{2.2}$$

where

$$\nabla f = (D_1 f,\ D_2 f, \cdots,\ D_m f), \qquad |\nabla f|^2 = (\sum_{i=1}^{m} |D_i f|^2)^{1/2}. \tag{2.3}$$

For a number k, we put

$$f^{(k)}(t,x) = \max\{f(t,x)-k,0\}, \qquad f^{(k)}(x) = \max\{f(x)-k,0\}. \tag{2.4}$$

$W_2^{1,0}(Q)$ is the Hilbert space with scalar product

$$(f,g)_{W_2^{1,0}(Q)} = \iint_Q \{fg+(\nabla f,\nabla g)\}dxdt, \tag{2.5}$$

where

$$(\nabla f,\nabla g) = \sum_{i=1}^{m} D_i f D_i g. \tag{2.6}$$

$V_2(Q)$ is the Banach space consisting of all elements of $W_2^{1,0}(Q)$ having a finite norm

$$|f|_Q = \sup\{||f(t,\cdot)||_{2,\Omega}:\ 0 < t < T\} + ||\nabla f||_{2,Q}. \tag{2.7}$$

In accordance with this, we handle local norms

$$|f|_{Q(\rho,\tau)} = \sup\{||f(t,\cdot)||_{\Omega_\rho(x_0)}: t_0-\tau < t < t_0\} + ||\nabla f||_{Q(\rho,\tau)}, \tag{2.8}$$

where $Q(\rho,\tau)$ is a local parabolic cylinder defined in (1.6).

Let W, γ, and δ be positive numbers and take positive numbers ρ_0 and τ_0. We say a function $f(t,x)$ belongs to a function space

$B_2(Q,W,\gamma,2(1+2/m),\delta,2/m)$(see to [4]),if $f(t,x)$ satisfies the conditions 1),2) and 3):

1) $f \in V_2(Q)$, (2.9)

2) $\sup\{|f(t,x)|:(t,x) \in Q\} \le W$, (2.10)

3) the functons $g = \pm f$ satisfy the following:

$$\sup\{||g^{(k)}(t,\cdot)||^2_{2,\Omega_{\rho-\sigma_1\rho}} : t_0-\tau < t < t_0\}$$

$$\le ||g^{(k)}(t_0-\tau,\cdot)||^2_{2,\Omega_\rho}+ \gamma\{(\sigma_1\rho)^{-2}||g^{(k)}||^2_{2,Q(\rho,\tau)}+\int_{t_0-\tau}^{t_0} |A_{k,\rho}(t)|dt\} \quad (2.11)$$

and

$$|g^{(k)}|^2_{Q(\rho-\sigma_1\rho,\tau-\sigma_2\tau)} \le\gamma[\{(\sigma_1\rho)^{-2}+ (\sigma_2\tau)^{-1}\}||g^{(k)}||^2_{2,Q(\rho,\tau)} + \int_{t_0-\tau}^{t_0} |A_{k,\rho}(t)|dt], \quad (2.12)$$

in which $Q(\rho,\tau)$ is any local parabolic cylinder; ρ and τ are arbitrary positive numbers satisfying $\rho \le \rho_0$ and $\tau \le \tau_0$; σ_1 and σ_2 are arbitrary numbers from the interval (0,1); k is an arbitrary number subject only to the condition: if $\bar{Q}(\rho,\tau) \subset Q$,

$$k \ge \sup\{g:Q(\rho,\tau)\} - \delta \text{ and } |k| \le W, \quad (2.13)$$

or if $\bar{Q}(\rho,\tau) \cap \Gamma \ne \phi$,

$$k > \sup\{g:\bar{Q}(\rho,\tau) \cap \Gamma\}; \quad (2.14)$$

$$A_{k,\rho}(t) = \{x \in \Omega_\rho(x_0);g(t,x) > k\}. \quad (2.15)$$

Let W, γ and δ be positive numbers and take a positive number ρ_0^*. We say a function $f(x)$ belongs to a function space $B_2(\Omega,W,\gamma,\delta,1/2m)$ (see to [3]), if $f(x)$ satisfies the following 1*), 2*) and 3*):

1*) $f \in W_2^1(\Omega)$, (2.16)

2*) $\sup\{|f(x)|: x \in \Omega\} \le W$, (2.17)

3*) the functions $g(x) = \pm f(x)$ satisfy the following:

$$\int_{A_{k,\rho-\sigma\rho}} |\nabla g|^2dx \le \gamma \{\sigma^{-2}\rho^{-1} \sup_{A_{k,\rho}}\{(g(x)-k)^2+1\}|A_{k,\rho}|^{1-1/m}, \quad (2.18)$$

in which ρ is an arbitrary positive number satisfying $\rho \le \rho_0^*$; σ is any positive number from the interval $(0,1)$; k is an arbitrary number subject only to the condition: if $B_\rho \subset \Omega$,

$$k \geq \sup \{|g(x)| : x \in B_\rho\} - \delta \quad \text{and} \quad |k| \leq W \tag{2.19}$$

or if $B_\rho \cap \partial\Omega \neq \phi$,

$$k > \sup \{|g(x)| : x \in B_\rho \cap \partial\Omega\}; \tag{2.20}$$

$$A_{k,\rho} = \{x \in \Omega(x_0): g(x) > k\}. \tag{2.21}$$

For the above two function spaces, there have been proved the followings(refer to [3],[4]):

