Research Report

KSTS/RR-91/004
September 1, 1991

Deformation Quantization of Poisson Algebras.

by

Hideki Omori, Yosiaki Maeda and Akira Yoshioka

Yosiaki Maeda

Department of Mathematics
Faculty of Science and Technology
Keio University

Hideki Omori and Akira Yoshioka

Department of Mathematics
Faculty of Science and Technology
Science University of Tokyo

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku, Yokohama, 223 Japan

KSTS/RR-91/004
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DEFORMATION QUANTIZATION OF POISSON ALGEBRAS

HIDEKI OMORI*), YOSIAKI MAEDA**), AKIRA YOSHIOKA*)

Dedicated to Professor Morio Obata for his 65th birthday

ABSTRACT. We give an obstruction theory for making an associative algebra deformed from a given Poisson algebra. The obstruction cocycle is obtained as a 3rd deRham-Chevalley cocycle of the Poisson algebra. Several examples of Poisson algebras without obstruction are given. These examples relate to non-commutative torus and quantum groups.

§0. Introduction

Let M be a C^∞ paracompact manifold, and $C^\infty(M)$ the commutative topological algebra over $\mathbb{C}$ with the C^∞ topology of all $\mathbb{C}$-valued C^∞ functions on M. In what follows, we denote $C^\infty(M)$ by $\mathfrak{a}$ for simplicity. We are now concerned with "deforming" this algebra to a non-commutative but an associative one.

By introducing a formal parameter ν, we consider the direct product

$$\mathfrak{a}[[\nu]] = \prod_{n=0}^{\infty} \nu^n \mathfrak{a}.$$

What we want is to define a product $*$ on $\mathfrak{a}[[\nu]]$ with the following properties:

(A. 1) $* : \mathfrak{a}[[\nu]] \times \mathfrak{a}[[\nu]] \to \mathfrak{a}[[\nu]]$ is a continuous and associative product.

(A. 2) ν commutes with any element of $\mathfrak{a}[[\nu]]$.

(A. 3) $1 * \tilde{f} = \tilde{f} * 1 = \tilde{f}$ for any $\tilde{f} \in \mathfrak{a}[[\nu]]$.

Given a product $*$ on $\mathfrak{a}[[\nu]]$ with (A. 1-3), we set for any $f, g \in \mathfrak{a}$,

$$f * g = \sum_{n=0}^{\infty} \nu^n \pi_n(f, g)$$

as the decomposition of $f * g$. So (A. 1-3) imply that, for any $f, g, h \in \mathfrak{a}$,

$$\text{(0.1)} \qquad \sum_{k+l=m} \pi_k(\pi_l(f,g),h) = \sum_{k+l=m} \pi_k(f, \pi_l(g,h)), \quad \text{for any} \quad m \geq 0,$$

$$\text{(0.2)} \qquad \pi_0(f,1) = \pi_0(1,f) = f, \quad \pi_m(f,1) = \pi_m(1,f) = 0 \quad \text{for any} \quad m > 0.$$

In this paper, we give the following notion on deformations of $\mathfrak{a}$.

*) Dept. of Math. Fac. of Science and Technology, Science Univ. of Tokyo, Noda, Chiba 278, Japan
**) Dept. of Math. Fac. of Science and Technology, Keio Univ., Hiyoshi, Yokohama 223, Japan .

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Definition 1. (0) $(\mathfrak{a}[[\nu]], *)$ is called an *associative deformation of* $\mathfrak{a}$ if $*$ satisfies (A.1-3) and

(A.4) $\qquad \pi_0(f,g) = fg \quad$ (the usual commutative product) for any $\quad f,g \in \mathfrak{a}$.

(1) An associative deformation $(\mathfrak{a}[[\nu]], *)$ of $\mathfrak{a}$ is called a *weak A-deformation* of $\mathfrak{a}$, if the following (A. 5) is satisfied:

(A.5) $\qquad \pi_m : \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathfrak{a}$ is a bidifferential operator. for any $m > 0$.

Here $\pi : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ is called a *bidifferential operator*, if $\pi(f,g)$ is a differential operator with respect to both f and g. The sum of the order of differentiations with respect to f and g will be called the *order* of π.

(2) Moreover, a weak A-deformation $(\mathfrak{a}[[\nu]], *)$ of $\mathfrak{a}$ is simply called an *A-deformation* of $\mathfrak{a}$, if

(A.6) $\qquad \pi_1(f,g) = -\pi_1(g,f)$

holds.

As it will be seen later (cf. Proposition 2.2), *any weak A-deformation can be changed into an A-deformation.* For an A-deformation $(\mathfrak{a}[[\nu]], *)$ of $\mathfrak{a}$, we set

$$\{f,g\} = -2\pi_1(f,g)$$

and call it the *Poisson bracket.* Put $[f,g] = f*g - g*f$. From the associativity of $(\mathfrak{a}[[\nu]], *)$, the identities

$$(0.3) \qquad \begin{cases} [f,g] = -[g,f], \\ [f, g*h] = [f,g]*h + g*[f,h], \\ [f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0, \end{cases}$$

for any $f,g,h \in \mathfrak{a}$, give the following relations for $\mathfrak{a}$ to be a Poisson algebra (cf.[W]):

$$\begin{cases} \{f,g\} = -\{g,f\}, \\ \{f,gh\} = \{f,g\}h + g\{f,h\}, \\ \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0. \end{cases}$$

Thus, a natural question arises as follows: *Given a Poisson manifold M with the Poisson bracket $\{\,,\,\}$, is there an A-deformation $(\mathfrak{a}[[\nu]], *)$ of $\mathfrak{a}$ such that $-2\pi_1(f,g) = \{f,g\}$?*

On M, we denote by $\cdot$ the usual commutative product on $\mathfrak{a} = C^\infty(M)$. The triplet $(\mathfrak{a}, \cdot, \{\,,\,\})$ is called the *Poisson algebra* of M.

Definition 2. For a given Poisson algebra $(\mathfrak{a}, \cdot, \{\,,\,\})$ of a Poisson manifold, $(\mathfrak{a}[[\nu]], *)$ is called a *Q-deformation* of $(\mathfrak{a}, \cdot, \{\,,\,\})$ if $(\mathfrak{a}[[\nu]], *)$ is an A-deformation such that $\{\,,\,\} = -2\pi_1$. The Poisson algebra $(\mathfrak{a}, \cdot, \{\,,\,\})$ is called to be *deformation quantizable* if it has a Q-deformation.

The purpose of this paper is to construct an obstruction theory for the deformation quantizability of Poisson manifolds. On any Poisson algebra $(\mathfrak{a}, \cdot, \{\,,\,\})$, $\{f,g\}(p)$, $p \in M$, depends only on $df(p)$, $dg(p)$. Thus, $\{\,,\,\}(p)$ defines a skew-symmetric bilinear mapping of $(T_p^*M)^{\mathbb{C}} \times (T_p^*M)^{\mathbb{C}}$ into $\mathbb{C}$, where $(T_p^*M)^{\mathbb{C}}$ is the complexification of the cotangent space T_p^*M at p. The rank of $\{\,,\,\}(p)$ will be called the *rank* of $\{\,,\,\}$ at p. M is a *symplectic manifold* if the rank of $\{\,,\,\}$ is equal to $\dim M$ at every point. It is known in [OMY],[DL] that if M is a symplectic manifold, then $(\mathfrak{a}, \cdot, \{\,,\,\})$ is deformation quantizable.

However, for Poisson algebras of non-constant rank, there is no general theory for the deformation quantizability. The following is a typical example of deformation quantizable Poisson algebras of non-constant rank:

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Ex.1. (cf. [B]) Let $\mathcal{G}^*$ be the dual space of a finite dimensional Lie algebra $\mathcal{G}$. Regarding $X \in \mathcal{G}$ as a linear function on $\mathcal{G}^*$, we define $\{X, Y\} = [X, Y]$, i.e for a linear basis $X_1, \cdots, X_n$ of $\mathcal{G}$, we set

$$\{X_i, X_j\} = \sum_{k=1}^{n} c_{ij}^k X_k$$

using the structure constants c_{ij}^k of $\mathcal{G}$. By the polynomial approximation theorem, the above procedure makes $C^\infty(\mathcal{G}^*)$ a Poisson algebra whose rank is not constant. $(C^\infty(\mathcal{G}^*), \cdot, \{\ ,\ \})$ is deformation quantizable and the Q-deformation is given by the closure of the universal enveloping algebra $\mathcal{U}_\nu(\mathcal{G})$ of $\mathcal{G}$ with the parameter ν, i.e. the algebra generated by $X_1, \cdots, X_n$ with the relations

$$[X_i, X_j] = -\nu \sum_{k=1}^{n} c_{ij}^k X_k.$$

Now, we define the following :

Definition 3. A Q-deformation $(\mathfrak{a}[[\nu]], *)$ is called *regular*, if π_m satisfies

$$\text{(A.7)} \qquad \pi_m(f, g) = (-1)^m \pi_m(g, f), \quad f, g \in \mathfrak{a}, \quad (m = 0, 1, 2, \cdots).$$

The product $*$ of a regular Q-deformation is sometimes called a $*$-*product*(cf.[B],[CG]).

Given a Poisson algebra $(\mathfrak{a}, \cdot, \{\ ,\ \})$, one can define a cohomology group $H^p(M, \cdot, \{\ ,\ \})$ by using the Chevalley coboundary operator defined on the space of alternative p-derivations (cf. §1), which is called the p-th *deRham-Chevalley cohomology group.* This is the same cohomology group which is said the pure 1-differentiable cohomology by Lichnerowicz [L1](see also [Va],[H],[LMR]). If M is a symplectic manifold, then $H^p(M, \cdot, \{\ ,\ \})$ is isomorphic to the usual p-th deRham cohomology group. The obstruction for $(\mathfrak{a}, \cdot, \{\ ,\ \})$ to be deformation quantizable appears in $H^3(M, \cdot, \{\ ,\ \})$.

Theorem 1. *Let M be a Poisson manifold. Assume $H^3(M, \cdot, \{\ ,\ \}) = 0$. Then, for any cohomology class $[\theta] \in H^2(M, \cdot, \{\ ,\ \})$, there exists a regular Q-deformation $(\mathfrak{a}[[\nu]], *_{[\theta]})$. Moreover, if for given two cohomology classes $[\theta]$, $[\theta'] \in H^2(M, \cdot, \{\ ,\ \})$, there exists an isomorphism*

$$\phi : (\mathfrak{a}[[\nu]], *_{[\theta]}) \cong (\mathfrak{a}[[\nu]], *_{[\theta']})$$

such that $\phi = 1\ (mod\, \nu^2)$ and $\phi(\nu) = \nu$, then $[\theta] = [\theta']$.

If the first obstruction cocycle R_4 (cf. (2.12)) is not a coboundary, then $(\mathfrak{a}, \cdot, \{\ ,\ \})$ has no regular Q-deformation (cf. 3.3, Remark 2). R_4 relates to the anormaly in the Jacobi identity of [VK]. However, we do not know whether there is a Poisson algebra with $R_4 \not\sim 0$.

In the case of $\dim M = 2$, $H^3(M, \cdot, \{\ ,\ \}) = 0$ trivially. Furthermore, we can observe that all obstructions vanish exactly for this case. Let $g^{(m)}$ $(m = 2, 3, \cdots)$ be a 2-contravariant C^∞ tensor fields on M such that

$$\text{(0.4)} \qquad g_{ij}^{(m)} = (-1)^m g_{ji}^{(m)},$$

where we write as

$$g^{(m)} = \sum_{ij=1}^{2} g_{ij}^{(m)} \partial_i \otimes \partial_j,$$

by using a local coordinate system (x_1, x_2). Following the proof of Theorem 1, we have

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Theorem 2. *Suppose* $\dim M = 2$. *Then, for any 2-contravariant* C^∞ *tensor fields* $g^{(m)}$ *with* (0.4) ($m = 2, 3, \cdots$), *there exists a regular Q-deformation of* $(\mathfrak{a}, \cdot, \{\ ,\ \})$ *such that* $\pi_m(x_i, x_j) = g_{ij}^{(m)}$.

Ex.2. Consider the symplectic form $\frac{1}{y^2} dx \wedge dy$ on the upper half plane H_+. This gives a Poisson algebra structure $\{,\}$ on $C^\infty(H_+)$ such that

$$\{f, g\} = y^2(\partial_x f \partial_y g - \partial_y f \partial_x g)$$

which can be extended to $C^\infty(\mathbb{R}^2)$. The above theorem shows that $(C^\infty(\mathbb{R}^2), \cdot, \{,\})$ has a regular Q-deformation. Since all π_m are bidifferential operators, the restriction $f * g|H_+$ depends only on $f|H_+, g|H_+$. Hence, any regular Q-deformation $(C^\infty(\mathbb{R}^2)[[\nu]], *)$ defines a $*$-product on $C^\infty(H_+)[[\nu]]$. Taking the cartesian coordinates $(x, y) \in \mathbb{R}^2$, we consider the quantized algebra $(C^\infty(\mathbb{R}^2)[[\nu]], *)$ obtained by setting $\pi_m(x, y) = 0$ for $m \geqq 2$. So we have the relation $[x, y] = -\nu y^2$ where $y^2 = yy = y * y$. This is equivalent to

$$y * x = (x + \nu y) * y$$

and the algebra $C^\infty(\mathbb{R}^2)[[\nu]]$ can be characterized only by this relation. Its restriction onto H_+ is isomorphic to the algebra of covariant symbol calculus given in [Be] [Mo]. Notice that our quantized Poisson algebra are obtained by a purely algebraic manner without using any operator representations.

Since any symplectic manifold is deformation quantizable [DL] ,[OMY], we see that the condition $H^3(M, \cdot, \{\ ,\ \}) = 0$ is not a necessary condition for a Poisson algebra to be deformation quantizable. Quantizability seems to relate to local structures of singularities of Poisson structure where the rank is changing.

The following theorem gives a generalization of the result of Lichinerowicz [L2](see also [G1] and [G2]):

Theorem 3. *If a Poisson algebra* $(\mathfrak{a}, \cdot, \{,\})$ *is deformation quantizable, and* $H^2(M, \cdot, \{,\}) = \{0\}$, *then any Q-deformation of* $(\mathfrak{a}, \cdot, \{,\})$ *is mutually isomorphic.*

Next three examples were found by means of our proofs of Theorems $1 - 3$:

Ex.3. Let x, y, z be the natural coordinate functions on $\mathbb{R}^3$. For any positive integers k, l, m, the relations

$$\{x, y\} = z^k, \quad \{y, z\} = x^l, \quad \{z, x\} = y^m$$

define a Poisson algebra structure on $C^\infty(\mathbb{R}^3)$, in which the function

$$f_0(x, y, z) = \frac{1}{l+1} x^{l+1} + \frac{1}{m+1} y^{m+1} + \frac{1}{k+1} z^{k+1}$$

Poisson-commutes with all elements of $C^\infty(\mathbb{R}^3)$ (i.e f_0 is in the center). The Poisson algebra $(C^\infty(\mathbb{R}^3), \cdot, \{,\})$ has a regular Q-deformation such that

$$\pi_j(x, y) = \pi_j(y, z) = \pi_j(z, x) = 0$$

for $j \geq 2$. The obtained Q-deformed algebra is characterized by the relations

$$[x, y] = -\nu z^k, \quad [y, z] = -\nu x^l, \quad [z, x] = -\nu y^m$$

where $z^k = (z\cdot)^k = (z*)^k$ etc.

It is remarkable that the obtained algebra has no nontrivial center.

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Ex.4. Let $x_1, x_2, \cdots, x_n$ be the natural coordinate functions on $\mathbb{R}^n$. For any skew-symmetric matrix $(a_{ij})_{1\leq i,j\leq n}$ and for any positive integers $p_1, \cdots, p_n$, the relations

$$\{x_i, x_j\} = a_{ij}x_i^{p_i}x_j^{p_j}, \quad (1 \leq i, j \leq n)$$

define a Poisson algebra structure on $C^\infty(\mathbb{R}^n)$. If $p_1 = \cdots = p_n = 1$, then $(C^\infty(\mathbb{R}^n), \cdot, \{,\})$ has a regular Q-deformation such that

$$\pi_k(x_i, x_j) = 0 \quad (1 \leq i, j \leq n) \quad \text{for } k \geq 2$$

or such that

$$\sum_{n=0}^{\infty} \nu^n \pi_n(x_i, x_j) = \sqrt{\frac{1 + \frac{\nu}{2}a_{ij}}{1 - \frac{\nu}{2}a_{ij}}} x_i x_j.$$

The latter relates to a non-commutative torus, for

$$x_i * x_j = \sqrt{\frac{1 + \frac{\nu}{2}a_{ij}}{1 - \frac{\nu}{2}a_{ij}}} x_i x_j, \quad x_j * x_i = \sqrt{\frac{1 - \frac{\nu}{2}a_{ij}}{1 + \frac{\nu}{2}a_{ij}}} x_i x_j,$$

hence $x_j * x_i = \frac{1-\frac{\nu}{2}a_{ij}}{1+\frac{\nu}{2}a_{ij}} x_i * x_j$. Thus, a non-commutative torus can be understood as a Q-deformation of Poisson algebra of this type .

Ex.5. Let $\mathfrak{g}$ be the algebra of the so called quantum group $Gl_q(2, R)$ (cf.[Wo],[D][M]). This is the algebra generated by x, y, u, v with the relations

$$\begin{cases} x * u = e^\nu u * x, \quad x * v = e^\nu v * x \\ u * y = e^\nu y * u, \quad v * y = e^\nu y * v \\ \qquad u * v = v * u \\ x * y - e^\nu u * v = y * x - e^{-\nu} u * v. \end{cases}$$

$\mathfrak{g}$ defines the structure of Poisson algebra on $C^\infty(M(2))$, where $M(2)$ is the space of 2×2 matrices, as follows:

$$\{x, u\} = xu, \quad \{x, v\} = xv, \quad \{x, y\} = 2uv,$$

$$\{u, v\} = 0, \quad \{u, y\} = uy, \quad \{v, y\} = vy.$$

This Poisson algebra $(C^\infty(M(2)), \cdot, \{\,,\,\})$ has a regular Q-deformation such that

$$\pi_m(linearfunction\,,\, linearfunction) = 0 \quad \text{for} \quad m \geq 2.$$

Indeed, by the similar computation as in Ex.4, we have that all obstructions vanish.

§1.Algebraic preliminaries

1.1. Hochschild coboundary operators. Let V be a vector space over a commutative ring $\mathcal{R}$. Denote by $\mathrm{C}^p(V)$, $p \geq 1$ p-linear mappings. We denote by $\mathrm{AC}^p(V)$ and $\mathrm{SC}^p(V)$ ($p \geq 1$) the set of the alternative and the symmetric p-linear mappings respectively. If p=0, we set $\mathrm{C}^0(V)$=$\mathrm{AC}^0(V)$= $\mathrm{SC}^0(V)$=V.

