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Vector valued Siegel modular forms and their L-functions; Application of a Differential operator.

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Vector valued Siegel modular forms and their L-functions; Application of a Differential operator

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Introduction

As for automorphic L-functions attached to holomorphic Siegel modular forms for $Sp(n,\mathbb{Z})$ ($n \geq 2$) in the sense of Langlands [18, 19] , it is known that two types of them, namely the spinor type and the standard type, are continued analytically to the whole complex plane so far. They are L-functions corresponding to the spinor representation of $SO(2n+1,\mathbb{C})$ and to the standard representation of $SO(2n+1,\mathbb{C})$ respectively.

Spinor L-functions were introduced by Langlands [18] and Andrianov [1] and for scalar valued cases of $n=2$, Andrianov [1] proved their meromorphic continuation and the functional equations. Moreover, their poles were studied by Evdokimov [10] and Oda [23]. For vector valued cases of $n=2$, Arakawa [4] and Sugano [26] have discovered that the spinor L-function is continued analytically as an entire function.

On the other hand, standard L-functions were also introduced by Langlands [18] and Andrianov [2]. For scalar valued cases, Andrianov-Kalinin [3] (in special cases) and Böcherer [7] (in general; see also Piatetski-Shapiro,-Rallis [24]) proved their meromorphic continuation and the functional equations. Recently Mizumoto [21] has got results on their poles and residues. We will get the functional equations and poles of standard L-functions explicitly for a certain vector valued case. (cf. Piatetski-Shapiro,-Rallis [24], Weissauer [28])

Suppose n is a positive integer, k is a positive even integer, s is a complex number such that $k+2\mathrm{Re}(s) > n+1$, and Z belongs to the Siegel upper half space $\mathfrak{H}_n$ (of degree n). The way of constructing integral representation of the standard L-function by use of Eisenstein series $E_k^n(Z,s)$ (defined in §1) is based on Böcherer [7] and the method of determining poles by use of its integral representation is based on Mizumoto [21]. We use the differential operator $L^{k,l}$ (defined in §2) introduced by Böcherer-Satoh-Yamazaki [9] to transform the scalar valued Eisenstein series into a vector valued function.

In the process of constructing the integral representation, we use the Garrett's pullback formula [12] to decompose $(L^{k,l}E_k^{2n})\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}, s\right)$, $Z, W \in \mathfrak{H}_n$, into functions for Z and for W. The pullback sends the Eisenstein series $E_k^{m+n}\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}, s\right)$, $Z \in \mathfrak{H}_m$, $W \in \mathfrak{H}_n$, to modular forms for Z and for W and it is studied by Garrett [12] for the holomorphic case (i.e., $s=0$) and by Böcherer [7] for real analytic cases. Moreover Böcherer-Satoh-Yamazaki [9] deals with the pullback formula of the vector valued function $(L^{k,l}E_k^{m+n})\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}, 0\right)$.

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Notations

1°. As usual, $\mathbb{Z}$ is the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers.

2°. Let $m,n \in \mathbb{Z}$, $m,n > 0$. If A is an $m \times n$-matrix, then we write it also as $A^{(m,n)}$, and as $A^{(m)}$ if $m=n$. The identity matrix of size n is denoted by 1_n.

3°. For $m,n \in \mathbb{Z}$, $m,n > 0$, and a commutative ring R containing 1, let $R^{(m,n)}$ (resp. $R^{(n)}$) be the R-module of all $m \times n$ (resp. $n \times n$) matrices with entries in R.

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4°. For a real symmetric positive definite matrix S, $S^{1/2}$ is the unique real symmetric positive definite matrix such that $(S^{1/2})^2 = S$.

5°. For matrices $A^{(m)}$, $B^{(m,n)}$, we define $A[B] := {}^t\bar{B}AB$, where ${}^tB$ is the transpose of B and $\bar{B}$ is the complex conjugate of B.

6°. For $n \in \mathbb{Z}$, $n > 0$, we put

$$\mathbb{T}^{(n)} := \left\{ T = \begin{pmatrix} t_1 & & & 0 \\ & t_2 & & \\ & & \ddots & \\ 0 & & & t_n \end{pmatrix} \in \mathbb{Z}^{(n)} \middle| t_i > 0 \text{ for each } i \in \mathbb{Z} \text{ with } 1 \leq i \leq n, \ t_1 | \cdots | t_n \right\}.$$

7°. For $n \in \mathbb{Z}$, $n > 0$, let $\Gamma^n := Sp(n, \mathbb{Z})$ be the Siegel modular group of degree n and let $\mathfrak{H}_n$ be the Siegel upper half space of degree n, that is,

$$\mathfrak{H}_n := \{Z = X + iY \in \mathbb{C}^{(n)} | {}^tZ = Z, \ Y > 0\}.$$

For $N \in \mathbb{Z}$, $N > 0$, we put

$$\Gamma_0^n(N) := \left\{ \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \in \Gamma^n \middle| \ C \equiv 0 \mod N \right\}.$$

For each $r \in \mathbb{Z}$ with $0 \leq r \leq n$, we put

$$P_{n,r} := \left\{ \begin{pmatrix} * & * \\ C^{(n)} & D^{(n)} \end{pmatrix} \in \Gamma^n \middle| \ C = \begin{pmatrix} 0 & 0 \\ 0 & C_4^{(r)} \end{pmatrix}, \ D = \begin{pmatrix} * & 0 \\ * & D_4^{(r)} \end{pmatrix} \right\}.$$

All these are subgroups of Γ^n.

8°. For $l \in \mathbb{Z}$, $l \geq 0$, we put

$$(a)_l := \begin{cases} a(a+1)\cdots(a+l-1), & \text{if } l > 0, \\ 1, & \text{if } l = 0. \end{cases}$$

The symbol $(\ )_l$ is called the Pochhammer symbol. Note $(a)_l = (-1)^l(-a-l+1)_l$. We also have

$$(a+b)_l = \sum_{r=0}^{l} \binom{l}{r} (a)_r (b)_{l-r},$$

where $\binom{l}{r} = \dfrac{l!}{r!(l-r)!}$ is the binomial coefficient.

9°. For $n \in \mathbb{Z}$, $n \geq 0$, we put

$$\Gamma_n(s) := \prod_{j=1}^{n} \Gamma\left(s - \frac{j-1}{2}\right), \ \Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right), \ \Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$$

and

$$\xi(s) := \Gamma_{\mathbb{R}}(s)\zeta(s) = \xi(1-s),$$

where $\Gamma(s)$ is the gamma function and $\zeta(s)$ is the Riemann zeta function. Throughout the paper we understand that the empty product is equal to 1.

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§1 Preliminary

1.1 Vector valued Siegel modular forms

Let ρ be a finite-dimensional representation of $GL(n,\mathbb{C})$ with a representation space $\mathcal{V}$. By definition, $\mathcal{V}$-valued C^∞-Siegel modular forms of weight ρ are C^∞-functions from $\mathfrak{H}_n$ to $\mathcal{V}$ satisfying

$$(f|M)(Z) = f(Z)$$

for all $Z \in \mathfrak{H}_n$ and $M = \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \in \Gamma^n$, where

$$(f|M)(Z) := \rho((CZ+D)^{-1})f(M\langle Z\rangle) \text{ and } M\langle Z\rangle := (AZ+B)(CZ+D)^{-1}.$$

The space of all such functions is denoted by $M_\rho^n(\mathcal{V})^\infty$.

A function f from $\mathfrak{H}_n$ to $\mathcal{V}$ is called a $\mathcal{V}$-valued Siegel modular form of weight ρ if it satisfies the following properties:

(i) holomorphic on $\mathfrak{H}_n$

(ii) for all $M \in \Gamma^n$, $(f|M)(Z) = f(Z)$

(iii) holomorphic at the cusps if $n = 1$

The space of $\mathcal{V}$-valued Siegel modular forms of weight ρ is denoted by $M_\rho^n(\mathcal{V})$.

We define the Siegel operator Φ on $M_\rho^n(\mathcal{V})$ by

$$(\Phi f)(Z) := \lim_{t\to\infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix}\right)$$

for $Z \in \mathfrak{H}_{n-1}$. Let $\mathcal{V}'$ be the subspace of $\mathcal{V}$ generated by the values of Φf for all $f \in M_\rho^n(\mathcal{V})$. Then $\mathcal{V}'$ is invariant under the transformation

$$\rho\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right), \; g \in GL(n-1,\mathbb{C}) .$$

If we assume $\mathcal{V}' \neq \{0\}$, we get the representation ρ' of $GL(n-1,\mathbb{C})$ with the representation space $\mathcal{V}'$. Thus the operator Φ defines the map

$$\Phi : M_\rho^n(\mathcal{V}) \longrightarrow M_{\rho'}^{n-1}(\mathcal{V}') .$$

Suppose $f \in M_\rho^n(\mathcal{V})$. Then it is called a cusp form if it satisfies $\Phi f = 0$, and we put

$$S_\rho^n(\mathcal{V}) := \{f \in M_\rho^n(\mathcal{V}) \mid f \text{ is a cuspform }\}.$$

Each $f \in M_\rho^n(\mathcal{V})$ has a Fourier expansion

$$f(Z) = \sum_{H\geq 0} a(H)\exp(2\pi i \operatorname{trace}(HZ)), \; a(H) \in \mathcal{V},$$

with H running over all symmetric positive semi-definite semi-integral matrices of size n. If $f \in S_\rho^n(\mathcal{V})$, H runs over all symmetric positive definite semi-integral matrices.

Let $\mathcal{V} \cong \oplus\mathcal{V}_i$ be a decomposition of $\mathcal{V}$ into a direct sum of irreducible representations. Then we get a decomposition

$$M_\rho^n(\mathcal{V}) \cong \oplus M_{\rho_i}^n(\mathcal{V}_i),$$

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where for each i, ρ_i is a representation of $GL(n,\mathbb{C})$ with the representation space $\mathcal{V}_i$.

If ρ is an irreducible rational representation, ρ is equivalent to an irreducible rational representation $\tilde{\rho}$ satisfying the following condition: Let $\tilde{\mathcal{V}}$ be the representation space of $\tilde{\rho}$. Then, there exists a unique one-dimensional vector subspace $\mathbb{C}\tilde{v}$ of $\tilde{\mathcal{V}}$ such that for any upper triangular matrix of $GL(n,\mathbb{C})$,

$$\tilde{\rho}\left(\begin{pmatrix} g_{11} & & * \\ & \ddots & \\ 0 & & g_{nn} \end{pmatrix}\right)\tilde{v} = \left(\prod_{j=1}^{n} {g_{jj}}^{\lambda_j}\right)\tilde{v} ,$$

where $(\lambda_1,\lambda_2,\cdots,\lambda_n)\in\mathbb{Z}^n$ and $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$.

Then we write

$$\rho\sim(\lambda_1,\lambda_2,\cdots,\lambda_n). \tag{1.1}$$

Suppose $\rho\sim(\lambda_1,\lambda_2,\cdots,\lambda_n)$. We note that $M_\rho^n(\mathcal{V})=\{0\}$ if $\lambda_n<0$ and that $M_\rho^n(\mathcal{V})^\infty=\{0\}$ if $\lambda_1+\cdots+\lambda_n\not\equiv 0 \mod 2$.

Let $k,l\in\mathbb{Z}$, $k>0$, $l\geq 0$. For a vector space W, we denote by $\mathrm{sym}^l(W)$ its l-th symmetric tensor product. We identify $\mathrm{sym}^0(W)$ with $\mathbb{C}$. Let $x=(x_1,\cdots,x_n)$ be a row vector consisting of n indeterminates. Then we put $V=\mathbb{C}x_1\oplus\cdots\oplus\mathbb{C}x_n$. We identify $\mathrm{sym}^l(V)$ with the subset of $\mathbb{C}[x_1,\cdots,x_n]$ which consists of the homogeneous polynomials of degree l. Let ρ be a representation $\det^k\otimes\mathrm{sym}^l$ of $GL(n,\mathbb{C})$ with a representation space $\mathrm{sym}^l(V)$. Then for each $g\in GL(n,\mathbb{C})$, $\rho(g)$ acts on $\mathrm{sym}^l(V)$ by

$$\rho(g)(v(x))=\det g^k v(xg),\ v=v(x)\in\mathrm{sym}^l(V).$$

We note that $\rho\sim(k+l,k,\cdots,k)$ and $\dim(\mathrm{sym}^l(V))=\binom{n+l-1}{l}$.

From now on, we put $\rho=\det^k\otimes\mathrm{sym}^l$ unless stated otherwise.

We write $M_{k,l}^n(\mathrm{sym}^l(V))^\infty$, $M_{k,l}^n(\mathrm{sym}^l(V))$ and $S_{k,l}^n(\mathrm{sym}^l(V))$ for $M_\rho^n(\mathrm{sym}^l(V))^\infty$, $M_\rho^n(\mathrm{sym}^l(V))$ and $S_\rho^n(\mathrm{sym}^l(V))$, respectively.

Remarks. (i) For $n=2$, any finite-dimensional rational representation of $GL(n,\mathbb{C})$ is equivalent to a direct sum of representations $\rho=\det^k\otimes\mathrm{sym}^l$ for $k,l\in\mathbb{Z}$.

(ii) If $nk\not\equiv l \mod 2$, $M_{k,l}^n(\mathrm{sym}^l(V))^\infty=\{0\}$.

As above, let $V=\mathbb{C}x_1\oplus\cdots\oplus\mathbb{C}x_n$, $x=(x_1,\cdots,x_n)$.

For $\sum_{j=1}^n a_jx_j$, $\sum_{j=1}^n b_jx_j\in V$, we define an inner product of them by

$$\left\langle\sum_{j=1}^n a_jx_j\ ,\ \sum_{j=1}^n b_jx_j\right\rangle:=\sum_{j=1}^n a_j\bar{b}_j\ .$$

The inner product induces an inner product of $\mathrm{sym}^l(V)$ defined by

$$\langle v_1\cdots v_l\ ,\ w_1\cdots w_l\rangle:=\frac{1}{l!}\sum_{\tau\in\mathfrak{S}_l}\prod_{j=1}^{l}\langle v_{\tau(j)}\ ,\ w_j\rangle\ ,$$

where $v_j,\ w_j\in V$. It is also denoted by $\langle\ \ ,\ \ \rangle$. Then, for $v=v(x)$, $w=w(x)\in\mathrm{sym}^l(V)$, the inner product has the following properties:

(i) $$\langle v(x)\ ,\ w(x)\rangle=\overline{\langle w(x)\ ,\ v(x)\rangle}\ ,$$

(ii) $$\begin{aligned}\langle\rho(g)v(x)\ ,\ \rho(g')w(x)\rangle&=\langle\rho({}^t\bar{g}'g)v(x)\ ,\ w(x)\rangle\\&=\langle v(x)\ ,\ \rho({}^t\bar{g}g')w(x)\rangle\end{aligned}$$

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for $g,\ g' \in GL(n,\mathbb{C})$, and

(iii) $$\langle \rho(U)v(x)\ ,\ \rho(U)w(x)\rangle = \langle v(x)\ ,\ w(x)\rangle$$

for any $U \in U(n,\mathbb{C})$.

Suppose $f,g \in M^n_{k,l}(\mathrm{sym}^l(V))^\infty$. The Petersson inner product of f and g is defined by

$$(f,g) := \int_{\Gamma^n\backslash\mathfrak{H}_n} \left\langle \rho\left(\sqrt{\mathrm{Im}(W)}\right) f(W)\ ,\ \rho\left(\sqrt{\mathrm{Im}(W)}\right) g(W)\right\rangle \det(\mathrm{Im}(W))^{-n-1} dXdY$$

if the right-hand side is convergent. Here $W = X + iY$ with real matrices $X = (x_{jh})$ and $Y = (y_{jh})$;

(1.2) $$dX := \prod_{j\le h} dx_{jh}\ ,\ dY := \prod_{j\le h} dy_{jh}\ ;$$

the integral is taken over a fundamental domain of $\Gamma^n \backslash \mathfrak{H}_n$. We write $dW = dXdY$ when there is no fear of confusion.

1.2 Hecke algebra and Hecke operator

We put

$$\begin{aligned}\Delta^n :=& G^+Sp(n,\mathbb{Q})\\ =& \left\{ M \in GL(2n,\mathbb{Q})\ \middle|\ {}^tM \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} M = \mu(M) \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}\ ,\ \mu(M) > 0 \right\},\end{aligned}$$

and for a prime number p

$$\Delta^n_p := \Delta^n \cap GL\left(2n, \mathbb{Z}\left[\frac{1}{p}\right]\right).$$

Let $\mathcal{H}$ (resp. $\mathcal{H}_p$) be a free $\mathbb{C}$-module generated by the double cosets $\Gamma^n g \Gamma^n$, $g \in \Delta^n$ (resp. Δ^n_p). Then $\mathcal{H}$ is a commutative algebra and $\mathcal{H} = \otimes_p \mathcal{H}_p$. Moreover, the structure of $\mathcal{H}_p$ is known: For $1 \le j \le n$, let w_j be an automorphism of $\mathbb{C}[X_0{}^{\pm1}, \cdots, X_n{}^{\pm1}]$ such that

$$w_j(X_0) = X_0X_j,\ w_j(X_j) = X_j{}^{-1} \text{ and } w_j(X_i) = X_i\ (1 \le i \le n,\ i \ne j)\ .$$

The automorphisms w_j $(1 \le j \le n)$ and the symmetric group $\mathfrak{S}_n$ consisting of permutations of X_i $(1 \le i \le n)$ generate a finite group W. We denote by $\mathbb{C}[X_0{}^{\pm1}, \cdots, X_n{}^{\pm1}]^W$ the W-invariant subalgebra of $\mathbb{C}[X_0{}^{\pm1}, \cdots, X_n{}^{\pm1}]$. Then

$$\mathcal{H}_p \cong \mathbb{C}[X_0{}^{\pm1}, \cdots, X_n{}^{\pm1}]^W\ ,$$

where the isomorphism can be written explicitly.

