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A RELATION BETWEEN THE LOGARITHMIC DERIVATIVES OF RIEMANN AND SELBERG ZETA FUNCTIONS AND A PROOF OF THE RIEMANN HYPOTHESIS UNDER AN ASSUMPTION ON A DISCRETE SUBGROUP OF $SL(2,\mathrm{R})$

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1. Introduction

Let $\zeta(s)$ be the Riemann's zeta function and $\eta(r)$ $(r = \sqrt{-1}(1/2 - s))$ the logarithmic derivative of ζ which is of the form:

$$\begin{aligned}\eta(r) &= \sum_{p \in Prim} \sum_{n \geq 1} (\log p) e^{-n(\log p)s} \\ &= \sum_{i \geq 1} \sum_{n \geq 1} a_{in} e^{-\sqrt{-1}n(\log p_i)r}, \end{aligned} \tag{1}$$

where $Prim = \{p_i; i \geq 1\}$ is the set of prime numbers and $a_{in} = (\log p_i)e^{-n(\log p_i)/2}$. This series converges absolutely and uniformly in any half plane $\Im(r) < -1/2 - \varepsilon$ $(\varepsilon > 0)$ and has meromorphic continuation to the whole complex plane. Then the Riemann Hypothesis that the roots of $\zeta(s)$ all do lie on $\Re(s) = 1/2$ is equivalent to showing that the non imaginaly poles of $\eta(r)$ all do lie on $\Im(r) = 0$.

Let G be a connected semisimple Lie group with finite center, K a maximal compact subgroup of G and Γ a discrete subgroup of G such that $\Gamma\backslash G$ is compact. Then for each character χ of a finite dimensional unitary representation of Γ, Gangolli[**G1**] investigates a zeta function $Z_\Gamma(s, \chi)$ of Selberg's type, Selberg[**S**] originally introduced into the case of $SL(2, \mathbf{R})$. The logarithmic derivative $\eta_G(r)$ of $Z_\Gamma(s, \chi)$ $(r = \sqrt{-1}(\rho_0 - s)$ and ρ_0 is a positive real number depending only on $(G, K))$ is of the form:

$$\eta_G(r) = \kappa \sum_{\delta \in Prim_\Gamma} \sum_{n \geq 1} \sum_{\lambda \in L} u_\delta m_\lambda \chi(\delta^n) \xi_\lambda(h(\delta))^{-n} e^{-nu_\delta s}, \tag{2}$$

where $Prim_\Gamma$ is a complete set of representatives for the conjugacy classes of prime elements in Γ and u_δ $(\delta \in Prim_\Gamma)$ the logarithm of the norm $N(\delta)$ of δ. For other

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notations refer to [G1]. This series converges absolutely and uniformly in any half plane $\Im(r) < -\rho_0 - \varepsilon$ ($\varepsilon > 0$) and has meromorphic continuation to the whole complex plane. Especially, the poles of $\eta_G(r)$ all do lie on $\Im(r) = 0$ or $\Re(r) = 0$, so the Riemann Hypothesis holds true for $Z_\Gamma(s, \chi)$. In what follows we shall rearrange the series as

$$\eta_G(r) = \sum_{i \geq 1} \sum_{n \geq 1} b_{in} e^{-\sqrt{-1} c_{in} u_{\delta_i} r} \tag{3}$$

for which the exponents satisfy $c_{in} u_{\delta_i} = c_{jm} u_{\delta_j}$ if and only if $i = j$ and $n = m$.

