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by

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THE UNIQUENESS OF STAR-PRODUCTS ON $P_n(\mathbf{C})$

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INTRODUCTION

It is fairly known that the classical mechanics is written by using functions on a phase manifold, and the quantum mechanics is written by using "operators" obtained through the correspondence principle from functions on the phase manifold. Here, "operators" are sometimes not genuine operators, densely defined on a Hilbert space, but a symbolic non-commutative objects corresponding to functions on the phase manifold.

Considering this vague character of the quantization procedure, Bayen et al. [B] proposed the idea, called *deformation quantization*, that quantization is not to make operators, but to deform the algebra of functions to a non-commutative, associative algebra. The advantage of this idea is to make us possible to consider the quantization purely algebraic without the representation theory.

First, let us recall the notion of deformation quantization briefly. Let (M,ω) be a C^∞ symplectic manifold and $C^\infty(M)$ the set of the smooth functions on M, which is denoted by $\mathfrak{a}$ simply. Consider the direct product

$$\mathfrak{a}[[\nu]] = \sum_{k\geq 0} \nu^k \mathfrak{a}.$$

An associative product $*$ defined on $\mathfrak{a}[[\nu]]$ is called a *deformation quantization* of $\mathfrak{a}$, or simply a $*$-product, if the following conditions are satisfied: For $f,g \in \mathfrak{a}$

(A.1) $(\nu^k f) * 1 = 1 * (\nu^k f) = \nu^k f,\ \ (\nu^k f) * (\nu^m g) = \nu^{k+m} f * g.$

(A.2) Let $\sum_{k\geq 0} \nu^k \pi_k(f,g),\ \ \pi_k(f,g) \in \mathfrak{a}$, be the decompostition of $f * g$. Then, $\pi_0(f,g) = fg$, the usual product, and $\pi_1(f,g) = -\{f,g\}/2$, where $\{\,,\,\}$ is the Poisson bracket on M.

We have already known in [DL], [OMY1] that there exists a $*$-product on any symplectic manifold. Thus, in this note, we shall concern how many $*$-products exist on a symplectic manifold. To consider this, we will call two algebras $(\mathfrak{a}[[\nu]], *)$ and $(\mathfrak{a}[[\nu]], *')$ *isomorphic*, if there is a linear isomorphism $\Phi\colon (\mathfrak{a}[[\nu]], *) \to (\mathfrak{a}[[\nu]], *')$ such that $\Phi(f * g) = \Phi(f) * \Phi(g)$. Note that $\Phi(\nu) = \nu$ is not requested.

For instance, let $P_n(\mathbf{C})$ be the complex projective n-space with the standard Kähler form. There are two different ways of making $*$-products on $P_n(\mathbf{C})$. The first one is intrinsic, initiated by Berezin [Be], who discussed on the 2-sphere, extended to $P_n(\mathbf{C})$ by Moreno [M], and to bounded domains or compact Kähler manifolds by Cahen, Gutt and Rawnsley [CGR]. Here the $*$-product is obtained at first by making operators corresponding to real analytic functions on $P_n(\mathbf{C})$, and the asymptotic symbol calculus similar to [K] gives the $*$-product on $C^\omega(P_n(\mathbf{C}))$. This can be extended easily on $C^\infty(P_n(\mathbf{C}))$. The second one was given in [OMY3] which is obtained by the reduction of the natural $*$-product on $\mathbf{C}^{n+1}$ via $\mathbf{C}^*$-action. Considering the natural projection $\mathbf{C}^{n+1}-\{0\}$ onto $P_n(\mathbf{C})$, one may regard $C^\infty(P_n(\mathbf{C}))$ as a subalgebra of $C^\infty(\mathbf{C}^{n+1}-\{0\})$ consisting of all f such that $f(z)=f(re^{i\theta})$ for any $re^{i\theta}$, $r>0$. Thus, let

$$\mathcal{A}=\{f\in C^\infty(\mathbf{C}^{n+1}-\{0\})[[\nu]] : f(z;\nu)=f(re^{i\theta}z;r^2\nu) \text{ for any } re^{i\theta}, r>0\}.$$

Then, $\mathcal{A}$ is a $*$-subalgebra of $C^\infty(\mathbf{C}^{n+1}-\{0\})[[\nu]]$. The center of $\mathcal{A}$ is $\mathbf{C}[[\frac{\nu}{R}]]$, the formal power series of $\frac{\nu}{R}$ and $R=(1/2)\sum_{1\leq i\leq n+1}|z|^2$. Regarding $\nu'=\frac{\nu}{R}$ as a deformation parameter, we showed in [OMY3] that $\mathcal{A}$ can be regarded as $C^\infty(P_n(\mathbf{C}))$, and hence $\mathcal{A}$ gives a $*$-product on $P_n(\mathbf{C})$.