Theorem P. If $f(t,x) \in B_2(Q,W,\gamma,2(1+2/m),\delta,2/m)$ satisfies a Hölder estimate on Γ_T: there exist some positive ε, ρ_1 and L such that

$$\operatorname{osc} \{f:\bar{Q}(\rho,\rho^2) \cap \Gamma_T\} \leq L\rho^{\varepsilon} \tag{2.22}$$

holds for any $0 < \rho \leq \rho_1$, then $f(t,x)$ satisfies a Hölder estimate in $\bar{Q}_T$. To be exact,

$$\operatorname{osc} \{f: \bar{Q}(\rho,\theta\rho^2)\} \leq C(\rho/\bar{\rho})^{\beta} \tag{2.23}$$

for

$$\rho \leq \bar{\rho} := \min \{\rho_0, \rho_1, \sqrt{\tau_0/\theta},\ \rho_1/\sqrt{\theta}\}, \tag{2.24}$$

where C,β,s,b and θ are positive numbers such that

$$\begin{aligned} &C = b^{\beta}\max\{\bar{w}, 2^{s}\bar{\rho}^{\beta}\},\ \bar{w} = \operatorname{osc}\{f:Q(\bar{\rho},\theta\bar{\rho}^2)\}, \\ &\beta = \beta(m,\gamma,L,\varepsilon,A),\ s = s(m,\gamma,\varepsilon,L), \\ &b > 1,\ \theta = \theta(m,\gamma,A). \end{aligned} \tag{2.25}$$

With a little modification, we may assume, instead of (2.12),

$$\begin{aligned} |g^{(k)}|^2_{Q(\rho-\sigma_1\rho,\tau-\sigma_2\tau)} \leq{} & \gamma\{(\sigma_1\rho)^{-2} + (\sigma_2\tau)^{-1}\}||g^{(k)}||^2_{2,Q(\rho,\tau)} \\ & + \sigma_2^{-2}\int_{t0-\tau}^{t_0} |(A_{k,\rho}(t)|dt \end{aligned} \tag{2.26}$$

for $\rho > \rho_2$, $\tau > \tau_2$ with some positive numbers ρ_2 and τ_2 such that $\rho_2 < \min\{ \rho_0,\rho_1\}$, $\tau_2 < \tau_0$ and for k as in (2.13) and (2.14). In fact, if, further,

$$\bar{\rho} := \min\{\rho_0, \rho_1, \sqrt{\tau_0/\theta}, \rho_1/\sqrt{\theta}\} > \max\{\rho_2, \sqrt{\tau_2/\theta}\} =: \underline{\rho}, \tag{2.27}$$

then estimate (2.23) does hold for ρ under the restriction

$$\underline{\rho} \le \rho \le \bar{\rho} . \tag{2.28}$$

Theorem E. If $f(x) \in B_2(\Omega, W, \gamma, \delta, 1/2m)$ satisfies a Hölder estimate on $\partial\Omega$: there exist some positive $\rho_1^*, \varepsilon^*,$ $0 < \varepsilon^* < 1$ and L^* such that

$$\operatorname{osc}\{f : B_\rho \cap \partial\Omega\} \le L^* \rho^{\varepsilon^*} \tag{2.29}$$

holds for any $0 < \rho \le \rho_1^*$, then $f(x)$ satisfies, for any $0 < \rho \le \bar{\rho}^* := \min\{\rho_0^*, \rho_1^*\}$,

$$\operatorname{osc}\{f : \bar{\Omega}_\rho\} \le C^* (\rho/\bar{\rho}^*)^{\beta^*}, \tag{2.30}$$

where C^*, β^*, s^* and b^* are positive numbers such that

$$C^* = b^{*\beta} \max\{\bar{w}^*, 2^s (\bar{\rho}^*)^{\beta^*}\}, \quad \bar{w}^* = \operatorname{osc}\{f : \Omega_{\bar{\rho}^*}\},$$

$$b^* > 1, \quad s^* = s^*(m, \gamma, \varepsilon^*, L), \quad \beta^* = \beta^*(m, \gamma, L^*, \varepsilon^*, A). \tag{2.31}$$

In order to treat solutions of equations (1.2), we now introduce two function spaces.

Let W, γ and δ be positive numbers and take positive numbers ρ_0 and τ_0. We shall say a function $f(t,x)$ belongs to a function class $B_2^h(Q, W, \gamma, 2(1+2/m), \delta, 2/m)$ if $f(t,x)$ satisfies (2.9) and (2.10) and, further, the functions $g(t,x) = \pm f(t,x)$ satisfy (2.11) and (2.26) for any ρ and τ with the restriction $0 < \rho \le \rho_0$, $\sqrt{h} \le \tau \le \tau_0$, and all $\sigma_1, \sigma_2 \in (0,1)$.

Let W, γ and δ be positive numbers and take positive numbers ρ_0^*. We shall say a function $f(x)$ belongs to a function class $B_2^h(\Omega, W, \gamma, \delta, 1/2m)$ if $f(x)$ satisfies (2.16) and (2.17) and, further, the functions $g(x) = \pm f(x)$ satisfy (2.18) for any positive ρ with the restriction $\rho < \rho_0^*$ and $\rho \le h$.

For the above function spaces, we have the following theorem, a variance of Theorem P. We now recall $w(t,x)$ is a function defined in (1.11).

Theorem P_h. Let $w(t,x)$ belong to the function class $B_2^h(Q,W,\gamma,2(1+2/m),\delta,2/m)$. If $w(t,x)$ satisfies a Hölder estimate on Γ_T: there exist some positive numbers ρ_1, ε, $0 < \varepsilon < 1$ and L such that

$$\operatorname{osc}\ \{\psi:\bar{Q}(\rho,\rho^2) \cap \Gamma_T\} \le L\rho^{\varepsilon} \tag{3.32}$$

holds for any positive $\rho \le \rho_1$, then $w(t,x)$ satisfies the estimate

$$\operatorname{osc}\ \{w:\bar{Q}(\rho,\theta\rho^2)\} \le C(\rho/\bar{\rho})^{\beta} \tag{2.33}$$

for any positive ρ satisfying $\theta^{-1/2}h^{1/4} \le \rho \le \bar{\rho}$, where C,θ,β are positive numbers defined as in (2.25) and $\bar{\rho}$ is a positive number defined as in (2.24).

For the proof of Theorem P_h we have only to make the following observation. We follow the proof of Theorem due to Ladyzhenskaya and Ural'tseva ([4]) with the use of local parabolic cylinders of only the special type $\Omega_\rho(x_0) \times (t_n - \tau, t_n)$, $x_0 \in \Omega, 1 \le n \le N$. Since (2.11) and (2.12) are satisfied for any positive τ, $\tau \ge \sqrt{h}$, it follows from Theorem P that (2.33) holds for any ρ, $\rho \ge \theta^{-1/2}h^{1/4}$.