We here note that (1) and (3) are quite similar in their forms. Therefore, if two distributions of $Prim$ and $Prim_\Gamma$ are similar in the logarithm of their norms, it is hoped that η and η_G have the same properties, especially, the Riemann Hypothesis holds for η and then, for ζ also. In this paper we let $G = SL(2, \mathbf{R})$ and make an assumption of magnitude and distance of $N(\delta)$ for $\delta \in Prim_\Gamma$, which guarntees the similarity between the distributions (see (**A**) in §2 and (**B**) in §5). Then, under a week assumption (**A**) we shall obatin an integral expression of η in terms of η_G such as

$$\eta(\nu) = \int_{\mathbf{R} - \sqrt{-1} y} \eta_G(x) H(\nu, x) dx \tag{4}$$

(see Proposition 3.3). Unfortunately, this formula is valid only for $\Im(\nu) \leq -L$ (L is a large positive number). Then, the Riemann Hypothesis is equivalent to showing that the right hand side of (4) has analytic continuation to $\Im(\nu) < 0$ except $\nu = -\sqrt{-1}/2$. Under a strong assumption (**B**) we shall obtain the continuation and prove the Riemann Hypothesis (see Theorem 5.4).

2. Notations

Let $G = SL(2, \mathbf{R})$ and let χ be the trivial character of Γ. Then $\rho_0 = 1/2$ and the explicit form of η_G is given by

$$\eta_G(r) = \sum_{i \geq 1} \sum_{n \geq 1} \frac{u_i/2}{\sinh(n u_i/2)} e^{-\sqrt{-1} n u_i r}, \tag{5}$$

where $u_i = u_{\delta_i}$, and in (3) $c_{in} = n$ and

$$b_{in}{}^{-1} = 2 u_i^{-1} \sinh(n u_i/2) \leq c e^{n u_i/2}. \tag{6}$$

For general references to the basic properties of η_G see [**G1**], [**H**] and [**S**]. We denote the increasing sequence of prime numbers as $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ and the one of the norms of elements in $Prim_\Gamma$ as $N(\delta_1), N(\delta_2), N(\delta_3), \ldots$ respectively. We define $u_i = \log N(\delta_i)$ and

$$\delta_{in} = \frac{1}{2} \inf_{\substack{(m,j) \in \mathbf{N}^2 \\ (m,j) \neq (n,i)}} |n u_i - m u_j| \tag{7}$$

for $i \geq 1$ and $n \geq 1$. Then, each δ_{in} is positive, because $\{u_i; i \geq 1\}$ does not have a finite point of accumulation (see **[G2]**, p.415). For each $\alpha, \beta \in \mathbf{R}$ and $C > 0$ we put

$$\varepsilon_{in} = \varepsilon_{in}(\alpha, \beta, C) = Ce^{-\alpha n(\log p_i)}e^{-\beta n u_i} \tag{8}$$

and throughout this paper we assume the following condition:

(A) There exist $\alpha, \beta \in \mathbf{R}$ and $C, A > 0$ for which

$$\varepsilon_{in}(\alpha, \beta, C) \leq \min(A, \delta_{in}) \quad \text{for all } i \text{ and } n \geq 1.$$

As said in §1, the Riemann Hypothesis holds for η_G. Actually, the poles of η_G are all simple and are as

$$\{\nu_j; j \in \mathbf{Z}\} \cup \{r_j; j \in \mathbf{N}\}, \tag{9}$$

where $\nu_j \in \mathbf{R}$ and $r_j \in \sqrt{-1}\mathbf{R}$ (cf. **[G1]**, Proposition 2.7 and **[H]**, p.68). Then it is known that $\nu_{-j} = -\nu_j$ and the poles of η_G which concentrate along $[-\sqrt{-1}/2, \sqrt{-1}/2]$ can be denoted as

$$\{\nu_0, r_j, \bar{r}_j; 1 \leq j \leq M\}, \tag{10}$$

where we let $r_1, r_2, \ldots, r_M$ be the poles of η_G which concentrate along $[-\sqrt{-1}/2, 0)$ and $\bar{r}_j = -r_j = r_{j+M}$. We denote the residues of η_G at ν_j and r_j by n_j and m_j respectively. Then, $n_{-j} = n_j$ and $m_j = m_{j+M} = 1$ for $1 \leq j \leq M$ (cf. **[H]**, Chap.2).

We fix sufficiently small (resp. large) positive numbers ε and δ (resp. E), and a positive number y such that $1/2 < y \leq 1/2 + \varepsilon$.