The following is our main result:

Theorem. *Any $*$-products on $P_n(\mathbf{C})$ are mutually isomorphic.*

The above theorem will be proved purely algebraically by using the facts that $\dim H^2(P_n(\mathbf{C}))=1$, and the symplectic 2-form is not exact.

1. Algebraic preliminaries

We recall several algebraic tools which have been developed in [OMY4]. Let M be a C^∞ paracompact manifold, and let $\mathrm{C}^p(\mathfrak{a})$ be the vector space of the p-linear mappings of $\mathfrak{a}\times\cdots\times\mathfrak{a}$ (p-times) into $\mathfrak{a}$. By $\mathrm{SC}^p(\mathfrak{a})$, (resp. $\mathrm{AC}^p(\mathfrak{a})$), we denote the subspace of $\mathrm{C}^p(\mathfrak{a})$ consisting of the symmetric (resp. alternative) p-linear mappings.

For any $\pi\in\mathrm{C}^2(\mathfrak{a})$, we define the *Hochschild coboundary operator* $\delta_\pi : \mathrm{C}^p(\mathfrak{a})\to\mathrm{C}^{p+1}(\mathfrak{a})$ as follows:

$$\begin{aligned}(\delta_\pi F)(v_1,\cdots,v_{p+1}) &= \pi(v_1,F(v_2,\cdots,v_{p+1}))\\ &\quad+\sum_{1\leq i\leq p}(-1)^iF(v_1,\cdots,\pi(v_i,v_{i+1}),\cdots,v_{p+1})\\ &\quad+(-1)^{p+1}\pi(F(v_1,\cdots,v_p),v_{p+1}).\end{aligned}$$

For any $\pi,\pi',\pi''\in\mathrm{C}^2(\mathfrak{a})$, we see that

$$\delta_\pi\pi'=\delta_{\pi'}\pi,\quad \sum_{\text{cyclic}}\delta_\pi\delta_{\pi'}\pi''=0, \tag{1}$$

where $\sum_{\text{cyclic}}$ means the cyclic sum with respect to π,π',π''. We shall denote δ_{π_0} by δ_0. The following is known in [OMY4]:

Lemma 1.1. (a) $\delta_\pi^2 = 0$ *is equivalent to* $\delta_\pi \pi = 0$.

(b) $\delta_\pi \pi = 0$ *if and only if* $(\mathfrak{a}, \pi)$ *is an associative algebra.*

(c) *For* $\pi \in \mathrm{C}^2(\mathfrak{a})$, $\delta_0 \pi = 0$ *if and only if* $\delta_0 \pi^+ = 0$ *and* $\delta_0 \pi^- = 0$, *where* $\pi^\pm(f,g) = (\pi(f,g) \pm \pi(g,f))/2$.

(d) *For* $\pi \in \mathrm{AC}^2(\mathfrak{a})$, $\delta_0 \pi = 0$, *if and only if* π *is a biderivation, i.e.* $\pi(fg,h) = f\pi(g,h) + \pi(f,h)g$.

(e) *Suppose* $\theta \in \mathrm{C}^2(\mathfrak{a})$ *is a bidifferential operator of order* k, *and* $\theta(1,*) = \theta(*,1) = 0$. *If* θ *is a Hochschild cocycle with respect to* δ_0, *then* θ *is a Hochschild coboundary, i.e.* $\theta = \delta_0 \xi$, *and* ξ *can be chosen as a linear differential operator of order* k.

On the other hand, for any $\pi \in \mathrm{AC}^2(\mathfrak{a})$, we define the *Chevalley coboundary operator* $d_\pi : \mathrm{AC}^2(\mathfrak{a}) \to \mathrm{AC}^{p+1}(\mathfrak{a})$ as follows:

$$
\begin{aligned}
(d_\pi F)(v_1, \cdots, v_{p+1}) = &\sum_{1 \le i \le p+1} (-1)^{i+1} \pi(v_i, F(v_1, \cdots, \overset{\vee}{v_i}, \cdots, v_{p+1})) \\
&+ \sum_{i<j} (-1)^{i+j} F(\pi(v_i, v_j), v_1, \cdots, \overset{\vee}{v_i}, \cdots, \overset{\vee}{v_j}, \cdots, v_{p+1}).
\end{aligned}
$$