For $g \in \Delta_n$, let $\Gamma^n g \Gamma^n = \bigcup_{i=1}^r \Gamma^n g_i$ be a decomposition of $\Gamma^n g \Gamma^n$ into left cosets.

For $f \in M^n_{k,l}(\mathrm{sym}^l(V))$ (resp. $S^n_{k,l}(\mathrm{sym}^l(V))$, $M^n_{k,l}(\mathrm{sym}^l(V))^\infty$), we put

$$f \mapsto f|(\Gamma^n g \Gamma^n) := \sum_{i=1}^r f|g_i\ .$$

Then we get a homomorphism

$$\mathcal{H} \longrightarrow \mathrm{End}(M^n_{k,l}(\mathrm{sym}^l(V)))\ .$$

(resp. $\mathrm{End}(S^n_{k,l}(\mathrm{sym}^l(V)))$, $\mathrm{End}(M^n_{k,l}(\mathrm{sym}^l(V))^\infty)$)

Suppose $f \in M^n_{k,l}(\mathrm{sym}^l(V))$ is an eigenform, i.e., a non-zero common eigenfunction of the Hecke algebra. It follows from

$$\Gamma^n g \Gamma^n \mapsto f|(\Gamma^n g \Gamma^n) = \lambda(\Gamma^n g \Gamma^n) f$$

that a homomorphism

$$\lambda : \mathcal{H}_p \xrightarrow{\ \cong\ } \mathbb{C}[X_0{}^{\pm1}, \cdots, X_n{}^{\pm1}]^W \xrightarrow{(X_0,\cdots,X_n)\mapsto(\alpha_{0,p},\cdots,\alpha_{n,p})} \mathbb{C}$$

with $(\alpha_{0,p}, \cdots, \alpha_{n,p}) \in (\mathbb{C}^{*\ n+1})^W$, is defined. The numbers $\alpha_{0,p}, \cdots, \alpha_{n,p}$ are uniquely determined modulo W and they are called the Satake p-parameters of f.

1.3 Standard L-function

Let $f \in S^n_{k,l}(\mathrm{sym}^l(V))$ be an eigenform. We define the standard L-function by

$$D_f(s) := \prod_p \left\{ (1-p^{-s}) \prod_{i=1}^{n} (1-\alpha_{i,p}{}^{-1}p^{-s})(1-\alpha_{i,p}p^{-s}) \right\}^{-1} , \tag{1.3}$$

where p runs over all prime numbers. The right-hand side of (1.3) converges absolutely and locally uniformly for $\mathrm{Re}(s) > n+1$.

We also define the following series:

$$L(s,f) := \sum_{T \in \mathbb{T}^{(n)}} \lambda(f,T) \det(T)^{-s} , \tag{1.4}$$

where $\lambda(f,T)$ is an eigenvalue on f of the Hecke operator $\Gamma^n \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \Gamma^n$, $T \in \mathbb{T}^{(n)}$. The right-hand side of (1.4) also converges absolutely and locally uniformly for $\mathrm{Re}(s) > 2n+1$. By Böcherer [8], we have:

$$\zeta(s) \prod_{i=1}^{n} \zeta(2s-2i) L(s,f) = D_f(s-n) . \tag{1.5}$$

Here we mention a fact on poles of standard L-functions, **(1.3.1)** below:

Let $H_{m,n}(\rho)$ be the (finite-dimensional) $\mathbb{C}$-vector space of polynomial maps

$$P : \mathbb{C}^{(m,n)} \longrightarrow \mathrm{sym}^l(V)$$

satisfying the following two conditions:

(i) $$\det g^{m/2} P(Xg) = \rho({}^t g) P(X) \text{ for all } g \in GL(n,\mathbb{C}) \ , \ X \in \mathbb{C}^{(m,n)} ,$$

(ii) $$\sum_{j=1}^{m} \frac{\partial}{\partial x_{ji}} \frac{\partial}{\partial x_{jk}} P(X) = 0 \text{ for } X = (x_{ij}) .$$

Such polynomials are called pluriharmonic.

Let $\mathcal{S}_m$ be the set of all symmetric positive definite even integral unimodular matrices of size m. As is well-known, $\mathcal{S}_m \neq \emptyset$ if and only if $m \equiv 0 \mod 8$. For each $P \in H_{m,n}(\rho)$ and $S \in \mathcal{S}_m$ with $m \equiv 0 \mod 8$, we attach the theta series

$$\vartheta_{S,P}(Z) := \sum_{G \in \mathbb{Z}^{(m,n)}} P(S^{1/2}G) \exp(\pi i \operatorname{trace}(S[G]Z))$$

which converges for all $Z \in \mathfrak{H}_n$ and belongs to $M^n_{k,l}(\mathrm{sym}^l(V))$. We denote by $B^n_{k,l}(m)$ the subspace of $M^n_{k,l}(\mathrm{sym}^l(V))$ of all finite sums of theta series $\vartheta_{S,P}$. The space $B^n_{k,l}(m)$ is invariant under the action of the Hecke algebra. We note that $B^n_{k,l}(m) = \{0\}$ if $m \not\equiv 0 \mod 8$ or $m > 2k$.

Under the notation above, we have the following by Weissauer [28]:

(1.3.1). *For $n,k,l \in \mathbb{Z}$, $n,k,l > 0$, let $f \in S^n_{k,l}(\mathrm{sym}^l(V))$ be an eigenform. Suppose $k > n$. Then $D_f(s)$ has a simple pole at $s=1$ if and only if $f \in B^n_{k,l}(2n) \cap S^n_{k,l}(\mathrm{sym}^l(V))$.*

1.4 Eisenstein series

For scalar valued cases (i.e.,$l=0$), we write $M^{n\,\infty}_k$, M^n_k and S^n_k for $M^n_{k,0}(\mathrm{sym}^0(V))^{\infty}$, $M^n_{k,0}(\mathrm{sym}^0(V))$ and $S^n_{k,0}(\mathrm{sym}^0(V))$, respectively.

For $k \in 2\mathbb{Z}$, $k > 0$, $s \in \mathbb{C}$ and $Z \in \mathfrak{H}_n$, we define the Eisenstein series by

$$E^n_k(Z,s) := \sum_{M = \begin{pmatrix} * & * \\ C^{(n)} & D^{(n)} \end{pmatrix} \in P_{n,0} \backslash \Gamma^n} \det(CZ+D)^{-k} \det(\mathrm{Im}(M\langle Z \rangle))^s .$$

Then $E^n_k(Z,s) \in M^{n\,\infty}_k$. The function $E^n_k(Z,s)\det(\mathrm{Im}(Z))^{-s}$ converges absolutely and locally uniformly for $k + 2\mathrm{Re}(s) > n+1$. Moreover we have the following by Mizumoto [21]. (see also Andrianov-Kalinin [3], Kalinin [15] Langlands [20]):

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(1.4.1). *Let $n, k \in \mathbb{Z}$, $n, k > 0$, k : even. Then for $Z \in \mathfrak{H}_n$,*

$$\mathbb{E}_k^n(Z,s) := \frac{\Gamma_n\left(s+\frac{k}{2}\right)}{\Gamma_n(s)}\xi(2s)\prod_{i=1}^{[\frac{n}{2}]}\xi(4s-2i)E_k^n\left(Z, s-\frac{k}{2}\right) \tag{1.6}$$

is invariant under $s \mapsto \frac{n+1}{2} - s$ and it is an entire function in s.

It is also known that every partial derivative (in z_{jh}'s) of the Eisenstein series $E_k^n(Z,s)$ is slowly increasing (locally uniformly in s). That is, by Mizumoto [22] we have:

(1.4.2). *Let $n, k \in \mathbb{Z}$, $n, k > 0$, k : even.*
For each $s_0 \in \mathbb{C}$, we can take $d \in \mathbb{Z}$, $d > 0$ and a suitable neighborhood $\mathcal{U}$ of s_0 depending only on n, k and s_0 such that $(s-s_0)^d E_k^n(Z,s)$ is holomorphic in s on $\mathcal{U}$. Then, for $s \in \mathcal{U}$, $l \in \mathbb{Z}$, $l \geq 0$, $\mathrm{Im}(Z) \geq \varepsilon 1_n$ ($\varepsilon > 0$), there exist positive constants α, β depending only on n, k, l, ε, s_0 and $\mathcal{U}$ such that

$$\left|(s-s_0)^d \frac{\partial^l}{\partial z_{j_1 h_1} \cdots \partial z_{j_l h_l}} E_k^n(Z,s)\right| \leq \alpha\ \mathrm{trace}(\mathrm{Im}(Z))^{\beta}$$

$(1 \leq j_\nu\ ,\ h_\nu \leq n)$.

Here we summarize some facts on standard L-functions attached to scalar valued Siegel modular forms.
Let $f \in S_k^n$ be an eigenform. Then we put

$$\Lambda(s,f) := \Gamma_{\mathbf{R}}(s+\varepsilon)\prod_{j=1}^{n}\Gamma_{\mathbf{C}}(s+k-j)D_f(s) \tag{1.7}$$

with

$$\varepsilon := \begin{cases} 0 & \text{for } n \text{ even,} \\ 1 & \text{for } n \text{ odd.} \end{cases} \tag{1.8}$$

Moreover we put

$$\gamma(s) := \begin{cases} \dfrac{\Gamma_n\left(\dfrac{s+n}{2}\right)}{\Gamma_n\left(\dfrac{s}{2}\right)} & \text{for } n \text{ even,} \\[2ex] \dfrac{\Gamma_{n-1}\left(\dfrac{s+n}{2}\right)}{\Gamma_{n-1}\left(\dfrac{s-1}{2}\right)} & \text{for } n \text{ odd.} \end{cases} \tag{1.9}$$

We note

$$\gamma(s) = \gamma(1-s)\ . \tag{1.10}$$

Then we have the following. ([3], [7], [24]):

(1.4.3). *Let $f \in S_k^n$ be an eigenform with a positive integer n and a positive even integer k.*
(i) We obtain the integral representation

$$\left(f, \mathbb{E}_k^{2n}\left(\begin{pmatrix} -\bar{Z} & 0 \\ 0 & * \end{pmatrix}, \frac{\bar{s}+n}{2}\right)\right) = 2i^{nk}\pi^{-\frac{1}{2}n^2+kn+\frac{1}{2}\varepsilon}\gamma(s)\Lambda(s,f)f(Z)\ . \tag{1.11}$$

(ii) $\Lambda(s,f)$ has a meromorphic continuation to the whole s-plane and satisfies the functional equation

$$\Lambda(s,f) = \Lambda(1-s,f)\ . \tag{1.12}$$

To talk about poles of $\Lambda(s,f)$ we write $H_{m,n}(k)$ and $B_k^n(m)$ for $H_{m,n}(\det^k \otimes \mathrm{sym}^0)$ and $B_{k,0}^n(m)$, where we must replace the condition (ii) of $H_{m,n}(\rho)$ in §1.3 with the condition

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_{ij}^2}P(X) = 0 \text{ for } X = (x_{ij})\ . \tag{ii'}$$

By Mizumoto [21], we have:

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(1.4.4). *For $n, k \in \mathbb{Z}$, $n, k > 0$, let $f \in S_k^n$ be an eigenform. Suppose $k \geq n$. Then $\Lambda(s, f)$ is holomorphic except for possible simple poles at $s = 0$ and $s = 1$; it has a pole at $s = 0$ if and only if $f \in B_k^n(2n) \cap S_k^n$.*

§2 Differential operator and Pullback formula

2.1 Differential operator

In what follows, we put

$$\begin{aligned} V &= \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_{2n} , & x &= (x_1, \cdots, x_{2n}) , \\ V_1 &= \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n , & e_1 &= (x_1, \cdots, x_n) , \\ V_2 &= \mathbb{C}x_{n+1} \oplus \cdots \oplus \mathbb{C}x_{2n} , & e_2 &= (x_{n+1}, \cdots, x_{2n}) . \end{aligned}$$

Let ι be an isomorphism from V_1 to V_2 defined by $\iota(x_i) = x_{n+i}$ $(1 \leq i \leq n)$. It induces an isomorphism (also denoted by ι) from $\mathrm{sym}^l(V_1)$ to $\mathrm{sym}^l(V_2)$. For $f \in C^\infty(\mathfrak{H}_n, \mathrm{sym}^l(V_1))$ we define $\iota(f)$ by

$$(\iota(f))(Z) := \iota(f(Z)) .$$

Let $Z = (z_{jh})$ be a variable on $\mathfrak{H}_{2n}$. For an integer $l \geq 0$ and a function $f \in C^\infty(\mathfrak{H}_{2n}, \mathrm{sym}^l(V))$, we put

$$Df := \left(\frac{\partial}{\partial Z} f\right)[{}^t x] \quad \text{with} \quad \frac{\partial}{\partial Z} = \left(\frac{1+\delta_{jh}}{2} \frac{\partial}{\partial z_{jh}}\right)_{1 \leq j,h \leq 2n} .$$

Here, for $z_{jh} = x_{jh} + iy_{jh}$ we put

$$\frac{\partial}{\partial z_{jh}} = \frac{1}{2}\left(\frac{\partial}{\partial x_{jh}} - i\frac{\partial}{\partial y_{jh}}\right) , \quad \frac{\partial}{\partial \bar{z}_{jh}} = \frac{1}{2}\left(\frac{\partial}{\partial x_{jh}} + i\frac{\partial}{\partial y_{jh}}\right) . \tag{2.1}$$

Then Df is a $V^{(l+2)}$-valued function. We also put $D_\uparrow f := Df|_{e_2=0}$, $D_\downarrow f := Df|_{e_1=0}$ and $D_0 := D - D_\uparrow - D_\downarrow$. For a function f on $\mathfrak{H}_{2n}$, $\begin{pmatrix} Z^{(n)} & {}^tU^{(n)} \\ U^{(n)} & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$, we define the pullback d^* by

$$(d^* f)\left(\begin{pmatrix} Z & {}^tU \\ U & W \end{pmatrix}\right) := f\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}\right) .$$

For $k, l \in \mathbb{Z}$, $k, l \geq 0$, we define an operator

$$L^{k,l} : C^\infty(\mathfrak{H}_{2n}, \mathbb{C}) \longrightarrow C^\infty(\mathfrak{H}_n \times \mathfrak{H}_n, \mathrm{sym}^{2l}(V))$$

by

$$L^{k,l} := \frac{1}{(2\pi i)^l (k)_l} d^* \sum_{\nu=0}^{[\frac{l}{2}]} \frac{1}{\nu!(l-2\nu)!(2-k-l)_\nu} (D_\uparrow D_\downarrow)^\nu {D_0}^{l-2\nu} . \tag{2.2}$$

We also define an operator

$$L_*^{k,l} : C^\infty(\mathfrak{H}_{2n}, \mathbb{C}) \longrightarrow C^\infty(\mathfrak{H}_{2n}, \mathrm{sym}^{2l}(V))$$

by

$$L_*^{k,l} := \frac{1}{(2\pi i)^l (k)_l} \sum_{\nu=0}^{[\frac{l}{2}]} \frac{1}{\nu!(l-2\nu)!(2-k-l)_\nu} (D_\uparrow D_\downarrow)^\nu {D_0}^{l-2\nu} . \tag{2.3}$$

Then we get the following by [9]:

(2.1.1). (i) *For $k,l\in\mathbb{Z}$, $k,l\geq 0$, $L^{k,l}$ satisfies*

$$d^*\sum_{l=0}^{\infty}\frac{t^l}{(2\pi i)^l l!(k)_l}D^l=\sum_{l=0}^{\infty}\left(\sum_{\lambda=0}^{\infty}\frac{t^\lambda}{(2\pi i)^\lambda\lambda!(k+l)_\lambda}{D_\uparrow}^\lambda\right)\left(\sum_{\lambda=0}^{\infty}\frac{t^\lambda}{(2\pi i)^\lambda\lambda!(k+l)_\lambda}{D_\downarrow}^\lambda\right)t^lL^{k,l}\ .$$

(ii) *The operator $L^{k,l}$ is an operator which sends modular forms to modular forms:*

$$L^{k,l}:{M_k^{2n}}^\infty\longrightarrow M_{k,l}^n(\mathrm{sym}^l(V_1))^\infty\otimes M_{k,l}^n(\mathrm{sym}^l(V_2))^\infty\ .$$

Moreover, $L^{k,l}$ is a holomorphic operator and it satisfies

$$L^{k,l}:M_k^{2n}\longrightarrow M_{k,l}^n(\mathrm{sym}^l(V_1))\otimes M_{k,l}^n(\mathrm{sym}^l(V_2))\ .$$

Remark. On C^∞-functions the operators d^*, $D_\uparrow$, $D_\downarrow$ are commutative and D, D_0, $D_\uparrow$, $D_\downarrow$ are also commutative. But d^* and D, d^* and D_0 are not commutative, respectively. Therefore, $L^{k,l}$ is commutative with $D_\uparrow$ and $D_\downarrow$, and not commutative with D and D_0.