3. Two Lemmata

The proof of our Theorem rests on the following two Lemmata, which are obtained by making combination of Theorem P with E.

Let $\{w_n\}(1 \le n \le N)$ be a family of weak solutions for (1.2) satisfying the prescribed data (1.9) and the condition (1.13). Defining a function $w(t,x)$ by the relation (1.11), we have the following Lemmata. Let δ be a positive number.

Lemma 1. There exists a positive number γ independent of h such that $w(t,x)$ belongs to $B_2^h(Q,M,\gamma,2(1+2/m),\delta,2/m)$.

Lemma 2. There exists a positive number γ independent of h such that $\{w_n(x)\}(1 \le n \le N)$ belong to $B_2^h(\Omega,M,\gamma,\delta,1/2m)$.

Proof of Lemma 1

According to the assumption on w_n, one finds that $w(t,x)$ satisfies the conditions (2.9) and (2.10) with $W = M$. Hence it remain to verify that $w(t,x)$ satisfies the conditions (2.11) and (2.26) with the restriction $\tau \ge \sqrt{h}$.

Let x_0 be any point in Ω and $t_n(1 \le n \le N)$ be a point defined in (1.1) and put $Q(\rho,\tau) = \Omega_\rho(x_0) \times (t_n - \tau, t_n)$. At first, we shall begin by considering the case: $Q(\rho,\tau) \subset Q$.

Defining a scalar-valued function $\xi \in C_0^\infty(R^n)$, for σ_1, $0 < \sigma_1 < 1$,

$$\xi(x) = \begin{cases} 1 & \text{for } |x - x_0| < (1 - \sigma_1)\rho, \\ 0 & \text{for } |x - x_0| > \rho, \end{cases} \tag{3.1}$$

$$0 \le \xi(x) \le 1, \quad |\nabla\xi(x)| \le 2/\sigma_1\rho.$$

We choose $hw_n^{(k)}\xi^2$ as a test function in (1.8). After thus, sum the resultant equality over n from n_1 to n_2 and one sees

$$\sum_{n=n_1}^{n_2} \int_{B_\rho} (w_n - w_{n-1}) w_n^{(k)} \xi^2 dx$$
$$= -h \sum_{n=n_1}^{n_2} \int_{B_\rho} a_{ij}(x) D_j w_n^{(k)} D_i w_n^{(k)} \xi^2 dx - 2h \sum_{n=n_1}^{n_2} \int_{B_\rho} a_{ij}(x) D_j w_n^{(k)} w_n^{(k)} \xi D_j \xi dx. \tag{3.2}$$

By virtue of

$$w_{n-1} - k \le w_{n-1}^{(k)} \quad \text{and} \quad (w_n - k) w_n^{(k)} = (w_n^{(k)})^2, \tag{3.3}$$

$$(w_n^{(k)} - w_{n-1}^{(k)})w_n^{(k)} \geq \{(w_n^{(k)})^2 - (w_{n-1}^{(k)})^2\}/2,$$

we have

$$\sum_{n=n_1}^{n_2} \int_{B_\rho} (w_n - w_{n-1}) w_n^{(k)} \xi^2 dx$$
$$= \sum_{n=n_1}^{n_2} \int_{B_\rho} \{(w_n - k) - (w_{n-1} - k)\} w_n^{(k)} \xi^2 dx$$
$$\geq \sum_{n=n_1}^{n_2} \int_{B_\rho} (w_n^{(k)} - w_{n-1}^{(k)}) w_n^{(k)} \xi^2 dx$$
$$\geq \frac{1}{2} \sum_{n=n_1}^{n_2} \int_{B_\rho} \{(w_n^{(k)})^2 - (w_{n-1}^{(k)})^2\} \xi^2 dx$$
$$= \frac{1}{2} \int_{B_\rho} (w_{n_2}^{(k)})^2 \xi^2 dx - \frac{1}{2} \int_{B_\rho} (w_{n_1-1}^{(k)})^2 \xi^2 dx. \tag{3.4}$$

On the other hand, it follows from Young's inequality that

$$-h \sum_{n=n_1}^{n_2} \int_{B_\rho} a_{ij}(x) D_j w_n^{(k)} D_i w_n^{(k)} \xi^2 dx \; -2h \sum_{n=n_1}^{n_2} \int_{B_\rho} a_{ij}(x) D_j w_n^{(k)} w_n^{(k)} \xi D_j \xi dx$$
$$\leq -\lambda h \sum_{n=n_1}^{n_2} \int_{B_\rho} |\nabla w_n^{(k)}|^2 \xi^2 dx \; - 2h \sum_{n=n_1}^{n_2} \int_{B_\rho} a_{ij}(x) D_j w_n^{(k)} w_n^{(k)} \xi D_j \xi dx$$
$$\leq -\frac{\lambda}{2} h \sum_{n=n_1}^{n_2} \int_{B_\rho} |\nabla w_n^{(k)}|^2 \xi^2 dx + 2\lambda^{-1} \mu h \sum_{n=n_1}^{n_2} \int_{B_\rho} (w_n^{(k)})^2 |\nabla \xi|^2 dx$$
$$\leq -\frac{\lambda}{2} h \sum_{n=n_1}^{n_2} \int_{B_\rho} |\nabla w_n^{(k)}|^2 \xi^2 dx + 8\lambda^{-1} \mu (\sigma_1 \rho)^{-2} h \sum_{n=n_1}^{n_2} \int_{B_\rho} (w_n^{(k)})^2 dx. \tag{3.5}$$

Combination of (3.4) and (3.5) with (3.2) gives the estimate

$$\int_{B_\rho} (w_{n_2}^{(k)})^2 \xi^2 dx + \lambda h \sum_{n=n_1}^{n_2} \int_{B_\rho} |\nabla w_n^{(k)}|^2 \xi^2 dx$$
$$\leq \int_{B_\rho} (w_{n_1-1}^{(k)})^2 \xi^2 dx + 16\lambda^{-1} \mu (\sigma_1 \rho)^{-2} h \sum_{n=n_1}^{n_2} \int_{B_\rho} (w_n^{(k)})^2 dx. \tag{3.6}$$