For any $\pi, \pi', \pi'' \in \mathrm{AC}^2(\mathfrak{a})$, we see that

$$
d_\pi \pi' = d_{\pi'} \pi, \quad \sum_{\text{cyclic}} d_\pi d_{\pi'} \pi'' = 0. \tag{2}
$$

Lemma 1.2. (cf. [OMY4]) (a) $d_\pi^2 = 0$ *is equivalent to* $d_\pi \pi = 0$

(b) $d_\pi \pi = 0$ *if and only if* $(\mathfrak{a}, \pi)$ *is a Lie algebra.*

$F \in \mathrm{C}^p(\mathfrak{a})$ will be called a *p-derivation* if $F(v_1, \cdots, v_p)$ satisfies the derivation rule for each v_i, that is

$$
F(v_1, \cdots, v_i w, \cdots, v_p) = v_i F(v_1, \cdots, w, \cdots, v_p) + F(v_1, \cdots, v_i, \cdots, v_p) w.
$$

By $\mathrm{Der}^p(\mathfrak{a})$, we denote the space of all p-derivations. Obviously, p-derivation is nothing but a $(p,0)$-tensor field on M. We set

$$
\mathfrak{A}^p(M) = \mathrm{Der}^p(\mathfrak{a}) \cap \mathrm{AC}^p(\mathfrak{a}).
$$

$\mathfrak{A}^p(M)$ is the dual space of p-forms. Note also that any Poisson bracket $\{\,,\,\}$ is an element of $\mathfrak{A}^2(M)$. If $\pi \in \mathfrak{A}^2(M)$, then it is easy to see that

$$
d_\pi \mathfrak{A}^p(M) \subset d_\pi \mathfrak{A}^{p+1}(M).
$$

Hence, one can define on any Poisson manifold, the cohomology group $H^p(M, \{\,,\,\})$ of the cochain complex $(\mathfrak{A}^p(M), d_1)$, where $d_1 = d_{\pi_1}$, and $\pi_1(f,g) = -\{f,g\}/2$.

Lemma 1.3. *Let (M,ω) be a symplectic manifold and $\{\,,\,\}$ its Poisson bracket. Then $H^p(M,\{\,,\,\})$ is isomorphic to p-th de Rham cohomology group.*

Thus, we will call $H^p(M,\{\,,\,\})$ the p-th *de Rham-Chevalley cohomology group.*

Now, let $(\mathfrak{a}[[\nu]],*)$ be any associative algebra such that

(B.1) $(\nu^k f)*1 = 1*(\nu^k f) = \nu^k f,\ \ (\nu^k f)*(\nu^m g) = \nu^{k+m} f*g.$

(B.2) Let $\sum_{k\geq 0}\nu^k\pi_k(f,g),\ \ f,g,\ \pi_k(f,g)\in\mathfrak{a}$, be the expression of $f*g$. Then, $\pi_0(f,g)=fg$, the usual product, and π_k is a bidifferential operator of order $2k$.

Remark. In §1 and §2 the condition $\pi_1=(-1/2)\{\,,\,\}$ is not assumed.

Since $(\mathfrak{a}[[\nu]],*)$ is associative, we have for each m,

$$\sum_{i+j=m}\delta_i\pi_j = 0, \quad \text{where} \quad \delta_i=\delta_{\pi_i}. \tag{3}$$

$$\sum_{i+j=m}\delta_i\delta_j = 0. \tag{4}$$

Now, let $\pi_i^{\pm}(f,g)=(\pi_i(f,g)\pm\pi_i(g,f))/2$. By the Jacobi identity, we have also the following for each integer $m\geq 0$:

$$\sum_{i+j=m} d_i^-\pi_j^- = 0, \quad \text{where} \quad d_i - = d_{\pi_i} -. \tag{5}$$

$$\sum_{i+j=m} d_i^- d_j^- = 0. \tag{6}$$

Note also that $\pi_0^-=0$.