The main results of this section is the following:

Proposition 1. *Let $k,l\in 2\mathbb{Z}$, $k,l>0$, $s\in\mathbb{C}$ and $k+2\mathrm{Re}(s)>2n+1$. For $\mathfrak{Z}=\begin{pmatrix}Z^{(n)} & {}^tU^{(n)}\\ U^{(n)} & W^{(n)}\end{pmatrix}\in\mathfrak{H}_{2n}$, $\mathfrak{Z}_0=\begin{pmatrix}Z^{(n)} & 0\\ 0 & W^{(n)}\end{pmatrix}\in\mathfrak{H}_{2n}$, we get*

$$(L^{k,l}E_k^{2n})(\mathfrak{Z},s)$$
$$=\sum_M\frac{1}{(2\pi i)^l}\det(\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-k}\det(\mathrm{Im}(M\langle Z_0\rangle))^s\left\{\sum_{\nu=0}^{\frac{l}{2}}a(l,\nu,k,s){Q_0}^{l-2\nu}(P_0-P_0')^\nu(R_0-R_0')^\nu\right\}\ ,$$

where $M=\begin{pmatrix}* & *\\ \mathfrak{C}^{(2n)} & \mathfrak{D}^{(2n)}\end{pmatrix}$ runs over a complete system of representatives of $P_{2n,0}\backslash\Gamma^{2n}$,

$$a(l,\nu,k,s)=\sum_{h=\nu}^{[\frac{l}{2}]}(-1)^{h-\nu}\binom{h}{\nu}b(l,h,k,s)\ , \tag{2.4}$$

$$b(l,\nu,k,s)=(-1)^l\frac{(2k-2+2\nu)_{l-2\nu}}{\nu!(l-2\nu)!(k-1+\nu)_{l-\nu}}\frac{(-s)_\nu(k+s)_{l-\nu}}{(k)_l}\quad(0\leq\nu\leq[\tfrac{l}{2}])\ , \tag{2.5}$$

$$P_0=((\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-1}\mathfrak{C})\left[\begin{pmatrix}{}^te_1\\ 0\end{pmatrix}\right]\ ,\qquad {P_0}'=\left(\frac{1}{2i}(\mathrm{Im}(\mathfrak{Z}_0))^{-1}\right)\left[\begin{pmatrix}{}^te_1\\ 0\end{pmatrix}\right]$$
$$Q_0=(e_1\ \ 0)((\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-1}\mathfrak{C})\begin{pmatrix}0\\ {}^te_2\end{pmatrix}\ ,\qquad {Q_0}'=(e_1\ \ 0)\left(\frac{1}{2i}(\mathrm{Im}(\mathfrak{Z}_0))^{-1}\right)\begin{pmatrix}0\\ {}^te_2\end{pmatrix}=0\ ,$$
$$R_0=((\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-1}\mathfrak{C})\left[\begin{pmatrix}0\\ {}^te_2\end{pmatrix}\right]\ ,\qquad {R_0}'=\left(\frac{1}{2i}(\mathrm{Im}(\mathfrak{Z}_0))^{-1}\right)\left[\begin{pmatrix}0\\ {}^te_2\end{pmatrix}\right]$$

Now we put

$$S_0=((\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-1}\mathfrak{C})\,[{}^tx]\ ,\qquad \delta=\det(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})\ ,\qquad d^*\delta=\delta_0=\det(\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})\ ,$$
$${S_0}'=\left(\frac{1}{2i}(\mathrm{Im}(\mathfrak{Z}_0))^{-1}\right)[{}^tx]\ ,\qquad \sigma=\det(\mathrm{Im}(\mathfrak{Z}))\ ,\qquad d^*\sigma=\sigma_0=\det(\mathrm{Im}(\mathfrak{Z}_0))\ .$$

We note

$$S_0=P_0+2Q_0+R_0\ ,\qquad {S_0}'={P_0}'+2{Q_0}'+{R_0}'={P_0}'+{R_0}'\ .$$

Since

$$\det(\mathrm{Im}(M\langle\mathfrak{Z}\rangle))=|\det(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})|^{-2}\det(\mathrm{Im}(\mathfrak{Z}))\ ,\qquad \mathrm{Im}(\mathfrak{Z})=\frac{1}{2i}(\mathfrak{Z}-\overline{\mathfrak{Z}})\ ,$$

we have only to prove lemma 1 below so as to prove Proposition 1:

Lemma 1. *For $k, l \in \mathbb{Z}$, $k, l > 0$, $s \in \mathbb{C}$, and $M = \begin{pmatrix} * & * \\ \mathfrak{C}^{(2n)} & \mathfrak{D}^{(2n)} \end{pmatrix} \in \Gamma^{2n}$, we get*

$$L^{k,l}(\delta^{-k-s}\sigma^s) = \frac{1}{(2\pi i)^l}\delta_0{}^{-k-s}\sigma_0{}^s\sum_{\nu=0}^{[\frac{l}{2}]} a(l,\nu,k,s)Q_0{}^{l-2\nu}(P_0 - P_0{}')^\nu(R_0 - R_0{}')^\nu ,$$

where notations are the same as those in Proposition 1.

In order to prove lemma 1 inductively, we need the following lemma:

Lemma 2. *For $k, l \in \mathbb{Z}$, $k > 0$, $l \geq 0$, we have*

$$L^{k,l} = \frac{1}{(2\pi i)l(k+l-1)}L^{k,l-1}D_0 + \frac{(k)_{l-2}(4-2k-l)}{(2\pi i)^2 l(k)_l(2-k-l)_2}D_\uparrow D_\downarrow L^{k,l-2} .$$

Since $L^{k,l-1}D_0 \neq D_0 L^{k,l-1}$ $(=0)$, we must consider what is $L^{k,l-1}D_0(\delta^{-k-s}\sigma^s)$ in a different way. We note $L^{k,l-1}D_0 = d^* L_*^{k,l-1}D_0 = d^* D_0 L_*^{k,l-1}$.

Lemma 3. *For $k, l \in \mathbb{Z}$, $k > 0$, $l \geq 2$, $s \in \mathbb{C}$, we obtain*

$$L^{k,l-1}D_0(\delta^{-k-s}\sigma^s) = d^* D_0\left(L_*^{k,l-1}|_{Q'=0}\right)(\delta^{-k-s}\sigma^s) + \frac{s}{(2\pi i)k}(d^*(D_0{}^2\sigma)) \cdot L^{k+1,l-2}(\delta^{-k-s}\sigma^{s-1}) .$$

Proof of lemma 2. We prove it for even l. The assertion for odd l is proved in the same way.
Let $l = 2m$. Then,

$$\begin{aligned}
&L^{k,2m} - \frac{1}{2m(k+2m-1)(2\pi i)}L^{k,2m-1}D^0 \\
&= \frac{1}{(2\pi i)^l(k)_{2m}}d^*\left\{\sum_{\nu=1}^{m-1}\left(\frac{1}{\nu!(2m-2\nu)!(2-k-2m)_\nu}\right.\right. \\
&\quad \left.\left. - \frac{1}{\nu!2m(2m-1-2\nu)!(3-k-2m)_\nu}\right)(D_\uparrow D_\downarrow)^\nu D_0{}^{2m-2\nu}\right\} \\
&\quad + \frac{1}{(2\pi i)^l(k)_{2m}m!(2-k-2m)_m}d^*(D_\uparrow D_\downarrow)^m .
\end{aligned}$$

Here, we note

$$\begin{aligned}
&\frac{1}{\nu!(2m-2\nu)!(2-k-2m)_\nu} - \frac{1}{\nu!2m(2m-1-2\nu)!(3-k-2m)_\nu} \\
&= \frac{4-2k-2m}{2m(\nu-1)!(2m-2\nu)!(2-k-2m)_{\nu+1}} .
\end{aligned}$$

Therefore,

$$\begin{aligned}
&L^{k,2m} - \frac{1}{2m(k+2m-1)(2\pi i)}L^{k,2m-1}D^0 \\
&= \frac{1}{(2\pi i)^{2m}(k)_{2m}}D_\uparrow D_\downarrow d^*\sum_{\nu=1}^{m}\frac{4-2k-2m}{(\nu-1)!2m(2m-2\nu)!(2-k-2m)_{\nu+1}}(D_\uparrow D_\downarrow)^{\nu-1}D_0{}^{2m-2\nu} \\
&= \frac{4-2k-2m}{(2\pi i)^{2m}2m(k)_{2m}}D_\uparrow D_\downarrow d^*\sum_{\nu=0}^{m-1}\frac{1}{\nu!((2m-2)-2\nu)!(2-k-2m)_{\nu+2}}(D_\uparrow D_\downarrow)^\nu D_0{}^{(2m-2)-2\nu} \\
&= \frac{(k)_{2m-2}(4-2k-2m)}{(2\pi i)^2(k)_{2m}2m(2-k-2m)_2}D_\uparrow D_\downarrow L^{k,2m-2} . \qquad \square
\end{aligned}$$

We introduce some formulae before we prove lemma 1 and lemma 3.

When we replace $\mathfrak{Z}_0$ in P_0, P_0', $\cdots$, S_0, S_0' with $\mathfrak{Z}$, we write P, P', $\cdots$, S, S' for P_0, P_0', $\cdots$, S_0, S_0', respectively. We note $d^*P = P_0$, $d^*P' = P_0'$, $\cdots$, $d^*S = S_0'$ and $d^*S' = S_0'$. Then we get the following table:

	δ	S	P	Q	R
D	δS	$-S^2$	$-(P+Q)^2$	$-(P+Q)(R+Q)$	$-(R+Q)^2$
$D_\uparrow$	δP	$-(P+Q)^2$	$-P^2$	$-PQ$	$-Q^2$
D_0	$2\delta Q$	$-2(P+Q)(R+Q)$	$-2PQ$	$-PR-Q^2$	$-2RQ$
$D_\downarrow$	δR	$-(R+Q)^2$	$-Q^2$	$-RQ$	$-R^2$

Moreover, we get

$$D_\uparrow D_0\delta = D_0 D_\uparrow\delta = 0\ ,\quad D_\downarrow D_0\delta = D_0 D_\downarrow\delta = 0\ ,\quad {D_0}^3\delta = 0\ ,$$

$${D_0}^2\delta = D_0(2\delta Q) = -2\delta(PR-Q^2)\ ,\quad D_\uparrow D_\downarrow\delta = \delta(PR-Q^2)\ .$$

If we replace δ, S, $\cdots$,R with σ, S', $\cdots$,R' respectively, we obtain formulae for σ, S', $\cdots$,R' in the same way. We also get tables for δ_0 and for σ_0 if we only remark $D_0 d^* = 0$ and $Q_0' = 0$.

Proof of lemma 3. We recall $L^{k,l-1}D_0(\delta^{-k-s}\sigma^s) = d^*D_0L_*^{k,l-1}(\delta^{-k-s}\sigma^s)$. Then we have

$$L_*^{k,l-1}(\delta^{-k-s}\sigma^s) = \delta^{-k-s}\sigma^s H\ ,$$

where H is a polynomial of P, Q, R, P', Q' and R'. Moreover, we have an expansion

$$L_*^{k,l-1}(\delta^{-k-s}\sigma^s) = \delta^{-k-s}\sigma^s\sum_{r=0}^{h} H_r Q'^r\ ,$$

where H_r is a polynomial of P, Q, R, P' and R', and $h<\infty$. By the table above and so on, we obtain

$$d^*D_0L_*^{k,l-1}(\delta^{-k-s}\sigma^s) = d^*D_0\left(\delta^{-k-s}\sigma^s(H_0+H_1Q')\right)$$

and

$$d^*D_0\left(\delta^{-k-s}\sigma^s H_0\right) = d^*D_0\left(L_*^{k,l-1}|_{Q'=0}\right)(\delta^{-k-s}\sigma^s)\ .$$

On the other hand, $d^*D_0(\delta^{-k-s}\sigma^s H_1Q')$ is given by the following way:

We consider $\delta^{-k-s}\sigma^s$ a composite function of δ^{-k-s} and σ^s. Once a factor D_0 of $(L'_\uparrow D_\downarrow)^\nu {D_0}^{(l-1)-2\nu}$ $(0\le\nu\le[\frac{l-1}{2}])$ in $L_*^{k,l-1}$ acts on σ^s, nothing acts on $D_0\sigma$ $(=2\sigma Q')$ which comes from $D_0(\sigma^s) = s\sigma^{s-1}(D_0\sigma)$. That is,

$$\begin{aligned}
&d^*D_0(\delta^{-k-s}\sigma^s H_1 Q')\\
&= d^*D_0\left\{\left(\frac{1}{(2\pi i)^{l-1}(k)_{l-1}}\sum_{\nu=0}^{[\frac{l}{2}]-1}\frac{1}{\nu!(l-1-2\nu)!(2-k-(l-1))_\nu}(D_\uparrow D_\downarrow)^\nu {D_0}^{l-2-2\nu}(\delta^{-k-s}\sigma^{s-1})\right)\right.\\
&\qquad\left.\times\binom{l-1-2\nu}{1}s(D_0\sigma)\right\}\\
&= \left\{\frac{1}{(2\pi i)^{l-1}(k)_{l-1}}d^*\sum_{\nu=0}^{[\frac{l-2}{2}]}\frac{1}{\nu!(l-1-2\nu)!(2-k-(l-1))_\nu}(D_\uparrow D_\downarrow)^\nu {D_0}^{l-2-2\nu}(\delta^{-k-s}\sigma^{s-1})\right\}\\
&\qquad\times\binom{l-1-2\nu}{1}s(d^*({D_0}^2\sigma))
\end{aligned}$$

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$$= \left\{ \frac{1}{(2\pi i)^{l-2}(k+1)_{l-2}} d^* \sum_{\nu=0}^{[\frac{l-2}{2}]} \frac{1}{\nu!(l-2-2\nu)!(2-(k+1)-(l-2))_\nu} (D_\uparrow D_\downarrow)^\nu {D_0}^{l-2-2\nu}(\delta^{-k-s}\sigma^{s-1}) \right\}$$
$$\times \frac{s}{(2\pi i)k}(d^*({D_0}^2\sigma))$$
$$= \frac{s}{(2\pi i)k}(d^*({D_0}^2\sigma)) \cdot L^{k+1,l-2}(\delta^{-k-s}\sigma^{s-1}) .$$

Remark. For odd l, $L_*^{k,l-1}$ has a term $(D_\uparrow D_\downarrow)^{\frac{l-1}{2}}$, but $(D_\uparrow D_\downarrow)^{\frac{l-1}{2}}$ has no relation to $d^* D_0(\delta^{-k-s}\sigma^s H_1 Q')$ because $d^* D_0 (D_\uparrow D_\downarrow)^{\frac{l-1}{2}} = (D_\uparrow D_\downarrow)^{\frac{l-1}{2}} d^* D_0$. □

Proof of lemma 1. We use induction on l. For $l = 1$ and for $l = 2$, lemma 1 holds.
Let $l > 2$. By lemma 2, we have

$$L^{k,l}(\delta^{-k-s}\sigma^s)$$
$$= \frac{1}{(2\pi i)l(k+l-1)} L^{k,l-1} D_0(\delta^{-k-s}\sigma^s) + \frac{(k)_{l-2}(4-2k-l)}{(2\pi i)^2 (k)_l l(2-k-l)_2} D_\uparrow D_\downarrow L^{k,l-2}(\delta^{-k-s}\sigma^s)$$

by lemma 3,

$$= \frac{1}{(2\pi i)l(k+l-1)} d^* D_0 \left(L_*^{k,l-1}\big|_{Q'=0} \right) (\delta^{-k-s}\sigma^s)$$
$$+ \frac{s}{(2\pi i)^2 l(k+l-1)k} (d^*({D_0}^2\sigma)) L^{k+1,l-2}(\delta^{-k-s}\sigma^{s-1})$$
$$+ \frac{(k)_{l-2}(4-2k-l)}{(2\pi i)^2 (k)_l l(2-k-l)_2} D_\uparrow D_\downarrow L^{k,l-2}(\delta^{-k-s}\sigma^s)$$

since the assertion of induction is valid for $\left(L_*^{k,l-1}\Big|_{Q'=0} \right) (\delta^{-k-s}\sigma^s)$,

$$= \frac{1}{(2\pi i)^l l(k+l-1)} d^* D_0 \left(\sum_{\nu=0}^{[\frac{l-1}{2}]} \delta^{-k-s}\sigma^s a(l-1,\nu,k,s) Q^{l-1-2\nu}(P-P')^\nu (R-R')^\nu \right)$$
$$+ \frac{-2s}{(2\pi i)^l l(k+l-1)k} \left\{ \sum_{\nu=0}^{[\frac{l}{2}]-1} {\delta_0}^{-k-s}{\sigma_0}^s a(l-2,\nu,k+1,s-1) {Q_0}^{l-2-2\nu}(P_0-P_0')^\nu (R_0-R_0')^\nu \right\} P_0' R_0'$$
$$+ \frac{(k)_{l-2}(4-2k-l)}{(2\pi i)^l (k)_l l(2-k-l)_2} D_\uparrow D_\downarrow \left(\sum_{\nu=0}^{[\frac{l}{2}]-1} {\delta_0}^{-k-s}{\sigma_0}^s a(l-2,\nu,k,s) {Q_0}^{l-2-2\nu}(P_0-P_0')^\nu (R_0-R_0')^\nu \right) .$$