The assumption $\tau \geq \sqrt{h}$ and $0 < h \leq 1$ imply $\tau \geq h$, thereby,

$$n_0 - [\tau/h] + 1 \leq n_0,$$

where $[\cdot]$ means, by convention, Gaussian symbol. Picking n_1 and n_2 n_2 in (3.6) as

$$n_1 = n_0 - [\tau/h] + 1, \quad n_0 - [\tau/h] + 1 \leq n_2 \leq n_0,$$

we have for $n_0 - [\tau/h] + 1 \le n_2 \le n_0$,

$$\int_{B_{\rho-\sigma_1\rho}} (w_{n_2}^{(k)})^2 dx \le \int_{B_\rho} (w_{n_0-[\tau/h]}^{(k)})^2 dx + 16\lambda^{-1}\mu(\sigma_1\rho)^{-2} h \sum_{n=n_1}^{n_2} \int_{B_\rho} (w_n^{(k)})^2 dx. \qquad (3.7)$$

While the inequality (3.7) with $n_2 = n_0 - [\tau/h]$ being trivially valid, we arrive at the estimate

$$\max_{n_0-[\tau/h]\le n\le n_0} \int_{B_{\rho-\sigma_1\rho}} (w_n^{(k)})^2 dx$$
$$\le \int_{B_\rho} (w_{n_0-[\tau/h]}^{(k)})^2 dx + 16\lambda^{-1} \mu(\sigma_1\rho)^{-2} h \sum_{n=n_1}^{n_2} \int_{B_\rho} (w_n^{(k)})^2 dx. \qquad (3.8)$$

Based on the definition of $w(t,x)$, we rewrite (3.8) in the form:

$$\sup_{t_{n_0}-\tau\le t\le t_{n_0}} \int_{B_{\rho-\sigma_1\rho}} (w^{(k)}(t,x))^2 dx$$
$$\le \int_{B_\rho} (w^{(k)}(t_{n_0} - \tau,x))^2 dx + 16\lambda^{-1}\mu(\sigma_1\rho)^{-2} \int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_\rho} (w^{(k)}(t,x))^2 dxdt. \qquad (3.9)$$

Consequently, $w(t,x)$ satisfies (2.11) with $\gamma = 16\lambda^{-1}\mu$.

We shall next show that $w(t,x)$ satisfies (2.26) for any τ with the relation $\tau \ge \sqrt{h}$. The assumption $\tau \ge \sqrt{h}$, together with (1.12), implies

$$\tau \ge \sqrt{h} > 6h. \qquad (3.10)$$

We now distinguish the following cases:

$$\sigma_2\tau \ge 4h, \qquad (3.11)$$

$$\sigma_2\tau < 4h. \qquad (3.12)$$

At first, we shall start from the case (3.11). Let $\xi(x)$ be the function defined as in (3.1).

For n_0, $1 \le n_0 \le N$, we introduce a step function $\eta(t)$ on $[0,T]$ as follows:

$$\eta(t) := \eta_n \qquad \text{for } t_{n-1} < t \le t_n (1 \le n \le N), \qquad (3.13)$$

$$\eta_n := \begin{cases} 1 & \text{for } n_0-[(1-\sigma_2)\tau/h] \le n \le n_0, \\ \{n-n_0+[\tau/h]-1\}/\{[\tau/h]-2-[(1-\sigma_2)\tau/h]\} & \\ \qquad \text{for } n_0-[\tau/h]+1 \le n \le n_0-[(1-\sigma_2)\tau/h]-1, & \\ 0 & \text{for } n \le n_0-[\tau/h]. \end{cases}$$

Particularly, in view of (3.11),

$$0 \le \eta_n - \eta_{n-1} \le 1/\{[\tau/h] - 2 - [(1-\sigma_2)\tau/h]\} \le 4h/\sigma_2\tau \quad (1 \le n \le N). \tag{3.14}$$

Under these preparations, we now take $w^{(k)}\xi^2\eta$ as a test function in (1.8) and then integrate the resultant equality over $[t_{n_0}-\tau, t_{n_0}]$,so that

$$\int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_\rho} \frac{w(t,x)-w(t-h,x)}{h}\, w^{(k)}(t,x)\xi^2(x)\eta(t)dxdt = -\int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_\rho} a_{ij}(x)\, D_j w D_i(w^{(k)}\xi^2)\eta dxdt. \tag{3.15}$$

We estimate the left-hand side of (3.15). From the definition (3.13) of $\eta(t)$, we have,by making a calculation analogous to that in (3.4),

$$\begin{aligned} &\int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_\rho} \frac{w(t,x)-w(t-h,x)}{h}\, w^{(k)}(t,x)\xi^2(x)\eta(t)dxdt \\ &\ge \int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_\rho} \frac{w^{(k)}(t,x)-w^{(k)}(t-h,x)}{h}\, w^{(k)}(t,x)\xi^2(x)\eta(t)dxdt \\ &= \sum_{n=n_0-[(1-\sigma_2)\tau/h]+1}^{n_0} \int_B (w_n^{(k)} - w_{n-1}^{(k)})w_n^{(k)}\xi^2 dx \\ &\quad + \sum_{n=n_0-[\tau/h]+1}^{n_0-[(1-\sigma_2)\tau/h]} \int_{B_\rho} (w_n^{(k)} - w_{n-1}^{(k)})w_n^{(k)}\eta_n\xi^2 dx, \end{aligned} \tag{3.16}$$

observing that $\eta_{n-[\tau/h]} = 0$, and, from (3.11), $n_0 - [(1-\sigma_2)\tau/h]-(n_0-[\tau/h]+1) \ge \sigma_2\tau/h \ge 4$. As seen in (3.6),

$$\begin{aligned} &\sum_{n=n_0-[(1-\sigma_2)\tau/h]+1}^{n_0} \int_{B_\rho} (w_n^{(k)} - w_{n-1}^{(k)})w_n^{(k)}\xi^2 dx \\ &\ge \frac{1}{2}\int_{B_\rho} (w_{n_0}^{(k)})^2\xi^2 dx - \frac{1}{2}\int_{B_\rho} (w_{n_0-[(1-\sigma_2)\tau/h]}^{(k)})^2\xi^2 dx. \end{aligned} \tag{3.17}$$