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For any $\pi \in C^2(V)$, we define the *Hochschild coboundary operator* $\delta_\pi : C^p(V) \to C^{p+1}(V)$, $p \geq 1$ by

$$(\delta_\pi F)(v_1, \cdots, v_{p+1}) = \pi(v_1, F(v_2, \cdots, v_{p+1})) + \sum_{i=1}^{p}(-1)^i F(v_1, \cdots, \pi(v_i, v_{i+1}), \cdots, v_p) + (-1)^{p+1}\pi(F(v_1, \cdots, v_p), v_{p+1}) \tag{1.1}$$

for $F \in C^p(V)$, and for p=0, we set for any $v \in V$,

$$(\delta_\pi v)(v_1) = \pi(v_1, v). \tag{1.2}$$

By a direct computation using the linearization, we have the following:

Lemma 1.1. *For any $\pi, \pi', \pi'' \in C^2(V)$, we have*

$$\delta_\pi \pi' = \delta_{\pi'}\pi, \qquad \delta_\pi I = \pi, \quad (I = identity) \tag{1.3}$$

$$\delta_\pi \delta_\pi \pi = 0, \tag{1.4}$$

$$\mathop{\mathfrak{S}}_{(\pi,\pi',\pi'')} \delta_\pi \delta_{\pi'} \pi'' = 0, \tag{1.5}$$

where $\mathop{\mathfrak{S}}_{(\pi,\pi',\pi'')}$ *means the cyclic summation with respect to π, π', π''.*

Regarding any $\pi \in C^2(V)$ as a bilinear product on V, we give the following:

Definition 1.2. $A \in C^1(V)$ is called a *derivation* of (V, π) if A satisfies

$$(\delta_\pi A)(v_1, v_2) = \pi(v_1, Av_2) - A(\pi(v_1, v_2)) + \pi(Av_1, v_2) = 0. \tag{1.6}$$

We denote by $Der(V, \pi)$ the set of derivations of (V,π).

For any $A, B \in C^1(V)$, we define $[A, B] \in C^1(V)$ by $AB - BA$. Note that

$$\begin{aligned}\delta_\pi[A, B](u, v) =& A(\delta_\pi B)(u, v) - (\delta_\pi B)(Au, v) - (\delta_\pi B)(u, Av) \\ &- B(\delta_\pi A)(u, v) + (\delta_\pi A)(Bu, v) + (\delta_\pi A)(u, Bv).\end{aligned} \tag{1.7}$$

So, we have $Der(V, \pi)$ is a Lie algebra.

For any $\pi \in C^2(V)$, $-\frac{1}{2}\delta_\pi \pi$ is called the *associator* of (V,π). Namely, we have

$$-\frac{1}{2}\delta_\pi \pi(u, v, w) = \pi(\pi(u, v), w) - \pi(u, \pi(v, w)). \tag{1.8}$$

Hence, $\delta_\pi \pi$=0, if and only if (V,π) is an *associative algebra.* If (V,π) is an associative algebra, then

$$\delta_\pi^2 F = 0, \tag{1.9}$$

for any $F \in C^p(V)$ (cf. [Mc]). In particular, $\delta_\pi^2 I = \delta_\pi \pi = 0$. Therefore, we have

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Lemma 1.3. *$\delta_\pi^2=0$ is equivalent to $\delta_\pi\pi=0$.*

1.2. Partial Hochschild coboundary operators. We will introduce the following notion:

Definition 1.4. Given $\pi \in \mathrm{C}^2(V)$, we define

$$\partial_i^\pi : \mathrm{C}^p(V) \longrightarrow \mathrm{C}^{p+1}(V) \quad i=1,\cdots,p, \quad p\geq 1 \tag{1.10}$$

by

$$\begin{aligned}(\partial_i^\pi F)(v_1,\cdots,v_{p+1}) &= \pi(v_i, F(v_1,\cdots,\hat{v}_i,\cdots,v_{p+1})) \\ &\quad - F(v_1,\cdots,\pi(v_i,v_{i+1}),\cdots,v_{p+1}) \\ &\quad + \pi(F(v_1,\cdots,\hat{v}_{i+1},\cdots,v_{p+1}),v_{i+1})\end{aligned} \tag{1.11}$$

for any $F\in \mathrm{C}^p(V)$. ∂_i^π, $(i=1,\cdots,p)$ are called the *partial Hochschild coboundary operators.*

Lemma 1.5. *Assume $\pi \in \mathrm{C}^2(\mathrm{V})$ is symmetric, i.e. $\pi \in \mathrm{SC}^2(V)$.*
(i) *For any $F\in \mathrm{C}^p(V)$, we have*

$$\delta_\pi F = \sum_{i=1}^{p}(-1)^{i-1}\partial_i^\pi F. \tag{1.12}$$

(ii) *If (V,π) is associative, i.e. $\delta_\pi\pi=0$, then*

$$(\partial_i^\pi - \partial_{i+1}^\pi)\partial_i^\pi = 0 \tag{1.13}$$

for $1\leq i\leq p$.

We define mappings σ_p, $\mathfrak{c}_p : \mathrm{C}^p(V)\longrightarrow \mathrm{C}^p(V)$ by

$$(\sigma_p F)(v_1,v_2,\cdots,v_{p-1},v_p) = F(v_p,v_{p-1},\cdots,v_2,v_1), \tag{1.14}$$

$$(\mathfrak{c}_p F)(v_1,v_2,\cdots,v_{p-1},v_p) = F(v_p,v_1,v_2,\cdots,v_{p-1}). \tag{1.15}$$

Obviously $\sigma_2=\mathfrak{c}_2$. Since $\mathfrak{c}_3^3=1$, we have

$$(1+\mathfrak{c}_3+\mathfrak{c}_3^2)(1-\mathfrak{c}_3)=0, \tag{1.16}$$

$$(1-\mathfrak{c}_3+\mathfrak{c}_3^2)(1+\mathfrak{c}_3)=2. \tag{1.17}$$

The following formulas are useful for later computations:

Lemma 1.6. (i) *For any $\pi \in \mathrm{C}^2(V)$ and $F\in \mathrm{C}^p(V)$, we have*

$$\delta_\pi\sigma_p F = (-1)^{p+1}\sigma_{p+1}\delta_{\sigma_2\pi}F, \tag{1.18}$$

$$\partial_j^\pi\mathfrak{c}_p F = \mathfrak{c}_{p+1}\partial_{j+1}^\pi F \quad (1\leq j\leq p-1), \quad \partial_p^\pi\mathfrak{c}_p F = \mathfrak{c}_{p+1}^2\partial_1^\pi F. \tag{1.19}$$

(ii) *In particular, if $\pi \in \mathrm{SC}^2(V)$, we have*

$$\partial_j^\pi\sigma_p F = \sigma_{p+1}\partial_{p+1-j}^\pi F \quad (1\leq j\leq p). \tag{1.20}$$

We call $F \in \mathrm{C}^p(V)$ a *p-derivation with respect to* π, if for any j, $(1 \le j \le p)$

$$\partial_j^\pi F = 0. \tag{1.21}$$

By $Der^p(V,\pi)$, we denote the space of all p-derivations with respect to π. We also set

$$\mathcal{A}^p(V,\pi) = \mathrm{AC}^p(V) \cap Der^p(V,\pi). \tag{1.22}$$

1.3. deRham-Chevalley cohomology. Let V be a vector space over a commutative ring $\mathcal{R}$. For any $\pi \in \mathrm{AC}^2(V)$, we define the *Chevalley coboudary operator*

$$d_\pi : \mathrm{AC}^p(V) \to \mathrm{AC}^{p+1}(V)$$

by

$$\begin{aligned}(d_\pi F)(v_1, \cdots, v_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i+1}\pi(v_i, F(v_1, \cdots, \hat{v}_i, \cdots, v_{p+1})) \\ &+ \sum_{i<j}(-1)^{i+j}F(\pi(v_i, v_j), v_1, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_{p+1}).\end{aligned} \tag{1.23}$$

By a direct computation using the linearization, we have

Lemma 1.7. *For any* $\pi, \pi', \pi'' \in \mathrm{AC}^2(V)$,

$$d_\pi \pi' = d_{\pi'}\pi, \quad d_\pi I = \pi, \qquad (I = \textit{identity}), \tag{1.24}$$

$$d_\pi d_\pi \pi = 0, \tag{1.25}$$

$$\sum_{(\pi,\pi',\pi'')} d_\pi d_{\pi'} \pi'' = 0. \tag{1.26}$$

Since $\pi \in \mathrm{AC}^2(V)$, we see that for any $A \in \mathrm{C}^1(V)$

$$d_\pi A = \delta_\pi A. \tag{1.27}$$

We have also

$$(d_\pi \pi)(u,v,w) = 2 \sum_{(u,v,w)} \pi(u, \pi(v,w)). \tag{1.28}$$

Thus, $d_\pi \pi = 0$ if and only if (V,π) is a Lie algebra. If (V,π) is a Lie algebra, then $d_\pi^2 F = 0$ for any $F \in \mathrm{AC}^p(V)$ (cf.[Ma]). Therefore,

Lemma 1.8. $d_\pi^2 = 0$ *is equivalent to* $d_\pi \pi = 0$.

In the following, we use the notations

$$\pi^{\pm}(u,v) = \frac{1}{2}\{\pi(u,v) \pm \pi(v,u)\}. \tag{1.29}$$

for $\pi \in \mathrm{C}^2(V)$.

We first remark the following:

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Lemma 1.9. *(V,π), $\pi \in C^2(V)$ is an associative algebra if and only if $\delta_\pi \pi \in \mathrm{AC}^3(V)$, and (V,π^-) is a Lie algebra.*

Proof. The necessity is obvious. To prove the sufficiency, note at first that $\delta_\pi\pi \in \mathrm{AC}^3(V)$ implies that (V,π) is an alternative algebra (cf. [S]). It is known in [S] p. 76 that

$$3\,\delta_\pi\pi(u,v,w) = 4 \sum_{(u,v,w)} \pi^-(u,\pi^-(v,w)). \tag{1.30}$$

Thus, if (V,π^-) is a Lie algebra, then $\delta_\pi\pi = 0$, hence (V,π) is associative. □

The following is not hard to prove:

Lemma 1.10. *For any $\pi \in \mathrm{C}^2(V)$, if $\pi' \in \mathcal{A}_2(V,\pi)$, then*

$$d_{\pi'}\mathcal{A}_p(V,\pi) \subset \mathcal{A}_{p+1}(V,\pi).$$

Consider two products π, π' on V such that $\pi \in \mathrm{C}^2(V)$, $\pi' \in \mathcal{A}_2(V,\pi)$. We give the following:

Definition 1.11. (i) A triplet (V,π,π') is called a *non-commutative Poisson algebra* if it satisfies

$$\delta_\pi\pi = 0, \quad d_{\pi'}\pi' = 0. \tag{V1}$$

(ii) Moreover, a non-commutative Poisson algebra (V,π,π') is called simply a *Poisson algebra* if

$$\pi \in \mathrm{SC}^2(V). \tag{V2}$$

Definition 1.12. For any non-commutative Poisson algebra (V,π,π'), we denote the p-th cohomology group of the cochain complex :

$$\cdots \longrightarrow \mathcal{A}_p(V,\pi) \xrightarrow{d_{\pi'}} \mathcal{A}_{p+1}(V,\pi) \longrightarrow \cdots \tag{1.31}$$

by $H^p(V,\pi,\pi')$. $H^*(V,\pi,\pi')$ will be called the *deRham-Chevalley cohomology group* of the (non-commutative) Poisson algebra.

§2. Deformation of $C^\infty(M)$

2.1. Associative deformations of $C^\infty(M)$. Let M be a paracompact smooth manifold. The usual multiplication $f \cdot g$ in $\mathfrak{a}=C^\infty(M)$ may be denoted sometimes by $\pi_0(f,g)$. Introducing a formal parameter ν, we consider the direct product

$$\mathfrak{a}[[\nu]] = \prod_{i=0}^{\infty} \nu^i \mathfrak{a}$$

with the direct product topology where $\mathfrak{a}$ is regarded as a vector space over $\mathbb{C}$ and $\mathfrak{a}[[\nu]]$ is a topological vector space over the coefficient ring $\mathbb{C}[[\nu]]$. By extending the coefficient ring,

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any $F \in C^p(\mathfrak{a})$ can be regarded as an element of $C^p(\mathfrak{a}[[\nu]])$. Hence, any $\tilde{F} \in C^p(\mathfrak{a}[[\nu]])$ can be decomposed as

$$\tilde{F} = \sum_{i=0}^{\infty} \nu^i F_i, \quad F_i \in C^p(\mathfrak{a}). \tag{2.1}$$

F_i will be called the *i-th component* in ν of $\tilde{F}$. By $\mathcal{C}^p(\mathfrak{a})$ (resp. $\mathcal{C}^p(\mathfrak{a}[[\nu]])$) we denote the vector space of all $F \in C^p(\mathfrak{a})$ (resp. $C^p(\mathfrak{a}[[\nu]])$) such that F is continuous. $\tilde{F}=\sum_{i=0}^{\infty} \nu^i F_i$ is an element of $\mathcal{C}^p(\mathfrak{a}[[\nu]])$ if and only if each $F_i \in \mathcal{C}^p(\mathfrak{a})$. Moreover, we put

$$A\mathcal{C}^p(\mathfrak{a}) = AC^p(\mathfrak{a}) \cap \mathcal{C}^p(\mathfrak{a}),$$

and

$$S\mathcal{C}^p(\mathfrak{a}) = SC^p(\mathfrak{a}) \cap \mathcal{C}^p(\mathfrak{a}).$$

Let $\tilde{\pi} \in \mathcal{C}^2(\mathfrak{a}[[\nu]])$ and $(\mathfrak{a}[[\nu]], \tilde{\pi})$ an associative deformation of $(\mathfrak{a}, \pi_0)$ (cf. Def. 1, §0). Let

$$\tilde{\pi} = \sum_{l=0}^{\infty} \nu^l \pi_l \tag{2.2}$$

be the decomposition of $\tilde{\pi}$. By (0.2), we have

$$\pi_m(1, *) = \pi_m(*, 1) = 0 \quad m \geq 1.$$

Note

$$\delta_{\tilde{\pi}} = \sum_{l=0}^{\infty} \nu^l \delta_l \quad (\delta_l = \delta_{\pi_l}). \tag{2.3}$$

If $\tilde{\pi}$ is an associative deformation of $(\mathfrak{a}, \pi_0)$, then $\{\pi_l\}$ satisfy

$$\sum_{i+j=m} \delta_i \pi_j = 0 \tag{2.4$_m$}$$

for each $m \geq 1$. By (1.3), this implies, in particular

$$\delta_0 \pi_1 = 0. \tag{2.5}$$

Lemma 2.1. (i) *If* $F \in Der^p(\mathfrak{a}, \pi_0)$, *then* $\delta_0 F=0$.
(ii) *For* $\pi \in AC^2(\mathfrak{a})$, $\delta_0\pi=0$ *if and only if* $\pi \in Der^2(\mathfrak{a}, \pi_0)$.

Proof. By (1.12), (i) and the sufficiency of (ii) are trivial. Suppose $\delta_0\pi=0$, $\pi \in AC^2(\mathfrak{a})$. By (1.12), (1.19), we have

$$(1 + \mathfrak{c}_3)\partial_2^0 \pi = (\partial_2^0 - \partial_1^0)\pi = -\delta_0 \pi = 0, \tag{2.6}$$

where $\partial_i^0 = \partial_i^{\pi_0}$. By (1.17), we have $\partial_2^0\pi=0$. $\partial_1^0\pi=0$ follows from (2.6) directly. □

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Proposition 2.2. *Let $\theta \in \mathcal{C}^2(\mathfrak{a})$ be a Hochschild 2-cocycle (i.e. $\delta_0\theta=0$) with $\theta(1,*)=\theta(*,1)=0$. If θ is a bidifferential operator and $\theta \in S\mathcal{C}^2(\mathfrak{a})$, then there exists a linear differential operator $\xi \in C^1(\mathfrak{a})$ such that $\theta=-\delta_0\xi$.*

Proof. Note that $\delta_0(\xi) \in S\mathcal{C}^2(\mathfrak{a})$ by Definition 1.1. Suppose $(U_\alpha, x_1, \cdots, x_n)$ is a local coordinate system on M. If $\theta=-\delta_0\xi_\alpha$ on each U_α, then using a partition of unity, $\{\phi_\alpha\}$, we see that $\theta= -\delta_0\sum_\alpha \phi_\alpha\xi_\alpha$. Thus, we have only to show that $\theta=-\delta_0\xi_\alpha$ on U_α. For any point $\mathbf{a}=(a_1,\cdots,a_n)$, and $f \in C^\infty(U_\alpha)$, we set $f_i^{\mathbf{a}}= f(a_1,\cdots,a_{i-1},x_i,\cdots,x_n)$. Obviously,

$$f(\mathbf{x})-f(\mathbf{a})=\sum_{i=1}^{n}\frac{f_i^{\mathbf{a}}-f_{i+1}^{\mathbf{a}}}{x_i-a_i}(x_i-a_i).$$

Notice that $\frac{f_i^{\mathbf{a}}-f_{i+1}^{\mathbf{a}}}{x_i-a_i}$ is C^∞ with respect to $(\mathbf{a},\mathbf{x}) \in U_\alpha \times U_\alpha$. Define $\xi_\alpha(f)$ by

$$\xi_\alpha(f)(\mathbf{a})=\sum_{i=1}^{n}\theta(\frac{f_i^{\mathbf{a}}-f_{i+1}^{\mathbf{a}}}{x_i-a_i},x_i-a_i)(\mathbf{a}).$$

Note that $\theta(*,1)=0$. If θ is a bidifferential operator of order k, $\xi_\alpha(f)$ is a linear differential operator of order k. Thus, we have

$$\begin{aligned}-(\delta_0\xi_\alpha)(f,g)(\mathbf{a}) &= (\xi_\alpha(fg)-f\xi_\alpha(g)-\xi_\alpha(f)g)(\mathbf{a})\\ &=\sum_{i=1}^{n}(\theta(f_i^{\mathbf{a}}\cdot\frac{g_i^{\mathbf{a}}-g_{i+1}^{\mathbf{a}}}{x_i-a_i},x_i-a_i)(\mathbf{a})-f(\mathbf{a})\theta(\frac{g_i^{\mathbf{a}}-g_{i+1}^{\mathbf{a}}}{x_i-a_i},x_i-a_i)(\mathbf{a}))\\ &+\sum_{i=1}^{n}(\theta(g_{i+1}^{\mathbf{a}}\cdot\frac{f_i^{\mathbf{a}}-f_{i+1}^{\mathbf{a}}}{x_i-a_i},x_i-a_i)(\mathbf{a})-g(\mathbf{a})\theta(\frac{f_i^{\mathbf{a}}-f_{i+1}^{\mathbf{a}}}{x_i-a_i},x_i-a_i)(\mathbf{a})).\end{aligned}$$

Since $\delta_0\theta=0$ implies

$$\theta(fg,h)-f\theta(g,h)=\theta(f,gh)-\theta(f,g)h,$$

we have by using $\theta(f,g)=\theta(g,f)$ and $\theta(*,1)=0$ that

$$\begin{aligned}-(\delta_0\xi_\alpha)(f,g)(\mathbf{a}) &=\sum_{i=1}^{n}(\theta(f_i^{\mathbf{a}},g_i^{\mathbf{a}}-g_{i+1}^{\mathbf{a}})+\theta(g_{i+1}^{\mathbf{a}},f_i^{\mathbf{a}}-f_{i+1}^{\mathbf{a}}))(\mathbf{a})\\ &=\sum_{i=1}^{n}(\theta(f_i^{\mathbf{a}},g_i^{\mathbf{a}})-\theta(f_{i+1}^{\mathbf{a}},g_{i+1}^{\mathbf{a}}))(\mathbf{a})\\ &=\theta(f,g)(\mathbf{a}). \quad \square\end{aligned}$$

We now give the following remark: Let $(\mathfrak{a}[[\nu]],*)$ be a weak A-deformation of $(\mathfrak{a},\pi_0)$. Note that $\delta_0\pi_1^+=0$ by (2.5) and (1.18), where π_1^+ is defined in (1.29). Then there exists $\xi \in \mathcal{C}^1(\mathfrak{a})$ such that $\pi_1^+ = -\delta_0\xi$. Set a $\mathbb{C}[[\nu]]$ -linear isomorphism ϕ: $\mathfrak{a}[[\nu]] \longrightarrow \mathfrak{a}[[\nu]]$ by

$$\phi(f)=f+\nu\xi(f)$$

where ξ is given by Proposition 2.2. Then, we have

$$\phi^{-1}(\phi(f)*\phi(g))=f\cdot g+\nu\pi_1^-(f,g) \pmod{\nu^2}.$$

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This implies that any weak A-deformation can be changed to an A-deformation via $\mathbb{C}[[\nu]]$-linear isomorphism.