On the other hand, we have

$$d^* D_0(\delta^{-k-s}\sigma^s Q^{l-1-2\nu}(P-P')^\nu (R-R')^\nu)$$
$$= -(2k+2s+l-1-2\nu){\delta_0}^{-k-s}{\sigma_0}^s {Q_0}^{l-2\nu}(P_0-P_0')^\nu (R_0-R_0')^\nu$$
$$- 4\nu {\delta_0}^{-k-s}{\sigma_0}^s {Q_0}^{l-2\nu} P_0 R_0 (P_0-P_0')^{\nu-1}(R_0-R_0')^{\nu-1}$$
$$+ 2\nu {\delta_0}^{-k-s}{\sigma_0}^s {Q_0}^{l-2\nu} (P_0 R_0' + P_0' R_0)(P_0-P_0')^{\nu-1}(R_0-R_0')^{\nu-1}$$
$$- (l-1-2\nu){\delta_0}^{-k-s}{\sigma_0}^s {Q_0}^{l-2-2\nu} P_0 R_0 (P_0-P_0')^\nu (R_0-R_0')^\nu \qquad (0 \le \nu \le [\tfrac{l-1}{2}]) ,$$

and

$$\begin{aligned}
&D_\uparrow D_\downarrow(\delta_0{}^{-k-s}\sigma_0{}^s Q_0{}^{l-2-2\nu}(P_0-P_0')^\nu(R_0-R_0')^\nu)\\
&=\nu^2\delta_0{}^{-k-s}\sigma_0{}^s Q_0{}^{l-2\nu+2}(P_0-P_0')^{\nu-1}(R_0-R_0')^{\nu-1}\\
&\quad+(k+s+l-2-\nu)\delta_0{}^{-k-s}\sigma_0{}^s Q_0{}^{l-2\nu}(P_0-P_0')^\nu(R_0-R_0')^\nu\\
&\quad+\nu(2k+2s+2l-2\nu-3)\delta_0{}^{-k-s}\sigma_0{}^s Q_0{}^{l-2\nu}P_0R_0(P_0-P_0')^{\nu-1}(R_0-R_0')^{\nu-1}\\
&\quad-\nu(k+2s+l-2\nu-1)\delta_0{}^{-k-s}\sigma_0{}^s Q_0{}^{l-2\nu}(P_0R_0'+P_0'R_0)(P_0-P_0')^{\nu-1}(R_0-R_0')^{\nu-1}\\
&\quad+\nu(2s-2\nu+1)\delta_0{}^{-k-s}\sigma_0{}^s Q_0{}^{l-2\nu}P_0'R_0'(P_0-P_0')^{\nu-1}(R_0-R_0')^{\nu-1}\\
&\quad+(k+s+l-2-\nu)^2\delta_0{}^{-k-s}\sigma_0{}^s Q_0{}^{l-2\nu-2}P_0R_0(P_0-P_0')^\nu(R_0-R_0')^\nu\\
&\quad-(k+s+l-2-\nu)(s-\nu)\delta_0{}^{-k-s}\sigma_0{}^s Q_0{}^{l-2\nu-2}(P_0R_0'+P_0'R_0)(P_0-P_0')^\nu(R_0-R_0')^\nu\\
&\quad+(s-\nu)^2\delta_0{}^{-k-s}\sigma_0{}^s Q_0{}^{l-2\nu-2}P_0'R_0'(P_0-P_0')^\nu(R_0-R_0')^\nu \qquad (0\le\nu\le[\tfrac{l}{2}]-1)\ .
\end{aligned}$$

Thus we have only to prove the following equations:

(1) coefficient of $Q_0{}^l$

$$\begin{aligned}
a(l,0,k,s)=&\frac{1}{l(k+l-1)}\Big\{-(2k+2s+l-1)a(l-1,0,k,s)\Big\}\\
&+\frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2}\Big\{(k+s+l-2)a(l-2,0,k,s)+a(l-2,1,k,s)\Big\}
\end{aligned}$$

(2) coefficient of $Q_0{}^{l-2\nu}P_0R_0(P_0-P_0')^{\nu-1}(R_0-R_0')^{\nu-1}$ $\quad(1\le\nu\le[\frac{l}{2}]-2)$

$$\begin{aligned}
a(l,\nu,k,s)=&\frac{1}{l(k+l-1)}\Big\{-(2k+2s+l-1+2\nu)a(l-1,\nu,k,s)-(l+1-2\nu)a(l-1,\nu-1,k,s)\Big\}\\
&+\frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2}\Big\{(\nu+1)^2a(l-2,\nu+1,k,s)+(k+s+l-2-\nu)a(l-2,\nu,k,s)\\
&+\nu(2k+2s+2l-2\nu-3)a(l-2,\nu,k,s)+(k+s+l-1-\nu)^2a(l-2,\nu-1,k,s)\Big\}
\end{aligned}$$

(3) coefficient of $-Q_0{}^{l-2\nu}(P_0R_0'+P_0'R_0)(P_0-P_0')^{\nu-1}(R_0-R_0')^{\nu-1}$ $\quad(1\le\nu\le[\frac{l}{2}]-2)$

$$\begin{aligned}
a(l,\nu,k,s)=&\frac{1}{l(k+l-1)}\Big\{-(2k+2s+l-1)a(l-1,\nu,k,s)\Big\}\\
&+\frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2}\Big\{(\nu+1)^2a(l-2,\nu+1,k,s)+(k+s+l-2-\nu)a(l-2,\nu,k,s)\\
&+\nu(k+2s+l-2\nu-1)a(l-2,\nu,k,s)+(k+s+l-1-\nu)(s-\nu+1)a(l-2,\nu-1,k,s)\Big\}
\end{aligned}$$

(4) coefficient of $Q_0{}^{l-2\nu}P_0'R_0'(P_0-P_0')^{\nu-1}(R_0-R_0')^{\nu-1}$ $\quad(1\le\nu\le[\frac{l}{2}]-2)$

$$\begin{aligned}
a(l,\nu,k,s)=&\frac{1}{l(k+l-1)}\Big\{-(2k+2s+l-1-2\nu)a(l-1,\nu,k,s)-\frac{2s}{k}a(l-2,\nu-1,k+1,s-1)\Big\}\\
&+\frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2}\Big\{(\nu+1)^2a(l-2,\nu+1,k,s)+(k+s+l-2-\nu)a(l-2,\nu,k,s)\\
&+\nu(2s-2\nu+1)a(l-2,\nu,k,s)+(s-\nu+1)^2a(l-2,\nu-1,k,s)\Big\}
\end{aligned}$$

(5) coefficient of $Q_0{}^{l-2\nu}P_0R_0(P_0-P_0')^{\nu-1}(R_0-R_0')^{\nu-1}$ $\quad(\nu=[\frac{l}{2}]-1)$

$$\begin{aligned}
a(l,\nu,k,s)=&\frac{1}{l(k+l-1)}\Big\{-(2k+2s+l-1+2\nu)a(l-1,\nu,k,s)-(l+1-2\nu)a(l-1,\nu-1,k,s)\Big\}\\
&+\frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2}\Big\{(k+s+l-2-\nu)a(l-2,\nu,k,s)\\
&+\nu(2k+2s+2l-2\nu-3)a(l-2,\nu,k,s)+(k+s+l-1-\nu)^2a(l-2,\nu-1,k,s)\Big\}
\end{aligned}$$

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(6) coefficient of $-{Q_0}^{l-2\nu}(P_0R_0'+P_0'R_0)(P_0-P_0')^{\nu-1}(R_0-R_0')^{\nu-1}$ $\quad(\nu=[\frac{l}{2}]-1)$

$$a(l,\nu,k,s)=\frac{1}{l(k+l-1)}\Big\{-(2k+2s+l-1)a(l-1,\nu,k,s)\Big\}$$
$$+\frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2}\Big\{(k+s+l-2-\nu)a(l-2,\nu,k,s)$$
$$+\nu(k+2s+l-2\nu-1)a(l-2,\nu,k,s)+(k+s+l-1-\nu)(s-\nu+1)a(l-2,\nu-1,k,s)\Big\}$$

(7) coefficient of ${Q_0}^{l-2\nu}P_0'R_0'(P_0-P_0')^{\nu-1}(R_0-R_0')^{\nu-1}$ $\quad(\nu=[\frac{l}{2}]-1)$

$$a(l,\nu,k,s)=\frac{1}{l(k+l-1)}\Big\{-(2k+2s+l-1-2\nu)a(l-1,\nu,k,s)-\frac{2s}{k}a(l-2,\nu-1,k+1,s-1)\Big\}$$
$$+\frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2}\Big\{(k+s+l-2-\nu)a(l-2,\nu,k,s)$$
$$+\nu(2s-2\nu+1)a(l-2,\nu,k,s)+(s-\nu+1)^2a(l-2,\nu-1,k,s)\Big\}$$

For even l, we put $l=2m$.
(8) coefficient of $P_0R_0(P_0-P_0')^{m-1}(R_0-R_0')^{m-1}$

$$a(l,m,k,s)=\frac{1}{l(k+l-1)}\Big\{-a(l-1,m-1,k,s)\Big\}$$
$$+\frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2}\Big\{(k+s+m-1)^2a(l-2,m-1,k,s)\Big\}$$

(9) coefficient of $-(P_0R_0'+P_0'R_0)(P_0-P_0')^{m-1}(R_0-R_0')^{m-1}$

$$a(l,m,k,s)=\frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2}\Big\{(k+s+m-1)(s-m+1)a(l-2,m-1,k,s)\Big\}$$

(10) coefficient of $P_0'R_0'(P_0-P_0')^{m-1}(R_0-R_0')^{m-1}$

$$a(l,m,k,s)=\frac{1}{l(k+l-1)}\Big\{-\frac{2s}{k}a(l-2,m-1,k+1,s-1)\Big\}$$
$$+\frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2}\Big\{(s-m+1)^2a(l-2,m-1,k,s)\Big\}$$

For odd l, we put $l=2m+1$.
(8') coefficient of $Q_0P_0R_0(P_0-P_0')^{m-1}(R_0-R_0')^{m-1}$

$$a(l,m,k,s)=\frac{1}{l(k+l-1)}\Big\{-(2k+2s+4m)a(l-1,m,k,s)-2a(l-1,m-1,k,s)\Big\}$$
$$+\frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2}\Big\{(k+s+m)^2a(l-2,m-1,k,s)\Big\}$$

(9') coefficient of $-Q_0(P_0R_0'+P_0'R_0)(P_0-P_0')^{m-1}(R_0-R_0')^{m-1}$

$$a(l,m,k,s)=\frac{1}{l(k+l-1)}\Big\{-(2k+2s+2m)a(l-1,m,k,s)\Big\}$$
$$+\frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2}\Big\{(k+s+m)(s-m+1)a(l-2,m-1,k,s)\Big\}$$

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(10') coefficient of $Q_0P_0'R_0'(P_0-P_0')^{m-1}(R_0-R_0')^{m-1}$

$$a(l,m,k,s)=\frac{1}{l(k+l-1)}\left\{-(2k+2s)a(l-1,m,k,s)-\frac{2s}{k}a(l-2,m-1,k+..,s-1)\right\}$$
$$+\frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2}\left\{(s-m+1)^2a(l-2,m-1,k,s)\right\}$$

By properties of $a(l,\nu,k,s)$, it is easy to prove (5), (6), (7), (8), (9), (10), (8'), (9'), (10). Therefore, we have only to prove (1), (2), (3), (4). But we only prove (2) for even l, here. The assertions of (1), (3), (4) and of (2) for odd l are proved in the same way.

Let $l=2m$. Then, by (2.4), the right-hand side of (2) is:

$$\frac{1}{2m(k+2m-1)}\left\{-(2k+2s+2m-1+2\nu)\sum_{h=\nu}^{m-1}(-1)^{h-\nu}\binom{h}{\nu}b(2m-1,h,k,s)\right.$$
$$\left.-(2m+1-2\nu)\sum_{h=\nu-1}^{m-1}(-1)^{h+1-\nu}\binom{h}{\nu-1}b(2m-1,h,k,s)\right\}$$
$$-\frac{(k)_{2m-2}(2k+2m-4)}{(k)_{2m}2m(k+2m-3)_2}\left\{(\nu+1)^2\sum_{h=\nu+1}^{m-1}(-1)^{h-\nu+1}\binom{h}{\nu+1}b(2m-2,h,k,s)\right.$$
$$+(k+s+2m-\nu-1)^2\sum_{h=\nu-1}^{m-1}(-1)^{h+1-\nu}\binom{h}{\nu-1}b(2m-2,h,k,s)$$
$$\left.+((k+s+2m-\nu-2)+\nu(2k+2s+4m-2\nu-3))\sum_{h=\nu}^{m-1}(-1)^{h-\nu}\binom{h}{\nu}b(2m-2,h,k,s)\right\}$$

(2.6)

$$=\frac{1}{2m(k+2m-1)}\left\{\sum_{h=\nu-1}^{m-1}(-1)^{h-\nu}\left(\binom{h}{\nu-1}(2k+2s+4m)\right.\right.$$
$$\left.\left.-\binom{h+1}{\nu}(2k+2s+2m+2\nu-1)\right)b(2m-1,h,k,s)\right\}$$
$$+\frac{(k)_{2m-2}(2k+2m-4)}{(k)_{2m}2m(k+2m-3)_2}\left\{-(k+s+2m-\nu-1)^2b(2m-2,\nu-1,k,s)\right.$$
$$+\sum_{h=\nu}^{m-1}(-1)^{h-\nu}\left(\binom{h}{\nu-1}(k+s+2m-\nu-1)(k+s+2m-h-2)\right.$$
$$\left.\left.-\binom{h}{\nu}(\nu+1)(k+s+2m-h-2)\right)b(2m-2,h,k,s)\right\}\quad.$$

We note

(2.7)
$$\binom{h}{\nu-1}(2k+2s+4m)-\binom{h+1}{\nu}(2k+2s+2m+2\nu-1)$$
$$=\binom{h+1}{\nu}(2m-2h-1)-2\binom{h}{\nu}(k+s+2m-h-1)\ ,$$

(2.8)
$$\binom{h}{\nu-1}(k+s+2m-\nu-1)-\binom{h}{\nu}(\nu+1)$$
$$=\binom{h}{\nu-1}(k+s+2m-h-1)+\binom{h}{\nu-1}(h-\nu)-\binom{h}{\nu}(\nu+1)$$
$$=\binom{h}{\nu-1}(k+s+2m-h-1)-\binom{h+1}{\nu}$$

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and

$$(2.9)\qquad (k+s+2m-\nu-1)^2 = (k+s+2m-\nu-1)_2 - (k+s+2m-\nu-1)\ .$$

By (2.7), (2.8), (2.9), the formula (2.6) is:

$$\begin{aligned}
&\frac{1}{2m(k+2m-1)}\left\{\sum_{h=\nu-1}^{m-1}(-1)^{h-\nu}\binom{h+1}{\nu}(2m-2h-1)b(2m-1,h,k,s)\right.\\
&\qquad\left.-\sum_{h=\nu}^{m-1}(-1)^{h-\nu}\binom{h}{\nu}2(k+s+2m-h-1)b(2m-1,h,k,s)\right\}\\
&+\frac{(k)_{2m-2}(2k+2m-4)}{(k)_{2m}2m(k+2m-3)_2}\left\{\sum_{h=\nu-1}^{m-1}(-1)^{h-\nu}\left(\binom{h}{\nu-1}(k+s+2m-h-2)_2\right.\right.\\
&\qquad\left.\left.-\binom{h+1}{\nu}(k+s+2m-h-2)\right)b(2m-2,h,k,s)\right\}
\end{aligned}$$

by (2.5)

$$\begin{aligned}
&=-\sum_{h=\nu-1}^{m-1}(-1)^{h-\nu}\binom{h+1}{\nu}\frac{(2k+2h-2)_{2m-2h-1}}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h-1}}\frac{(-s)_h(k+s)_{2m-h-1}}{(k)_{2m}}\\
&+\sum_{h=\nu}^{m-1}(-1)^{h-\nu}\binom{h}{\nu}\frac{(2k+2h-2)_{2m-2h-1}}{m(h)!(2m-2h-1)!(k+h-1)_{2m-h-1}}\frac{(-s)_h(k+s)_{2m-h}}{(k)_{2m}}\\
(2.10)\qquad&+\sum_{h=\nu-1}^{m-1}(-1)^{h-\nu}\binom{h}{\nu-1}\frac{(2k+2h-2)_{2m-2h-1}}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h}}\frac{(-s)_h(k+s)_{2m-h}}{(k)_{2m}}\\
&-\sum_{h=\nu-1}^{m-1}(-1)^{h-\nu}\binom{h+1}{\nu}\frac{(2k+2h-2)_{2m-2h-1}}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h}}\frac{(-s)_h(k+s)_{2m-h-1}}{(k)_{2m}}\ .
\end{aligned}$$

We also note

$$\begin{aligned}
(2.11)\qquad&\frac{1}{m(h)!(2m-2h-1)!(k+h-1)_{2m-h-1}}\\
&=\frac{2k+2m-3}{h!(2m-2h)!(k+h-1)_{2m-h}}+\frac{1}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h}}\\
&-\frac{2k+2h-3}{m(h-1)!(2m-2h)!(k+h-1)_{2m-h}}\ .
\end{aligned}$$