Because of (3.14) and the identity

$$(a_n - a_{n-1})b_n = (a_nb_n - a_{n-1}b_{n-1}) - a_{n-1}(b_n - b_{n-1}), \tag{3.18}$$

we have

$$\sum_{n=n_0-[\tau/h]+1}^{n_0-[(1-\sigma_2)\tau/h]} \int_{B_\rho}(w_n^{(k)} - w_{n-1}^{(k)})w_n^{(k)}\eta_n\xi^2dx$$
$$\geq \frac{1}{2}\sum_{n=n_0-[\tau/h]+1}^{n_0-[(1-\sigma_2)\tau/h]} \int_{B_\rho}\{(w_n^{(k)})^2-(w_{n-1}^{(k)})^2\}\eta_n\xi^2dx$$
$$= \frac{1}{2}\sum_{n=n_0-[\tau/h]+1}^{n_0-[(1-\sigma_2)\tau/h]} \int_{B_\rho}\{(w_n^{(k)})^2\eta_n-(w_{n-1}^{(k)})^2\eta_{n-1}\}\ \xi^2dx$$
$$-\frac{1}{2}\sum_{n=n_0-[\tau/h]+1}^{n_0-[(1-\sigma_2)\tau/h]} \int_{B_\rho}(w_{n-1}^{(k)})^2(\eta_n-\eta_{n-1})\xi^2\ dx$$
$$\geq \frac{1}{2}\int_{B_\rho}(w^{(k)}_{n_0-[(1-\sigma_2)\tau/h]})^2\xi^2\ dx$$
$$- 2(\sigma_2\tau)^{-1}h\sum_{n=n_0-[\tau/h]}^{n_0-[(1-\sigma_2)\tau/h]-1}\int_{B_\rho}(w_n^{(k)})^2\xi^2\ dx\ . \tag{3.19}$$

Hence, substituting estimates (3.17) and (3.19) into (3.16) yields

$$\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho}\frac{w(t,x)-w(t-h,x)}{h}\ w^{(k)}(t,x)\xi^2(x)\eta(t)dxdt$$
$$\geq \frac{1}{2}\int_{B_{\rho-\sigma_1\rho}}(w_{n_0}^{(k)})^2dx - 2(\sigma_2\tau\ )^{-1}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho}(w^{(k)})^2dxdt. \tag{3.20}$$

We also have

$$-\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho}a_{ij}(x)D_jwD_i(w^{(k)}\xi^2)\eta dxdt$$
$$\leq -\frac{\lambda}{2}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho}|\nabla w^{(k)}|^2\xi^2dxdt$$
$$+8\lambda^{-1}\mu(\sigma_1\rho)^{-2}\int_{t_n-\tau}^{t_{n_0}}\int_{B_\rho}(w_n^{(k)})^2dxdt. \tag{3.21}$$

Substitution (3.20) and (3.21) into (3.15) leads to the inequalities

$$\lambda\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_{\rho-\sigma_1\rho}}|\nabla w^{(k)}|^2dxdt$$

$$\leq \max\{4, 16\lambda^{-1}\mu\}\{(\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1}\}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho}(w^{(k)}(t,x))^2 dxdt \qquad (3.22)$$

and similarly

$$\sup_{t_{n_0}-(1-\sigma_2)\tau\leq t\leq t_{n_0}}\int_{B_{\rho-\sigma_1\rho}}(w^{(k)}(t,x))^2 dx$$

$$\leq \max\{4, 16\lambda^{-1}\mu\}\{(\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1}\}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho}(w^{(k)}(t,x))^2 dxdt. \qquad (3.23)$$

(3.22) and (3.23) imply that w satisfies (3.26) as long as $\tau \geq \sqrt{h}$ and $\sigma_2\tau \geq 4h$.

We shall next deal with the case (3.12):$\sigma_2\tau < 4h$. From this assumption with $\tau \geq \sqrt{h}$,

$$\sqrt{h} \leq \tau < 4h/\sigma_2 \qquad (3.24)$$

and therefore

$$1 < 16h\sigma_2^{-2}. \qquad (3.25)$$

From the uniform boundedness of w_n, $|w(t,x)| \leq M$, (2.13) and (3.25),

$$\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho}\frac{w(t,x)-w(t-h,x)}{h}\, w^{(k)}(t,x)\ \xi^2(x)dxdt$$

$$\leq 32\delta\sigma_2^{-2}M\int_{t_{n_0}-\tau}^{t_{n_0}}|A_{k,\rho}(t)|\ dt, \qquad (3.26)$$

where

$$A_{k,\rho}(t) = \{x \in B_\rho : w(t,x) > k\}. \qquad (3.27)$$

Putting $\eta(t) \equiv 1$ in (3.15) and bearing in mind (3.21) and (3.26), we have (3.22).

Let us now calculate the term

$$\int_{B_{\rho-\sigma_1\rho}}(w_n^{(k)})^2 dx \quad \text{for} \quad n_0 - [(1-\sigma_2)\tau/h] \leq n \leq n_0.$$

To this end, we classify the cases:

$$[(1-\sigma_2)\tau/h] \geq 1, \qquad (3.28)$$

$$[(1 - \sigma_2)\tau/h] = 0). \tag{3.29}$$

We first treat the case (3.28). With the aid of (3.25) and (2.13) in $Q(\rho,\rho^2)$, we have for $n_0 - [(1-\sigma_2)\tau/h] \le n \le n_0$

$$\int_{B_{\rho-\sigma_1\rho}} (w_n^{(k)})^2 dx \le 16\sigma_2^{-2} h \int_{B_{\rho(1-\sigma_1)}} (w_n^{(k)})^2 dx$$

$$\le 16\delta^2\sigma_2^{-2} h |A_{k,\rho}^n| \le 16\delta^2\sigma_2^{-2} h \sum_{n=n_0-[(1-\sigma_2)\tau/h]}^{n_0} |A_{k,\rho}^n|$$

$$\le 16\delta^2 \sigma_2^{-2} \int_{t_{n_0}-\tau}^{t_{n_0}} |A_{k,\rho}(t)| dt, \tag{3.30}$$

where

$$A_{k,\rho}^n = \{x \in B_\rho : w_n(x) > k\} \text{ and } A_{k,\rho}(t) = \{x \in B_\rho : w(t,x) > k\}. \tag{3.31}$$