In Proposition 2.6 (i), we shall show that $d_{\pi_1^-}\pi_1^-=0$ is another necessary condition for $(\mathfrak{a},\pi_0)$ to be deformed as an associative algebra. However, in this stage, we restrict our attention to the Hochschild coboundary operator.

Note that to make a weak A-deformation of π_0 is to make $\{\pi_i\}_{i\geq 1}$ satisfying $(2.4)_m$ for all m. Suppose that $\pi_1, \cdots, \pi_{k-1}$ are obtained so that $(2.4)_m$ holds for any $m \leq k-1$. To make π_k, one has to solve $(2.4)_k$ with respect to π_k. For that purpose, we rewrite $(2.4)_k$ as follows by using (1.3):

$$\delta_0\pi_k = -\frac{1}{2}\sum_{i+j=k,i,j\geq 1}\delta_i\pi_j. \tag{2.7}$$

Since $\delta_0^2=0$ by the associativity of π_0, if (2.7) can be solved, then the right hand side must satisfy

$$\sum_{i+j=k,i,j\geq 1}\delta_0\delta_i\pi_j = 0. \tag{2.8}$$

At the first glance, (2.8) looks like another necessary condition for $(\mathfrak{a},\pi_0)$ to be associatively deformed, but in fact (2.8) is fulfilled automatically. Namely, we have

Proposition 2.3. *Let $(\mathfrak{a},\pi_0)$ be any associative algebra. If $\pi_0,\pi_1,\cdots,\pi_{k-1}\in C^2(\mathfrak{a})$ satisfy $\sum_{i+j=l}\delta_i\pi_j=0$ for any integer l such that $0\leq l\leq k-1$, then $\pi_0,\cdots,\pi_{k-1}$ satisfy also (2.8).*

Proof. By the assumption, $\hat{\pi}=\pi_0+\nu\pi_1+\cdots+\nu^{k-1}\pi_{k-1}$ satisfies $\delta_{\hat{\pi}}\hat{\pi}=0 \pmod{\nu^k}$, hence $\hat{\pi}$ defines an associative algebra structure on $\mathfrak{a}\oplus\cdots\oplus\nu^{k-1}\mathfrak{a}$ with $\nu^k=0$. Since $\delta_{\hat{\pi}}^2=0$ by Lemma 1.3, we have $\sum_{i+j=l}\delta_i\delta_j=0$ for any l such that $0\leq l\leq k-1$. Hence, for any l, $1\leq l\leq k-1$, we have

$$\sum_{i+j=l}\delta_i\delta_j\pi_{k-l}=0.$$

It follows

$$\sum_{i+j=k,i,j\geq 1}\delta_0\delta_i\pi_j+\sum_{i+j=k,i,j\geq 1}\delta_i\delta_0\pi_j+\sum_{a+b+c=k,a,b,c\geq 1}\delta_a\delta_b\pi_c=0. \tag{2.9}$$

By (1.5), the third term vanishes, hence by (1.3) we see

$$\sum_{i+j=k,i,j\geq 1}\delta_0\delta_i\pi_j=-\sum_{i+j=k,i,j\geq 1}\delta_i\delta_j\pi_0. \tag{2.10}$$

On the other hand, since $\delta_{\hat{\pi}}\hat{\pi}=0 \pmod{\nu^k}$ by the associativity, we have $\sum_{i+j=l}\delta_i\pi_j=0$ for any l, $0\leq l\leq k-1$. Hence for any l, $1\leq l\leq k-1$, we have

$$\sum_{i+j=l}\delta_{k-l}\delta_i\pi_j=0.$$

It follows

$$\sum_{i+j=k,i,j\geq 1}\delta_i\delta_0\pi_j+\sum_{i+j=k,i,j\geq 1}\delta_i\delta_j\pi_0+\sum_{a+b+c=k,a,b,c\geq 1}\delta_a\delta_b\pi_c=0. \tag{2.11}$$

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Since the third term of (2.11) vanishes by (1.5), (2.11) together with (2.10) gives Proposition 2.3. □

Definition 2.4. Let $\pi_0, \cdots, \pi_{k-1} \in \mathcal{C}^2(\mathfrak{a})$ satisfying $\pi_0^- = 0$ and $(2.4)_m$ for each $1 \leq m \leq k-1$. For simplicity, we set $d_i^- = d_{\pi_i^-}$ and

$$\begin{cases} Q_k &= \frac{1}{2}\sum_{i+j=k,i,j\geq 1}\delta_i\pi_j, \\ R_k &= \frac{1}{2}\sum_{i+j=k,i,j\geq 1} d_i^-\pi_j^-. \end{cases} \tag{2.12}$$

Remark. If k is odd and $\pi_j(f,g)=(-1)^j\pi_j(g,f)$ is satisfied for $0 \leq j \leq k-1$, then $R_k=0$.

By Proposition 2.3, we have $\delta_0 Q_k=0$ if $\pi_0, \pi_1, \cdots, \pi_{k-1}$ satisfy $(2.4)_m$ for any m, $1 \leq m \leq k-1$. By a similar manner, we have the following:

Proposition 2.5. *Let $(\mathfrak{a}, \pi_0, \pi_1)$ be any Poisson algebra. If $\pi_0, \cdots, \pi_{k-1} \in \mathcal{C}^2(\mathfrak{a})$ satisfy $\sum_{i+j=l}\delta_i\pi_j=0$ for any integer l such that $0 \leq l \leq k-1$, then $d_1^- R_k=0$.*

Proof. By the assumption, $\hat{\pi}= \pi_0+\nu\pi_1+\cdots+\nu^{k-1}\pi_{k-1}$ defines an associative product on $\mathfrak{a}\oplus\cdots\oplus\nu^{k-1}\mathfrak{a}$ with $\nu^k=0$. By Lemma 1.9, it follows that $\hat{\pi}^-= \nu\pi_1^-+\cdots+\nu^{k-1}\pi_{k-1}^-$ gives a Lie algebra structure on the same space. By using Lemma 1.8, we have for any m, $2 \leq m \leq k-1$, that

$$\sum_{i+j=m} d_i^-\pi_j^- = 0, \tag{2.13}$$

$$\sum_{i+j=m} d_i^- d_j^- = 0. \tag{2.14}$$

Since $d_1^- d_1^-=0$ by the assumption, and (1.24) holds, we have only to show that

$$\sum_{i+j=k,i,j\geq 2} d_1^- d_i^-\pi_j^- = 0.$$

By (2.13), we have

$$\sum_{i+j=m} d_{k-m}^- d_i^-\pi_j^- = 0 \quad \text{for} \quad 2 \leq m \leq k-2.$$

It follows that

$$\sum_{i+j=k,i,j\geq 2} d_i^- d_1^-\pi_j^- + \sum_{i+j=k,i,j\geq 2} d_i^- d_j^-\pi_1^- + \sum_{a+b+c=k,a,b,c\geq 2} d_a^- d_b^-\pi_c^- = 0. \tag{2.15}$$

The third term of (2.15) vanishes by (1.26). Hence by (1.24) we have

$$\sum_{i+j=k,i,j\geq 2} d_i^- d_1^-\pi_j^- = 0.$$

On the other hand, by (2.14) we have

$$\sum_{i+j=m} d_i^- d_j^-\pi_{k-m} = 0 \quad \text{for} \quad 2 \leq m \leq k-2.$$

Thus, we have

$$\sum_{i+j=k,i,j\geq 2} d_1^- d_i^-\pi_j^- + \sum_{i+j=k,i,j\geq 2} d_i^- d_1^-\pi_j^- + \sum_{a+b+c=k,a,b,c\geq 2} d_a^- d_b^-\pi_c^- = 0. \tag{2.16}$$

Since the third term of (2.16) vanishes, we have $\sum_{i+j=k,i,j\geq 2} d_1^- d_i^-\pi_j^-=0$. □

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Proposition 2.6. *Let $(\mathfrak{a},\pi_0,\pi_1)$ be a Poisson algebra and $\pi_2,\cdots,\pi_k \in \mathcal{C}^2(\mathfrak{a})$.*
(i) *If $\pi_0,\pi_1,\cdots,\pi_k$ satisfy $\sum_{i+j=l}\delta_i\pi_j=0$ for $0\le l\le k$, then $R_l=0$ for $1\le l\le k$.*
(ii) *If $\pi_0,\pi_1,\cdots,\pi_k$ satisfy for $0\le l\le k$ that*

$$\sum_{i+j=l}\delta_i\pi_j=\frac{2}{3}\sum_{i+j=l,i,j\geq 1}d_i^-\pi_j^- \quad (\text{cf. } (1.28), (1.30)), \tag{2.17}$$

then for any integer l, $1\le l\le k$,

$$\sum_{i+j=l}(R_i(\pi_j^-(f,g),h,t)+R_i(f,g,\pi_j^-(h,t))), \quad f,g,h,t\in\mathfrak{a} \tag{2.18}$$

is alternative with respect to (f,g,h,t) (see also Remark 4, §3.3).

Proof. $\hat{\pi}=\pi_0+\nu\pi_1+\cdots+\nu^k\pi_k$ defines an associative product on $\mathfrak{a}\oplus\cdots\oplus\nu^k\mathfrak{a}$ with $\nu^{k+1}=0$. Thus, $\hat{\pi}^-$ gives the Lie algebra structure on the same space. Hence, $d_{\hat{\pi}^-}\hat{\pi}^-=0$. It follows that

$$\sum_{i+j=l}d_i^-\pi_j^-=0, \quad 1\le l\le k.$$

Since $d_0^-=0$, $\pi_0^-=0$, the above equality shows $R_l=0$.

For the second assertion, note that $\hat{\pi}=\pi_0+\nu\pi_1+\cdots+\nu^k\pi_k$ defines an alternative algebra structure on $\mathfrak{a}\oplus\cdots\oplus\nu^k\mathfrak{a}$, with $\nu^{k+1}=0$. For simplicity, we shall denote $\hat{\pi}(f,g)$ by $f*g$. By $[f,g]_*$ and $\{f,g,h\}_*$, we denote the commutator and the associator respectively,i.e.

$$[f,g]_*=f*g-g*f, \quad \{f,g,h\}_*=(f*g)*h-f*(g*h).$$

Consider $F(f,g,h,t)$ defined by

$$F(f,g,h,t)=\{[f,g]_*,h,t\}_*+\{f,g,[h,t]_*\}_*.$$

It is known in [BK], Lemma 2.1 that $F(f,g,h,t)$ is alternative.

By (1.8) combined with the definition (2.12) of R_l, we see

$$\{\,,\,,\,\}_*=-\frac{1}{2}\delta_{\hat{\pi}}\hat{\pi}=-\frac{2}{3}\sum_{l=1}^{k}\nu^l R_l.$$

Since $[\,,\,]_*=2\sum_{l=1}^{k}\nu^l\pi_l^-$, the alternativity of F gives the desired result. □

§3. Jacobi identities

3.1. Associativity for $\hat{\pi}$. For a Poisson manifold M, consider the Poisson algebra $(\mathfrak{a},\pi_0,\pi_1)$, where $\mathfrak{a}=C^\infty(M)$, $\pi_0(f,g)=f\cdot g$ and $\pi_1(f,g)=-\frac{1}{2}\{f,g\}$. Suppose $\pi_2,\cdots,\pi_{k-1}$ are given so that $\sum_{i+j=l}\delta_i\pi_j=0$ for any l, such that $0\le l\le k-1$.

What we have to consider is to make $\pi_k\in\mathcal{C}^2(\mathfrak{a})$ such that $\sum_{i+j=k}\delta_i\pi_j=0$. As we have already seen in Proposition 2.6 (i), a necessary condition for the existence of π_k is $R_k=0$. This is in fact a necessary condition for $\hat{\pi}=\pi_0+\nu\pi_1+\cdots+\nu^k\pi_k$ to define an associative product on $\mathfrak{a}\oplus\cdots\oplus\nu^k\mathfrak{a}$ with $\nu^{k+1}=0$. In this section, we shall investigate the equation $\sum_{i+j=k}\delta_i\pi_j=0$ more precisely.

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By (2.12), this can be rewritten as

$$\delta_0 \pi_k = -Q_k. \tag{3.1}$$

By Proposition 2.3, we see that $\delta_0 Q_k = 0$. Write $\pi_k^{\pm} = \frac{1}{2}(1 \pm \sigma_2)\pi_k$. Remarking $\sigma_2 = \mathfrak{c}_2$, and using (1.18)-(1.20), we have

$$\delta_0 \pi_k^+ = \frac{1}{2}(1 - \sigma_3)\delta_0 \pi_k = -(1 - \mathfrak{c}_3)\partial_2^0 \pi_k^+, \tag{3.2}$$

$$\delta_0 \pi_k^- = \frac{1}{2}(1 + \sigma_3)\delta_0 \pi_k = -(1 + \mathfrak{c}_3)\partial_2^0 \pi_k^-, \tag{3.3}$$

where $\partial_i^{\pi_0} = \partial_i^0$. By (1.17), the equation (3.1) splits into two equations:

$$\partial_2^0 \pi_k^- = \frac{1}{4}(1 - \mathfrak{c}_3 + \mathfrak{c}_3^2)(1 + \sigma_3)Q_k, \tag{3.4}$$

$$(1 - \mathfrak{c}_3)\partial_2^0 \pi_k^+ = \frac{1}{2}(1 - \sigma_3)Q_k. \tag{3.5}$$

Assume that (3.1) has a solution π_k. By applying (1.13) and (1.16) to (3.4) and (3.5) respectively, Q_k must satisfy in addition to $\delta_0 Q_k = 0$ the following consistency conditions for (3.4) and (3.5):

$$(\partial_2^0 - \partial_3^0)(1 - \mathfrak{c}_3 + \mathfrak{c}_3^2)(1 + \sigma_3)Q_k = 0, \tag{3.6}$$

$$(1 + \mathfrak{c}_3 + \mathfrak{c}_3^2)(1 - \sigma_3)Q_k = 0. \tag{3.7}$$

However, (3.6) is not a new condition as one can see below:

Lemma 3.1. *If $\delta_0 Q = 0$ for $Q \in \mathcal{C}^3(\mathfrak{a})$, then $(\partial_2^0 - \partial_3^0)(1 - \mathfrak{c}_3 + \mathfrak{c}_3^2)(1 + \sigma_3)Q = 0$.*

Proof. If $\delta_0 Q = 0$, then $\delta_0(1 + \sigma_3)Q = 0$ by (1.18). Set $Q^+ = \frac{1}{2}(1 + \sigma_3)Q$. Note that $\delta_0 = \partial_1^0 - \partial_2^0 + \partial_3^0$ by Lemma 1.5 (i). So, we have

$$(\partial_2^0 - \partial_3^0)Q^+ = \partial_1^0 Q^+.$$

Using (1.19), we have $(\partial_2^0 - \partial_3^0)\mathfrak{c}_3^2 = \mathfrak{c}_4^3(\partial_1^0 - \partial_2^0)$. Then, we have

$$(\partial_2^0 - \partial_3^0)\mathfrak{c}_3^2 Q^+ = -\mathfrak{c}_4^3 \partial_3^0 Q^+.$$

Hence,

$$(\partial_2^0 - \partial_3^0)(1 - \mathfrak{c}_3 + \mathfrak{c}_3^2)Q^+ = \partial_1^0 Q^+ - (\partial_2^0 - \partial_3^0)\mathfrak{c}_3 Q^+ - \mathfrak{c}_4^3 \partial_3^0 Q^+. \tag{3.8}$$

By substituting (f, g, h, t), we compute the right hand side of (3.8) directly. Thus, we have

$$(3.9)\qquad f\cdot \underset{\blacktriangle}{Q^+(g,h,t)} - Q^+(f\cdot g,h,t) + \underline{Q^+(f,h,t)\cdot g}$$
$$- g\cdot Q^+(t,f,h) + Q^+(t,f,g\cdot h) - \underline{Q^+(h\cdot t,f,g) + Q^+(h,f,g)\cdot t}$$
$$- t\cdot Q^+(f,h,g) + Q^+(g,h,t\cdot f) - \underset{\blacktriangle}{Q^+(g,h,t)}\cdot f,$$

where $f\cdot g = \pi_0(f,g)$. The terms marked by ▲ are trivially cancelled. Use $\sigma_3 Q^+ = Q^+$, $\delta_0 Q = 0$, to the underlined terms of (3.9). Then, these terms are changed into $Q^+(g\cdot f,h,t) - Q^+(g,f\cdot h,t)$. Hence (3.9) is

$$-Q^+(g,f\cdot h,t) - g\cdot Q^+(t,f,h) + Q^+(t,f,g\cdot h) - t\cdot Q^+(f,h,g) + Q^+(g,h,t\cdot f).$$

Using $\sigma_3 Q^+ = Q^+$ to $Q^+(g,h,t\cdot f)$, we see that (3.9) is $-(\delta_0 Q^+)(t,f,h,g) \;= 0$. □

Next, we consider (3.7), the consistency condition for (3.5).

Lemma 3.2. $(1+\mathfrak{c}_3+\mathfrak{c}_3^2)(1-\sigma_3)Q_k = 4R_k$.

Proof. Since $\delta_i = \delta_i^+ + \delta_i^-$, where $\delta_i^\pm = \delta_{\pi_i^\pm}$, we see by the definition of Q_k, (2.12),

$$(3.10)\qquad Q_k = \frac{1}{2}\sum_{i+j=k,i,j\geq 1}(\delta_i^+\pi_j^+ + \delta_i^-\pi_j^-) + \sum_{i+j=k,i,j\geq 1}\delta_i^+\pi_j^-.$$

Note $\sigma_3\delta_i^+\pi_j^- = \delta_i^+\pi_j^-$, $\sigma_3\delta_i^+\pi_j^+ = -\delta_i^+\pi_j^+$, $\sigma_3\delta_i^-\pi_j^- = -\delta_i^-\pi_j^-$ by (1.18). Then, we have

$$(3.11)\qquad \begin{cases} Q_k - \sigma_3 Q_k &= \sum_{i+j=k,i,j\geq 1}(\delta_i^+\pi_j^+ + \delta_i^-\pi_j^-),\\ Q_k + \sigma_3 Q_k &= 2\sum_{i+j=k,i,j\geq 1}\delta_i^+\pi_j^-.\end{cases}$$

By (1.28) and (3.11), we have

$$(1+\mathfrak{c}_3+\mathfrak{c}_3^2)(1-\sigma_3)Q_k(f,g,h) = 4\sum_{i+j=k,i,j\geq 1}\underset{(f,g,h)}{\mathfrak{S}}\pi_i^-(f,\pi_j^-(g,h)) = 4R_k(f,g,h).\qquad □$$

Remark. By Lemma 3.2, $R_k = 0$ is a necessary condition for (3.5) to be solved. $R_k = 0$ may be called *Jacobi identities* (cf. (2.12)).