3. Transition from η_G to η

Let ϕ be a C^∞ compactly supported function on $\mathbf{R}$ satisfying

$$\begin{aligned} &(i) \quad supp(\phi) \subset (-1, 1), \\ &(ii) \quad \phi(0) = 1, \\ &(iii) \quad \phi^{(k)}(0) = 0 \quad (1 \leq k \leq 2M) \end{aligned}$$

and let

$$h_{in}(t) = \frac{a_{in}}{b_{in}}\phi\left(\frac{t - n(\log p_i)}{\varepsilon_{in}}\right) \quad (t \in \mathbf{R}) \tag{11}$$

for $i \geq 1$ and $n \geq 1$. Then it is easy to see that h_{in} satisfies the following conditions.

$$\begin{aligned} &(i) \quad supp(h_{in}) \subset (n(\log p_i) - \varepsilon_{in}, n(\log p_i) + \varepsilon_{in}), \\ &(ii) \quad h_{in}(n(\log p_i)) = \frac{a_{in}}{b_{in}}, \\ &(iii) \quad h_{in}^{(k)}(n(\log p_i)) = 0 \quad (1 \leq k \leq 2M). \end{aligned} \tag{12}$$

Without loss of generality we may assume that $\varepsilon_{11} \leq 1/2\log 2$ and thus, $supp(h_{in}) \subset [1/2\log 2, \infty)$ for all i and $n \geq 1$. Here we put $\hat{h}_{in}(x) = (2\pi)^{-1}\int_{\mathbf{R}} h_{in}(z)e^{-\sqrt{-1}xz}dz$ and

$$H(\nu, x) = \sum_{i,n\geq 1} e^{\sqrt{-1}(nu_i - n(\log p_i))x}\hat{h}_{in}(\nu - x) \tag{13a}$$

$$= \sum_{i,n\geq 1} e^{-\sqrt{-1}(n(\log p_i)\nu - nu_i x)}\frac{a_{in}}{b_{in}}\varepsilon_{in}\hat{\phi}(\varepsilon_{in}(\nu - x)). \tag{13b}$$

Then, we consider a condition for which the series (13) converges. Let $\theta \geq 0$ and $1 \leq p, q \leq \infty$ such that $1/p + 1/q = 1$. We now suppose that ν and x satisfy

$$(a_E) \quad -E \leq \Im(\nu), \Im(x) \leq E,$$

$$(b_\theta^{p,q}) \quad \begin{cases} \Im(\nu) - 1/2 - (1-\theta)\alpha \leq -1/p - \delta \\ -\Im(x) + 1/2 - (1-\theta)\beta \leq -1/q - \delta, \end{cases}$$

where δ is a fixed sufficiently small positive number (see §2). Then, substituting the definition of a_{in} and b_{in} (see (1) and (6)) for (13b), we see that $|\nu - x|^\theta |H(\nu, x)|$ is dominated by

$$c\sum_{i,n\geq 1} \log p_i e^{(\Im(\nu)-1/2)n(\log p_i)}e^{(-\Im_A(x)+1/2)nu_i}\varepsilon_{in}^{1-\theta}|(\varepsilon_{in}(\nu - x))^\theta\hat{\phi}(\varepsilon_{in}(\nu - x))|.$$

Since $\hat{\phi}$ is rapidly decreasing and is holomorphic of exponential type ≤ 1 (cf. **[Su]**, p.146), for each $N \in \mathbf{N}$ there exists $C_N > 0$ for which

$$|\hat{\phi}(x)| \leq C_N(1 + |x|)^{-N}e^{|\Im(x)|} \quad (x \in \mathbf{C}).$$

Therefore, it follows from (**A**), (a_E) and (8) that $|\nu - x|^\theta |H(\nu, x)|$ is dominated by

$$cC^{1-\theta}C_{[\theta]+1}e^{2EA}\sum_{i,n\geq 1} \log p_i e^{(\Im(\nu)-1/2-(1-\theta)\alpha)n(\log p_i)}e^{(-\Im(x)+1/2-(1-\theta)\beta)nu_i},$$

where $[\theta]$ is the greatest integer not exceeding θ. Then, this series converges absolutely and uniformly by $(b_\theta^{p,q})$ and the Hölder's inequality.

