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A Remark on Wagner's Ring of Normal Numbers.

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A remark on Wagner's ring of normal numbers

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Wagner [3] is the first who constructed rings of normal and nonnormal numbers. In [1], the author jointly with I. Shiokawa gave a new construction of rings of the same properties. In this paper, we shall extend the construction of Wagner by the method developed in [1].

Theorem. Let a and b be integers with $a, b \geqq 2$ and $(a, b) = 1$, and let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be increasing sequences of positive integers with

$$(\mathrm{G1}) \quad \lim_{n\to\infty} \frac{\alpha_n}{\beta_{n-1}} = \infty, \qquad (\mathrm{G2}) \quad \lim_{n\to\infty} \frac{\log \beta_n}{\alpha_n} = \infty.$$

Let R be the ring generated by the set of all numbers

$$\prod_{n=1}^{\infty} \left(1 + \frac{\varepsilon_n}{a^{\alpha_n} b^{\beta_n}}\right), \quad \varepsilon_n \in \{-1, 1\}.$$

Then R has the following properties:

(a) R is uncountable,

(b) all numbers $x \in R$, $x \neq 0$ are normal to base b, and

(c) all numbers $x \in R$ are nonnormal to base ab.

In Wagner's theorem in [3] it is assumed that a is prime and $\lim_{n\to\infty} \frac{\alpha_n}{n\beta_{n-1}} = \infty$ instead of (G1). To prove our Theorem, we use the following lemma.

Lemma ([1] Theorem 3). Let a, $b > 1$ be integers with $(a, b) = 1$, let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ be sequences of positive integers which are increasing and

$$\beta_n \geqq a^{\alpha_n}$$

for all large n, and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of integers such that

$$|A_n| < a^{\alpha_n - \alpha_{n-1}}$$

for all large n and $A_n \neq 0$ for infinitely many n. Then the number

$$x = \sum_{n=1}^{\infty} \frac{A_n}{a^{\alpha_n} b^{\beta_n}}$$

is normal to base b and nonnormal to base ab.

Proof of Theorem. Since the proof of (a) is easy, we prove (b) and (c). For any given polynomial

$$F(x_1, \cdots, x_h) = \sum_{\lambda=1}^{l} \sum_{m=1}^{M_\lambda} v_{\lambda m} x_1^{e_{m1}} \cdots x_h^{e_{mh}} \quad (e_{m1} + \cdots + e_{mh} = \lambda)$$

of h variables $x_1, \cdots, x_h$ of total degree $l \geqq 1$ with integral coefficients and generators

$$y_k = \prod_{n=1}^{\infty} \left(1 + \frac{\varepsilon_{nk}}{a^{\alpha_n} b^{\beta_n}}\right), \quad \varepsilon_{nk} \in \{-1, 1\}, \quad k = 1, \cdots, h$$

we have to prove $x = F(y_1, \cdots, y_h)$ is normal to base b and nonnormal to base ab. We can write as in [3]

$$x = A(0) + \sum_{n_1 > \cdots > n_r \geqq 1} \frac{A(n_1, \cdots, n_r; \mu_1, \cdots, \mu_r)}{a^{\sum_{j=1}^{n} \mu_j \alpha_{n_j}} b^{\sum_{j=1}^{n} \mu_j \beta_{n_j}}}$$

with

$$A(0) = \sum_{\lambda=1}^{l} \sum_{m=1}^{M_\lambda} v_{\lambda m}, \quad 1 \leqq \mu_j \leqq l,$$

where

$$|A(n_1, \cdots, n_r; \mu_1, \cdots, \mu_r)| \leqq 2^{n_1 l} \sum_{\lambda=1}^{l} \sum_{m=1}^{M_\lambda} |v_{\lambda m}| \ll 2^{n_1 l}.$$

Hence we have

$$x = A(0) + \sum_{n=1}^{\infty} \sum_{m=1}^{N_n} \frac{A(n, m; \overset{(m)}{\mu}_{n1}, \cdots, \overset{(m)}{\mu}_{nn})}{a^{\sum_{i=1}^{n} \overset{(m)}{\mu}_{ni} \alpha_i} b^{\sum_{i=1}^{n} \overset{(m)}{\mu}_{ni} \beta_i}}$$

with

$$|A(n, m; \overset{(m)}{\mu}_{n1}, \cdots, \overset{(m)}{\mu}_{nn})| \ll 2^{nl}, \quad 0 \leqq \overset{(m)}{\mu}_{ni} \leqq l, \quad \overset{(m)}{\mu}_{nn} \neq 0, \quad N_n \leqq l(l+1)^{n-1}. \qquad (1)$$