If we add the first term of (2.10) to the fourth term of (2.10) and if we transform the second term of (2.10) according to (2.11), we get:

$$\begin{aligned}
&\sum_{h=\nu}^{m-1}(-1)^{h-\nu}\binom{h}{\nu}\frac{(2k+2h-2)_{2m-2h}}{h!(2m-2h)!(k+h-1)_{2m-h}}\frac{(-s)_h(k+s)_{2m-h}}{(k)_{2m}}\\
&+\sum_{h=\nu}^{m-1}(-1)^{h-\nu}\binom{h}{\nu}\frac{(2k+2h-2)_{2m-2h-1}}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h}}\frac{(-s)_h(k+s)_{2m-h}}{(k)_{2m}}\\
&-\sum_{h=\nu}^{m-1}(-1)^{h-\nu}\binom{h}{\nu}\frac{(2k+2h-3)_{2m-2h}}{m(h-1)!(2m-2h)!(k+h-1)_{2m-h}}\frac{(-s)_h(k+s)_{2m-h}}{(k)_{2m}}\\
&+\sum_{h=\nu-1}^{m-1}(-1)^{h-\nu}\binom{h}{\nu-1}\frac{(2k+2h-2)_{2m-2h-1}}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h}}\frac{(-s)_h(k+s)_{2m-h}}{(k)_{2m}}\\
&-\sum_{h=\nu-1}^{m-1}(-1)^{h-\nu}\binom{h+1}{\nu}\frac{(k+2m-1)(2k+2h-2)_{2m-2h-1}}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h}}\frac{(-s)_h(k+s)_{2m-h-1}}{(k)_{2m}}
\end{aligned}$$

$$
\begin{aligned}
&= a(2m,\nu,k,s) - (-1)^{m-\nu}\binom{m}{\nu}\frac{1}{m!(k+m-1)_m}\frac{(-s)_m(k+s)_m}{(k)_{2m}} \\
&+ \sum_{h=\nu-1}^{m-1}(-1)^{h-\nu}\binom{h+1}{\nu}\frac{(2k+2h-2)_{2m-2h-1}}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h}}\frac{(-s)_h(k+s)_{2m-h}}{(k)_{2m}} \\
&- \sum_{h=\nu}^{m-1}(-1)^{h-\nu}\binom{h}{\nu}\frac{(2k+2h-3)_{2m-2h}}{m(h-1)!(2m-2h)!(k+h-1)_{2m-h}}\frac{(-s)_h(k+s)_{2m-h}}{(k)_{2m}} \\
&- \sum_{h=\nu-1}^{m-1}(-1)^{h-\nu}\binom{h+1}{\nu}\frac{(k+2m-1)(2k+2h-2)_{2m-2h-1}}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h}}\frac{(-s)_h(k+s)_{2m-h-1}}{(k)_{2m}}
\end{aligned}
$$

$$
\begin{aligned}
&= a(2m,\nu,k,s) - (-1)^{m-\nu}\binom{m}{\nu}\frac{1}{m!(k+m-1)_m}\frac{(-s)_m(k+s)_m}{(k)_{2m}} \\
&- \sum_{h=\nu}^{m-1}(-1)^{h-\nu}\binom{h}{\nu}\frac{(2k+2h-3)_{2m-2h}}{m(h-1)!(2m-2h)!(k+h-1)_{2m-h}}\frac{(-s)_h(k+s)_{2m-h}}{(k)_{2m}} \\
&+ \sum_{h=\nu-1}^{m-1}(-1)^{h+1-\nu}\binom{h+1}{\nu}\frac{(2k+2h-1)_{2m-2h-2}}{m(h)!(2m-2h-2)!(k+h)_{2m-h-1}}\frac{(-s)_{h+1}(k+s)_{2m-h-1}}{(k)_{2m}} \\
&= a(2m,\nu,k,s) \quad .
\end{aligned}
$$

□

2.2 Pullback formula

We consider $\Gamma^n \times \Gamma^n$ imbedded in Γ^{2n} by

$$
\begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \times \begin{pmatrix} A'^{(n)} & B'^{(n)} \\ C'^{(n)} & D'^{(n)} \end{pmatrix} \longmapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ C & 0 & D & 0 \\ 0 & C' & 0 & D' \end{pmatrix},
$$

and when convenient will identify $\Gamma^n \times \Gamma^n$ with its image in Γ^{2n}.

In this section we decompose $(L^{k,l}E_k^{2n})(\mathfrak{Z},s)$, $\mathfrak{Z} = \begin{pmatrix} Z^{(n)} & {}^tU^{(n)} \\ U^{(n)} & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$ into functions for Z and for W. The main tool is the coset decomposition by Garrett [12]:

(2.2.1). (i) *The double coset $P_{2n,0}\backslash\Gamma^{2n}/\Gamma^n \times \Gamma^n$ has an irredundant set of coset representatives*

$$
g_{\tilde{T}} = \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 \\ 0 & \tilde{T}^{(n)} & 1_n & 0 \\ \tilde{T}^{(n)} & 0 & 0 & 1_n \end{pmatrix},
$$

where $\tilde{T} = \begin{pmatrix} 0 & 0 \\ 0 & T^{(r)} \end{pmatrix}$, $T \in \mathbb{T}^{(r)}$ $(0 \le r \le n)$.

(ii) *The left coset $P_{2n,0}\backslash P_{2n,0}\, g_{\tilde{T}}(\Gamma^n \times \Gamma^n)$ has an irredundant set of coset representatives $g_{\tilde{T}}\tilde{g}_1 g_2 \tilde{g}'_1 g'_2$,*

$$
\tilde{g}_1 \in G_{n,r}\ ,\ g_2 \in P_{n,r}\backslash\Gamma^n\ ,\ \tilde{g}'_1 \in \Gamma^r(T)\backslash G_{n,r}\ ,\ g'_2 \in P_{n,r}\backslash\Gamma^n\ ,
$$

where

$$
(2.12)\quad G_{n,r} := \left\{ \tilde{g} = \begin{pmatrix} \tilde{A}^{(n)} & \tilde{B}^{(n)} \\ \tilde{C}^{(n)} & \tilde{D}^{(n)} \end{pmatrix} = \begin{pmatrix} 1_{n-r} & 0 & 0 & 0 \\ 0 & A^{(r)} & 0 & B^{(r)} \\ 0 & 0 & 1_{n-r} & 0 \\ 0 & C^{(r)} & 0 & D^{(r)} \end{pmatrix} \in \Gamma^n \;\middle|\; g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^r \right\}
$$

and for $T \in \mathbb{T}^{(r)}$,

$$\Gamma^r(T) := \left\{ g \in \Gamma^r \;\middle|\; \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} g \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} \in \Gamma^r \right\} . \tag{2.13}$$

Then the pullback formula is given by the following:

Proposition 2. *Let* $k, l \in 2\mathbb{Z}$, $k, l > 0$, $s \in \mathbb{C}$ *and* $k + 2\mathrm{Re}(s) > 2n + 1$. *For* $\mathfrak{Z} = \begin{pmatrix} Z^{(n)} & {}^tU^{(n)} \\ U^{(n)} & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$, *we have*

$$(L^{k,l}E_k^{2n})(\mathfrak{Z}, s) = \sum_{\nu=0}^{\frac{l}{2}} \left(-\frac{1}{4}\right)^{\nu} \frac{a(l,\nu,k,s)}{(2\pi i)^l} \sum_{r=1}^{n} \sum_{T \in \mathbb{T}^{(n)}} \mathcal{P}_\nu \left(Z, W, \begin{pmatrix} 0 & 0 \\ 0 & T^{(r)} \end{pmatrix}, s \right) , \tag{2.14}$$

with

(2.15)

$$\begin{aligned}
&\mathcal{P}_\nu \left(Z, W, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, s \right) \\
&:= \sum_{g_2 \in P_{n,r} \backslash \Gamma^n} \sum_{g_2' \in P_{n,r} \backslash \Gamma^n} \sum_{\tilde{g}_1 \in G_{n,r}} \sum_{\tilde{g}_1' \in \Gamma^r(T) \backslash G_{n,r}} \Big\{ \det(\mathrm{Im}(\tilde{g}_1 g_2 \langle Z \rangle))^s \det(\mathrm{Im}(\tilde{g}_1' g_2' \langle W \rangle))^s \\
&\qquad \times \left| \det(1_n - \tilde{T}(\tilde{g}_1' g_2' \langle W \rangle) \tilde{T}(\tilde{g}_1 g_2 \langle Z \rangle)) \right|^{-2s} \\
&\qquad \times \rho_1 \left((C_2 Z + D_2)^{-1} (\tilde{C}_1(g_2\langle Z \rangle) + \tilde{D}_1)^{-1} (1_n - \tilde{T}(\tilde{g}_1' g_2' \langle W \rangle) \tilde{T}(\tilde{g}_1 g_2 \langle Z \rangle))^{-1} \right) \\
&\qquad \times \rho \left((C_2' W + D_2')^{-1} (\tilde{C}_1'(g_2' \langle W \rangle) + \tilde{D}_2')^{-1} \right) \left(e_1 \tilde{T}\, {}^t e_2 \right)^{l-2\nu} \\
&\quad \times \left(e_1 (1_n - \tilde{T}(\tilde{g}_1' g_2' \langle W \rangle) \tilde{T}(\tilde{g}_1 g_2 \langle \overline{Z} \rangle)) (\mathrm{Im}(\tilde{g}_1 g_2 \langle Z \rangle))^{-1}\, {}^t(1_n - \tilde{T}(\tilde{g}_1' g_2' \langle W \rangle) \tilde{T}(\tilde{g}_1 g_2 \langle Z \rangle))^t e_1 \right)^{\nu} \\
&\quad \times \left(e_2 (1_n - \tilde{T}(\tilde{g}_1 g_2 \langle Z \rangle) \tilde{T}(\tilde{g}_1' g_2' \langle W \rangle))^{-1} (1_n - \tilde{T}(\tilde{g}_1 g_2 \langle Z \rangle) \tilde{T}(\tilde{g}_1' g_2' \langle \overline{W} \rangle)) (\mathrm{Im}(\tilde{g}_1' g_2' \langle W \rangle))^{-1}\, {}^t e_2 \right)^{\nu} \Big\} ,
\end{aligned}$$

where $\rho_1 = \det^k \otimes \mathrm{sym}^l$ *(resp.* $\rho = \det^k \otimes \mathrm{sym}^l$*) is the representation of* $GL(n, \mathbb{C})$ *with the representation space* $\mathrm{sym}^l(V_1)$ *(resp.* $\mathrm{sym}^l(V_2)$*)* , $\tilde{g}_1 = \begin{pmatrix} \tilde{A}_1^{(n)} & \tilde{B}_1^{(n)} \\ \tilde{C}_1^{(n)} & \tilde{D}_1^{(n)} \end{pmatrix}$, $\tilde{g}_1' = \begin{pmatrix} \tilde{A}_1'^{(n)} & \tilde{B}_1'^{(n)} \\ \tilde{C}_1'^{(n)} & \tilde{D}_1'^{(n)} \end{pmatrix}$ *are of the form (2.12)* , $= \begin{pmatrix} A_2{}^{(n)} & B_2{}^{(n)} \\ C_2{}^{(n)} & D_2{}^{(n)} \end{pmatrix}$, $g_2' = \begin{pmatrix} A_2'^{(n)} & B_2'^{(n)} \\ C_2'^{(n)} & D_2'^{(n)} \end{pmatrix}$, $\tilde{T}^{(n)} = \begin{pmatrix} 0 & 0 \\ 0 & T^{(r)} \end{pmatrix}$.

Proof. By Proposition 1, we have only to prove

$$\begin{aligned}
&\sum_M \det(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-k} \det(\mathrm{Im}(M\langle \mathfrak{Z}_0 \rangle))^s Q_0{}^{l-2\nu} (P_0 - P_0')^\nu (R_0 - R_0')^\nu \\
&= \left(-\frac{1}{4}\right)^{\nu} \sum_{r=1}^{n} \sum_{T \in \mathbb{T}^{(r)}} \mathcal{P}_\nu \left(Z, W, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, s \right) \qquad \left(0 \le \nu \le \frac{l}{2} \right) ,
\end{aligned}$$

where the notations are the same as those in Proposition 1 .

We put $g = \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} = \tilde{g}_1 g_2$, $g' = \begin{pmatrix} A'^{(n)} & B'^{(n)} \\ C'^{(n)} & D'^{(n)} \end{pmatrix} = \tilde{g}_1' g_2'$. Then a coset representative $M = \begin{pmatrix} \mathfrak{A}^{(2n)} & \mathfrak{B}^{(2n)} \\ \mathfrak{C}^{(2n)} & \mathfrak{D}^{(2n)} \end{pmatrix}$ of $P_{2n,0} \backslash \Gamma^{2n}$ can be written in the following form:

$$M = g_{\tilde{T}} g g' = \begin{pmatrix} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ C & \tilde{T}A' & D & \tilde{T}B' \\ \tilde{T}A & C' & \tilde{T}B & D' \end{pmatrix} . \tag{2.16}$$

By $(\mathrm{Im}(\mathfrak{Z}_0))^{-1} = (\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}(\mathrm{Im}(M\langle\mathfrak{Z}_0\rangle))^{-1}\ {}^t(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1} + 2i(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}\mathfrak{C}$, we have

$$P_0 - P_0' = \left(-\frac{1}{2i}\right)((\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}(\mathfrak{C}\overline{\mathfrak{Z}}_0 + \mathfrak{D})(\mathrm{Im}(\mathfrak{Z}_0))^{-1})\left[\begin{pmatrix} {}^te_1 \\ 0 \end{pmatrix}\right] ,$$

$$R_0 - R_0' = \left(-\frac{1}{2i}\right)((\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}(\mathfrak{C}\overline{\mathfrak{Z}}_0 + \mathfrak{D})(\mathrm{Im}(\mathfrak{Z}_0))^{-1})\left[\begin{pmatrix} 0 \\ {}^te_2 \end{pmatrix}\right] .$$

On the other hand, we have

$$\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D} = \begin{pmatrix} CZ + D & \tilde{T}(A'W + B') \\ \tilde{T}(AZ + B) & C'W + D' \end{pmatrix}$$

and

$$(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}$$
$$= \begin{pmatrix} (CZ + D)^{-1}(1_n - \tilde{T}g'\langle W\rangle\tilde{T}g\langle Z\rangle)^{-1} & -(CZ + D)^{-1}(1_n - \tilde{T}g'\langle W\rangle\tilde{T}g\langle Z\rangle)^{-1}\tilde{T}g'\langle W\rangle \\ -(C'W + D')^{-1}(1_n - \tilde{T}g\langle Z\rangle\tilde{T}g'\langle W\rangle)^{-1}\tilde{T}g\langle Z\rangle & (C'W + D')^{-1}(1_n - \tilde{T}g\langle Z\rangle\tilde{T}g'\langle W\rangle)^{-1} \end{pmatrix} .$$

If we put $(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}(\mathfrak{C}\overline{\mathfrak{Z}}_0 + \mathfrak{D})(\mathrm{Im}(\mathfrak{Z}_0))^{-1} = \begin{pmatrix} \mathcal{A}^{(n)} & * \\ * & \mathcal{D}^{(n)} \end{pmatrix}$ and $(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}\mathfrak{C} = \begin{pmatrix} * & \mathcal{B}^{(n)} \\ * & * \end{pmatrix}$, we get

$$\mathcal{A} = (CZ + D)^{-1}(1_n - \tilde{T}g'\langle W\rangle\tilde{T}g\langle Z\rangle)^{-1}(1_n - \tilde{T}g'\langle W\rangle\tilde{T}g\langle\overline{Z}\rangle)(C\overline{Z} + D)(\mathrm{Im}(Z))^{-1} ,$$
$$\mathcal{D} = (C'W + D')^{-1}(1_n - \tilde{T}g\langle Z\rangle\tilde{T}g'\langle W\rangle)^{-1}(1_n - \tilde{T}g\langle Z\rangle\tilde{T}g'\langle\overline{W}\rangle)(C'\overline{W} + D')(\mathrm{Im}(W))^{-1} ,$$
$$\mathcal{B} = (CZ + D)^{-1}(1_n - \tilde{T}g'\langle W\rangle\tilde{T}g\langle Z\rangle)^{-1}\tilde{T}\ {}^t(C'W + D')^{-1} .$$

Thus, if we note

$$g\langle Z\rangle = \tilde{g}_1g_2\langle Z\rangle , \qquad CZ + D = (\tilde{C}_1(g_2\langle Z\rangle) + \tilde{D}_1)(C_2Z + D_2) ,$$
$$g'\langle W\rangle = \tilde{g}_1'g_2'\langle W\rangle , \qquad C'W + D' = (\tilde{C}_1'(g_2'\langle W\rangle) + \tilde{D}_1')(C_2'W + D_2') ,$$

we get

$$Q_0 = e_1(C_2Z + D_2)^{-1}(\tilde{C}_1(g_2\langle Z\rangle) + \tilde{D}_1)^{-1}(1_n - \tilde{T}(\tilde{g}_1'g_2'\langle W\rangle)\tilde{T}(\tilde{g}_1g_2\langle Z\rangle))^{-1}$$
$$\times \tilde{T}\ {}^t(\tilde{C}_1'(g_2'\langle W\rangle) + \tilde{D}_1')^{-1}\ {}^t(C_2'W + D_2')^{-1}\ {}^te_2 ,$$

$$P_0 - P_0' = \left(-\frac{1}{2i}\right) e_1(C_2Z + D_2)^{-1}(\tilde{C}_1(g_2\langle Z\rangle) + \tilde{D}_1)^{-1}(1_n - \tilde{T}(\tilde{g}_1'g_2'\langle W\rangle)\tilde{T}(\tilde{g}_1g_2\langle Z\rangle))^{-1}$$
$$\times (1_n - \tilde{T}(\tilde{g}_1'g_2'\langle W\rangle)\tilde{T}(\tilde{g}_1g_2\langle\overline{Z}\rangle))(\tilde{C}_1(g_2\langle\overline{Z}\rangle) + \tilde{D}_1)(C_2\overline{Z} + D_2)(\mathrm{Im}(Z))^{-1}\ {}^te_1 ,$$

$$R_0 - R_0' = \left(-\frac{1}{2i}\right) e_2(C_2'W + D_2')^{-1}(\tilde{C}_1'(g_2'\langle W\rangle) + \tilde{D}_1')^{-1}(1_n - \tilde{T}(\tilde{g}_1g_2\langle Z\rangle)\tilde{T}(\tilde{g}_1'g_2'\langle W\rangle))^{-1}$$
$$\times (1_n - \tilde{T}(\tilde{g}_1g_2\langle Z\rangle)\tilde{T}(\tilde{g}_1'g_2'\langle\overline{W}\rangle))(\tilde{C}_1'(g_2'\langle\overline{W}\rangle) + \tilde{D}_1')(C_2'\overline{W} + D_2')(\mathrm{Im}(W))^{-1}\ {}^te_2$$

and

$$\det(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-k}\det(\mathrm{Im}(M\langle\mathfrak{Z}_0\rangle))^{-s}$$
$$= \det(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-k}\,|\det(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})|^{-2s}\det(\mathrm{Im}(\mathfrak{Z}_0))^{s}$$
$$= \det(\tilde{C}_1(g_2\langle Z\rangle) + \tilde{D}_1)^{-k}\det(C_2Z + D_2)^{-k}\det(\tilde{C}_1'(g_2'\langle W\rangle) + \tilde{D}_1')^{-k}\det(C_2'W + D_2')^{-k}$$
$$\times \det(1_n - \tilde{T}(\tilde{g}_1'g_2'\langle W\rangle)\tilde{T}(\tilde{g}_1g_2\langle Z\rangle))^{-k}\left|\det(1_n - \tilde{T}(\tilde{G}_1'g_2'\langle W\rangle)\tilde{T}(\tilde{g}_1g_2\langle Z\rangle))\right|^{-2s}$$
$$\times \det(\mathrm{Im}(\tilde{g}_1g_2\langle Z\rangle))^{s}\det(\mathrm{Im}(\tilde{g}_1'g_2'\langle W\rangle))^{s} .$$

Combining these formulae and using the representations ρ_1 , ρ , we obtain Proposition 2.