3.2. Cohomological property for R_k. To simplify the notations, we shall use the following notations:

$$(3.12)\qquad \begin{cases} f\cdot g &= \pi_0(f,g),\quad \langle f,g\rangle_m^\pm = \pi_m^\pm(f,g),\quad (m\geq 1),\\ \langle f,\langle g,h\rangle^\pm\rangle_m^\pm &= \sum_{i+j=m,i,j\geq 1}\pi_i^\pm(f,\pi_j^\pm(g,h))\qquad (m\geq 2),\\ \langle\langle f,\langle g,h\rangle^\pm\rangle^\pm,t\rangle_m^\pm &= \sum_{a+b+c=m,a,b,c\geq 1}\pi_a^\pm(\pi_b^\pm(f,\pi_c^\pm(g,h)),t)\quad (m\geq 3),\\ \langle\langle f,g\rangle^\pm,\langle h,t\rangle^\pm\rangle_m^\pm &= \sum_{a+b+c=m,a,b,c,\geq 1}\pi_a^\pm(\pi_b^\pm(f,g),\pi_c^\pm(h,t))\quad (m\geq 4).\end{cases}$$

In what follows, we shall prove the following:

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Theorem 3.3. *Let $(\mathfrak{a}, \pi_0, \pi_1)$ be a Poisson algebra. Suppose $\pi_2, \cdots, \pi_{k-1} \in \mathcal{C}^2(\mathfrak{a})$ are given so that they may satisfy $\sum_{i+j=l} \delta_i \pi_j = 0$ for any l, $(0 \leq l \leq k-1)$. Then,*

$$\partial_j^0 R_k = 0, \quad \text{for} \quad j = 1, 2, 3, \ \text{i.e.} \ R_k \in \mathcal{A}_3(\mathfrak{a}, \pi_0).$$

Hence, by Proposition 2.5 R_k is a deRham-Chevalley 3-cocycle.

Proof. By using notations (3.12), R_k is written as

$$\begin{aligned} R_k(f,g,h) &= \langle f, \langle g,h\rangle^- \rangle_k^- + \langle g, \langle h,f\rangle^- \rangle_k^- + \langle h, \langle f,g\rangle^- \rangle_k^- \\ &= \frac{1}{4} \underset{(f,g,h)}{\mathfrak{S}} \sum_{i+j=k} \delta_i^- \pi_j^-(f,g,h). \end{aligned} \tag{3.13}$$

Now, suppose the hypothesis of Theorem 3.3 is fulfilled. For any $m, 1 \leq m \leq k-1$, $\partial_2 \pi_m^-$ is given by (3.4), and this is equivalent to the following:

$$\begin{aligned} &\pi_m^-(f, g\cdot h) \\ =&g \cdot \pi_m^-(f,h) + \pi_m^-(f,g)\cdot h + \langle\langle f,g\rangle^-, h\rangle_m^+ + \langle\langle f,h\rangle^-, g\rangle_m^+ - \langle f, \langle g,h\rangle^+\rangle_m^-. \end{aligned} \tag{3.14}$$

We now compute the following quantity:

$$R_k(f\cdot g, h, t) = \langle f\cdot g, \langle h,t\rangle^- \rangle_k^- + \langle h, \langle t, f\cdot g\rangle^- \rangle_k^- + \langle t, \langle f\cdot g, h\rangle^- \rangle_k^- . \tag{3.15}$$

By using (3.14), (3.15) can be rewritten as

$$\begin{aligned} & f\cdot \langle g, \langle h,t\rangle^- \rangle_k^- + g\cdot \langle f, \langle h,t\rangle^- \rangle_k^- + \langle\langle f, \langle h,t\rangle^- \rangle^-, g\rangle_k^+ \\ & + \langle\langle g, \langle h,t\rangle^- \rangle^-, f\rangle_k^+ + \underset{\blacktriangle}{\langle\langle h,t\rangle^-, \langle f,g\rangle^+\rangle_k^-} \\ & + \underline{\langle h, \langle t,f\rangle^- \cdot g\rangle_k^-} + \underline{\langle h, \langle t,g\rangle^- \cdot f\rangle_k^-} + \langle h, \langle\langle t,f\rangle^-, g\rangle^+\rangle_k^- \\ & + \langle h, \langle\langle t,g\rangle^-, f\rangle^+\rangle_k^- + \underset{\blacktriangle}{\langle h, \langle\langle f,g\rangle^+, t\rangle^-\rangle_k^-} \\ & - \underline{\langle t, \langle h,f\rangle^- \cdot g\rangle_k^-} - \underline{\langle t, \langle h,g\rangle^- \cdot f\rangle_k^-} - \langle t, \langle\langle h,f\rangle^-, g\rangle^+\rangle_k^- \\ & - \langle t, \langle\langle h,g\rangle^-, f\rangle^+\rangle_k^- - \underset{\blacktriangle}{\langle t, \langle\langle f,g\rangle^+, h\rangle^-\rangle_k^-} . \end{aligned} \tag{3.16}$$

The three terms marked by ▲ vanish by virtue of Proposition 2.5, for setting $A_l = \langle f,g\rangle_l^+$ we see that these terms are

$$\begin{aligned} & \sum_{l=1}^{k-1} \{\langle\langle h,t\rangle^-, A_l\rangle_{k-l}^- + \langle\langle t, A_l\rangle^-, h\rangle_{k-l}^- + \langle\langle A_l, h\rangle^-, t\rangle_{k-l}^-\} \\ & = -\sum_{l=1}^{k-1} R_{k-l}(A_l, h, t) = 0. \end{aligned} \tag{3.17}$$

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Computing the underlined 4 terms of (3.16) by using (3.14), we have

$$
\begin{array}{l}
R_k(f\cdot g,h,t)\\
\quad = f\cdot\{\langle g,\langle h,t\rangle^-\rangle_k^- + \langle h,\langle t,g\rangle^-\rangle_k^- + \langle t,\langle g,h\rangle^-\rangle_k^-\}\\
\quad + g\cdot\{\langle f,\langle h,t\rangle^-\rangle_k^- + \langle h,\langle t,f\rangle^-\rangle_k^- + \langle t,\langle f,h\rangle^-\rangle_k^-\}\\
\quad + \langle h,g\rangle^-\cdot\langle t,f\rangle^- + \langle h,f\rangle^-\cdot\langle t,g\rangle^- - \langle t,g\rangle^-\cdot\langle h,f\rangle^- - \langle t,f\rangle^-\cdot\langle h,g\rangle^-\\
\quad +\langle\langle\langle t,f\rangle^-\underset{\blacktriangle}{,}g\rangle^+,h\rangle_k^- \quad +\langle\langle h,\langle t,f\rangle^-\rangle^-,g\rangle_k^+ \quad +\langle\langle h,g\rangle^-\underset{\bigstar}{,}\langle t,f\rangle^-\rangle_k^+\\
\quad +\langle\langle\langle t,g\rangle^-\underset{\blacktriangle}{,}f\rangle^+,h\rangle_k^- \quad +\langle\langle h,\langle t,g\rangle^-\rangle^-,f\rangle_k^+ \quad +\langle\langle h,f\rangle^-\underset{\bigstar}{,}\langle t,g\rangle^-\rangle_k^+\\
\quad -\langle\langle\langle h,f\rangle^-\underset{\blacktriangle}{,}g\rangle^+,t\rangle_k^- \quad -\langle\langle t,\langle h,f\rangle^-\rangle^-,g\rangle_k^+ \quad -\langle\langle t,g\rangle^-\underset{\bigstar}{,}\langle h,f\rangle^-\rangle_k^+\\
\quad -\langle\langle\langle h,g\rangle^-\underset{\blacktriangle}{,}f\rangle^+,t\rangle_k^- \quad -\langle\langle t,\langle h,g\rangle^-\rangle^-,f\rangle_k^+ \quad -\langle\langle t,f\rangle^-\underset{\bigstar}{,}\langle h,g\rangle^-\rangle_k^+\\
\hline
\quad +\langle h,\langle\langle t,f\rangle^-\underset{\blacktriangle}{,}g\rangle^+\rangle_k^-\\
\quad +\langle h,\langle\langle t,g\rangle^-\underset{\blacktriangle}{,}f\rangle^+\rangle_k^-\\
\quad -\langle t,\langle\langle h,f\rangle^-\underset{\blacktriangle}{,}g\rangle^+\rangle_k^- \quad +\langle\langle f,\langle h,t\rangle^-\rangle^-,g\rangle_k^+ \quad +\langle\langle g,\langle h,t\rangle^-\rangle^-,f\rangle_k^+\\
\quad -\langle t,\langle\langle h,g\rangle^-\underset{\blacktriangle}{,}f\rangle^+\rangle_k^-
\end{array}
\tag{3.18}
$$

where six terms below the line of (3.18) come directly from (3.16), and $A^-\cdot B^-$ means $\sum_{i+j=k,i,j\geq 1} A_i^-\cdot B_j^-$. Note that the terms marked by ★ and ▲ vanish by themselves, and the third line of the right hand side of (3.18) also vanish by itself. Hence, we have

$$
\begin{aligned}
R_k(f\cdot g,h,t)&\\
= f\cdot R_k(g,h,t) &+ g\cdot R_k(f,h,t)\\
+ \langle\langle h,\langle t,g\rangle^-\rangle^-,f\rangle_k^+ &+ \langle\langle t,\langle g,h\rangle^-\rangle^-,f\rangle_k^+ + \langle\langle g,\langle h,t\rangle^-\rangle^-,f\rangle_k^+\\
+ \langle\langle h,\langle t,f\rangle^-\rangle^-,g\rangle_k^+ &+ \langle\langle t,\langle f,h\rangle^-\rangle^-,g\rangle_k^+ + \langle\langle f,\langle h,t\rangle^-\rangle^-,g\rangle_k^+.
\end{aligned}
\tag{3.19}
$$

The last six terms vanish by virtue of Proposition 2.6. Hence, we have $\partial_1^0 R_k = 0$. As R_k is alternative, we have $\partial_j^0 R_k = 0$ $(j = 1,2,3)$. Then, Theorem 3.3 is obtained. □

3.3. Remarks on R_k. For later use, we give several remarks on R_k.

Remark 1. Let U be an open set of $\mathbf{R}^n$ with the coordinate functions $x_1,\cdots,x_n$. Consider $M = U$. If $R_k(x_i,x_j,x_k) = 0$, then the 3-derivation property given in the above theorem yields easily $R_k = 0$ together with the continuity of R_k and the polynomial approximation theorem.

In the later section, we shall show that we can set always $\pi_l(x_i, x_j) = 0$ for $l \geq 2$. If this is the case, we have only to check the quantities

$$R_k(x_i, x_j, x_k) = \sum_{(i,j,k)} \pi^-_{k-1}(x_i, \pi^-_1(x_j, x_k)). \tag{3.20}$$

R_2 always vanishes because $d_{\pi_1}\pi_1 = 0$. Hence, if $\pi_1(x_i, x_j) = c_{ij} + \sum_k c^k_{ij}x_k$, then $R_k = 0$ for any $k \geq 2$. This will be the reason why Poisson algebras of constant rank, and linearizable Poisson algebras (cf. [W]) are deformation quantizable.

Remark 2. In §4 and §5, we shall show that $R_k = 0$ is necessary and sufficient condition for $\hat{\pi} = \pi_0 + \nu\pi_1 + \cdots + \nu^k\pi_k$ to be an associative algebra on $\mathfrak{a} \oplus \nu\mathfrak{a} \oplus \cdots \oplus \nu^k\mathfrak{a}$ with $\nu^{k+1} = 0$. However, notice that the solution of (2.7) is not unique. One may replace π_k by $\pi_k + \theta_k$ such that $\delta_0\theta_k = 0$. In Theorem 3.3, if R_k is deRham-Chevalley 3-coboundary, we can modify π_{k-1} so that $R_k = 0$ (cf. see the proof in 6.1).

If one considers only regular Q-deformations, θ_k must satisfy $\theta_k(f, g) = (-1)^k\theta_k(g, f)$. Hence, if k =even, then $\theta_k = \delta_0 c$ by Proposition 2.2, and if k =odd, then θ_k must be a 2-derivation by virtue of Lemma 2.1.

If one replace π^+_{2k-2} by $\pi^+_{2k-2} + \delta_0 c$, then π^-_{2k-1} is influenced by this replacement. One has to replace π^-_{2k-1} by $\pi^-_{2k-1} + d_1 c$, but this replacement does not change R_{2k}. If one changes π^-_{2k-1} by $\pi^-_{2k-1} + \theta_{2k-1}$ furthermore, then R_{2k} is changed by $R_{2k} + d_1\theta_{2k-1}$. Thus the cohomology class of R_{2k} does not change.

Therefore, if a Poisson algebra $(\mathfrak{a}, \cdot, \{\ ,\ \})$ is given, then the cohomology class of the first obstruction cocycle R_4 is determined only by $(\mathfrak{a}, \cdot, \{\ ,\ \})$. If there exists $(\mathfrak{a}, \cdot, \{\ ,\ \})$ such that $[R_4] \neq 0$, then such a Poisson algebra has no regular Q-deformation.

Remark 3. If we relax the associativity of $\hat{\pi}$, and request that $\hat{\pi}$ defines an alternative algebra instead, then the equation corresponding to (2.7) is given by

$$\delta_0\pi_k = -\frac{1}{2}\sum_{i+j=k, i,j\geq 1} \delta_i\pi_j + \frac{2}{3}R_k \quad (= -Q_k + \frac{2}{3}R_k). \tag{3.21}$$

By the same manner as above, (3.21) splits into two equations as follows:

$$\partial^0_2\pi^-_k = \frac{1}{4}(1 - \mathfrak{c}_3 + \mathfrak{c}_3^2)(1 + \sigma_3)Q_k \tag{3.22}$$

$$(1 - \mathfrak{c}_3)\partial^0_2\pi^+_k = \frac{1}{2}(1 - \sigma_3)Q_k - \frac{2}{3}R_k \tag{3.23}$$

because $(1 + \sigma_3)R_k = 0$. Note that (3.22) is same as (3.4). Since $\delta_0 Q_k = 0$ by Proposition 2.3, the consistency condition for (3.22) is fulfilled by Lemma 3.1.

Consider the consistency condition for (3.23). Recall Lemma 3.2:

$$(1 + \mathfrak{c}_3 + \mathfrak{c}_3^2)\frac{1}{2}(1 - \sigma_3)Q_k = 2R_k.$$

Since $R_k \in AC^3(\mathfrak{a})$ by definition of R_k, (2.12), we have $(1 + \mathfrak{c}_3 + \mathfrak{c}_3^2)R_k = 3R_k$. Hence the consistency condition for (3.23) is fulfilled automatically. However, instead of this, another

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necessary condition appears for π_{k+1} to be made so that $\pi_0 + \nu\pi_1 + \cdots + \nu^{k+1}\pi_{k+1}$ defines an alternative algebra with $\nu^{k+2} = 0$. Namely, by Proposition 2.6, we must have that

$$R_k(\pi_1^-(f,g),h,t)) + R_k(f,g,\pi_1^-(h,t)), \quad f,g,h,t \in \mathfrak{a} \tag{3.24}$$

is alternative with respect to (f,g,h,t).

Remark 4. The alternativity of $R_k(\pi_1^-(f,g),h,t) + R_k(f,g,\pi_1^-(h,t))$ looks like a strong condition. Since this is equivalent to

$$R_k(\pi_1^-(f,g),g,t) + R_k(f,g,\pi_1^-(g,t)) = 0,$$

we replace f by f^2. By the derivation properties of π_1^-, R_k, the above equality yields

$$R_k(f,g,t) \cdot \pi_1^-(f,g) = 0.$$

Hence, if $\pi_1^-(f,g) \neq 0$, then $R_k(f,g,t) = 0$ for any t. (cf. [BK].) It is not known whether there exists a non-associative, alternative deformation of $\mathfrak{a}$.

§4. Construction of π_{even}

Let M be a Poisson manifold and $(\mathfrak{a}, \cdot, \{\ ,\ \})$ a Poisson algebra where $\mathfrak{a} = C^\infty(M)$.
In this section, we impose the following:

Assumptions.
(HE.1) Set $\pi_0(f,g) = f \cdot g$, $\pi_1(f,g) = -\frac{1}{2}\{f,g\}$. Furthermore, $\pi_2, \cdots, \pi_{2k-1} \in \mathcal{C}^2(\mathfrak{a})$ are given and they satisfy $\sum_{i+j=l} \delta_i \pi_j = 0$ for any $l, 0 \le l \le 2k-1$.
(HE.2) $\pi^+_{\text{odd}} = \pi^-_{\text{even}} = 0$ for $\pi_0, \pi_1, \cdots, \pi_{2k-1}$.
(HE.3) π_m are bidifferential operator of order $2m$ for $0 \le m \le 2k-1$.

The goal of this section is as follows.

Theorem 4.1. *Assume* (HE.1) - (HE.3). *There exists π_{2k} such that*
(a) $\pi^-_{2k} = 0$, *and π_{2k} is a bidifferential operator.*
(b) $\sum_{i+j=2k} \delta_i \pi_j = \frac{1}{3} \sum_{i+j=2k} \mathfrak{S}\, \delta_i^- \pi_j^- (= \frac{4}{3} R_{2k})$ (*cf.* (3.13) *and Remark 3, §3.3*).
In particular, if $R_{2k} = 0$, then $\hat{\pi} = \pi_0 + \nu\pi_1 + \cdots + \nu^{2k}\pi_{2k}$ gives an associative product on $\mathfrak{a} \oplus \nu\mathfrak{a} \oplus \cdots \oplus \nu^{2k}\mathfrak{a}$ (mod ν^{2k+1}).