In what follows, we shall often use linear isomorphisms

$$\phi\colon\mathfrak{a}[[\nu]]\to\mathfrak{a}[[\nu]] \quad \text{such that} \quad \phi(f)=f+\nu^k\rho(f),\ \ k\geq 1,$$

where ρ is a linear differential operator such that $\rho(\nu)=\nu$, and consider a new $*$-product given by $\phi^{-1}(\phi(f)*\phi(g))$. Since ϕ can be regarded as a change of the decomposition $\mathfrak{a}[[\nu]]=\sum_{k\geq 0}\nu^k\mathfrak{a}$, the new $*$-product gives a new expression of the same $*$-product by a new decomposition. For $k\geq 2$, then $f*g$ in the new expression is written as follows:

$$\begin{aligned} f*g = {} & fg+\nu\pi_1(f,g)+\cdots+\nu^{k-1}\pi_{k-1}(f,g) \\ & +\nu^k(\pi_k+\delta_0\rho)(f,g)+\nu^{k+1}(\pi_{k+1}+d_1\rho)(f,g)+\cdots. \end{aligned} \tag{7}$$

We use also the replacement of ν by $\nu(1+a\nu^k)$, $a\in\mathbf{C}$. Since these are in the center, this replacement gives an isomorphism of $\mathfrak{a}[[\nu]]$.

2. Skew-symmetric parts of π_1, π_{even} and symmetric parts of π_{odd}

Let $(\mathfrak{a}[[\nu]], *)$ be an associative algebra with (B.1), (B.2) in §1. By (1) and (3), we have $\delta_0\pi_1 = 0$, and hence $\delta_0\pi_1^{\pm} = 0$ by Lemma 1.1, (c). Therefore, by Lemma 1.1, (e), there is a differential operator ξ of order 2 such that $\delta_0\xi = \pi_1^+$. Now, consider the linear isomorphism

$$\phi(f) = f - \nu\xi(f), \quad \phi(\nu) = \nu,$$

and the new expression of $f * g$ is as follows:

$$f * g = fg + \nu\pi_1^-(f,g) + \cdots + \nu^k\pi_k(f,g) + \cdots$$

where π_j, for $j \geq 2$ are changed from the original ones.

Next, we shall consider π_2. In this stage one may assume that $\pi_1^- = \pi_1$. By (1) and (3), we have that

$$\delta_0\pi_2 = -\frac{1}{2}\delta_1\pi_1. \tag{8}$$

By a little complicated calculation, (8) is equivalent to the following:

$$(1-\mathfrak{c})\partial_2^0\pi_2^+ = \frac{1}{2}\delta_1\pi_1, \quad \partial_2^0\pi_2^- = 0, \tag{9}$$

where $\mathfrak{c}$ is the cyclic operator defined by $(\mathfrak{c}F)(f,g,h) = F(h,f,g)$, and

$$(\partial_2^0 F)(f,g,h) = gF(f,h) - F(f,gh) + F(f,g)h.$$

The second equality of (9) implies also that π_2^- is an element of $\mathfrak{A}^2(M)$ and hence $\delta_0\pi_2^- = 0$. Moreover, by (2) and (5), we have $d_1\pi_2^- = 0$. Thus, π_2^- determines a cohomology class $[\pi_2^-]$ of $H^2(M, \{\ ,\ \})$.

In the case that $\dim H^2(M, \{\ ,\ \}) = 1$ and $[\pi_1] \neq 0$ such as $P_n(\mathbf{C})$, there is $a \in \mathbf{C}$, and $\eta \in \mathfrak{A}^1(M)$ such that $\pi_2^- = a\pi_1 + d_1\eta$. Consider also the linear isomorphism

$$\phi(f) = f - \nu\eta(f), \quad \phi(\nu) = \nu.$$

Since $\delta_0\eta = 0$, we have, by using (7), that the new expression of $f * g$ is as follows:

$$f * g = fg + \nu\pi_1(f,g) + \nu^2(\pi_2^+ + a\pi_1)(f,g) + \cdots.$$

Consider now the replacement of ν by $\nu(1 - a\nu)$. This replacement gives an isomorphism of $\mathfrak{a}[[\nu]]$ onto itself, and the new expression of $f * g$ is

$$f * g = fg + \nu\pi_1(f,g) + \nu^2\pi_2^+(f,g) + \cdots,$$

where π_j, for $j \geq 3$ are changed from the original ones.

Now, we prove the following:

Theorem 2.1. *If* $\dim H^2(M) = 1$, *and* $[\pi_1] \neq 0$, *then by a suitable isomorphism* $\phi\colon \mathfrak{a}[[\nu]] \to \mathfrak{a}[[\nu]]$, *the expression of* $f * g$ *satisfies* $\pi^+_{odd} = 0$, $\pi^-_{even} = 0$.