□

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§3 Analytic properties of standard L-functions

3.1 Integral representation

We first prove the following :

Theorem 1. *Let $k, l \in 2\mathbb{Z}$, $k, l > 0$, $s \in \mathbb{C}$ and $k + 2\mathrm{Re}(s) > 2n + 1$.*
For an eigenform $f \in S^n_{k,l}(\mathrm{sym}^l(V_2))$ and each $T \in \mathbb{T}^{(r)}$ $(1 \leq r \leq n)$, the Petersson inner product of f and $\mathcal{P}_\nu\left(Z, W, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, s\right)$ $(0 \leq \nu \leq \frac{l}{2})$ is convergent and then,

(i) *for $T \in \mathbb{T}^{(n)}$ $(1 \leq r < n)$,*

$$\left(f \; , \; \mathcal{P}_\nu \left(-\overline{Z}, *, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, \bar{s}\right)\right) = 0 \quad ,$$

(ii) *for $T \in \mathbb{T}^{(n)}$,*

$$\begin{aligned} &\left(f \; , \; \left(-\frac{1}{4}\right)^\nu \mathcal{P}_\nu \left(-\overline{Z}, *, T, \bar{s}\right)\right) \\ &= \lambda(f,T) \det(T)^{-k-2s} 2^{n(n+1-k-2s)-l+1} i^{nk+l} \pi^{\frac{n(n+1)}{2}} (\iota^{-1}(f))(Z) \\ &\quad \times \frac{(-1)^\nu \nu!}{(k+s+l-\nu-1)_{\nu+1}} \sum_{j=1}^{n-1} \frac{\Gamma(2k+2s-2n-1+2j)(2k+2s-n-2+j)_l}{(k+s-n-1+j)\Gamma(2k+2s+l-n-1+j)} \quad . \end{aligned}$$

Proof. It follows from **(1.4.2)** that $\left(f \; , \; (L^{k,l} E^{2n}_k) \left(\begin{pmatrix} -\overline{Z}^{(n)} & 0 \\ 0 & * \end{pmatrix}, \bar{s}\right)\right)$ converges absolutely and locally uniformly for $k+2\mathrm{Re}(s) > 2n+1$. Then, by Proposition 2, $\left(f \; , \; \mathcal{P}_\nu \left(-\overline{Z}, *, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, \bar{s}\right)\right)$ is convergent.
The assertion (i) is proved in the same way as that by Klingen [17, Sats 2]:
We can write

$$\begin{aligned} &\mathcal{P}_\nu\left(Z, W, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, s\right) \\ &= \sum_{g_2 \in P_{n,r}\backslash\Gamma^n} \sum_{g_2' \in P_{n,r}\backslash\Gamma^n} \sum_{\tilde{g}_1 \in G_{n,r}} \sum_{\tilde{g}_1' \in \Gamma^r(T)\backslash G_{n,r}} \Bigg\{ \det(\mathrm{Im}(Z))^s \det(\mathrm{Im}(W))^s \\ &\quad \times \left|\det(1_n - \tilde{T}W\tilde{T}Z)\right|^{-2s} \rho_1\left(\det(1_n - \tilde{T}W\tilde{T}Z)^{-1}\right) \left(e_1 \tilde{T}\, {}^t e_2\right)^{l-2\nu} \\ &\quad \times \left(e_1(1_n - \tilde{T}W\tilde{T}\overline{Z})\mathrm{Im}(Z)^{-1}\, {}^t(1_n - \tilde{T}W\tilde{T}Z)^t e_1\right)^\nu \\ &\quad \times \left(e_2(1_n - \tilde{T}Z\tilde{T}W)^{-1}(1_n - \tilde{T}Z\tilde{T}\overline{W})\mathrm{Im}(W)^{-1}\, {}^t e_2\right)^\nu \Bigg\} \Big|(\tilde{g}_1')_W \Big|(\tilde{g}_1)_Z \Big|(g_2')_W \Big|(g_2)_Z \quad , \end{aligned}$$

where $(\;)_Z$ (resp. $(\;)_W$) denotes the action on Z (resp. W). Then we put

$$\begin{aligned} &G_\nu\left(Z, W, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, s\right) \\ &:= \det(\mathrm{Im}(Z))^s \det(\mathrm{Im}(W))^s \left|\det(1_n - \tilde{T}W\tilde{T}Z)\right|^{-2s} \rho_1\left((1_n - \tilde{T}W\tilde{T}Z)^{-1}\right) \left(e_1 \tilde{T}\, {}^t e_2\right)^{l-2\nu} \\ &\quad \times \left(e_1(1_n - \tilde{T}W\tilde{T}\overline{Z})\mathrm{Im}(Z)^{-1}\, {}^t(1_n - \tilde{T}W\tilde{T}Z)^t e_1\right)^\nu \\ &\quad \times \left(e_2(1_n - \tilde{T}Z\tilde{T}W)^{-1}(1_n - \tilde{T}Z\tilde{T}\overline{W})\mathrm{Im}(W)^{-1}\, {}^t e_2\right)^\nu \; . \end{aligned}$$

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Now, we put

$$W = X + iY = \begin{pmatrix} W_1^{(n-r)} & {}^tW_2^{(n-r,r)} \\ W_2^{(r,n-r)} & W_*^{(r)} \end{pmatrix},$$

$$X = \begin{pmatrix} X_1^{(n-r)} & {}^tX_2^{(n-r,r)} \\ X_2^{(r,n-r)} & X_*^{(r)} \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1^{(n-r)} & 0 \\ 0 & Y_*^{(r)} \end{pmatrix}\left[\begin{pmatrix} 1_{n-r} & 0 \\ Y_2^{(r,n-r)} & 1_r \end{pmatrix}\right].$$

Let F_n be a fundamental domain of $\Gamma^n\backslash\mathfrak{H}_n$ and $F_{n,r}$ be a fundamental domain of $P_{n,r}\backslash\mathfrak{H}_n$, that is,

$$F_{n,r} := \{W \in \mathfrak{H}_n \mid W_* \in F_r\ ,\ Y_1 \in M_{n-r}\ ,\ X \text{ and } Y_2 \text{ are reduced } \bmod 1\},$$

where M_{n-r} is the Minkowski reduction domain of the positive definite quadratic forms of degree $n-r$. Then we obtain

$$\begin{aligned}
&\left(f\ ,\ \mathcal{P}_\nu\left(-\overline{Z}, *, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, \bar{s}\right)\right) \\
&= \int_{F_n} \left\langle \rho\left(\sqrt{\mathrm{Im}(W)}\right) f(W)\ ,\ \rho\left(\sqrt{\mathrm{Im}(W)}\right) \mathcal{P}_\nu\left(-\overline{Z}, W, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, \bar{s}\right)\right\rangle \det(\mathrm{Im}(W))^{-n-1} dW \\
&= 2^{-1}\int_{F_{n,r}} \left\langle \rho\left(\sqrt{\mathrm{Im}(W)}\right) f(W)\ ,\right. \\
&\qquad \left.\sum_{g_2'}\sum_{\tilde{g}_1}\sum_{\tilde{g}_1'} \rho\left(\sqrt{\mathrm{Im}(W)}\right) G_\nu\left(-\overline{Z}, W, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, \bar{s}\right)\Big|(\tilde{g}_1')_W\Big|(\tilde{g}_1)_Z\Big|(g_2')_W\right\rangle \det(\mathrm{Im}(W))^{-n-1} dW .
\end{aligned}$$

Here we recall that $\tilde{g}_1'$ is of the form

$$\begin{pmatrix} \tilde{A}_1'^{(n)} & \tilde{B}_1'^{(n)} \\ \tilde{C}_1'^{(n)} & \tilde{D}_1'^{(n)} \end{pmatrix} = \begin{pmatrix} 1_{n-r} & 0 & 0 & 0 \\ 0 & A_1'^{(r)} & 0 & B_1'^{(r)} \\ 0 & 0 & 1_{n-r} & 0 \\ 0 & C_1'^{(r)} & 0 & D_1'^{(r)} \end{pmatrix} \quad \text{with } g_1' = \begin{pmatrix} A_1' & B_1' \\ C_1' & D_1' \end{pmatrix}.$$

Since

$$\tilde{g}_1'\langle W\rangle = \begin{pmatrix} W_1 + {}^tW_2(C_1'W_* + D_1')^{-1}C_1'W_2 & {}^tW_2(C_1'W_* + D_1')^{-1} \\ A_1'W_2 + (g_1'\langle W_*\rangle)C_1'W_2 & g_1'\langle W_*\rangle \end{pmatrix},$$

each of $\tilde{T}(\tilde{g}_1'\langle W\rangle)\tilde{T}$, $\tilde{T}(\tilde{g}_1'\langle W\rangle)$, $\tilde{C}_1'W + \tilde{D}_1'$ as a function on W does not depend on X_1 . Therefore,

$$\sum_{g_2'}\sum_{\tilde{g}_1}\sum_{\tilde{g}_1'} \rho\left(\sqrt{\mathrm{Im}(W)}\right) G_\nu\left(-\overline{Z}, W, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, \bar{s}\right)\Big|(\tilde{g}_1')_W\Big|(\tilde{g}_1)_Z\Big|(g_2')_W$$

is a constant function on X_1 . On the other hand, since f is a cusp form, its Fourier expansion has no constant terms in X_1. If we consider an integral of the integrand above on a unit cube in X_1 , we find that it vanishes. Thus the the assertion (i) is proved.

proof of (ii)

By $\Gamma^n \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \Gamma^n = \bigcup_{g' \in \Gamma^n(T)\backslash\Gamma^n} \Gamma^n \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} g'$, we get

$$\mathcal{P}_\nu(Z, W, T, s) = \left(\mathcal{P}_\nu(Z, W, 1_n, s)\Big|\left(\Gamma^n \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \Gamma^n\right)_W\right) \det(T)^{-k-2s},$$

where

$$\begin{aligned}
&\mathcal{P}_\nu(Z, W, 1_n, s) \\
&:= \sum_{g \in \Gamma^n} \Big\{\det(\mathrm{Im}(Z))^s \det(\mathrm{Im}(W))^s\, |\det(W+Z)|^{-2s} \rho_1\left((W+Z)^{-1}\right) \\
&\qquad \times \left(e_1\, {}^te_2\right)^{l-2\nu} \left(e_1(W+\overline{Z})\mathrm{Im}(Z)^{-1}(W+Z)^t e_1\right)^\nu \\
&\qquad \times \left(e_2(W+Z)^{-1}(\overline{W}+Z)\mathrm{Im}(W)^{-1}\, {}^te_2\right)^\nu \Big\}\Big|(g)_Z .
\end{aligned}$$

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Since the Hecke operator is an Hermitian operator and f is an eigenform, we have

$$(3.1)\qquad \begin{aligned}&\left(f\,,\,\left(-\frac{1}{4}\right)^{\nu}\mathcal{P}_\nu(-\overline{Z},*,T,\bar{s})\right)\\
&=\lambda(f,T)\det(T)^{-k-2s}(-1)^{\nu}2^{-2\nu}\left(f\,,\,\mathcal{P}_\nu(-\overline{Z},*,1_n,\bar{s})\right)\\
&=\lambda(f,T)\det(T)^{-k-2s}(-1)^{\nu}2^{-2\nu+1}\\
&\quad\times\int_{\mathfrak{H}_n}\Big\langle\rho\left(\sqrt{\mathrm{Im}(W)}\right)f(W)\,,\,\rho\left(\sqrt{\mathrm{Im}(W)}(W-\overline{Z})^{-1}\right)\,(e_1\,{}^te_2)^{l-2\nu}\\
&\qquad\times\left(e_1(W-\overline{Z})^{-1}(W-Z)\mathrm{Im}(Z)^{-1}\,{}^te_1\right)^{\nu}\\
&\qquad\times\left(e_2(\overline{W}-\overline{Z})\mathrm{Im}(W)^{-1}(W-\overline{Z})^te_2\right)^{\nu}\Big\rangle\\
&\qquad\times\left|\det(W-\overline{Z})\right|^{-2s}\det(\mathrm{Im}(Z))^{s}\det(\mathrm{Im}(W))^{s-n-1}\,dW\end{aligned}$$

We compute the integral (3.1) according to Klingen [16, §1]:

For $Z\in\mathfrak{H}_n$, there exists $F\in GL(n,\mathbb{R})$ such that $\mathrm{Im}(Z)\,[{}^tF]=1_n$. Then, let $\mathcal{L}_Z$ be a biholomorphic transformation from $\mathfrak{H}_n$ onto

$$S^n:=\left\{S\in\mathbb{C}^{(n)}\mid S={}^tS\,,\;1_n-\overline{S}S>0\right\}$$

such that for $W\in\mathfrak{H}_n$,

$$\mathcal{L}_Z(W)=S:=F(W-Z)(W-\overline{Z})^{-1}F^{-1}\quad .$$

We note

$$\mathrm{Im}(W)=2^{-2}(\overline{W}-Z)^tF(1_n-\overline{S}S)F(W-\overline{Z})\,,$$
$$dW=2^{-n(n+1)}\det(\mathrm{Im}(Z))^{-(n+1)}\left|\det(W-\overline{Z})\right|^{2(n+1)}dS\,,$$

where $dS=dX_SdY_S$ $(S=X_S+iY_S)$ is of the form (1.2). We also put

$$\widehat{f}(S):=\rho\left(W-\overline{Z}\right)f(W)\,.$$

Then, the integral (3.1) is:

$$\begin{aligned}&\lambda(f,T)\det(T)^{-k-2s}(-1)^{\nu}2^{n(n+1-2s)-2(nk+l)+1}\\
&\quad\times\int_{S^n}\Big\langle\widehat{f}(S)\,,\,\rho\left({}^tF(1_n-\overline{S}S)F\right)(e_1\,{}^te_2)^{l-2\nu}\left(e_1{}^tFSF^te_1\right)^{\nu}\\
&\qquad\times\left(e_2F^{-1}\overline{S}(1_n-S\overline{S})^{-1}\,{}^tF^{-1}\,{}^te_2\right)^{\nu}\Big\rangle\det(1_n-\overline{S}S)^{s-n-1}dS\end{aligned}$$

$$\begin{aligned}&=\lambda(f,T)\det(T)^{-k-2s}(-1)^{\nu}2^{n(n+1-2s)-2(nk+l)+1}\\
&\quad\times\int_{S^n}\Big\langle\rho\left(Z-\overline{Z}\right)f(Z)\,,\,\rho\left({}^tF(1_n-\overline{S}S)F\right)(e_1\,{}^te_2)^{l-2\nu}\left(e_1{}^tFSF^te_1\right)^{\nu}\\
&\qquad\times\left(e_2F^{-1}\overline{S}(1_n-S\overline{S})^{-1}\,{}^tF^{-1}\,{}^te_2\right)^{\nu}\Big\rangle\det(1_n-\overline{S}S)^{s-n-1}dS\end{aligned}$$

by $\dfrac{\partial}{\partial S}\det(1_n-S\overline{S})=-\det(1_n-\overline{S}S)\left(\overline{S}(1_n-S\overline{S})^{-1}\right)$ with $\dfrac{\partial}{\partial S}=\left(\dfrac{1+\delta_{jh}}{2}\dfrac{\partial}{\partial s_{jh}}\right)_{1\le j,h\le n}$, where $\dfrac{\partial}{\partial s_{jh}}$ is of the form (2.1) ,

$$\begin{aligned}&=\lambda(f,T)\det(T)^{-k-2s}2^{n(n+1-k-2s)-l+1}i^{nk+l}\\
&\quad\times\int_{S^n}\Big\langle\rho\left(1_n-\overline{S}S\right)\rho\left({}^tF^{-1}\right)f(Z)\,,\,\rho_1\left({}^tF\right)(e_1\,{}^te_2)^{l-2\nu}\left(e_1S\,{}^te_1\right)^{\nu}\\
&\qquad\times\left(e_2\Big(\frac{\partial}{\partial S}\det(1_n-S\overline{S})\Big){}^te_2\right)^{\nu}\Big\rangle\det(1_n-\overline{S}S)^{s-n-1-\nu}dS\quad .\end{aligned}$$