We put

$$\alpha'_{k\lambda}=\lambda\alpha_k+l\sum_{i=1}^{k-1}\alpha_i,\quad \beta'_{k\lambda}=\lambda\beta_k+l\sum_{i=1}^{k-1}\beta_i$$

and define new sequences $\{\alpha''_n\}_{n=1}^{\infty}$ and $\{\beta''_n\}_{n=1}^{\infty}$ by

$$\{\alpha''_n\}_{n=1}^{\infty}=\{\alpha'_{11},\cdots,\alpha'_{1l},\alpha'_{21},\cdots,\alpha'_{2l},\cdots\cdots,\alpha'_{k1},\cdots,\alpha'_{kl},\cdots\cdots\},$$

$$\{\beta''_n\}_{n=1}^{\infty}=\{\beta'_{11},\cdots,\beta'_{1l},\beta'_{21},\cdots,\beta'_{2l},\cdots\cdots,\beta'_{k1},\cdots,\beta'_{kl},\cdots\cdots\}.$$

Using these symbols we may write

$$x=\sum_{k=1}^{\infty}\sum_{\lambda=1}^{l}\frac{A'_{k\lambda}}{a^{\alpha'_{k\lambda}}b^{\beta'_{k\lambda}}}=\sum_{n=1}^{\infty}\frac{A''_n}{a^{\alpha''_n}b^{\beta''_n}} \tag{2}$$

with

$$A''_n=A'_{k\lambda}=\sum_{m=1}^{N_n}A(n,m;\overset{(m)}{\mu}_{n1},\cdots,\overset{(m)}{\mu}_{nn})\,a^{\sum_{i=1}^{n-1}(l-\overset{(m)}{\mu}_{ni})\alpha_i}\,b^{\sum_{i=1}^{n-1}(l-\overset{(m)}{\mu}_{ni})\beta_i},$$

where

$$|A''_n|=|A'_{k\lambda}|\leqq N_n\max A(n,m;\overset{(m)}{\mu}_{n1},\cdots,\overset{(m)}{\mu}_{nn})a^{\sum_{i=1}^{n-1}(l-\overset{(m)}{\mu}_{ni})\alpha_i}b^{\sum_{i=1}^{n-1}(l-\overset{(m)}{\mu}_{ni}\beta_i)}$$
$$\ll(l+1)^{n-1}2^{nl}a^{2l\alpha_{n-1}}b^{2l\beta_{n-1}},$$

using (1), (G1), and (G2). Hence we get

$$\log_a|A''_n|\ll n+\alpha_{n-1}+\beta_{n-1}.$$

Thus we have by (G1)

$$\log_a|A''_n|<\alpha_n. \tag{3}$$

for all large n. Therefore, noticing that

$$\alpha''_n-\alpha''_{n-1}\geqq\alpha_n,$$

we have by (3)

$$|A''_n|<a^{\alpha''_n-\alpha''_{n-1}}. \tag{4}$$

We have also by (G1) and (G2)

$$a^{\alpha''_n}\leqq a^{\lambda\alpha_k+l\sum_{i=1}^{k-1}\alpha_i}\leqq a^{2l\alpha_k}\leqq\beta_n. \tag{5}$$

Finally we remark that $A''_n\neq 0$ for infinitely many n by Corollary of Lemma 2 in [3]. Hence we may apply Lemma to the number x defined by (2) with (4) and (5), and find that x is normal to base b and nonnormal to base ab.

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References

[1] H. Kano and I. Shiokawa, "Rings of normal and nonnormal numbers", preprint.

[2] A. N. Korobov, "Continued fraction of certain normal numbers", Math. Notes 47 (1990) No. 2, pp. 128–132 = Matematicheskie Zametki 47 (1990) No. 2, pp. 28–33.

[3] G. Wagner, "On rings of normal and nonnormal numbers", preprint.

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