Lemma 3.1. *If ν and x satisfy (a_E) and $(b_0^{p,q})$, then the series $H(\nu, x)$ converges absolutely and uniformly, and is holomorphic of ν and x. Moreover, if $(b_\theta^{p,q})(\theta \geq 0)$ is satisfied, there exists a positive constant $C = C_{E,A,\theta}$ such that*

$$|H(\nu, x)| \leq C|\nu - x|^{-\theta}.$$

Let $-y \leq -y_0 \leq E$ and

$$(b_{\theta,y_0}^{p,q}) \quad \begin{cases} \Im(\nu) - 1/2 - (1-\theta)\alpha \leq -1/p - \delta, \\ y_0 + 1/2 - (1-\theta)\beta \leq -1/q - \delta. \end{cases}$$

Then, if ν satisfies (a_E) and $(b^{p,q}_{\theta+1,y_0})(\theta \in \mathbf{N})$, it follows similarly as above that

$$\begin{aligned}&\int_{\mathbf{R}-\sqrt{-1}y_0} |x|^\theta |H(\nu,x)|dx \\ \leq &c \sum_{i,n\geq 1} \log p_i e^{(\Im(\nu)-1/2)n(\log p_i)} e^{(y_0+1/2)nu_i} \varepsilon_{in}^{-\theta} \left[\varepsilon_{in} \int_{\mathbf{R}-\sqrt{-1}y_0} |(\varepsilon_{in}x)^\theta \hat{\phi}(\varepsilon_{in}(\nu-x))|dx\right]\end{aligned}$$

and by letting $x = (x-\nu)+\nu$,

$$\leq cC^{-\theta}C_{\theta+2}e^{2EA}P_{A,\theta}(|\nu|) \sum_{i,n\geq 1} \log p_i e^{(\Im(\nu)-1/2+\theta\alpha)n(\log p_i)} e^{(y_0+1/2+\theta\beta)nu_i},$$

where $P_{A,\theta}$ is a polynomial of degree θ with coefficients depending only on A and θ. Then this series converges absolutely and uniformly by $(b^{p,q}_{\theta+1,y_0})$ and the Hölder's inequality.

Lemma 3.2. *Let ν be in a compact set S in the tube domain defined by (a_E) and $(b^{p,q}_{\theta+1,y_0})$ $(\theta \in \mathbf{N}$ and $-y \leq -y_0 \leq E)$. Let f be a function on $\mathbf{R}-\sqrt{-1}y_0$ such that $f(x) = O(|x|^\theta)$. Then, there exists a positive constant $C = C_{E,A,\theta,f,S}$ for which $\int_{\mathbf{R}-\sqrt{-1}y_0} |f(x)H(\nu,x)|dx \leq C$. Especially,*

$$T_{y_0}f(\nu) = \int_{\mathbf{R}-\sqrt{-1}y_0} f(x)H(\nu,x)dx$$

is well-defined and is holomorphic of ν satisfying (a_E) and $(b^{p,q}_{\theta+1,y_0})$.

Proposition 3.3. *Let P be a polynomial of degree $k(0 \leq k \leq 2M)$ and ν satisfy (a_E) and $(b^{p,q}_{k+1,y})$. Then,*

$$\begin{aligned}(i) \quad & P(\nu)\eta(\nu) = T_y(P\eta_G)(\nu) \\ &\qquad = \int_{\mathbf{R}-\sqrt{-1}y} P(x)\eta_G(x)H(\nu,x)dx, \\ (ii) \quad & \qquad 0 = \int_{\mathbf{R}-\sqrt{-1}y} P(x)\eta_G(x)H(\nu,-x)dx.\end{aligned}$$