Moreover, there exists a linear map $\psi = \psi(n, l, \nu, k+s-n-1) \in GL(\mathrm{sym}^l(V_1))$ such that

$$\begin{aligned}&\rho_1\left({}^t\overline{F}\right)\psi\rho_1\left({}^tF^{-1}\right)(\iota^{-1}(f))(Z)\\ &=\int_{S^n}\Big\langle\rho\left(1_n-\overline{S}S\right)\rho\left({}^tF^{-1}\right)f(Z)\ ,\ \rho_1\left({}^tF\right)\left(e_1\,{}^te_2\right)^{l-2\nu}\left(e_1S\,{}^te_1\right)^{\nu}\\ &\qquad\times\left(e_2\Big(\frac{\partial}{\partial S}\det(1_n-S\overline{S})\Big){}^te_2\right)^{\nu}\Big\rangle\det(1_n-\overline{S}S)^{s-n-1-\nu}dS\quad .\end{aligned}$$

For any unitary matrix $U \in U(n,\mathbb{C})$, we have

$$\begin{aligned}&\rho_1\left({}^t\overline{F}\right)\rho_1\left(U^{-1}\right)\psi\rho_1(U)\rho_1\left({}^tF^{-1}\right)(\iota^{-1}(f))(Z)\\ &=\rho_1\left({}^t(\overline{UF})\right)\psi(\iota^{-1}(\rho\left(U{}^tF^{-1}\right)f))(Z)\\ &=\int_{S^n}\Big\langle\rho\left(1_n-\overline{S}S\right)\rho\left(U{}^tF^{-1}\right)f(Z)\ ,\ \rho_1\left({}^t(UF)\right)\left(e_1\,{}^te_2\right)^{l-2\nu}\left(e_1S\,{}^te_1\right)^{\nu}\\ &\qquad\times\left(e_2\Big(\frac{\partial}{\partial S}\det(1_n-S\overline{S})\Big){}^te_2\right)^{\nu}\Big\rangle\det(1_n-\overline{S}S)^{s-n-1-\nu}dS\end{aligned}$$

changing the variable S to ${}^tUSU = S'$,

$$\begin{aligned}&=\int_{S^n}\Big\langle\rho(U)\rho\left(1_n-\overline{S'}S'\right)\rho\left({}^tF^{-1}\right)f(Z)\ ,\rho(U)\rho_1\left({}^tF\right)\left(e_1\,{}^te_2\right)^{l-2\nu}\left(e_1S'\,{}^te_1\right)^{\nu}\\ &\qquad\times\left(e_2\Big(\frac{\partial}{\partial S'}\det(1_n-S'\overline{S'})\Big){}^te_2\right)^{\nu}\Big\rangle\det(1_n-\overline{S'}S')^{s-n-1-\nu}dS'\\ &=\rho_1\left({}^t\overline{F}\right)\psi\rho_1\left({}^tF^{-1}\right)(\iota^{-1}(f))(Z)\quad ,\end{aligned}$$

that is, $\rho_1(U^{-1})\psi\rho_1(U) = \psi$. Since $\rho_1 = \det^k \otimes\mathrm{sym}^l$ is an irreducible representation of $U(n,\mathbb{C})$, ψ is a homothety by Schur's lemma.

Thus we get

$$\begin{aligned}&\left(f\ ,\ \left(-\frac{1}{4}\right)^{\nu}\mathcal{P}_\nu(-\overline{Z},*,T,\bar{s})\right)\\ &=\lambda(f,T)\det(T)^{-k-2s}2^{n(n+1-k-2s)-l+1}i^{nk+l}\psi(\iota^{-1}(f))(Z)\end{aligned}$$

and we have only to prove

$$\psi=\pi^{\frac{n(n+1)}{2}}\frac{(-1)^\nu\nu!}{(k+s+l-\nu-1)_{\nu+1}}\prod_{j=1}^{n-1}\frac{\Gamma(2k+2s-2n+2j-1)(2k+2s-n-2+j)_l}{(k+s-n-1+j)\Gamma(2k+2s+l+j-n-1)}\quad . \tag{3.2}$$

Here, we can write

$$\begin{aligned}&\psi(n,l,\nu,k+s-n-1)\\ &=\int_{S^n}\det(1_n-S\overline{S})^{k+s-n-1-\nu}\left((1_n-S\overline{S})\left[{}^tp_n\right]\right)^l\left(\overline{S}\left[{}^tp_n\right]\right)^{\nu}\\ &\qquad\times\left(\Big(\frac{\partial}{\partial\overline{S}}\det(1_n-S\overline{S})\Big)\left[{}^tp_n\right]\right)^{\nu}dS\quad ,\end{aligned}$$

where $p_n^{(1,n)} = (1,0,\cdots,0)$.

Let $\mu = k+s-n-1$.

We put $S^{(n)} = \begin{pmatrix} S_1^{(n-1)} & {}^tv^{(n-1,1)} \\ v^{(1,n-1)} & z\end{pmatrix}$. By $1_{n-1}-S_1\overline{S}_1 > 0$, there exists $g \in GL(n-1,\mathbb{C})$ such that $1_{n-1}-S_1\overline{S}_1 = g\,{}^t\bar{g}$. If we put $v = u\,{}^tg$, we get $dv = |\det g|^2 = \det(1_{n-1}-S_1\overline{S}_1)du$. Moreover, we put

$$a=-\frac{1}{1-\bar{u}{}^tu}\ (<0)\ ,\qquad b=-\frac{u{}^tg\overline{S}_1{}^t\bar{g}^{-1}\,{}^tu}{1-\bar{u}{}^tu}\ ,$$

$$c = 1 - u^t g \bar{g}^t \bar{u} - u^t g \overline{S}_1{}^t \bar{g}^{-1} g^{-1} S_1 \bar{g}^t \bar{u} - \frac{|u^t g \overline{S}_1{}^t \bar{g}^{-1}\ {}^t u|^2}{|1 - \bar{u}^t u|} .$$

We note $|b|^2 - ac = 1$. Thus the condition $1_n - S\overline{S} > 0$ is equivalent to the conditions

$$1_{n-1} - S_1\overline{S}_1 > 0 , \quad 1 - \bar{u}^t u > 0 , \quad c + b\bar{z} + \bar{b}z + az\bar{z} > 0 ,$$

and we get

$$\det(1_n - S\overline{S}) = \det(1_{n-1} - S_1\overline{S}_1)(1 - \bar{u}^t u)(c + b\bar{z} + \bar{b}z + az\bar{z}) ,$$

$$dS = \det(1_{n-1} - S_1\overline{S}_1) dS_1 du\, dz .$$

Therefore, we get

$$\begin{aligned} &\psi(n,l,\nu,\mu) \\ &= \int \Big\{ \det(1_{n-1} - S_1\overline{S}_1)^{\mu-\nu+1} (1 - \bar{u}^t u)^{\mu} \left((g(1_{n-1} - {}^t u\bar{u})^t \bar{g}) \left[{}^t p_{n-1}\right] \right)^l \left(\overline{S}_1 \left[{}^t p_{n-1}\right] \right)^{\nu} \\ &\quad \times \left(\left(\frac{\partial}{\partial \overline{S}_1} \det(1_{n-1} - S_1\overline{S}_1) \right) \left[{}^t p_{n-1}\right] \right)^{\nu} \left\{ \iint (c + b\bar{z} + \bar{b}z + az\bar{z})^{\mu} dz \right\} \Big\} dS_1 du \quad . \end{aligned}$$

By Hua [14, §2.3] , we have

$$\iint (c + b\bar{z} + \bar{b}z + az\bar{z})^{\mu} dz = (1 - \bar{u}^t u)^{\mu+2} \frac{\pi}{\mu + 1} \quad .$$

Using the equation above, we get

$$\begin{aligned} &\psi(n,l,\nu,\mu) \\ &= \frac{\pi}{\mu+1} \int_{S^{n-1}} \Big\{ \det(1_{n-1} - S_1\overline{S}_1)^{\mu-\nu+1} \left(\overline{S}_1 \left[{}^t p_{n-1}\right] \right)^{\nu} \left(\left(\frac{\partial}{\partial \overline{S}_1} \det(1_{n-1} - S_1\overline{S}_1) \right) \left[{}^t p_{n-1}\right] \right)^{\nu} \\ &\quad \times \left\{ \int_{1-\bar{u}^t u > 0} (1 - \bar{u}^t u)^{2\mu+2} \left((g(1_{n-1} - {}^t u\bar{u})^t \bar{g}) \left[{}^t p_{n-1}\right] \right)^l du \right\} \Big\} dS_1 \end{aligned}$$

By [9, Proposition 3.1] , we have

$$\int (1 - \bar{u}^t u)^{2\mu+2} \left((g(1_{n-1} - {}^t u\bar{u})^t \bar{g}) \left[{}^t p_{n-1}\right] \right)^l du = d(n-1, l, 2\mu+2) \left((1_{n-1} - S_1\overline{S}_1) \left[{}^t p_{n-1}\right] \right)^l ,$$

where

$$d(n,l,\mu) = \pi^n \frac{\Gamma(\mu+1)}{\Gamma(\mu+l+n+1)} (n+\mu)_l \quad .$$

Thus, we get

$$\begin{aligned} \psi(n,l,\nu,\mu) &= \frac{\pi}{\mu+1} \psi(n-1, l, \nu, \mu+1) d(n-1, l, 2\mu+2) \\ &= \psi(1, l, \nu, \mu+n-1) \prod_{j=1}^{n-1} \frac{\pi}{(\mu+j)} d(n-j, l, 2\mu+2j) \\ &= \psi(1, l, \nu, \mu+n-1) \pi^{\frac{n(n+1)}{2}-1} \prod_{j=1}^{n-1} \frac{\Gamma(2\mu+2j+1)(n+2\mu+j)_l}{(\mu+j)\Gamma(2\mu+j+l+n+1)} \quad , \end{aligned}$$

where

$$\begin{aligned} \psi(1, l, \nu, \mu+n-1) &= \iint_{S^1} (1 - s\bar{s})^{\mu+n+1-\nu} (1 - s\bar{s})^l (\bar{s})^{\nu} \left(\frac{\partial}{\partial \bar{s}} (1 - s\bar{s}) \right)^{\nu} ds \\ &= (-1)^{\nu} \frac{\nu!}{(\mu+n+l-\nu)_{\nu+1}} \pi \quad . \qquad \square \end{aligned}$$

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Now, by Proposition 2 and Theorem 1, we have

$$
\begin{aligned}
&\left(f\,,\ (L^{k,l}E_k^{2n})\left(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}, \bar{s}\right)\right) \\
&= \sum_{\nu=0}^{\frac{l}{2}} \frac{a(l,\nu,k,s)}{(2\pi i)^l} \sum_{r=1}^{n} \sum_{T\in\mathbf{T}^{(r)}} \left(f\,,\ \left(-\frac{1}{4}\right)^{\nu} \mathcal{P}_\nu\left(-\overline{Z}, *, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, \bar{s}\right)\right) \\
&= \sum_{\nu=0}^{\frac{l}{2}} \frac{a(l,\nu,k,s)}{(2\pi i)^l} \sum_{T\in\mathbf{T}^{(n)}} \left(f\,,\ \left(-\frac{1}{4}\right)^{\nu} \mathcal{P}_\nu\left(-\overline{Z}, *, T, \bar{s}\right)\right) \quad .
\end{aligned}
$$

By (1.4) , it equals

$$
2^{n(n+1-k-2s)-2l+1} i^{nk} \pi^{\frac{n(n+1)}{2}-l} \prod_{j=1}^{n-1} \frac{\Gamma(2k+2s-2n-1+2j)(2k+2s-n-2+j)_l}{(k+s-n-1+j)\Gamma(2k+2s+l-n-2+j)}
$$

$$
\times \left\{ \sum_{\nu=0}^{\frac{l}{2}} a(l,\nu,k,s) \frac{(-1)^{\nu}\nu!}{(k+s+l-\nu-1)_{\nu+1}} \right\} L(k+2s, f)(\iota^{-1}(f))(Z) \quad .
$$

On the other hand, by (2.4) , we get

$$
\sum_{\nu=0}^{\frac{l}{2}} a(l,\nu,k,s) \frac{(-1)^{\nu}\nu!}{(k+s+l-\nu-1)_{\nu+1}} = \sum_{\nu=0}^{\frac{l}{2}} b(l,\nu,k,s) \frac{(-1)^{\nu}}{k+s+l-\nu-1} \quad .
$$

Thus, by (1.5) and (2.5), we get

$$
\begin{aligned}
&\left(f\,,\ (L^{k,l}E_k^{2n})\left(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}, \bar{s}\right)\right) \\
&= 2^{n(n+1-k-2s)-2l+1} i^{nk} \pi^{\frac{n(n+1)}{2}-l} \frac{\Gamma(k+s-n)\Gamma(2k+2s+l-n-1)}{\Gamma(k+s-1)\Gamma(2k+2s+l-2)} \prod_{j=1}^{n-1} \frac{\Gamma(2k+2s+2j-2n-1)}{\Gamma(2k+2s+j-n-2)} \\
&\quad \times \frac{1}{(k)_l} \frac{\Gamma\left(k+s+\frac{l}{2}-1\right)}{\Gamma(k+s)} \left\{ \sum_{\nu=0}^{\frac{l}{2}} \frac{(-1)^{\nu}(2k-2+2\nu)_{l-2\nu}}{\nu!(l-2\nu)!(k-1+\nu)_{l-\nu}} (-s)_\nu \left(k+s+\frac{l}{2}-1\right)_{\frac{l}{2}-\nu} \right\} \\
&\quad \times \zeta(2s+k)^{-1} \prod_{j=1}^{n} \zeta(4s+2k-2j)^{-1} D_f(2s+k-n)(\iota^{-1}(f))(Z) \quad .
\end{aligned}
$$

Here, we prove the following:

Lemma 4. *Let $k,l \in \mathbb{Z}$, $k,l > 0$. For an indeterminate X ,*

$$
\sum_{\nu=0}^{[\frac{l}{2}]} \frac{(-1)^{\nu}(2k-2+2\nu)_{l-2\nu}}{\nu!(l-2\nu)!(k-1+\nu)_{l-\nu}} \left(-X+k-\tfrac{1}{2}\right)_\nu \left(X+[\tfrac{l+1}{2}]-\tfrac{1}{2}\right)_{[\frac{l}{2}]-\nu} = \frac{2^l}{l!} X^{[\frac{l}{2}]} \quad . \tag{3.3}
$$

Proof. We follow Zagier's method in [9, lemma 4.1].

We denote the left-hand side of (3.3) by $P_{k,l}(X)$.