4.1. Induction for constructing π_{ev}. Under the assumptions (HE. 1-2), the equations for $\pi_{2k} = \pi^+_{2k} + \pi^-_{2k}$ given by (3.22-23) are rewritten as follows:

$$(1 - \mathfrak{c}_3)\partial_2 \pi^+_{2k} = +\frac{1}{2} \sum_{i+j=2k, i,j \ge 1} (\delta_i^+ \pi_j^+ + \delta_i^- \pi_j^-) - \frac{2}{3} R_{2k}, \tag{4.1}$$

$$\partial_2 \pi^-_{2k} = 0, \tag{4.2}$$

where we used (3.11). By (4.2), one can set

$$\pi^-_{2k} = 0, \tag{4.3}$$

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for this is a solution of (4.2). Now, by a little careful computation together with the definition of $\delta_i^+\pi_j^+$, $\delta_i^-\pi_j^-$, and (3.13), we see that (4.1) is equivalent to the following:

$$\pi_{2k}^+(f,gh) - \pi_{2k}^+(h,gf) = E_{2k}(f,g,h). \tag{4.4}$$

Here, we put

$$\begin{aligned} E_{2k}(f,g,h) = \pi_{2k}^+(f,g)h - \pi_{2k}^+(h,g)f \\ + \langle\langle f,g\rangle^+, h\rangle_{2k}^+ - \langle\langle h,g\rangle^+, f\rangle_{2k}^+ \\ - \langle\langle h,f\rangle^-, g\rangle_{2k}^- + \frac{1}{3}R_{2k}(f,g,h), \end{aligned} \tag{4.5}$$

where $\langle\ ,\ \rangle_{2k}^{\pm}$ is defined by (3.12). Note that

$$E_{2k}(f,g,h) = -E_{2k}(h,g,f). \tag{4.6}$$

To construct π_{2k}^+, we consider at first on a local coordinate neighborhood $(U, x_1, \cdots, x_n)$, and we set

$$\pi_{2k}^+(x_i,x_j) = g_{ij}^{(2k)}, \tag{4.7}$$

where $g_{ij}^{(2k)}$ is arbitrary element of $C^\infty(U)$ such that $g_{ij}^{(2k)} = g_{ji}^{(2k)}$. For multi-indices α, β, we construct $\pi_{2k}^+(x^\alpha, x^\beta)$ inductively. At the same time, let $\zeta_i = x_i - x_i(p)$ for any fixed $p \in U$. Since $\pi_{2k}^+(f,1) = 0$ by the normalizing condition, we see $\pi_{2k}^+(\zeta_i,\zeta_j) = g_{ij}^{(2k)}$. Thus, one can construct $\pi_{2k}^+(\zeta^\alpha,\zeta^\beta)$ by the same procedure. We shall show that for any $p \in U$ and α, β such that $|\alpha+\beta| \geq 4k$,

$$\pi_{2k}^+(\zeta^\alpha,\zeta^\beta) \in \sum_{\gamma, |\gamma| = |\alpha+\beta|-4k} \zeta^\gamma C^\infty(U). \tag{4.8}$$

(4.8) implies that if $|\alpha+\beta| - 4k > 0$, then $\pi_{2k}^+(\zeta^\alpha,\zeta^\beta)(p) = 0$. Hence by Taylor expansions at p, we have

$$\pi_{2k}^+(f,g)(p) = \sum_{\substack{\alpha,\beta \\ |\alpha|+|\beta| \leq 4k}} \frac{1}{\alpha!\beta!}(\partial^\alpha f)(p)(\partial^\beta g)(p)\pi_{2k}^+(\zeta^\alpha,\zeta^\beta)(p). \tag{4.9}$$

This is a bidifferential operator of order $4k$. Thus, to show (a) of Theorem 4.1, it is enough to show (4.8). Since $p \in U$ is fixed arbitrarily, we have only to construct $\pi_{2k}^+(\zeta^\alpha,\zeta^\beta)$ instead of $\pi_{2k}^+(x^\alpha,x^\beta)$. After that, we shall define $\pi_{2k}^+(f,g)$ by (4.9).

To obtain $\pi_{2k}^+(\zeta^\alpha,\zeta^\beta)$, we shall use induction. So, assume the following:

(B)$_s$ $\pi_{2k}^+(\zeta^\alpha,\zeta^\beta)$ are obtained for any $\zeta^\alpha, \zeta^\beta$ such that $|\alpha+\beta| \leq s$, and these satisfy (4.4), and $\pi_{2k}^+(\zeta^\alpha,\zeta^\beta) = \pi_{2k}^+(\zeta^\beta,\zeta^\alpha)$.

In what follows, we put unknown quantities $\pi_{2k}^+(\zeta^\alpha,\zeta^\beta)$ by $\varpi_{2k}^+(\zeta^\alpha,\zeta^\beta)$ for $|\alpha+\beta| = s+1$. Under the assumption (B)$_s$, we want at first to obtain $\varpi_{2k}^+(\zeta_i,\zeta^\gamma)$ for $|\gamma|+1 = s+1$.

We fix ζ^μ arbitrarily such that $|\mu| = s+1$.

In what follows, we shall use the notations

$$(\zeta^\alpha) \in \zeta^\mu, \quad (\zeta^\alpha, \zeta^\beta, \zeta^\gamma) \in \zeta^\mu \quad \text{etc}$$

if there exist $\zeta^\delta, \zeta^{\delta'}$ such that $\zeta^\alpha\zeta^\delta = \zeta^\mu$, $\zeta^\alpha\zeta^\beta\zeta^\gamma\zeta^{\delta'} = \zeta^\mu$ etc.

Now, for any $(\zeta_i, \zeta^\beta, \zeta_j)$ such that $\zeta_i\zeta_j\zeta^\beta = \zeta^\mu$, (4.4) is read as follows:

$$\varpi^+_{2k}(\zeta_i, \zeta^\beta\zeta_j) - \varpi^+_{2k}(\zeta_j, \zeta^\beta\zeta_i) = E_{2k}(\zeta_i, \zeta^\beta, \zeta_j), \tag{4.10}$$

where E_{2k} is defined by (4.5). Set the right hand side $A_{ij}(= -A_{ji})$. Under the assumption $(\mathrm{B})_s$, A_{ij}'s are known quantities.

4.2. Left extremals. We now assume that ζ^μ is fixed as $|\mu| = s+1$. $\varpi^+_{2k}(\zeta_i, \zeta^\beta\zeta_j)$ depends only on i such that $(\zeta_i) \in \zeta^\mu$. Thus, we set

$$T_i = \varpi^+_{2k}(\zeta_i, \zeta^\beta\zeta_j). \tag{4.11}$$

Then, (4.10) is nothing but an over determined system

$$T_i - T_j = A_{ij} \quad \text{for} \quad (\zeta_i, \zeta_j) \in \zeta^\mu.$$

This can be solved if and only if A_{ij} satisfy

$$A_{ij} + A_{jh} + A_{hi} = 0 \quad \text{for any} \quad (\zeta_i, \zeta_j, \zeta_h) \in \zeta^\mu. \tag{4.12}$$

If (4.12) is satisfied, then T_i is given by

$$T_i = \frac{1}{n(\mu)} \sum_l A_{il} + K_{2k}(\zeta^\mu). \tag{4.13}$$

where $n(\mu)$ is the number of (l) such that $(\zeta_l) \in \zeta^\mu$, hence $1 \le n(\mu) \le n$, and

$$K_{2k}(\zeta^\mu) = \text{arbitrary element of} \quad C^\infty(U) \quad \text{depending only on} \quad \zeta^\mu. \tag{4.14}$$

For the later use, we choose K_{2k} as a linear differential operator of order $4k$, or simply $K_{2k} = 0$. If $n(\mu) = 1$, then $T_i = K_{2k}(\zeta^\mu)$. In Proposition 4.2, we shall show that (4.12) is satisfied under the assumptions (HE.1-2).

For that purpose, we shall investigate (4.4) more precisely. For any fixed (f,g,h), (4.4) can be regarded as a linear system with unknowns $\pi^+_{2k}(f,gh)$, $\pi^+_{2k}(g,hf)$, $\pi^+_{2k}(h,fg)$:

$\pi^+_{2k}(f,gh)$	$\pi^+_{2k}(g,hf)$	$\pi^+_{2k}(h,fg)$	
1	0	-1	$:\kappa(f,g,h)$
-1	1	0	$:\kappa(g,h,f)$
0	-1	1	$:\kappa(h,f,g)$

where

$$\kappa(f,g,h) =$$

$$\pi^+_{2k}(f,g)h - \pi^+_{2k}(h,g)f + \langle\langle f,g\rangle^+, h\rangle^+_{2k} - \langle\langle h,g\rangle^+, f\rangle^+_{2k} + \langle\langle f,h\rangle^-, g\rangle^-_{2k} + \frac{1}{3}R_{2k}(f,g,h)$$

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The solvability condition of the above linear system is satisfied by virtue of the term R_{2k}. Set

$$S_{2k}(f,g,h) = \sum_{(f,g,h)} \pi^+_{2k}(f,gh). \tag{4.15}$$

Then $S_{2k} \in SC^3(C^\infty(U))$, and the solution of the linear system (4.15) is written as follows:

$$\begin{aligned}\pi^+_{2k}(f,gh) &= \frac{1}{3}S_{2k}(f,g,h) + \frac{1}{3}\pi^+_{2k}(f,g)h + \frac{1}{3}\pi^+_{2k}(f,h)g - \frac{2}{3}f\pi^+_{2k}(g,h) \\ &+ \frac{1}{3}\langle\langle f,g\rangle^+,h\rangle^+_{2k} + \frac{1}{3}\langle\langle f,h\rangle^+,g\rangle^+_{2k} - \frac{2}{3}\langle\langle g,h\rangle^+,f\rangle^+_{2k} \\ &+ \frac{1}{3}\langle\langle f,g\rangle^-,h\rangle^-_{2k} + \frac{1}{3}\langle\langle f,h\rangle^-,g\rangle^-_{2k}.\end{aligned} \tag{4.16}$$

Note that R_{2k} does not appear in the expression. All others are obtained by the cyclic permutation of (f,g,h). Note also that (4.16) can be applied for π^+_m such that $m \le 2k-1$, where $R_m = 0$.

In this subsection, we shall show the following:

Proposition 4.2. *For any fixed ζ^μ such that $|\mu| = s+1$, the solvability condition* (4.12) *is satisfied, and hence $\varpi^+_{2k}(\zeta_i,\zeta^\alpha)$ are obtained by* (4.13) *for any (ζ_i,ζ^α) such that $\zeta_i\zeta^\alpha = \zeta^\mu$.*

Suppose $(\zeta_i,\zeta_j,\zeta_h) \in \zeta^\mu$, i.e. there is a monomial g such that $\zeta_i\zeta_j\zeta_h g = \zeta^\mu$. By (4.5), we have

$$\begin{aligned}&A_{ij} + A_{jh} + A_{hi} \\ &= \sum_{(i,j,h)} [\pi^+_{2k}(\zeta_i,g\zeta_h)\zeta_j - \pi^+_{2k}(\zeta_j,g\zeta_h)\zeta_i \\ &\quad + \langle\langle\zeta_i,g\zeta_h\rangle^+,\zeta_j\rangle^+_{2k} - \langle\langle\zeta_j,g\zeta_h\rangle^+,\zeta_i\rangle^+_{2k} \\ &\quad + \langle\langle\zeta_i,\zeta_j\rangle^-,g\zeta_h\rangle^-_{2k} + \frac{1}{3}R_{2k}(\zeta_i,g\zeta_h,\zeta_j)] \\ &= (1)+(2)+(3),\end{aligned} \tag{4.17}$$

where

$$\begin{aligned}(1) &= \sum_{(i,j,h)} \zeta_i\{\pi^+_{2k}(\zeta_h,g\zeta_j) - \pi^+_{2k}(\zeta_j,g\zeta_h)\} = \zeta_i E_{2k}(\zeta_h,g,\zeta_j) \\ (2) &= \sum_{(i,j,h)} \langle\zeta_i,\langle\zeta_h,g\zeta_j\rangle^+ - \langle\zeta_j,g\zeta_h\rangle^+\rangle^+_{2k} \\ (3) &= \sum_{(i,j,h)} \{\langle\langle\zeta_i,\zeta_j\rangle^-,g\zeta_h\rangle^-_{2k} - \frac{1}{3}R_{2k}(\zeta_i,\zeta_j,g)\zeta_h\} - gR_{2k}(\zeta_i,\zeta_j,\zeta_h).\end{aligned}$$

Here, to compute (3), we have applied Theorem 3.3 to the term $R_{2k}(\zeta_i,\zeta_j,\zeta_h g)$. Recalling (3.13) and using (3.14) for the first term of (3), we have

$$\begin{aligned}(3) &= \sum\zeta_h\langle\langle\zeta_i,\zeta_j\rangle^-,g\rangle^-_{2k} - \sum\frac{1}{3}\zeta_h R_{2k}(\zeta_i,\zeta_j,g) \\ &+ \sum\langle\langle\langle\zeta_i,\zeta_j\rangle^-,g\rangle^-,\zeta_h\rangle^+_{2k} - \sum\langle\langle\zeta_i,\zeta_j\rangle^-,\langle g,\zeta_h\rangle^+\rangle^-_{2k},\end{aligned} \tag{4.18}$$

where we used

$$\sum\langle\langle\langle\zeta_i,\zeta_j\rangle^-,\zeta_h\rangle^-,g\rangle^+_{2k} = \sum_{a+b=2k,a,b\ge 1} \pi^+_a(R_b(\zeta_i,\zeta_j,\zeta_h),g) = 0.$$

From (4.5), we have

$$(4.19) \qquad (1) = \sum \zeta_i\{\langle\langle\zeta_h, g\rangle^+, \zeta_j\rangle^+_{2k} - \langle\langle\zeta_j, g\rangle^+, \zeta_h\rangle^+_{2k}\} + \sum \zeta_i\langle\langle\zeta_h, \zeta_j\rangle^-, g\rangle^-_{2k} + \frac{1}{3}\zeta_i R_{2k}(\zeta_h, g, \zeta_j).$$

Note that in (1) + (3) the last two terms of (4.19) and the first two terms of (4.18) are cancelled out. Use (4.10) to (2), and remark that $R_m = 0$ for $m \le 2k-1$. Then, we see

(4.20)

$$\begin{aligned}
&A_{ij} + A_{jh} + A_{hi} \\
&\quad = \sum \langle\langle g, \zeta_h\rangle^+, \langle\zeta_i, \zeta_j\rangle^-\rangle^-_{2k} + \sum \langle\langle\langle\zeta_i, \zeta_j\rangle^-, g\rangle^-, \zeta_h\rangle^+_{2k} \\
&\quad + \sum \zeta_i\{\langle\langle\zeta_h, g\rangle^+, \zeta_j\rangle^+_{2k} - \langle\langle\zeta_j, g\rangle^+, \zeta_h\rangle^+_{2k}\} + \sum \langle\zeta_i, \langle\zeta_h, g\rangle^+\zeta_j - \langle\zeta_j, g\rangle^+\zeta_h\rangle^+_{2k} \\
&\quad + \sum \langle\zeta_i, \langle\langle\zeta_h, g\rangle^+, \zeta_j\rangle^+ - \langle\langle\zeta_j, g\rangle^+, \zeta_h\rangle^+\rangle^+_{2k} + \sum \langle\zeta_i, \langle\langle\zeta_h, \zeta_j\rangle^-, g\rangle^-\rangle^+_{2k}.
\end{aligned}$$

Note that the second term and the last term of the right hand side of (4.20) are cancelled out. We now use (4.16) to the second line in (4.20). After a little complicated rearrangement of the terms, we have

(4.21)

$$\begin{aligned}
&A_{ij} + A_{jh} + A_{hi} \\
&\quad = \sum \zeta_i\langle\langle\zeta_h, g\rangle^+, \zeta_j\rangle^+_{2k} - \sum \zeta_i\langle\langle\zeta_j, g\rangle^+, \zeta_h\rangle^+_{2k} + \sum \langle\langle g, \zeta_h\rangle^+, \langle\zeta_i, \zeta_j\rangle^-\rangle^-_{2k} \\
&\quad + \sum \langle\zeta_i, \langle\zeta_j, \langle\zeta_h, g\rangle^+\rangle^+\rangle^+_{2k} - \sum \langle\zeta_i, \langle\zeta_h, \langle\zeta_j, g\rangle^+\rangle^+\rangle^+_{2k} \\
&\quad + \frac{1}{3}\sum \underset{\bigstar}{\langle S_{2k}(\zeta_j, g, \zeta_h), \zeta_i\rangle^+_{2k}} - \frac{1}{3}\sum \underset{\bigstar}{\langle S_{2k}(\zeta_h, g, \zeta_j), \zeta_i\rangle^+_{2k}} \\
&\quad + \frac{1}{3}\sum \underset{\blacktriangle}{\langle\zeta_i, \zeta_j\rangle^+ \cdot \langle\zeta_h, g\rangle^+} + \frac{1}{3}\sum \zeta_j \cdot \langle\zeta_i, \langle\zeta_h, g\rangle^+\rangle^+_{2k} - \frac{2}{3}\sum \zeta_i \cdot \langle\zeta_j, \langle\zeta_h, g\rangle^+\rangle^+_{2k} \\
&\quad - \frac{1}{3}\sum \underset{\blacktriangle}{\langle\zeta_i, \zeta_h\rangle^+ \cdot \langle\zeta_j, g\rangle^+} - \frac{1}{3}\sum \zeta_h \cdot \langle\zeta_i, \langle\zeta_j, g\rangle^+\rangle^+_{2k} + \frac{2}{3}\sum \zeta_i \cdot \langle\zeta_h, \langle\zeta_j, g\rangle^+\rangle^+_{2k} \\
&\quad + \frac{1}{3}\sum \underset{\blacklozenge}{\langle\langle\zeta_i, \zeta_j\rangle^+, \langle\zeta_h, g\rangle^+\rangle^+_{2k}} + \frac{1}{3}\sum \langle\langle\zeta_i, \langle\zeta_h, g\rangle^+\rangle^+, \zeta_j\rangle^+_{2k} - \frac{2}{3}\sum \langle\zeta_i, \langle\zeta_j, \langle\zeta_h, g\rangle^+\rangle^+\rangle^+_{2k} \\
&\quad - \frac{1}{3}\sum \underset{\blacklozenge}{\langle\langle\zeta_i, \zeta_h\rangle^+, \langle\zeta_j, g\rangle^+\rangle^+_{2k}} - \frac{1}{3}\sum \langle\langle\zeta_i, \langle\zeta_j, g\rangle^+\rangle^+, \zeta_h\rangle^+_{2k} + \frac{2}{3}\sum \langle\zeta_i, \langle\zeta_h, \langle\zeta_j, g\rangle^+\rangle^+\rangle^+_{2k} \\
&\quad + \frac{1}{3}\sum \langle\langle\zeta_i, \zeta_j\rangle^-, \langle\zeta_h, g\rangle^+\rangle^-_{2k} + \frac{1}{3}\sum \langle\langle\zeta_i, \langle\zeta_h, g\rangle^+\rangle^-, \zeta_j\rangle^-_{2k} \\
&\quad - \frac{1}{3}\sum \langle\langle\zeta_i, \zeta_h\rangle^-, \langle\zeta_j, g\rangle^+\rangle^-_{2k} - \frac{1}{3}\sum \langle\langle\zeta_i, \langle\zeta_j, g\rangle^+\rangle^-, \zeta_h\rangle^-_{2k},
\end{aligned}$$

where $A^+ \cdot B^+$ means $\sum_{a+b=2k, a,b\ge 1} A^+_a B^+_b$. The terms marked by ▲, ★, ♦ are cancelled out respectively. Since

$$\sum \zeta_i \cdot \langle\langle\zeta_h, g\rangle^+, \zeta_j\rangle^+_{2k} = \sum \zeta_i \cdot \langle\zeta_j, \langle\zeta_h, g\rangle^+\rangle^+_{2k} = \sum \zeta_h \cdot \langle\zeta_i, \langle\zeta_j, g\rangle^+\rangle^+_{2k},$$

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the four terms involving $\cdot$ of (4.21) are cancelled out. Note also that

(4.22) $$\mathfrak{S}\ \langle\langle\zeta_i,\langle\zeta_j,g\rangle^+\rangle^+,\zeta_h\rangle^+_{2k}=\mathfrak{S}\ \langle\zeta_i,\langle\zeta_j,\langle\zeta_h,g\rangle^+\rangle^+\rangle^+_{2k}$$

$$\mathfrak{S}\ \langle\langle\zeta_i,\zeta_h\rangle^-,\langle\zeta_j,g\rangle^+\rangle^-_{2k}=-\mathfrak{S}\ \langle\langle\zeta_i,\zeta_j\rangle^-,\langle\zeta_h,g\rangle^+\rangle^-_{2k}.$$

Then, the last terms that remain are computed as follows:

(4.23)
$$\begin{aligned}
&-\frac{1}{3}\mathfrak{S}\ \langle\langle\zeta_i,\zeta_j\rangle^-,\langle\zeta_h,g\rangle^+\rangle^-_{2k}+\frac{1}{3}\mathfrak{S}\langle\langle\zeta_i,\langle\zeta_h,g\rangle^+\rangle^-,\zeta_j\rangle^-_{2k}\\
&\quad-\frac{1}{3}\mathfrak{S}\langle\langle\zeta_i,\langle\zeta_j,g\rangle^+\rangle^-,\zeta_h\rangle^-_{2k}\\
&=-\frac{1}{3}\underset{(i,j,k)}{\mathfrak{S}}\{\langle\langle\zeta_i,\zeta_j\rangle^-,\langle\zeta_h,g\rangle^+\rangle^-_{2k}+\langle\langle\langle\zeta_h,g\rangle^+,\zeta_i\rangle^-,\zeta_j\rangle^-_{2k}+\langle\langle\zeta_i,\langle\zeta_j,g\rangle^+\rangle^-,\zeta_h\rangle^-_{2k}\}\\
&=-\frac{1}{3}\sum_{a+b=2k,a,b\geq 1}\underset{(i,j,k)}{\mathfrak{S}}R_a(\zeta_i,\zeta_j,\langle\zeta_h,g\rangle^+_b)=0.
\end{aligned}$$

So, $\varpi^+_{2k}(\zeta_i,\zeta^\alpha)$ is obtained by (4.13) for any (ζ_i,ζ^α) such that $\zeta_i\zeta^\alpha=\zeta^\mu$. Thus, Proposition 4.2 is proved. □

4.3. Bridges. For a fixed μ such that $|\mu|=s+1$, we define a set of pairs of multi-indices by

(4.24) $$S_\mu=\{(\alpha,\beta);\alpha+\beta=\mu,|\alpha|\geq 1,|\beta|\geq 1\}.$$

For any i, $1\leq i\leq n$, we denote $<i>=(0,\cdots,1,\cdots,0)$. An element $(<i>,\ \mu-<i>)$ (resp. $(\mu-<i>,<i>)$) will be called a *left extremal point* (resp. *right extremal point*) of S_μ.