We next handle the case (3.31). In order to do this, we distinguish the sub-cases:

$$[\tau/h] > [(1 - \sigma_2)\tau/h], \tag{3.32}$$

$$[\tau/h] = [(1 - \sigma_2)\tau/h]). \tag{3.33}$$

In case (3.32): Since $[\tau/h] \ge 1$,

$$t_{n_0} - (t_{n_0} - \tau) = \tau \ge [\tau/h]h \ge h.$$

Namely, by (3.25), we have

$$\int_{B_{\rho(1-\sigma_1)}} (w_{n_0}^{(k)})^2 dx \le 16\sigma_2^{-2} h \int_{B_{\rho(1-\sigma_1)}} (w_{n_0}^{(k)})^2 dx \tag{3.34}$$

$$\le 16\delta^2\sigma_2^{-2} h |A_{k,\rho}^{n_0}| \le 16\delta^2\sigma_2^{-2} \int_{t_{n_0}-\tau}^{t_{n_0}} |A_{k,\rho}(t)| dt.$$

In case (3.33): From $[\tau/h] = [(1 - \sigma_2)\tau/h]$, we have ,for n, n_0 - $n_0 - [(1-\sigma_2)\tau/h] \le n \le n_0$,

$$\int_{B_{\rho(1-\sigma_1)}} (w_n^{k)})^2 dx \le \tau^{-1} \int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_{\rho(1-\sigma_1)}} (w_n^{(k)})^2 dx dt \tag{3.35}$$

Combining (3.34) with (3.35), we obtain

$$\int_{B_{\rho(1-\sigma_1)}}(w^{(k)}(t,x))^2dx \le (\sigma_2\tau)^{-1}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho}(w^{(k)})^2dxdt + 16\delta^2\sigma_2^{-2}\int^{t_{n_0}}|A_{k,\rho}(t)|dt \tag{3.36}$$

for $t_{n_0} - (1 - \sigma_2)\tau \le t \le t_{n_0}$.

Collecting (3.21),(3.26) and (3.36), we have (3.23).

As rergards the case:$Q(\rho,\tau)\cap\Gamma_T \ne \phi$, in all the same way as just described, we can prove that $w(t,x)$ satisfies (2.11) and (2.26) for

$k > \max \{w:\bar{Q}(\rho,\tau) \cap \Gamma \}$.

So we can for $-w(t,x)$. Hence we have completed the proof of Lemma 1.

Proof of Lemma 2

From the definition of the weak solutions of (1.2) and assumptions (1.8),$w_n(x)$ satisfies (2.16) and (2.17). It remains only to show (2.18) with $\rho \le h$.

At first,we consider the case $B_\rho \subset \Omega$. Let ξ be a function defined by (3.1) and k a positive number subject to (2.19). In the relation (1.8), we insert $w_n^{(k)}\xi^2 \in \overset{0}{W}{}_2^1(\Omega)$ into test function φ to obtain

$$\int_\Omega\{\frac{w_n - w_{n-1}}{h} w_n^{(k)}\xi^2 + a_{ij}(x)D_jw_nD_i(w_n^{(k)}\xi^2)\}dx = 0, \tag{3.37}$$

where $w^{(n)}$ is the function defined as in (2.4). By the assumptions $\rho \le h$, we infer, κ_m being the area of the unit sphere in R^m,

$$1/h \le 1/\rho = \kappa_m^{1/m}|B_\rho|^{-1/m}, \tag{3.38}$$

By virtue of $|w_n| \le M$ and $|w_n^{(k)}| \le \delta$ in B_ρ,

$$|\int_{B_\rho}\frac{w_n - w_{n-1}}{h} w_n^{(k)}\xi^2dx| \le 2\delta Mh^{-1}\int_{A^n_{k,\rho}}\xi^2dx$$

$$\leq 2\delta M\ \kappa_m^{1/m}|B_\rho|^{-1/m}|A^n_{k,\rho}|, \tag{3.39}$$

where $A^n_{k,\rho}$ is the set defined in (3.31).

On the other hand,by making use of Young's inequality, we estimate the second term of (3.37):

$$\int_{B_\rho} a_{ij}(x)D_j w_n D_i(w_n^{(k)}\xi^2)dx$$
$$\geq \frac{\lambda}{2}\int_{B_\rho}|\nabla w_n^{(k)}|^2\xi^2 dx - 16\lambda^{-1}\mu(\sigma\rho)^{-2}\int_{B_\rho}(w_n^{(k)})^2 dx. \tag{3.40}$$

Consequently, from (3.41),(3.43) and (3.44),

$$\int_{B_{\rho-\sigma_1\rho}}|\nabla w_n^{(k)}|^2 dx \leq 32\lambda^{-2}\mu(\sigma\rho)^{-2}\int_{B_\rho}(w_n^{(k)})^2 dx$$
$$+ 4\delta\lambda^{-1}M\kappa_m^{1/m}|B_\rho|^{-1/m}\ |A^n_{k,\rho}|^{1-1/m}$$
$$\leq \{32\lambda^{-2}\mu(\sigma\rho)^{-2}\sup(w_n - k)^2|A^n_{k,\rho}|^{1/m}$$
$$+ 4\delta\lambda^{-1}M\kappa_m^{1/m}\}|A^n_{k,\rho}|^{1-1/m}$$
$$\leq \kappa_m^{1/m}\max(32\lambda^{-2}\mu, 4\delta\lambda^{-1}M)\{\sigma^{-2}\rho^{-1}\sup(w_n - k)^2 + 1\}|A^n_{k,\rho}|^{1-1/m}, \tag{3.41}$$

which is the required inequality (2.18) with

$$\gamma = \kappa_m^{1/m}\max\{32\lambda^{-1}\mu, 4\delta\lambda^{-1}M\}. \tag{3.42}$$

In case $B_\rho \cap \partial\Omega \neq \Phi$, one can prove (2.18), in the same manner as the arguements just carried out, for

$$k > \max\ \{w_n(x) : x \in B_\rho \cap \partial\Omega\},$$

the difference being,however,that the balls B_ρ are to replaced by Ω_ρ.