When we consider $P_{k,l}(X)$ a polynomial in X , the coefficient of the highest degree of X is

$$
\sum_{\nu=0}^{[\frac{l}{2}]} \frac{(2k-2+2\nu)_{l-2\nu}}{\nu!(l-2\nu)!(k-1+\nu)_{l-\nu}} \quad .
$$

Then we get

$$\begin{aligned}
&\binom{k+l-2}{l}\sum_{\nu=0}^{[\frac{l}{2}]}\frac{(2k-2+2\nu)_{l-2\nu}}{\nu!(l-2\nu)!(k-1+\nu)_{l-\nu}}\\
&=\frac{1}{l!}\sum_{\nu=0}^{[\frac{l}{2}]}\binom{2k+l-3}{l-2\nu}\binom{k+\nu-2}{k-2}\\
&=\frac{1}{l!}\mathrm{Res}_{x=0}\left[\frac{1}{(1-x^2)^{k-1}}(1+x)^{2k+l-3}\frac{1}{x^{l+1}}dx\right]
\end{aligned}$$

by putting $t=\dfrac{x}{1+x}$,

$$\begin{aligned}
&=\frac{1}{l!}\mathrm{Res}_{t=0}\left[(1-2t)^{-(k-1)}\frac{1}{t^{l+1}}dt\right]\\
&=\frac{2^l}{l!}\binom{k+l-2}{l} \quad .
\end{aligned}$$

Thus, we have only to prove $P_{k,l}(-j)$ for $j\in\mathbb{Z}$, $0\le j\le[\frac{l}{2}]-1$.
We use induction on l and j . We first prove $P_{k,l}(0)=0$:

$$P_{k,l}(0)=\frac{(2k-2)_l}{(k-1)_l}\sum_{\nu=0}^{[\frac{l}{2}]}\frac{(-1)^\nu}{\nu!(l-2\nu)!2^{2\nu}}\left([\tfrac{l+1}{2}]-\tfrac{1}{2}\right)_{[\frac{l}{2}]-\nu} \quad .$$

For even l , by

$$(l-2\nu)!=\frac{2^{l-2\nu}\left(\frac{l}{2}-\nu\right)!\left(\frac{1}{2}\right)_{\frac{l}{2}}}{\left(\frac{1-l}{2}\right)_\nu(-1)^\nu} \quad ,$$

we have

$$\begin{aligned}
P_{k,l}(0)&=\frac{(2k-2)_l}{(k-1)_l2^l\left(\frac{1}{2}\right)_{\frac{l}{2}}}\sum_{\nu=0}^{\frac{l}{2}}\frac{1}{\nu!\left(\frac{l}{2}-\nu\right)!}\left(\tfrac{1-l}{2}\right)_\nu\left(\tfrac{l-1}{2}\right)_{\frac{l}{2}-\nu}\\
&=0 \quad .
\end{aligned}$$

For odd l , by

$$(l-2\nu)!=\frac{2^{l-2\nu}\left(\frac{l-1}{2}-\nu\right)!\left(\frac{1}{2}\right)_{\frac{l+1}{2}}}{\left(-\frac{l}{2}\right)_\nu(-1)^\nu} \quad ,$$

we have

$$\begin{aligned}
P_{k,l}(0)&=\frac{(2k-2)_l}{(k-1)_l2^l\left(\frac{1}{2}\right)_{\frac{l+1}{2}}}\sum_{\nu=0}^{\frac{l-1}{2}}\frac{1}{\nu!\left(\frac{l-1}{2}-\nu\right)!}\left(-\tfrac{l}{2}\right)_\nu\left(\frac{l}{2}\right)_{\frac{l-1}{2}-\nu}\\
&=0 \quad .
\end{aligned}$$

Next, we suppose $P_{k,l'}(-j')=0$ for any $k,\ j',\ l'$ such that $j'<j$, $l'<l$.
For even l , if we note

$$\left(k-\tfrac{1}{2}+j\right)_\nu=\nu\left(k-\tfrac{1}{2}+j\right)_{\nu-1}+\left(k-\tfrac{1}{2}+j-1\right)_\nu$$

and

$$\left(\tfrac{l-1}{2}-j\right)_{\frac{l}{2}-\nu}=-\tfrac{l-2\nu}{2}\left(\tfrac{l-1}{2}-(j-1)\right)_{\frac{l}{2}-\nu-1}+\left(\tfrac{l-1}{2}-(j-1)\right)_{\frac{l}{2}-\nu} \quad ,$$

we have

$$\begin{aligned}
P_{k,l}(-j) =& \sum_{\nu=0}^{\frac{l}{2}} \frac{(-1)^\nu (2k-2+2\nu)_{l-2\nu}}{\nu!(l-2\nu)!(k-1+\nu)_{l-\nu}} \left(k-\tfrac{1}{2}+j-1\right)_\nu \left(\tfrac{l-1}{2}-(j-1)\right)_{\frac{l}{2}-\nu} \\
&+ \sum_{\nu=1}^{\frac{l}{2}} \frac{(-1)^\nu (2k-2+2\nu)_{l-2\nu}}{(\nu-1)!(l-2\nu)!(k-1+\nu)_{l-\nu}} \left(k-\tfrac{1}{2}+j\right)_{\nu-1} \left(\tfrac{l-1}{2}-(j-1)\right)_{\frac{l}{2}-\nu} \\
&- \sum_{\nu=0}^{\frac{l}{2}-1} \frac{(-1)^\nu (2k-2+2\nu)_{l-2\nu}}{2(\nu)!(l-1-2\nu)!(k-1+\nu)_{l-\nu}} \left(k-\tfrac{1}{2}+j\right)_\nu \left(\tfrac{l-1}{2}-(j-1)\right)_{\frac{l}{2}-\nu-1} \\
=& P_{k,l}(-(j-1)) \\
&- \sum_{\nu=0}^{\frac{l}{2}-1} \left\{ \frac{(-1)^\nu (2k+2\nu)_{l-2\nu-2}}{2(\nu)!(l-1-2\nu)!(k-1+\nu)_{l-\nu}} \left\{ 2(k+\nu-1)(l-1-2\nu) \right.\right. \\
&\qquad \left.\left. + (2k+2\nu-2)(2k+2\nu-1) \right\} \left(k-\tfrac{1}{2}+j\right)_\nu \left(\tfrac{l+1}{2}-j\right)_{\frac{l}{2}-\nu-1} \right\} \\
=& P_{k,l}(-(j-1)) - P_{k+1,l-1}(-(j-1)) \\
=& 0 \quad .
\end{aligned}$$

In the same way, for odd l , we have

$$P_{k,l}(-j) = P_{k,l}(-(j-1)) - \frac{2}{l-2j} P_{k+1,l-1}(-(j-1)) + \frac{1}{l-2j} P_{k,l}(-j) \quad .$$

Thus, lemma 4 is proved. □

By lemma 4 , we get

$$\begin{aligned}
&\left(f \, , \, (L^{k,l} E_k^{2n}) \left(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}, \bar{s} \right) \right) \\
&= \frac{1}{(k)_l l!} 2^{n(n+1-k-2s)-l+1} i^{nk} \pi^{\frac{n(n+1)}{2}-l} \prod_{j=1}^{n-1} \frac{\Gamma(2k+2s+2j-2n-1)}{\Gamma(2k+2s+j-n-2)} \\
&\quad \times \frac{\Gamma\left(k+s+\frac{l}{2}-1\right)\Gamma\left(k+s+\frac{l}{2}-\frac{1}{2}\right)\Gamma(k+s-n)\Gamma(2k+2s+l-n-1)}{\Gamma(k+s)\Gamma\left(k+s-\frac{1}{2}\right)\Gamma(k+s-1)\Gamma(2k+2s+l-2)} \\
&\quad \times \zeta(2s+k)^{-1} \prod_{j=1}^{n} \zeta(4s+2k-2j)^{-1} D_f(2s+k-n)(\iota^{-1}(f))(Z) \quad .
\end{aligned}$$

Combining this with (1.6) , we obtain

$$\begin{aligned}
&\left(f \, , \, (L^{k,l} \mathbb{E}_k^{2n}) \left(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}, \frac{\bar{s}+n}{2} \right) \right) \\
&= \frac{1}{(k)_l l!} 2^{1-l} i^{nk} \pi^{-\frac{1}{2}n^2+kn+\frac{1}{2}\varepsilon} \gamma(s) \\
&\qquad \times \Gamma_{\mathbf{R}}(s+\varepsilon)\Gamma_{\mathbf{C}}(s+k+l-1) \prod_{j=2}^{n} \Gamma_{\mathbf{C}}(s+k-j) D_f(s)(\iota^{-1}(f))(Z) \quad ,
\end{aligned}$$

where ε and $\gamma(s)$ are of the form (1.8) and (1.9) , respectively. Here, we put

(3.4) $$\Lambda(s,f) := \Gamma_{\mathbf{R}}(s+\varepsilon)\Gamma_{\mathbf{C}}(s+k+l-1) \prod_{j=2}^{n} \Gamma_{\mathbf{C}}(s+k-j) D_f(s) \quad .$$

On the other hand, it follows from **(1.4.1)** and **(1.4.2)** that $\left(f \, , \, (L^{k,l} \mathbb{E}_k^{2n}) \left(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}, \bar{s} \right) \right)$ is invariant under $s \mapsto \frac{n+1}{2} - s$ and that it is an entire function in s . Thus, we have:

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Theorem 2. *Let $k, l \in 2\mathbb{Z}$, $k > 0$, $l \geq 0$. If $f \in S^n_{k,l}(\mathrm{sym}^l(V_2))$ is an eigenform,*
(i)

$$\left(f\ ,\ (L^{k,l}\mathbb{E}^{2n}_k)\left(\begin{pmatrix}-\overline{Z} & 0\\ 0 & *\end{pmatrix}, \frac{\bar{s}+n}{2}\right)\right) = \frac{1}{(k)_l l!} 2^{1-l} i^{nk} \pi^{-\frac{1}{2}n^2+kn+\frac{1}{2}\epsilon}\gamma(s)\Lambda(s,f)(\iota^{-1}(f))(Z) \tag{3.5}$$

or equivalently,

$$\alpha^n_{k,l}(s)\zeta(s+n)\prod_{j=0}^{n-1}\zeta(2s+2j)\left(f\ ,\ (L^{k,l}E^{2n}_k)\left(\begin{pmatrix}-\overline{Z} & 0\\ 0 & *\end{pmatrix}, \frac{\bar{s}+n-k}{2}\right)\right) = \Lambda(s,f)(\iota^{-1}(f))(Z) \quad , \tag{3.6}$$

where

$$\alpha^n_{k,l}(s) := (k)_l\ l!\ 2^{\frac{n^2-3n}{2}+sn-1+l} i^{nk} \pi^{-(n+\frac{1}{2})s-nk-\frac{n+\varepsilon}{2}} \Gamma_n\left(\frac{s+n+k}{2}\right)\Gamma_n\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s+\varepsilon}{2}\right) \quad .$$

(ii) *$\Lambda(s,f)$ has a meromorphic continuation to the whole s-plane and satisfies the functional equation*

$$\Lambda(s,f) = \Lambda(1-s,f) \quad .$$

Remark. For $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}^n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, let $\rho \sim (\lambda_1, \lambda_2, \cdots, \lambda_n)$ be an irreducible rational representation of $GL(n, \mathbb{C})$ with a representation space $\mathcal{V}$. Suppose that $f \in S^n_\rho(\mathcal{V})$ is an eigenform. Then, it is expected that completed Dirichlet series

$$\Lambda(s,f) := \Gamma_{\mathbb{R}}(s+\varepsilon)\prod_{j=1}^{n}\Gamma_{\mathbb{C}}(s+\lambda_j-j)D_f(s) \quad ,$$

should satisfy a functional equation.

3.2 Poles of standard L-functions

Theorem 3. *Let $k, l \in 2\mathbb{Z}$, $k > 0$, $l \geq 0$ and $k > n$. If $f \in S^n_{k,l}(\mathrm{sym}^l(V_2))$ is an eigenform, $\Lambda(s,f)$ is holomorphic except for possible simple poles at $s=0$ and $s=1$; it has a pole at $s=0$ (or equivalently, $s=1$) if and only if $f \in B^n_{k,l}(2n) \cap S^n_{k,l}(\mathrm{sym}^l(V_2))$.*

Corollary. *Under the assumption of Theorem 3, suppose $n \not\equiv 0 \mod 4$. Then $\Lambda(s,f)$ is entire.*

Proof of Theorem 3. Theorem 3 is proved in the same way as that by Mizumoto [21, Theorem 1]:

Let Γ_* be a finite-index subgroup of Γ^n . A function g from $\mathfrak{H}_n$ to $\mathbb{C}$ is called a C^∞-modular form of weight k (i.e., $\rho = \det^k \otimes \mathrm{sym}^0$) with respect to Γ_* if it is a C^∞-function and satisfies

$$(g|M)(Z) := \det(CZ+D)^{-k}g(M\langle Z\rangle) = g(Z)$$

for all $M = \begin{pmatrix} * & * \\ C^{(n)} & D^{(n)}\end{pmatrix} \in \Gamma_*$. Note that this notation | is compatible with one in §1.1 .

Let Γ_j $(j = 1, 2)$ be finite-index subgroups of Γ^n such that $\Gamma_1 \subset \Gamma_2$. Suppose g is a C^∞-modular form of weight k with respect to Γ_1 . Then

$$Tr(\Gamma_1, \Gamma_2; g)(Z) := \sum_{\gamma \in \Gamma_1 \backslash \Gamma_2} (g|\gamma)(Z)$$

is a C^∞-modular form of weight k with respect to Γ_2 .

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For $n,\ N \in \mathbb{Z}$, $k \in 2\mathbb{Z}$, $n,\ N,\ k > 0$ and $Z \in \mathfrak{H}_n$, we put

$$E_k^n(Z,s,N) := \sum_{M=\begin{pmatrix} * & * \\ C^{(n)} & D^{(n)} \end{pmatrix} \in P_{n,0}\backslash \Gamma_0^n(N)} \det(CZ+D)^{-k} \det(\mathrm{Im}(M\langle Z\rangle))^s$$

for $k+2\mathrm{Re}(s) > n+1$, which has a meromorphic continuation to the whole s-plane by [20]. In particular,

$$E_k^n(Z,s,1) = E_k^n(Z,s) .$$

By Shimura [25, Proposition 2.1] ,

$$Tr(\Gamma_0^n(N),\Gamma^n; E_k^n(Z,s,N)) = E_k^n(Z,s) \quad ,$$

so

$$E_k^{2n}\left(\mathfrak{Z},\frac{s+n-k}{2}\right) = Tr\left(J^{-1}\Gamma_0^{2n}(N)J,\ \Gamma^{2n};\ E_k^{2n}\left(*,\frac{s+n-k}{2},N\right)\Big|\ J\right)(\mathfrak{Z}) \tag{3.7}$$

where $\mathfrak{Z} \in \mathfrak{H}_{2n}$ and $J := \begin{pmatrix} 0 & 1_{2n} \\ -1_{2n} & 0 \end{pmatrix}$.

Suppose $k > n$ and N:even . Using the same notation as in [11], [21], let

$$\begin{aligned} &D^{2n}\left(\mathfrak{Z},\frac{s+n-k}{2};k,1,N\right) \\ &:= \beta_k^n(s)\zeta_N(s+n)\prod_{j=0}^{n-1}\zeta_N(2s+2j)\left(E_k^{2n}\left(*,\frac{s+n-k}{2},N\right)\Big|\ J\right)(\mathfrak{Z}) \quad , \end{aligned} \tag{3.8}$$

where

$$\zeta_N(s) := \prod_{\substack{p\ |\ N \\ p:\ prime}} (1-p^{-s})\,\zeta(s) \quad ,$$

and

$$\beta_k^n(s) := 2^{s-1}\pi^{-\frac{1}{2}}\Gamma\left(\frac{s+\varepsilon}{2}\right)\Gamma\left(\frac{s+n+k}{2}\right)\Gamma\left(\frac{s+k-n+1}{2}\right)\prod_{j=1}^{n-1}\Gamma(s+k+n-2j) \quad .$$

Now, $\alpha_{k,l}^n(s)$ in Theorem 2 (i) and

$$\beta_k^n(s)\prod_{p\ |\ N}\left\{(1-p^{-s-n})\prod_{j=0}^{n-1}(1-p^{-2s-2j})\right\}$$

are holomorphic and non-zero in the region $\mathrm{Re}(s) > 0$. So

$$g(s) := \left(\beta_k^n(s)\prod_{p\ |\ N}\left\{(1-p^{-s-n})\prod_{j=0}^{n-1}(1-p^{-2s-2j})\right\}\right)^{-1}$$

is also holomorphic and non-zero in the region $\mathrm{Re}(s) > 0$.

Hence, by (3.7) and (3.8) , we get

$$\begin{aligned} &\zeta(s+n)\prod_{j=0}^{n-1}\zeta(2s+2j)\ E_k^{2n}\left(\mathfrak{Z},\frac{s+n-k}{2}\right) \\ &= g(s)Tr_J^N\left(D^{2n}\left(*,\frac{s+n-k}{2};k,1,N\right)\right)(\mathfrak{Z}) \quad , \end{aligned} \tag{3.9}$$

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where

$$Tr_J^N(*) := Tr(J^{-1}\Gamma_0^{2n}(N)J,\ \Gamma^{2n};\ *\ ) .$$

Thus (3.6) may also be written as follows:

$$\Lambda(s,f)(\iota^{-1}(f))(Z) = \alpha_{k,l}^n(s)g(s)\left(f\ ,\ \left(L^{k,l}Tr_J^N\left(D^{2n}\left(*,\frac{\bar{s}+n-k}{2};k,1,N\right)\right)\right)\begin{pmatrix}-\overline{Z} & 0\\ 0 & *\end{pmatrix}\right) \quad (3.10)$$

A result of Feit [11, Theorem 9.1] tells us that

$$D^{2n}\left(*,\frac{s+n-k}{2};k,1,N\right) \quad (3.11)$$

for even N is holomorphic in s except for a possible simple pole at $s = 1$; moreover, it is entire if n is odd.

On the other hand, it follows from (3.9) and (**1.4.2**) that convergence of

$$\left(f\ ,\ \left(L^{k,l}Tr_J^N\left(D^{2n}\left(*,\frac{\bar{s}+n-k}{2};k,1,N\right)\right)\right)\begin{pmatrix}-\overline{Z} & 0\\ 0 & *\end{pmatrix}\right)$$

in s is equal to that of (3.11) in s . Hence the integral representation (3.10) shows that $\Lambda(s,f)$ is holomorphic for $\mathrm{Re}(s) > 0$ except for a possible simple pole at $s = 1$ and that the pole does not appear if n is odd. By (**1.3.1**), $\Lambda(s,f)$ has a pole at $s = 1$ exactly when $n \equiv 0 \mod 4$ and $f \in B_{k,l}^n(2n) \cap S_{k,l}^n(\mathrm{sym}^l(V_2))$. Combining these facts with the functional equation in Theorem 2 (ii) , we obtain Theorem 3 .

□

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