Proof. Since $\eta_G(x) = O(1)$ for $x \in \mathbf{R}-\sqrt{-1}y$ (see **[H]**, Proposition 6.7) and $(b^{p,q}_{k+1,y})$ implies $(b^{p,q}_{k+1,-y})$, the right hand sides of (i) and (ii) are well-defined and are holomorphic of ν satisfying (a_E) and $(b^{p,q}_{k+1,y})$ (see Lemma 3.2). Therefore, we may suppose that $\Im(\nu) \leq -y$. Since $mu_j > 0$ for all $m,j \geq 1$, it follows that

$$\begin{aligned}&\int_{\mathbf{R}-\sqrt{-1}y} e^{-\sqrt{-1}mu_j x} H(\nu,x)dx \\ = &\int_{\mathbf{R}} e^{-\sqrt{-1}mu_j x} H(\nu,x)dx.\end{aligned}$$

Then, substituting the definition of $H(\nu,x)$ (see (13a)), we see formally that

$$= \sum_{k,l\geq 1} \int_{\mathbf{R}} e^{-\sqrt{-1}mu_j x} e^{\sqrt{-1}(lu_k - l(\log p_k))x} \hat{h}_{kl}(\nu - x)dx$$
$$= \sum_{k,l\geq 1} e^{-\sqrt{-1}(mu_j - lu_k + l(\log p_k))\nu} \int_{\mathbf{R}} e^{\sqrt{-1}(mu_j - lu_k + l(\log p_k))x} \hat{h}_{kl}(x)dx$$
$$= \sum_{k,l\geq 1} e^{-\sqrt{-1}(mu_j - lu_k + l(\log p_k))\nu} h_{kl}(mu_j - lu_k + l(\log p_k)).$$

Since each support of h_{kl} is disjointed from the others, it is easy to see that the condition that $\Im(\nu) \leq -y$ guarantees the validity of the above calculation. Moreover, since the support of h_{kl} is contained in $(l(\log p_k) - \varepsilon_{kl}, l(\log p_k) + \varepsilon_{kl})$ and $h_{kl}(l(\log p_k)) = a_{kl}b_{kl}^{-1}$ (see (12)(i) and (ii)), it follows from ($\mathbf{A}$) and the definition of δ_{kl} (see (7)) that

$$= \epsilon_{kj}\epsilon_{lm} h_{kl}(l(\log p_k)) e^{-\sqrt{-1}l(\log p_k)\nu}$$
$$= \epsilon_{kj}\epsilon_{lm} a_{kl} b_{kl}^{-1} e^{-\sqrt{-1}l(\log p_k)\nu},$$

where $\epsilon_{ij} = 1$ if $i = j$ and 0 otherwise. Therefore, we can deduce that

$$\begin{aligned} T_y\eta_G(\nu) &= \int_{\mathbf{R}-\sqrt{-1}y} \eta_G(x)H(\nu,x)dx \\ &= \sum_{j,m\geq 1} b_{jm} \int_{\mathbf{R}-\sqrt{-1}y} e^{-\sqrt{-1}mu_j x} H(\nu,x)dx \\ &= \sum_{j,m\geq 1} a_{jm} e^{-\sqrt{-1}m(\log p_j)\nu} \qquad (14) \\ &= \eta(\nu). \end{aligned}$$

Here we rewrite $P(\nu)$ as

$$P(\nu) = R_\nu(\nu - x) + P(x),$$

where R_ν is a polynomial of degree k with coefficients depending only on k and ν. Then the formula (i) follows from (14) provided that

$$\int_{\mathbf{R}-\sqrt{-1}y} (\nu - x)^l \eta_G(x)H(\nu,x)dx = 0 \quad (1 \leq l \leq k). \qquad (15)$$

We now show (15). If we define $H^{(l)}(\nu,x)$ by replacing h_{in} in (13a) with $(\sqrt{-1})^{-l}h_{in}^{(l)}$, we easily see that the left hand side of (15) is equal to

$$\int_{\mathbf{R}-\sqrt{-1}y} \eta_G(x)H^{(l)}(\nu,x)dx.$$

Obviously, this integral is finite by the condition $(b_{k+1,y}^{p,q})$. Then, applying the same argument that deduces (14), especially, by using (12)(iii) instead of (12)(ii), we can show that this integral is equal to 0. The formula (ii) follows by the quite same way. □

We now let ε and δ (resp. E) sufficiently small (resp. large). Then, we can deduce the following,

Corollary 3.4. *The equations (i) and (ii) in Proposition 3.3 hold for ν satisfying*

$$\begin{cases} \Im(\nu) - 1/2 + k\alpha < -1/p \\ 1 + k\beta < -1/q, \end{cases}$$

where $1 \le p, q \le \infty$ and $1/p + 1/q = 1$.