Proof. Suppose $\pi^+_{2k-1} = \pi^-_{2k} = 0$ for $2k-1, 2k \leq 2r$. For $m = 2r+1$, we have

$$\sum_{i+j=m} \delta_i \pi_j = 0, \quad \text{where} \quad \delta_i = \delta_{\pi_i}. \tag{10}$$

By a little complicated calculation (cf. [OMY4]), the above equality yields the following:

$$\begin{aligned} &\delta_0 \pi^+_m = 0, \\ &\partial^0_2 \pi^-_m = \frac{1}{8}(1 - \mathfrak{c} + \mathfrak{c}^2)(1+\sigma) \sum_{i+j=m, i,j\geq 1} \delta_i \pi_j \end{aligned} \tag{11}$$

where $(\sigma F)(f,g,h) = F(h,g,f)$.

By Lemma 1.1,(e), there is a differential operator ξ of order $2m$ such that $\delta_0 \xi = \pi^+_m$. Now, consider the linear isomorphism

$$\phi(f) = f - \nu^m \xi(f), \quad \phi(\nu) = \nu,$$

and the new expression of $f * g$ is as follows:

$$f * g = fg + \nu\pi_1(f,g) + \cdots + \nu^{m-1}\pi_{m-1}(f,g) + \nu^m \pi^-_m(f,g) + \cdots.$$

Thus, one may assume in what follows that $\pi_{2r+1} = \pi^-_{2r+1}$.

Next, we shall consider π_m for $m = 2r+2$. By a little complicated calculation (cf. [OMY4]), we have

$$\begin{aligned} &(1-\mathfrak{c})\partial^0_2 \pi^+_m = \frac{1}{2} \sum_{i+j=m, i,j\geq 1} \delta_i \pi_j, \\ &\partial^0_2 \pi^-_m = 0. \end{aligned} \tag{12}$$

The second equality of (12) implies also that π^-_m is an element of $\mathfrak{A}^2(M)$ and hence $\delta_0 \pi^-_m = 0$. Moreover, by (2) and (5), we have $d_1 \pi^-_m = 0$, because $m+1$ is odd. Thus, π^-_m determines a cohomology class of $H^2(M)$. So, by the same procedure as in the case of π_2, one can eliminate the skew-symmetric part of π_m in the expression. The condition $\dim H^2(M) = 1$ is used at this stage. Repeating these procedures, we obtain the result. □

3. The proof of Theorem

Let $C^\infty(P_n(\mathbf{C}))$ denote $\mathfrak{a}$. Now, suppose we have two associative algebras $(\mathfrak{a}[[\nu]], *)$, $(\mathfrak{a}[[\nu]], *')$ satisfying (A.1) and (A.2). Moreover, by the above theorem, one may assume that $\pi^+_{odd} = 0$, $\pi^-_{even} = 0$ and $\pi'^+_{odd} = 0$, $\pi'^-_{even} = 0$.

Suppose $\pi_j = \pi'_j$ for $j, 0 \le j \le m-1$. Then $\delta_0(\pi_m - \pi'_m) = 0$ by (3). If m is even, then by Lemma 1.1,(e), there is a differential operator ξ of order $2m$ such that $\delta_0\xi = \pi_m - \pi'_m$. Now, consider the linear isomorphism

$$\phi(f) = f - \nu^m\xi(f), \quad \phi(\nu) = \nu.$$

Then, one can assume $\pi_m = \pi'_m$ in the new expression.

If m =odd, we have by (5) that $d_1(\pi_m - \pi'_m) = 0$. Since $\dim H^2(M) = 1$, there is $a \in \mathbf{C}$, and $\eta \in \mathfrak{A}^1(M)$ such that $\pi_m - \pi'_m = a\pi_1 + d_1\eta$.

Consider the linear isomorphism

$$\phi(f) = f - \nu^m\eta(f), \quad \phi(\nu) = \nu.$$

Since $\delta_0\eta = 0$, we may assume, by using (7), that $\pi_m - \pi'_m = a\pi_1$. Thus, the replacement of ν by $\nu(1 + a\nu^{m-1})$ gives an isomorphism of $\mathfrak{a}[[\nu]]$ onto itself, and yields $\pi_m = \pi'_m$. Repeating these we have the desired isomorphism.

Remark. If $H^2(M) = 0$, then we obtain the same conclusion without using the replacement of ν. The result has been known as the theorem of Lichnerowicz. Also, Gutt [G] discussed the case $\dim H^2(M) = 1$, and $[\pi_1] = 0$. One can show this case can really happen on a cotangent bundle.

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