In what follows, for the pair of multi-indices (α,β) with $\alpha+\beta=\mu$, we shall construct $\varpi^+_{2k}(\zeta^\alpha,\zeta^\beta)$.

Definition 4.3. Given a multi-index γ, the pair of multi-indices (α,β) and (α',β') is said to have the *bridge relation* if they satisfies the following:

(B.1) $\alpha'=\alpha+\gamma$, $\beta'=\beta-\gamma$ and $\alpha+\beta=\alpha'+\beta'=\mu$.

(B.2) $\varpi^+_{2k}(\zeta^\alpha,\zeta^\beta)$ and $\varpi^+_{2k}(\zeta^{\alpha'},\zeta^{\beta'})$ have the relation

$(4.25)_\gamma$ $$\varpi^+_{2k}(\zeta^{\alpha'},\zeta^{\beta'})-\varpi^+_{2k}(\zeta^\alpha,\zeta^\beta)=-E_{2k}(\zeta^\alpha,\zeta^\gamma,\zeta^{\beta'}),$$

where

$$\begin{aligned}
E_{2k}(\zeta^\alpha,\zeta^\gamma,\zeta^{\beta'})&=\pi^+_{2k}(\zeta^\alpha,\zeta^\gamma)\zeta^{\beta'}-\zeta^\alpha\pi^+_{2k}(\zeta^\gamma,\zeta^{\beta'})\\
&\quad+\langle\langle\zeta^\alpha,\zeta^\gamma\rangle^+,\zeta^{\beta'}\rangle^+_{2k}-\langle\zeta^\alpha,\langle\zeta^\gamma,\zeta^{\beta'}\rangle^+\rangle^+_{2k}\\
&\qquad-\langle\zeta^\gamma,\langle\zeta^\alpha,\zeta^{\beta'}\rangle^-\rangle^-_{2k}+\frac{1}{3}R_{2k}(\zeta^\alpha,\zeta^\gamma,\zeta^{\beta'}),\quad\text{(cf. (4.5))}.
\end{aligned}$$

If (α,β), $(\alpha',\beta')\in S_\mu$ have the bridge relation $(4.25)_\gamma$, we denote by $(\alpha,\beta)\overset{\gamma}{\rightsquigarrow}(\alpha',\beta')$ (or $(\zeta^\alpha,\zeta^\beta)\overset{\gamma}{\rightsquigarrow}(\zeta^{\alpha'},\zeta^{\beta'})$).

Note that if $(\alpha,\beta)\overset{\gamma}{\rightsquigarrow}(\alpha',\beta')$, then $(\beta',\alpha')\overset{\gamma}{\rightsquigarrow}(\beta,\alpha)$, which is called the *dual bridge relation* to $(\alpha,\beta)\overset{\gamma}{\rightsquigarrow}(\alpha',\beta')$. The following lemma shows that any chain of bridges from a point of S_μ to another can be replaced by a direct bridge:

Lemma 4.4. *For $(\alpha,\beta+\gamma+\gamma')$, $(\alpha+\gamma,\beta+\gamma')$, $(\alpha+\gamma+\gamma',\beta)\in S_\mu$, the relations $(\alpha,\beta+\gamma+\gamma')\overset{\gamma}{\rightsquigarrow}(\alpha+\gamma,\beta+\gamma')$ and $(\alpha+\gamma,\beta+\gamma')\overset{\gamma'}{\rightsquigarrow}(\alpha+\gamma+\gamma',\beta)$ generates the relation $(\alpha,\beta+\gamma+\gamma')\overset{\gamma+\gamma'}{\rightsquigarrow}(\alpha+\gamma+\gamma',\beta)$.*

Proof. Let $f=\zeta^\alpha, g=\zeta^\gamma, h=\zeta^{\gamma'}, k=\zeta^\beta$ for the simplicity, and set

$$\hat{Q}(a,b,c)=\langle a,\langle b,c\rangle^+\rangle^+_{2k}-\langle\langle a,b\rangle^+,c\rangle^+_{2k}+\langle b,\langle a,c\rangle^-\rangle^-_{2k}+\frac{1}{3}R_{2k}(a,b,c). \tag{4.26}$$

By Proposition 2.3 and Theorem 3.3, we see that $\delta_0\hat{Q}=0$. Using (3.21) and (3.23), we have

$$\hat{Q}(a,b,c)=\left(-\frac{1}{2}\sum_{i+j=2k,i,j\geq 1}(\delta_i^+\pi_j^++\delta_i^-\pi_j^-)+\frac{2}{3}R_{2k}\right)(a,b,c).$$

The bridge equations $(4.25)_\gamma$, $(4.25)_{\gamma'}$, $(4.25)_{\gamma+\gamma'}$ are written as follows:

$$-f\pi^+_{2k}(g,ht)+\varpi^+_{2k}(fg,ht)-\varpi^+_{2k}(f,ght)+\pi^+_{2k}(f,g)ht=\hat{Q}(f,g,ht),$$

$$-fg\pi^+_{2k}(h,t)+\varpi^+_{2k}(fgh,t)-\varpi^+_{2k}(fg,ht)+\pi^+_{2k}(fg,h)t=\hat{Q}(fg,h,t), \tag{4.27}$$

$$-f\pi^+_{2k}(gh,t)+\varpi^+_{2k}(fgh,t)-\varpi^+_{2k}(f,ght)+\pi^+_{2k}(f,gh)t=\hat{Q}(f,gh,t).$$

Thus, computing $-(4.25)_\gamma-(4.25)_{\gamma'}+(4.25)_{\gamma+\gamma'}$, we have

$$\begin{aligned}f(\delta_0\pi^+_{2k})(g,h,t)+&(\delta_0\pi^+_{2k})(f,g,h)t\\&=-\hat{Q}(f,g,ht)-\hat{Q}(fg,h,t)+\hat{Q}(f,gh,t).\end{aligned} \tag{4.28}$$

By the assumption $(\mathrm{B})_s$, we have

$$(\delta_0\pi^+_{2k})(g,h,t)=-\hat{Q}(g,h,t),\quad(\delta_0\pi^+_{2k})(f,g,h)=-\hat{Q}(f,g,h).$$

Hence, (4.28) is

$$-f\hat{Q}(g,h,t)-\hat{Q}(f,g,h)t=-\hat{Q}(fg,h,t)+\hat{Q}(f,gh,t)-\hat{Q}(f,g,ht).$$

This holds because of $\delta_0\hat{Q}=0$. □

Note that by (4.26),(4.27) and (2.12), we see easily that

$$\sum_{(f,g,h)}\hat{Q}(f,g,h)=0. \tag{4.29}$$

By a similar manner as above combined with (4.29), we have

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Lemma 4.5. *If there are relations* $(< i >, \mu - < i >) \overset{\gamma}{\rightsquigarrow} (\alpha, \beta)$, $(< j >, \mu - < j >) \overset{\gamma'}{\rightsquigarrow} (\alpha, \beta)$, *then* $\varpi^+_{2k}(\zeta^\alpha, \zeta^\beta)$ *computed by* $(4.25)_\gamma$ *and by* $(4.25)_{\gamma'}$ *coincides.*

Proof. One may assume that $i \neq j$. Since there are bridges, $(\zeta^\alpha, \zeta^\beta)$ must be given in the shape $(\zeta_i\zeta_j h, \zeta^\beta)$. We set $t = \zeta^\beta$ for simplicity. Then, $(4.25)_\gamma$, $(4.25)_{\gamma'}$ are written as follows:

$$\varpi^+_{2k}(\zeta_i\zeta_j h, t) = \varpi^+_{2k}(\zeta_i, \zeta_j h t) + \zeta_i \pi_{2k}(\zeta_j h, t) - \pi_{2k}(\zeta_i, \zeta_j h)t + \hat{Q}(\zeta_i, \zeta_j h, t), \tag{4.30}$$

$$\varpi^+_{2k}(\zeta_j\zeta_i h, t) = \varpi^+_{2k}(\zeta_j, \zeta_i h t) + \zeta_j \pi_{2k}(\zeta_i h, t) - \pi_{2k}(\zeta_j, \zeta_i h)t + \hat{Q}(\zeta_j, \zeta_i h, t). \tag{4.31}$$

We have only to show the right hand side of (4.30) – (4.31) vanishes. Note that $\varpi^+_{2k}(\zeta_i, \zeta^\alpha)$ satisfies (4.10). Computing the right hand side by using

$$\begin{aligned} &\varpi^+_{2k}(\zeta_i, ht\zeta_j) - \varpi^+_{2k}(\zeta_j, ht\zeta_i) \\ &\qquad = -\zeta_i \pi^+_{2k}(ht, \zeta_j) + \pi^+_{2k}(\zeta_i, ht)\zeta_j - \hat{Q}(\zeta_i, ht, \zeta_j), \end{aligned} \tag{4.32}$$

which is obtained by (4.4), (4.5) and (4.26), we have that (4.30) – (4.31) is

$$\begin{aligned} &\zeta_i(\pi^+_{2k}(\zeta_j h, t) - \pi^+_{2k}(ht, \zeta_j)) \\ &\quad + \zeta_j(\pi^+_{2k}(\zeta_i, ht) - \pi^+_{2k}(\zeta_i h, t)) \\ &\quad + t(\pi^+_{2k}(\zeta_j, \zeta_i h) - \pi^+_{2k}(\zeta_i, \zeta_j h)) \\ &\quad + \hat{Q}(\zeta_i, \zeta_j h, t) - \hat{Q}(\zeta_j, \zeta_i h, t) - \hat{Q}(\zeta_i, ht, \zeta_j). \end{aligned} \tag{4.33}$$

By the assumption (B_s), the above quantity is

$$\begin{aligned} &\zeta_i \hat{Q}(\zeta_j, h, t) - \zeta_j \hat{Q}(\zeta_i, h, t) - t\hat{Q}(\zeta_j, h, \zeta_i) \\ &\quad + \hat{Q}(\zeta_i, \zeta_j h, t) + \hat{Q}(t, \zeta_i h, \zeta_j) + \hat{Q}(\zeta_j, ht, \zeta_i). \end{aligned}$$

Using (4.29), we see that the above quantity is

$$(\delta_0 \hat{Q})(\zeta_i, \zeta_j, h, t) - (\delta_0 \hat{Q})(\zeta_j, \zeta_i, h, t) = 0. \qquad \square \tag{4.34}$$

4.4. Right extremals. As we have shown in 4.2, we got $\varpi^+_{2k}(\zeta_i, \zeta^\alpha)$ for $\alpha + < i >= \mu$, $|\mu| = s + 1$. Next, we shall determine $\varpi^+_{2k}(\zeta^\alpha, \zeta_i)$ for $\alpha + < i >= \mu$, $|\mu| = s + 1$. Given (ζ^α, ζ_i), there are a pair (ζ_j, ζ^β) and a multi-index γ such that $(\zeta_j, \zeta^\beta) \overset{\gamma}{\rightsquigarrow} (\zeta^\alpha, \zeta_i)$. Thus, we can get $\varpi^+_{2k}(\zeta^\alpha, \zeta_i)$ by $(4.25)_\gamma$. By using Lemma 4.5, $\varpi^+_{2k}(\zeta^\alpha, \zeta_i)$ is independent of the choice of γ and (ζ_j, ζ^β). We now show that $\varpi^+_{2k}(\zeta_i, \zeta^\alpha) = \varpi^+_{2k}(\zeta^\alpha, \zeta_i)$.

First of all, we easily have

Lemma 4.6. *If* $(< i >, \mu - < i >), \overset{\mu - 2<i>}{\rightsquigarrow} (\mu - < i >, < i >)$ *then*

$$\varpi^+_{2k}(\zeta^{\mu - <i>}, \zeta_i) = \varpi^+_{2k}(\zeta_i, \zeta^{\mu - <i>}). \tag{4.35}$$

Proof. By definition 4.3 and $R_{2k}(\zeta_i, \zeta^{\mu - 2<i>}, \zeta_i) = 0$, we have (4.35) $\square$

Lemma 4.7. *For any i, j and a multi-index α, we have*

$$\varpi_{2k}^{+}(\zeta^{\alpha}\zeta_i, \zeta_j) = \varpi_{2k}^{+}(\zeta_j, \zeta^{\alpha}\zeta_i). \tag{4.36}$$

Proof. Consider a bridge relation $(< i >, \alpha + < j >) \overset{\alpha}{\rightsquigarrow} (\alpha + < i >, < j >)$ and we have

$$\varpi_{2k}^{+}(\zeta^{\alpha}\zeta_i, \zeta_j) = \varpi_{2k}^{+}(\zeta_i, \zeta^{\alpha}\zeta_j) - E_{2k}(\zeta_i, \zeta^{\alpha}, \zeta_j) \tag{4.37}$$

by $(4.25)_{\alpha}$. On the other hand, we write down (4.10) for $(\zeta_j, \zeta^{\alpha}\zeta_i)$:

$$\varpi_{2k}^{+}(\zeta_j, \zeta^{\alpha}\zeta_i) = \varpi_{2k}^{+}(\zeta_i, \zeta^{\alpha}\zeta_j) + A_{ji}. \tag{4.38}$$

Combining (4.37) with (4.38), we have (4.36). □

Using Lemma 4.6 and Lemma 4.7, we have

Proposition 4.8. *For any i and α, we have*

$$\varpi_{2k}^{+}(\zeta_i, \zeta^{\alpha}) = \varpi_{2k}^{+}(\zeta^{\alpha}, \zeta_i). \tag{4.39}$$

4.5. Determination for $\varpi_{2k}^{+}(\zeta^{\alpha}, \zeta^{\beta})$. To determine $\varpi_{2k}^{+}(\zeta^{\alpha}, \zeta^{\beta})$, we choose an left extremal point $(\zeta_i, \zeta^{\delta})$ such that $(\zeta_i, \zeta^{\delta}) \overset{\gamma}{\rightsquigarrow} (\zeta^{\alpha}, \zeta^{\beta})$. Thus, we put $\varpi_{2k}^{+}(\zeta^{\alpha}, \zeta^{\beta})$ by $(4.25)_{\gamma}$, which also does not depend on the choice of γ and $(\zeta_i, \zeta^{\delta})$.

We now prove

Proposition 4.9. *Under the assumptions* (HE.1-2), *$\varpi_{2k}^{+}(\zeta^{\alpha}, \zeta^{\beta})$ can be constructed so that they may satisfy* (4.4), *and $\varpi_{2k}^{+}(\zeta^{\alpha}, \zeta^{\beta}) = \varpi_{2k}^{+}(\zeta^{\beta}, \zeta^{\alpha})$*

Proof. Using the bridge relation

$$\begin{aligned} \varpi_{2k}^{+}(\zeta^{\gamma+<i>}, \zeta^{\beta}) - \varpi_{2k}^{+}(\zeta_i, \zeta^{\gamma+\beta}) &= -E_{2k}(\zeta_i, \zeta^{\gamma}, \zeta^{\beta}) \\ \varpi_{2k}^{+}(\zeta^{\gamma+\beta}, \zeta_i) - \varpi_{2k}^{+}(\zeta^{\beta}, \zeta^{\gamma+<i>}) &= -E_{2k}(\zeta^{\beta}, \zeta^{\gamma}, \zeta_i) \end{aligned} \tag{4.40}$$

By (4.6), we have $\varpi_{2k}^{+}(\zeta^{\alpha}, \zeta^{\beta}) = \varpi_{2k}^{+}(\zeta^{\beta}, \zeta^{\alpha})$ for $|\alpha + \beta| = s + 1$. This implies that for any α, β, γ such that $\alpha + \beta + \gamma = \mu$, the equation $(4.25)_{\gamma}$ is equal to that of (4.4) substituted by $f = \zeta^{\alpha}, g = \zeta^{\gamma}, h = \zeta^{\beta}$. Then, we get Proposition 4.8. □

We now put $\pi_{2k}^{+}(\zeta^{\alpha}, \zeta^{\beta}) = \varpi_{2k}^{+}(\zeta^{\alpha}, \zeta^{\beta})$. As a byproduct of the proof, we have also the following:

Corollary 4.10. *Under the assumptions* (HE.1-3), *the obtained $\pi_{2k}^{+}(\zeta^{\alpha}, \zeta^{\beta})$ satisfy*

$$\pi_{2k}^{+}(\zeta^{\alpha}, \zeta^{\beta}) \in \sum_{\gamma} \zeta^{\gamma} C^{\infty}(U), \quad |\gamma| = |\alpha + \beta| - 4k.$$

Proof. By (HE.3), we have $\pi_m^{\pm}(\zeta^{\alpha}, \zeta^{\beta}) \in \sum_{\gamma} \zeta^{\gamma} C^{\infty}(U)$, $|\gamma| = |\alpha + \beta| - 2m$ for any $m \leq 2k - 1$. Thus, $\langle \zeta^{\alpha} \langle \zeta^{\beta}, \zeta^{\gamma} \rangle^{\pm} \rangle_{2k}^{\pm} \in \sum_{\gamma'} \zeta^{\gamma'} C^{\infty}(U)$, $|\gamma'| = |\alpha + \beta + \gamma| - 4k$. Since K_{2k} in (4.14) is a differential operator of order $4k$, we see by induction that $\pi_{2k}^{+}(\zeta_i, \zeta^{\alpha}) \in \sum_{\gamma} \zeta^{\gamma} C^{\infty}(U)$, $|\gamma| = |\alpha| - 4k$, by using (4.10) and (4.13). Hence by using $(4.25)_{\gamma}$, we see $\pi_{2k}^{+}(\zeta_i, \zeta^{\alpha}) \in \sum_{\gamma} \zeta^{\gamma} C^{\infty}(U)$, $|\gamma| = |\alpha + \beta| - 4k$. □

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4.6. Proof of Theorem 4.1. Let $\{U_\lambda\}_\lambda$ be a locally finite coordinate covering of M. by using Corollary 4.9, π^+_{2k} is a differential operator. Thus, we define $\pi^{+(\lambda)}_{2k}(f,g)=\pi^{+(\lambda)}_{2k}(g,f)$ on each U_λ. Since $\pi^+_{2k}(\zeta^\alpha,\zeta^\beta)$ satisfies (4.4), $\pi^{+(\lambda)}_{2k}(f,g)$ must satisfy (4.4) by polynomial approximation theorem. Thus, $\pi^{+(\lambda)}_{2k}$ satisfies on each U_λ the property (b) in the statement of Theorem 4.1.