4. Proof of Theorem

On the basis of the fact that

$$w_n|_{\partial\Omega} = \psi(t_n,\cdot) \in C^{\alpha}o(\{t_n\} \times \partial\Omega)$$

and

$$\operatorname{osc}\ \{w : Q(\rho,\rho^2) \cap \Gamma\}$$
$$\leq C\ \sup\ \{((n - n')h)^{\alpha/2} + |x - y|^{\alpha} : (x,t_n),(y,t_{n'}) \in Q(\rho,\rho^2)\cap\Gamma\}, \tag{4.1}$$

(2.22) in Theorem P and (2.29) in Theorem E are valid. Via

Lemma 1 and 2,

$$w \in B_2^h(Q,M,\gamma,2(1+2/m),\delta,2/m)$$

and

$$w_n \in B_2^h(\Omega,M,\gamma,\delta,1/2m)$$

for some positive γ independent of h.

Let $n (1 \le n \le N)$ and $x_0 \in \Omega$ be arbitrary taken and fixed. We put $Q(\rho,\theta\rho^2) = (t_n - \theta\rho^2, t_n) \times \Omega_\rho(x_0)$, where θ is a positive number determined in (2.25). In the light of Theorem P and E, we establish the estimations of $w(t,x)$ and $w_n(x)(1 \le n \le N)$:

$$\text{osc } \{w:Q(\rho,\theta\rho^2)\} \le C\rho^\beta \text{ for } \theta^{-1/2}h^{1/4} \le \rho \le \bar{\rho} \tag{4.2}$$

and

$$\text{osc } \{w_n:\Omega_\rho(x_0)\} \le \bar{C}_n^*(\rho/h)^{\beta^*} \quad \text{for} \quad \rho \le h(1 \le n \le N), \tag{4.3}$$

where

$$\bar{C}_n^* = b^{\beta^*}\max \{\bar{w}_n^*, 2^{s^*}h^{\beta^*}\}, \ \bar{w}_n^* = \text{osc } \{w_n:\Omega_h(x_0)\}(1 \le n \le N) \tag{4.4}$$

and the constants $C,\theta,\rho,\beta,\beta^*$ and s^* are positive numbers, independent of h, given in Lemma 1 and 2. By setting $\rho = \theta^{-1/2}h^{1/4}$, we derive from (4.2)

$$\text{osc } \{w:Q(\theta^{-1/2}h^{1/4},h^{1/2})\} \le C\theta^{-\beta/2}h^{\beta/4}. \tag{4.5}$$

We now fix positive numbers ρ and bear in mind that assumption $h \le h \le \theta^{-2/3}$ $(0 < \theta < 1)$ of (1.12) gives

$$h < \theta^{-1/2}h^{1/4}. \tag{4.6}$$

To proceed the proof, we now classify the relations between h and ρ:

$$\rho < h, \tag{4.7}$$

$$h \le \rho \le \theta^{-1/2}h^{1/4}, \tag{4.8}$$

$$\theta^{-1/2}h^{1/4} < \rho. \tag{4.9}$$

The case (4.7): Upon putting

$$Q(\theta^{-1/2}h^{1/4},h^{1/2}) = \Omega_{\theta^{-1/2}h^{1/4}}(x_0) \times (t_n - h^{1/2}, t_n),$$

we deduce from (4.5) and (4.6) that

$$\bar{w}_n^* = \text{osc}\,\{w_n : \Omega_h(x_0)\} \le \text{osc}\,\{w(t,x) : x \in \Omega_{\theta^{-1/2}h^{1/4}}(x_0), t = t_n\}$$
$$\le \text{osc}\,\{w : Q(\theta^{-1/2}h^{1/4}, h^{1/2}\} \le C\theta^{-\beta/2}h^{\beta/4}, \qquad (4.10)$$

in which

$$\alpha = \min\{\beta/4, \beta^*\}. \qquad (4.11)$$

To continue the proof, we further classify the sub-cases.

Case 1. Suppose $2^{s*}h^{\beta*} \ge \bar{w}_n^*$. Then, from (4.4),

$$\bar{C}_n^* = b^{\beta^*}2^{s^*}h^{\beta^*}.$$

Since $\rho \le h$ (< 1), from (4.3),

$$\text{osc}\,\{w_n : \Omega_\rho(x_0)\} \le b^{\beta^*}2^{s^*}h^{\beta^*}(\rho/h)^{\beta^*} \le b^{\beta^*}2^{s^*}\rho^\alpha. \qquad (4.12)$$

Case 2. Suppose $2^{s^*}h^{\beta^*} < \bar{w}_n^*$. We have from (4.4) and (4.10)

$$\bar{C}_n^* \le b^{\beta^*}\bar{w}_n^* \le C\theta^{-\beta/2}b^{\beta^*}h^{\beta/4}. \qquad (4.13)$$

Thus ,it follows from (4.3) and (4.13) that

$$\text{osc}\,\{w_n : \Omega_\rho(x_0)\} \le C\theta^{-\beta/2}b^{\beta^*}h^{\beta/4}(\rho/h)^{\beta^*}. \qquad (4.14)$$

To proceed an estimate of (4.14), we here notice the following inequality holds:

$$h^{\beta/4}(\rho/h)^{\beta^*} \le \rho^\alpha \quad \text{for} \quad \rho \le h. \qquad (4.15)$$

Actually,

In case $\beta^* \ge \beta/4$,

$$h^{\beta/4-\beta^*}\rho^{\beta^*} \le \rho^{\beta/4-\beta^*}\rho^{\beta^*} = \rho^{\beta/4} = \rho^\alpha.$$

In case $\beta^* < \beta/4$,

$$h^{\beta/4-\beta^*}\rho^{\beta^*} \le \rho^{\beta^*} = \rho^\alpha.$$

Estimates (4.14) and (4.15) imply

$$\text{osc}\,\{w_n : \Omega_\rho(x_0)\} \le C\theta^{-\beta/4}b^{\beta^*}\rho^\alpha \quad \text{for} \quad \rho \le h. \qquad (4.16)$$

The case (4.8): On account of assumption $h < \theta^2$,

$$h \le \rho \le \theta^{-1/2}h^{1/4} \le 1.$$

Hence, by taking

$$Q(\theta^{-1/2}h^{1/4},h^{1/2}) = \Omega_{\theta^{-1/2}h^{1/4}}(x_0) \times (t_n - h^{1/2},t_n),$$

we obtain from (4.5)

$$\begin{aligned} \operatorname{osc} \{w_n:\Omega_\rho(x_0)\} &\le \operatorname{osc} \{w(t,x):x \in \Omega_{\theta^{-1/2}h^{1/4}}(x_0),t = t_n\} \\ &\le \operatorname{osc} \{w:Q(\theta^{-1/2}h^{1/4},h^{1/2}\} \le C\theta^{-\beta/2}h^{\beta/4} \\ &\le C\theta^{-\beta/2}\rho^{\beta/4} \le C\theta^{-\beta/2}\rho^{\alpha}. \end{aligned} \tag{4.17}$$

The case of (4.9): By putting

$$Q(\rho,\theta\rho^2) = \Omega_\rho(x_0) \times (t_n - \theta\rho^2,t_n),$$

and noticing (4.9), we have from (4.2)

$$\begin{aligned} \operatorname{osc} \{w_n:\Omega_\rho(x_0)\} &= \operatorname{osc} \{w(t,x):x \in \Omega_\rho(x_0),t = t_n\} \\ &\le \operatorname{osc} \{w:Q(\rho,\theta\rho^2)\} \le C\rho^\beta \le C\rho^\alpha. \end{aligned} \tag{4.18}$$

Thus, summing up the estimates (4.16), (4.17) and (4.18) obtained above, we obtain

$$\operatorname{osc} \{w_n:\Omega_\rho(x_0)\} \le \tilde{C}\rho^\alpha \quad \text{for} \quad \rho \le \bar{\rho} \ (1 \le n \le N), \tag{4.19}$$

where

$$\tilde{C} = \max \{C,C\theta^{-\beta/2},C\theta^{-\beta/2}b^{\beta^*}\}, \tag{4.20}$$

which is not independent of h.

We shall close the proof of Theorem by proving estimate (1.12). We take two positive integers n and n' with

$$(n - n')h < 1 \ (n > n'). \tag{4.21}$$

For such n and n' taken as above, we choose a positive number $\tilde{\rho}$ satisfying

$$h \le (n - n')h = \theta^2\tilde{\rho}^4 < 1, \tag{4.22}$$

where θ is the positive number from (2.25). From (4.22), there holds

$$\theta^{-1/2}h^{1/4} \le \tilde{\rho}. \tag{4.23}$$

For each $x \in \Omega$, we take $x_0 \in \Omega$ satisfing $x \in \Omega_{\tilde{\rho}}(x_0)$. Putting

$$Q(\tilde{\rho},\theta\tilde{\rho}^2) = \Omega_{\tilde{\rho}}(x_0) \times (t_n - \theta\tilde{\rho}^2,t_n),$$

we make use of (4.2) to obtain

$$\operatorname{osc}\{w:Q(\tilde{\rho},\theta\tilde{\rho}^2)\} \le C\tilde{\rho}^{\beta}. \tag{4.24}$$

By (4.22),

$$\theta^2\tilde{\rho}^4 < \theta\tilde{\rho}^2, \tag{4.25}$$

so we see

$$t_n - t_{n'} = (n - n')h = \theta^2\tilde{\rho}^4 < \theta\tilde{\rho}^2. \tag{4.26}$$

Hence, we obtain from (4.24) and (4.26) that

$$|w_n(x) - w_{n'}(x)| \le \sup\{|w(t_n,y) - w(t_{n'},y)| : y \in \Omega_{\tilde{\rho}}(x_0)\}$$
$$\le \operatorname{osc}\{w:Q(\tilde{\rho},\theta^2\tilde{\rho}^4)\} \le C\theta^{-\beta/2}((n - n')h)^{\beta/4}. \tag{4.27}$$

We reach the desired estimate (2.15).

These complete the proof of Theorem.

References

[1] DeGiorgi,E.,Sulla differeziabilita e l'analiticita delle estremalidegli integrali multipli regolari,Mem.Accad.Sci.Trino.Cl.Sci. Fis.Mat.Nat.(3)3(1957),25-43.

[2] Kikuchi,N.,A construction of a solution for $\partial u/\partial t = a(\Delta u), a(0) = 0$, $a \in C^1(-\infty,\infty), a' > 0$) with initial and boundary conditions of class $C^{3+\alpha}(0 < \alpha < 1)$,Ann.Mat.Pura Appl.(4)155(1989),261-269.

[3] Ladyzhenskaya,O.A. and N.N.Ural'tseva,"Linear and quasilinear elliptic equation",Academic Press,New York,1968.

[4] Ladyzhenskaya,O.A.,V.A.Solonnikov and N.N.Ural'tseva,"Linear and quasi-linear equation of parabolic type",Translation of Mathematical Monographs 23,1968.

[5] Moser,J.,A new proof of deGiorgi's theorem concerning the regularity problem for elliptic differential equations,Comm.Pure Appl. Math.13(1960)457-468.

[6] Moser,J.,A Harnack inequality for parabolic differential

equations,Comm.Pure Appl.Math.17(1964),101-134,Errata,ibid;20(1967),
231-236.

[7] Nash,J.,Continuity of solutions of parabolic differential equations,Comm.Pure Appl.Math.17(1964),101-134.