Let $\{\phi_\lambda\}$ be a partition of unity subordinate to $\{U_\lambda\}$. We set

$$\pi^+_{2k}(f,g)=\sum_\lambda \phi_\lambda \pi^{+(\lambda)}_{2k}(f,g). \tag{4.41}$$

Since $\pi^+_{2k}(f,g)=\pi^+_{2k}(g,f)$, and $\delta_0\pi^+_{2k}=\sum_\lambda \phi_\lambda\delta_0\pi^{+(\lambda)}_{2k}$, we see that π^+_{2k} satisfies (2.7), i.e.

$$2\delta_0\pi^+_{2k}=-\sum_{i+j=2k,i,j\geq 1}\delta_i\pi_j. \tag{4.42}$$

By Proposition 2.2, the ambiguity of π^+_{2k} is only in $\delta_0\mathcal{C}^1(\mathfrak{a})$.

§5. Construction of π_{odd}

5.1. Construction of π_{odd}. Set $\pi_0(f,g)=fg$, $\pi_1(f,g)=-\frac{1}{2}\{f,g\}$. As in the previous section, we assume the following throughout this section:

Assumptions.

(HO. 1) $\pi_2,\cdots,\pi_{2l}\in\mathcal{C}^2(\mathfrak{a})$ are given, and $\pi_0,\cdots,\pi_{2l}$ satisfy

$$\sum_{i+j=m}\delta_i\pi_j=0$$

for any m such that $0\leq m\leq 2l-1$, and

$$\sum_{i+j=2l}\delta_i\pi_j=\frac{1}{3}\sum_{i+j=2l}\mathfrak{S}\,\delta_i^-\pi_j^-.$$

(HO. 2) $\pi^+_{\text{odd}}=\pi^-_{\text{even}}=0$ for $\pi_0,\pi_1,\cdots,\pi_{2l}$.

(HO. 3) π_m are bidifferential operator of order $2m$ for $0\leq m\leq 2l$.

In this section, we prove the following:

Theorem 5.1. *Under the assumptions* (HO. 1-3), *there exists a bidifferential operator of order* $2(2l+1)$, π_{2l+1} *such that*

(a) $\sum_{i+j=2l+1,i,j\geq 0}\delta_i\pi_j=0$.

(b) $\pi^+_{2l+1}=0$, *if and only if* $R_{2l}(\pi_1^-(f,g),h,t)+R_{2l}(f,g,\pi_1^-(h,t))$ *is alternative with respect to* (f,g,h,t).

Notice at first that $R_m=0$ for $m\leq 2l-1$ by (HO. 1). By (HO. 2), we see $R_{2l+1}=0$ (cf. Definition 2.4, Remark). Under the assumptions (HO. 1-2), the equations (3.22), (3.23) are changed into

$$\partial_2^0\pi^-_{2l+1}=\frac{1}{8}(1-\mathfrak{c}_3+\mathfrak{c}_3^2)(1+\sigma_3)\sum_{i+j=2l+1,i,j\geq 1}\delta_i\pi_j, \tag{5.1}$$

$$(5.2)\qquad (1-c_3)\partial_2^0\pi_{2l+1}^+ = \frac{1}{2}\sum_{i+j=2l+1,i,j\geq 1}(\delta_i^+\pi_j^+ + \delta_i^-\pi_j^-). \quad (\text{cf. } (3.5),(3.11))$$

Notice that R_{2l} does not appear in the equations. By (HO. 2), the right hand side of (5.2) vanishes. In what follows we set

$$(5.3)\qquad \pi_{2l+1}^+ = 0.$$

To treat the equation (5.1), we shall consider at first on a local coordinate neighborhood $(U; x_1, \cdots, x_n)$ and set

$$(5.4)\qquad \pi_{2l+1}^-(x_i, x_j) = g_{ij}^{(2l+1)},$$

where $g_{ij}^{(2l+1)}$ is an arbitrary element of $C^\infty(U)$ such that $g_{ij}^{(2l+1)} = -g_{ji}^{(2l+1)}$. By the normalization condition $\pi_{2l+1}^-(1,f) = \pi_{2l+1}^-(f,1) = 0$, we see that

$$\pi_{2l+1}^-(x_i, x_j) = \pi_{2l+1}^-(\zeta_i, \zeta_j) = g_{ij}^{(2l+1)},$$

where $\zeta_i = x_i - x_i(p)$, $p \in U$.

By (3.11) and (3.14) we see that (5.1) is equivalent to

$$(5.5)\qquad \begin{aligned} &g\pi_{2l+1}^-(f,h) - \pi_{2l+1}^-(f,gh) + \pi_{2l+1}^-(f,g)h \\ &= -\langle\langle f,g\rangle^-, h\rangle_{2l+1}^+ - \langle\langle f,h\rangle^-, g\rangle_{2l+1}^+ + \langle f, \langle g,h\rangle^+\rangle_{2l+1}^-. \end{aligned}$$

If one regards f in (5.5) as a parameter, then (5.5) has been already solved by Proposition 2.2, that is, for any fixed $f \in C^\infty(U)$, there exists $\tilde{\pi}_f(g)$ such that

$$(\delta_0\tilde{\pi}_f)(g,h) = -\langle\langle f,g\rangle^-, h\rangle_{2l+1}^+ - \langle\langle f,h\rangle^-, g\rangle_{2l+1}^+ + \langle f, \langle g,h\rangle^+\rangle_{2l+1}^-,$$

because the consistency condition is satisfied by Lemma 3.1.

For any coordinate function x_i, $1 \leq i \leq n$, we define $\pi_{2l+1}^-(x_i, h)$ by

$$(5.6)\qquad \pi_{2l+1}^-(x_i,h)(p) = \sum_{j=1}^n (g_{ij}^{(2l+1)} - \tilde{\pi}_{x_i}(\zeta_j))(p)\frac{\partial h}{\partial x_j}(p) + \tilde{\pi}_{x_i}(h)(p), \quad p \in U,$$

where $\zeta_i = x_i - x_i(p)$. (5.6) is the solution of (5.5) for $f = x_i$ such that $\pi_{2l+1}^-(x_i, x_j) = g_{ij}^{(2l+1)}$.

Define $\pi_{2l+1}^-(h, x_i)$ by

$$(5.7)\qquad \pi_{2l+1}^-(h, x_i) = -\pi_{2l+1}^-(x_i, h).$$

For any fixed $f \in C^\infty(U)$, we define $\pi_{2l+1}^-(f,h)$ by

$$(5.8)\qquad \pi_{2l+1}^-(f,h)(p) = \sum_{j=1}^n (\pi_{2l+1}^-(f,x_j) - \tilde{\pi}_f(\zeta_j))(p)\frac{\partial h}{\partial x_j}(p) + \tilde{\pi}_f(h)(p), \quad p \in U.$$

This is the solution of (5.5) for the fixed f such that $\pi_{2l+1}^-(f, x_j)$ is the prescribed one. Thus, we obtain $\pi_{2l+1}^-(f,h)$ for any $f, h \in C^\infty(U)$. However, we only see that $\pi_{2l+1}^-(\zeta_i, \zeta_j)$ is skew-symmetric.

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5.2. Skew-symmetricity of π_{2l+1}. To get Theorem 5.1, we shall show the following:

Proposition 5.2. *If $R_{2l}(\pi_1^-(f,g),h,t) + R_{2l}(f,g,\pi_1^-(h,t))$ is alternative, then $\pi_{2l+1}^-(f,h)$ given by (5.8) is skew-symmetric.*

In what follows, we assume the following:

$$(\mathrm{S})_s \qquad \pi_{2l+1}^-(\zeta^\alpha,\zeta^\beta) = -\pi_{2l+1}^-(\zeta^\beta,\zeta^\alpha) \quad \text{for any} \quad \alpha,\beta \quad \text{such that} \quad |\alpha+\beta| \le s.$$

Consider $\pi_{2l+1}^-(\zeta^\alpha,\zeta^\beta)$ such that $|\alpha+\beta| = s+1$. If one of $|\alpha|,|\beta|$ is 1, then (5.7) shows the skew-symmetricity. We now show $(\mathrm{S})_{s+1}$ for $|\alpha|,|\beta| \ge 2$. Since π_{2l+1}^- is a continuous bilinear mapping, it is enough to show that

$$\pi_{2l+1}^-(\zeta^\alpha\zeta^{\alpha'},\zeta^\beta\zeta^{\beta'}) = -\pi_{2l+1}^-(\zeta^\beta\zeta^{\beta'},\zeta^\alpha\zeta^{\alpha'}) \quad \text{for} \quad |\alpha|,|\alpha'|,|\beta|,|\beta'| \ge 1.$$

For simplicity, set $f=\zeta^\alpha$, $g=\zeta^{\alpha'}$, $h=\zeta^\beta$, $t=\zeta^{\beta'}$. By the assumption $(\mathrm{S})_s$,one obtains

$$(5.9) \qquad \pi_{2l+1}^-(fg,h) = -\pi_{2l+1}^-(h,fg), \quad \pi_{2l+1}^-(f,gh) = -\pi_{2l+1}^-(gh,f), \quad \text{etc}$$

By (5.5), we have

$$\pi_{2l+1}^-(fg,ht) = \pi_{2l+1}^-(fg,h)t + \pi_{2l+1}^-(fg,t)h + \langle\langle fg,h\rangle^-,t\rangle_{2l+1}^+$$
$$+ \langle\langle fg,t\rangle^-,h\rangle_{2l+1}^+ - \langle fg,\langle h,t\rangle^+\rangle_{2l+1}^-.$$

Using (5.9), and (5.5), we have

(5.10)
$$\pi_{2l+1}^-(fg,ht) = \pi_{2l+1}^-(f,h)gt + \pi_{2l+1}^-(g,h)ft + \pi_{2l+1}^-(f,t)gh + \pi_{2l+1}^-(g,t)fh$$
$$- t\langle\langle h,f\rangle^-,g\rangle_{2l+1}^+ - t\langle\langle h,g\rangle^-,f\rangle_{2l+1}^+ + t\langle h,\langle f,g\rangle^+\rangle_{2l+1}^-$$
$$- h\langle\langle t,f\rangle^-,g\rangle_{2l+1}^+ - h\langle\langle t,g\rangle^-,f\rangle_{2l+1}^+ + h\langle t,\langle f,g\rangle^+\rangle_{2l+1}^-$$
$$+ \langle\langle fg,h\rangle^-,t\rangle_{2l+1}^+ + \langle\langle fg,t\rangle^-,h\rangle_{2l+1}^+ - \langle fg,\langle h,t\rangle^+\rangle_{2l+1}^-.$$

The first line of the right hand side of (5.10) is skew-symmetric under the permutation of $(f,g,h,t) \to (h,t,f,g)$, which we shall denote by σ. Let $\mathfrak{S}$ denote $1+\sigma$. Then, using (3.14)

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to the last line of (5.10), we have the following:

$$
\begin{array}{llll}
\multicolumn{4}{l}{\mathfrak{S}\pi^-_{2l+1}(fg,ht) =} \\
 & -\mathfrak{S}t\langle\langle h,f\rangle^-,g\rangle^+_{2l+1} & -\mathfrak{S}t\langle\langle h,g\rangle^-,f\rangle^+_{2l+1} & +\underset{\blacktriangle}{\mathfrak{S}t\langle h,\langle f,g\rangle^+\rangle^-_{2l+1}} \\
 & -\mathfrak{S}h\langle\langle t,f\rangle^-,g\rangle^+_{2l+1} & -\mathfrak{S}h\langle\langle t,g\rangle^-,f\rangle^+_{2l+1} & +\underset{\blacktriangle}{\mathfrak{S}h\langle t,\langle f,g\rangle^+\rangle^-_{2l+1}} \\
 & & +\underset{\blacktriangle}{\mathfrak{S}f\langle g,\langle h,t\rangle^+\rangle^-_{2l+1}} & +\underset{\blacktriangle}{\mathfrak{S}g\langle f,\langle h,t\rangle^+\rangle^-_{2l+1}} \\
 & +\underset{\blacklozenge}{\mathfrak{S}\langle\langle f,g\rangle^+,\langle h,t\rangle^+\rangle^-_{2l+1}} & & \\
(5.11) & & +\underset{\blacktriangledown}{\mathfrak{S}\langle\langle\langle h,t\rangle^+,f\rangle^-,g\rangle^+_{2l+1}} & +\underset{\blacktriangledown}{\mathfrak{S}\langle\langle\langle h,t\rangle^+,g\rangle^-,f\rangle^+_{2l+1}} \\
 & -\mathfrak{S}\langle\langle\langle h,f\rangle^-,g\rangle^+,t\rangle^+_{2l+1} & -\mathfrak{S}\langle\langle\langle h,g\rangle^-,f\rangle^+,t\rangle^+_{2l+1} & -\underset{\blacktriangledown}{\mathfrak{S}\langle\langle\langle f,g\rangle^+,h\rangle^-,t\rangle^+_{2l+1}} \\
 & -\mathfrak{S}\langle\langle\langle t,f\rangle^-,g\rangle^+,h\rangle^+_{2l+1} & -\mathfrak{S}\langle\langle\langle t,g\rangle^-,f\rangle^+,h\rangle^+_{2l+1} & -\underset{\blacktriangledown}{\mathfrak{S}\langle\langle\langle f,g\rangle^+,t\rangle^-,h\rangle^+_{2l+1}} \\
 & +\mathfrak{S}\langle f\langle g,h\rangle^-,t\rangle^+_{2l+1} & +\mathfrak{S}\langle g\langle f,h\rangle^-,t\rangle^+_{2l+1} & \\
 & +\mathfrak{S}\langle f\langle g,t\rangle^-,h\rangle^+_{2l+1} & +\mathfrak{S}\langle g\langle f,t\rangle^-,h\rangle^+_{2l+1} &
\end{array}
$$

The terms marked by ▲, ▼, ◆ are cancelled out. If we denote by σ_{12},σ_{34} the permutations $(f,g,h,t)\to(g,f,h,t)$, $(f,g,h,t)\to(f,g,t,h)$ respectively, then the above quantity can be written as follows:

$$
\begin{aligned}
(5.12)\qquad & \mathfrak{S}\pi^-_{2l+1}(fg,ht) \\
&= -\mathfrak{S}(1+\sigma_{34})(1+\sigma_{12})\{t\langle\langle h,f\rangle^-,g\rangle^+_{2l+1} \\
&\qquad + \langle\langle\langle h,f\rangle^-,g\rangle^+,t\rangle^+_{2l+1} - \langle f\langle g,h\rangle^-,t\rangle^+_{2l+1}\}.
\end{aligned}
$$

Substitute (4.16) to the last term after remarking that (4.16) is valid for any π^+_m such that $m\le 2l$. Note that

$$
(5.13)\qquad \mathfrak{S}(1+\sigma_{34})(1+\sigma_{12})S_m(f,\langle g,h\rangle^-,t)=0,\quad 1\le m\le 2l.
$$

After a little complicated calculation, we have

$$
\begin{aligned}
(5.14)\qquad \mathfrak{S}\pi^-_{2l+1}(fg,ht) &= -\frac{1}{3}\mathfrak{S}(1+\sigma_{34})(1+\sigma_{12})\langle f,\langle t,\langle g,h\rangle^-\rangle^-\rangle^-_{2l+1} \\
&= \frac{1}{3}\langle t,\langle f,\langle g,h\rangle^-\rangle^-\rangle^-_{2l+1} - \frac{1}{3}\langle f,\langle t,\langle g,h\rangle^-\rangle^-\rangle^-_{2l+1} \\
&+ \frac{1}{3}\langle t,\langle g,\langle f,h\rangle^-\rangle^-\rangle^-_{2l+1} - \frac{1}{3}\langle g,\langle t,\langle f,h\rangle^-\rangle^-\rangle^-_{2l+1} \\
&+ \frac{1}{3}\langle h,\langle f,\langle g,t\rangle^-\rangle^-\rangle^-_{2l+1} - \frac{1}{3}\langle f,\langle h,\langle g,t\rangle^-\rangle^-\rangle^-_{2l+1} \\
&+ \frac{1}{3}\langle h,\langle g,\langle f,t\rangle^-\rangle^-\rangle^-_{2l+1} - \frac{1}{3}\langle g,\langle h,\langle f,t\rangle^-\rangle^-\rangle^-_{2l+1}.
\end{aligned}
$$

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Since $R_m = 0$ for $m \leq 2l-1$ and $R_{2l+1} = 0$, we see by (3.13) that

$$\langle t, \langle f, \langle g,h\rangle^- \rangle^- \rangle^-_{2l+1} - \langle f, \langle t, \langle g,h\rangle^- \rangle^- \rangle^-_{2l+1} = -\langle \langle g,h\rangle^-, \langle t,f\rangle^- \rangle^-_{2l+1} + R_{2l}(t,f,\pi_1^-(g,h)).$$

Substituting these to (5.14), we have

$$\mathfrak{S}\pi^-_{2l+1}(fg,ht) = \frac{1}{3}R_{2l}(t,f,\pi_1^-(g,h)) + \frac{1}{3}R_{2l}(\pi_1^-(t,f),g,h) + \frac{1}{3}R_{2l}(t,g,\pi_1^-(f,h)) + \frac{1}{3}R_{2l}(\pi_1^-(t,g),f,h). \tag{5.15}$$

Thus, we have $\mathfrak{S}\pi^-_{2l+1}(fg,ht) = 0$. Proposition 5.2 is thereby proved.

Recall Proposition 2.6. We see that π^-_{2l+1} is obtained as a skew-symmetric bilinear form if and only if $R_{2l}(f,g,\pi_1^-(h,t)) + R_{2l}(\pi_1^-(f,g),h,t)$ is alternative. Hence, to complete the proof of Theorem 5.1, we have only to show the following:

Lemma 5.3. *π^-_{2l+1} given by (5.8) is a bidifferential operator of order $2(2l+1)$.*

Proof. Obviously $\pi_1^- = -\frac{1}{2}\{f,g\}$ is a bidifferential operator of order 2, for π_1^- is a biderivation. Suppose that $\pi_m(f,g)$ are bidifferential operator of order $2m$ for $1 \leq m \leq 2l$. It follows that at any $p \in U$, letting $\zeta_i = x_i - x_i(p)$,

$$\langle \zeta^\alpha, \langle \zeta^\beta, \zeta^\gamma\rangle^\pm \rangle^\pm_{2l+1} \in \sum_\gamma \zeta^\delta C^\infty(U), \quad |\delta| = |\alpha+\beta+\gamma| - 4l - 2. \tag{5.16}$$

Rewrite (5.5) as follows:

$$\begin{aligned}\pi^-_{2l+1}(f,gh) &= g\pi^-_{2l+1}(f,h) + \pi^-_{2l+1}(f,g)h + \langle\langle f,g\rangle^-, h\rangle^+_{2l+1} \\ &\quad + \langle\langle f,h\rangle^-, g\rangle^+_{2l+1} - \langle f, \langle g,h\rangle^+\rangle^-_{2l+1}.\end{aligned}$$

One can show inductively by using (5.16) and the skew-symmetricity of π^-_{2l+1} that

$$\pi^-_{2l+1}(\zeta^\alpha, \zeta^\beta) \in \sum_\gamma \zeta^\gamma C^\infty(U), \quad |\gamma| = |\alpha+\beta| - 2(2l+1).$$

This implies that π^-_{2l+1} is a bidifferential operator of order $2(2l+1)$. □

Theorem 5.1 is easily proved by using a partition of unity.

Let $(\mathfrak{a}, \cdot, \{\ ,\ \})$ be a Poisson algebra. Let $\pi_0(f,g) = f\cdot g$, $\pi_1(f,g) = -\frac{1}{2}\{f,g\}$. Since $R_2 = 0$ by the Jacobi identity of $\{\ ,\ \}$, we see the following by combining Theorem 4.1 and Theorem 5.1:

Theorem 5.4. *For a Poisson algebra $(\mathfrak{a}, \cdot, \{\ ,\ \})$, there are $\pi_2, \pi_3, \pi_4 \in \mathcal{C}^2(\mathfrak{a})$ such that*

(a) $\pi^+_{\text{odd}} = \pi^-_{\text{even}} = 0$,
(b) π_m *are bidifferential operators of order* $2m$ $(m \leq 4)$,
(c) $\sum_{i+j=m} \delta_i \pi_j = 0$ *for* $m = 2,3$,

(d) $\sum_{i+j=4}\delta_i\pi_j = \frac{1}{3}\sum_{i+j=4}\sum \delta_i^-\pi_j^-$, $(=\frac{4}{3}R_4)$.

The above theorem shows that $\hat\pi = \pi_0 + \nu\pi_1 + \nu^2\pi_2 + \nu^3\pi_3$ defines an associative algebra structure on $\mathfrak{a}\oplus\nu\mathfrak{a}\oplus\nu^2\mathfrak{a}\oplus\nu^3\mathfrak{a}$, and $\hat\pi' = \pi_0+\nu\pi_1+\nu^2\pi_2+\nu^3\pi_3+\nu^4\pi_4$ defines an alternative algebra structure on $\mathfrak{a}\oplus\cdots\oplus\nu^4\mathfrak{a}$. If $R_4 = 0$, then $\hat\pi'$ is associative, and one can obtain π_5 and π_6 such that $\pi_0+\cdots+\nu^5\pi_5$ is an associative deformation with $\nu^6 = 0$, and $\pi_0+\cdots+\nu^6\pi_6$ is an alternative deformation with $\nu^7 = 0$. If R_4 is not a coboundary, then there is no regular Q-deformation of $(\mathfrak{a},\cdot,\{\ ,\ \})$ (cf. Remark 2, §3.3).

§6. Proofs of Theorems 1-3 and Examples 3 and 4

6.1. Proof of Theorem 1. Assume $H^3(M,\cdot,\{\ ,\ \}) = \{0\}$. Suppose $\pi_0,\pi_1,\cdots,\pi_{2l-1}$ are given with Assumptions (HE. 1-3), where $\pi_0(f,g) = f\cdot g$, $\pi_1(f,g) = -\frac{1}{2}\{f,g\}$. By Theorem 3.3, $R_{2l}\in\mathcal{A}_3(\mathfrak{a},\pi_0)$. Moreover, by Proposition 2.5, we see that $d_1^- R_{2l} = 0$. Hence, by the assumption, there is $\pi'\in\mathcal{A}_2(\mathfrak{a},\pi_0)$ such that $d_1^-\pi' = R_{2l}$. Since π' is a biderivation, we see that $\delta_0\pi' = 0$ by Lemma 2.1.

By setting $\pi'_{2l-1} = \pi_{2l-1}-\pi'$, we see π'_{2l-1} is skew-symmetric and $\pi_0,\pi_1,\cdots,\pi_{2l-2},\pi'_{2l-1}$ satisfy (HE. 1-3). Moreover

$$R'_{2l} = d_1^-(\pi^-_{2l-1}-\pi') + \frac{1}{2}\sum_{i+j=2l,i,j\geq 2} d_i^-\pi_j^- = R_{2l} - d_1^-\pi' = 0. \tag{6.1}$$

However, note that $\pi'(x_i,x_j)\neq 0$ in general, hence $\pi'_{2l-1}(x_i,x_j)\neq\pi_{2l-1}(x_i,x_j)$. Thus, one may not give $\pi_{2l-1}(x_i,x_j)$ arbitrarily. By loosing these freedom, one obtains π_m, $m\geq 2$ such that $\sum_{i+j=m}\delta_i\pi_j = 0$ for any m, and $\pi^+_{\text{odd}} = \pi^-_{\text{even}} = 0$.

To prove the first assertion, let $(\mathfrak{a}[[\nu]],*)$ be a regular Q-deformation and let $f*g = \sum_{m=0}^\infty \nu^m\pi_m(f,g)$. For any $[\theta]\in H^2(M,\cdot,\{\ ,\ \})$, we set

$$\pi'_3 = \pi_3+\theta. \tag{6.2}$$

Since $\delta_0\theta = 0$, $d_1\theta = 0$, we see that $R_4 = R'_4 = 0$. R'_6 may not vanish, but by the assumption, one can replace π_5 so that R'_6 vanishes. Keeping this procedure, one obtains another $*$-product, which we shall denote by $*_\theta$.

Let $(\mathfrak{a}[[\nu]],*_\theta)$, $(\mathfrak{a}[[\nu]],*_{\theta'})$ be two regular Q-deformations of $(\mathfrak{a},\cdot,\{\ ,\ \})$ such that $\pi_0(f,g) = \pi'_0(f,g) = f\cdot g$, $\pi_1(f,g) = \pi'_1(f,g) = -\frac{1}{2}\{f,g\}$. Since they are regular Q-deformations, we see that $R_4 = d_1\pi_3 = 0$, $R'_4 = d_1\pi'_3 = 0$. Now, suppose there is an isomorphism

$$\phi : (\mathfrak{a}[[\nu]],*_\theta)\to(\mathfrak{a}[[\nu]],*_{\theta'}), \tag{6.3}$$

such that $\phi(\nu) = \nu$ and

$$\phi(f) = f+\nu^2\phi_2(f)+\nu^3\phi_3(f)+\cdots,\quad f\in\mathfrak{a}. \tag{6.4}$$

Then it is easy to see that

$$\begin{aligned}\delta_0\phi_2 &= \pi_2-\pi'_2,\\ \delta_0\phi_3+d_1\phi_2 &= \pi_3-\pi'_3.\end{aligned} \tag{6.5}$$

Since $\pi_3-\pi'_3$ is skew-symmetric, (6.5) implies $\delta_0\phi_3 = 0$, $d_1\phi_2 = \pi_3-\pi'_3$. Remark that $\pi_3-\pi'_3 = \theta-\theta'$. We see $[\theta] = [\theta']$.

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6.2. Proof of Theorem 2. Suppose $\dim M = 2$, and let $(\mathfrak{a}, \cdot, \{\ ,\ \})$ be any Poisson algebra. We set $\pi_0(f,g) = f \cdot g$, $\pi_1(f,g) = -\frac{1}{2}\{f,g\}$. Suppose $\pi_2, \cdots, \pi_{2l-1}$ are given with Assumptions (HE. 1-3) in §4. Since $\dim M = 2$, we have $R_{2l}(x_i, x_j, x_k) = 0$ on any coordinate neighborhood $(U; x_1, x_2)$. Since R_{2l} is a 3-derivation by Theorem 3.3, this implies $R_{2l} = 0$. The same conclusion is obtained also by using Artin's theorem [S], that is, any 2-generated alternative algebra is associative.

As $R_{2l} = 0$, one can construct π_{2l}, π_{2l+1} with the properties (HE. 1-3) and (HO. 1-3) by using Theorems 4.1, 5.1, and $R_{2(l+1)} = 0$ by the same reasons. In these construction, one can give $g_{ij}^{(m)}(x_i, x_j)$, m=2,3,4,$\cdots$, arbitrary, whenever $g_{ij}^{(m)} = (-1)^m g_{ji}^{(m)}$.

6.3. Proof of Theorem 3. Let $(\mathfrak{a}[[\nu]], *)$, $(\mathfrak{a}[[\nu]], *')$ be any Q-deformation of $(\mathfrak{a}, \cdot, \{\ ,\ \})$ By (3.4), we see $\partial_2^0 \pi_2^- = \partial_2^0 \pi_2'^- = 0$. Hence, $\pi_2^-, \pi_2'^- \in \mathcal{A}_2(\mathfrak{a}, \cdot)$ by the skew-symmetricity. As $R_3 = 0$ we have $d_1^- \pi_2^- = d_1^- \pi_2'^- = 0$ by (1.24) and (2.12). By the assumption, there must exist $\psi_1, \psi_1' \in \mathcal{A}_1(\mathfrak{a}, \cdot)$ such that $\pi_2^- = d_1^- \psi_1$, $\pi_2'^- = d_1^- \psi_1'$, $\delta_0 \psi_1 = \delta_0 \psi_1' = 0$.

Change the decomposition $\mathfrak{a}[[\nu]] = \sum \nu^n \mathfrak{a}$ by isomorphisms $\psi, \psi' : \mathfrak{a}[[\nu]] \to \mathfrak{a}[[\nu]]$ given by

$$\psi(f) = f - \nu\psi_1(f), \quad \psi'(f) = f - \nu\psi_1'(f).$$

In the new expression of $*, *'$ we see $\pi_1 (= \pi_1')$and $\pi_2^+, \pi_2'^+$ are not changed, but $\pi_2^-, \pi_2'^-$ disappear. Thus, one may assume $\pi_2^- = \pi_2'^- = 0$. Since $\pi_2' - \pi_2$ is symmetric, there is $\phi_2 \in C^1(\mathfrak{a})$ by Proposition 2.2 such that $\delta_0 \phi_2 = \pi_2' - \pi_2$. (One may assume that ϕ_2 is a linear differential operator of order 4, if π_2, π_2' are bidifferential operator of order 4.) For any $\xi \in \mathcal{A}_1(\mathfrak{a}, \cdot)$ one may replace ϕ_2 by $\phi_2 + \xi$.

Since $R_4 = R_4' = 0$ and $\pi_2^- = \pi_2'^- = 0$, we see $d_1^- \pi_3'^- = d_1^- \pi_3^- = 0$. Hence by the assumption, there is $\xi \in \mathcal{A}_1(\mathfrak{a}, \cdot)$ such that $d_1^-(\xi + \phi_2) = \pi_3'^- - \pi_3^-$. By the isomorphism

$$\phi(f) = f + \nu^2(\phi_2 + \xi)(f),$$

we see that one may assume that $\pi_2 = \pi_2'$, $\pi_3^- = \pi_3'^-$. Repeating this procedure, we see that there is an isomorphism ψ of $(\mathfrak{a}[[\nu]], *)$ onto $(\mathfrak{a}[[\nu]], *')$ such that $\psi(f) = f \pmod{\nu}$.

6.4 Proof of Example 3. Let x, y, z be the coordinate functions on $\mathbf{R}^3$. Set $\pi_0(f,g) = f \cdot g$, $\pi_1(f,g) = \frac{1}{2}\{f,g\}$. Suppose $\pi_2, \cdots, \pi_{2p-1}$ are given with (HE. 1)-(HE. 3) and with the additional conditions:

$$\pi_m(x_i, x_j) = 0 \quad \text{for} \quad 2 \le m \le 2p-1, \quad \text{where} \quad x_1 = x, x_2 = y, x_3 = z, \tag{6.6}$$

$$K_m = 0 \quad \text{for} \quad 0 \le m \le 2p-1, \quad \text{and } m \text{ is even} \quad (\text{cf. } (4.14)). \tag{6.7}$$

To prove $R_{2p} = 0$, we have only to show $R_{2p}(x,y,z) = 0$ because $R_{2p}(x,y,y)$, etc vanish by the alternativity of R_{2p}. Since

$$R_{2p}(x,y,z) = \pi_{2p-1}^-(x, x^l) + \pi_{2p-1}^-(y, y^m) + \pi_{2p-1}^-(z, z^k) \tag{6.8}$$

by (6.6), we have only to show that $\pi_{2p-1}^-(x, x^l) = 0$, etc.

It is enough to prove $\pi_{2p-1}^-(x^c, x^d) = 0$ for any c, d. This will be shown by induction. So assume that $\pi_s^-(x^a, x^b) = 0$ for $s \le 2p-2$, $a + b \le r$, or for $s \le 2p-1$, $a + b \le r - 1$, for a fixed integer r. By (5.5), we have

$$\pi_{2p-1}^-(x^c, x^d) = -\langle x^c, \langle x^a, x^b \rangle^+ \rangle_{2p-1}^-, \quad a + b = d, \ c + d = r. \tag{6.9}$$

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Hence, we have only to show that $\pi_{2q}^{+}(x^a, x^b) = 0$ for $2q \leq 2p-2$, $a+b \leq r-1$.

On the other hand, since $K_{2q} = 0$ by (6.7), we have by (4.13) that

$$\pi_{2q}^{+}(x, x^c) = 0 \quad \text{for any} \quad c.$$

By the bridge equation $(4.25)_\gamma$ in §4, we see $\pi_{2q}^{+}(x^a, x^b) = 0$ for $a+b \leq r-1$.

Thus, we see that $(C^\infty(\mathbf{R}^3), \cdot, \{\ ,\ \})$ is deformation quantizable. Since $\pi_{2p-1}^{-}(x^c, x^d) = 0$, etc, in the Q-deformation, the following equations hold:

$$[x,y] = -\nu z^k, \quad [y,z] = -\nu x^l, \quad [z,x] = -\nu y^m. \tag{6.10}$$

Suppose there is an element $\tilde{f} \in C^\infty(\mathbf{R}^3)[[\nu]]$, $\tilde{f} = f_0 + \nu f_1 + \cdots + \nu^p f_p + \cdots$, such that $[\tilde{f}, \tilde{g}] = 0$ for any $\tilde{g} \in C^\infty(\mathbf{R}^3)[[\nu]]$. If $\tilde{f} \neq 0$, then multiplying a suitable non-zero constant, one may assume that

$$f_0 = \frac{1}{l+1}x^{l+1} + \frac{1}{m+1}y^{m+1} + \frac{1}{k+1}z^{k+1}. \tag{6.11}$$

Thus, f_1 must satisfy $f_1 = \lambda f_0$, and f_2 must satisfy

$$\frac{1}{2}\{x, f_2\} = \pi_3^{-}(x, f_0), \quad \frac{1}{2}\{y, f_2\} = \pi_3^{-}(y, f_0), \quad \frac{1}{2}\{z, f_2\} = \pi_3^{-}(z, f_0). \tag{6.12}$$

Therefore, we have

$$\begin{cases} z^k \partial_y f_2 - y^m \partial_z f_2 & = 2\pi_3^{-}(x, f_0), \\ -z^k \partial_x f_2 + x^l \partial_z f_2 & = 2\pi_3^{-}(y, f_0), \\ y^m \partial_x f_2 - x^l \partial_y f_2 & = 2\pi_3^{-}(z, f_0). \end{cases} \tag{6.13}$$

Computing $\pi_3^{-}(x, f_0)$ by using (5.5), (4.10) together with (6.7) and (6.12), we see that $2\pi_3^{-}(x, f_0)$ involves the term $\lambda x^{2l} y^{m-2} z^{k-2}$. This implies that (6.13) has no solution. Hence, the quantized Poisson algebra has the trivial center.

6.5 Proof of Example 4. Let $(C^\infty(\mathbf{R}^3), \cdot, \{\ ,\ \})$ be the Poisson algebra such that $\{x_i, x_j\} = a_{ij} x_i x_j$. Set $\pi_0(f,g) = f \cdot g$, $\pi_1(f,g) = -\frac{1}{2}\{f,g\}$. Suppose $\pi_2, \cdots, \pi_{2l+1}$ are given with (HE. 1-3) for $l \geq 1$. We assume also that

$$\pi_m(x_i, x_j) = 0 \quad \text{for} \quad 1 \leq m \leq 2l+1, \tag{6.14}$$

$$K_m = 0 \quad \text{for} \quad 1 \leq m \leq 2l+1, \quad \text{and } m \text{ is even} \quad (\text{cf.}(4.14)). \tag{6.15}$$

We have only to show that $R_{2l+2} = 0$. By (6.14), we see that

$$R_{2l+2}(x_i, x_j, x_k) = \sum_{(i,j,k)} \pi_{2l+1}^{-}(x_i, \pi_1^{-}(x_j, x_k)) = \sum_{(i,j,k)} a_{jk} \pi_{2l+1}^{-}(x_i, x_j x_k).$$

Since $l \geq 1$, by (5.5) and $a_{ij} = -a_{ji}$, we see that

$$-2R_{2l+2}(x_i, x_j, x_k) = \sum_{(i,j,k)} a_{ij} a_{jk} \{\pi_{2l}^{+}(x_k, x_i x_j) - \pi_{2l}^{+}(x_i, x_k x_j)\}. \tag{6.16}$$

Using (4.4) to the right hand side of (6.16), we have

$$4R_{2l+2}(x_i, x_j, x_k) = -a_{ij}a_{jk}a_{ki} \sum_{(i,j,k)} \pi^-_{2l-1}(x_i, x_k x_j). \tag{6.17}$$

By (5.5), if $l > 1$, then the right hand side of (6.17) vanishes. If $l = 1$, then $\sum_{(i,j,k)} \{x_i, x_k x_j\} = 0$ by using Leibniz identity.

We have easily that $x_i * x_j = x_i x_j + \frac{\nu}{2} a_{ij} x_i x_j$.

To obtain the case

$$\sum_{n=0}^{\infty} \nu^n \pi_n(x_i, x_j) = \sqrt{\frac{1 + \frac{\nu}{2}a_{ij}}{1 - \frac{\nu}{2}a_{ij}}} x_i x_j,$$

change the decomposition $\mathfrak{a}[[\nu]] = \sum \nu^n \mathfrak{a}$ of the above algebra by an isomorphism $\psi : \mathfrak{a}[[\nu]] \to \mathfrak{a}[[\nu]]$, $\psi(\nu) = \nu$,

$$\psi = 1 + \sum_{i<j} \left(\frac{1}{\sqrt{1 - (\frac{\nu}{2}a_{ij})^2}} - 1 \right) x_i x_j \frac{\partial^2}{\partial x_i \partial x_j}.$$

Then obviously $\psi^{-1}(x_j) = x_j$, but

$$\psi(\psi^{-1}(x_i) * \psi^{-1}(x_j)) = (1 + \frac{\nu}{2}a_{ij})\psi(x_i x_j) = \sqrt{\frac{1 + \frac{\nu}{2}a_{ij}}{1 - \frac{\nu}{2}a_{ij}}} x_i x_j.$$

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