Remark 3.5. If there exists a positive constant B such that

$$u_i \le B \log p_i \quad (\text{resp. } u_i \ge B \log p_i) \text{ for all } i \ge 1,$$

then we can replace $(b_\theta^{p,q})$ as

$$(b_{\theta,B,\gamma}^{p,q}) \quad \begin{cases} \Im(\nu) - 1/2 - (1-\theta)\alpha + \gamma \le -1/p - \delta \\ -\Im(x) + 1/2 - (1-\theta)\beta - \gamma/B \le -1/q - \delta, \end{cases}$$

and moreover, the condition of ν in Corollary 3.4 as

$$\begin{cases} \Im(\nu) - 1/2 + k\alpha + \gamma < -1/p \\ 1 + k\beta - \gamma/B < -1/q, \end{cases}$$

where $\gamma \ge 0$ (resp. $\gamma \le 0$), $1 \le p, q \le \infty$ and $1/p + 1/q = 1$.

4. A relation between η and the poles of η_G

We keep the notations and the assumption (**A**). We first recall that η_G satisfies the functional equation:

$$\eta_G(x) + \eta_G(-x) = cx \tanh \pi x \tag{16}$$

(see [**H**], Proposition 4.26). In this section we shall express η as the sum of an integral of $x \tanh \pi x$ and the residues of η_G.

Lemma 4.1. *Let P be a polynomial of degree $k (0 \le k \le 2M)$ and let ν be in a compact set S satisfying $\Im(S) < 0$, (a_E) and $(b_{k+6,0}^{p,q})$. Then the series $\sum_{j\in\mathbf{Z}} n_j P(\nu_j)\ H(\nu, \nu_j)$ converges absolutely and uniformly. Especially, $\sum_{j\in\mathbf{Z}} n_j P(\nu_j)\ H(\nu,\nu_j)$ is well-defined and is holomorphic of ν satisfying $\Im(S) < 0$, (a_E) and $(b_{k+6,0}^{p,q})$.*

Proof. Since $\nu_j \in \mathbf{R}$ and $\nu \in S$, Lemma 3.1 implies that for $x \in \mathbf{R}$

$$|H(\nu, x)| \le C|\nu - x|^{-(k+6)} \sim (1 + |x|)^{-(k+6)}.$$

Then, noting the fact that

$$\sum_{\{j;\nu_j^2 \le x\}} n_j \quad \sim \quad x^2 \quad (x \to \infty)$$

(see §2 and [**G1**], Proposition1.2), we see that

$$\begin{aligned}
&\sum_{j\in\mathbf{Z}} n_j|P(\nu_j)H(\nu,\nu_j)| \\
&\sim \sum_{j\in\mathbf{Z}} n_j(1+|\nu_j|)^{-6} \\
&\sim \sum_{k=0}^{\infty} \sum_{k\le|\nu_j|<k+1} n_j(1+|\nu_j|)^{-6} \\
&\sim \sum_{k=0}^{\infty}(1+k)^{-2} < \infty. \qquad \square
\end{aligned}$$

We now suppose that ν satisfies $\Im(\nu) < 0$, (a_E) and $(b^{p,q}_{6,y})$. We note that, if $|\Im(x)| \le \varepsilon$, then $x\tanh\pi x = O(|x|)$ and $\eta_G(x) = O(|x|)$ (see [**H**], Proposition 6.7). Therefore, since $(b^{p,q}_{6,y})$ implies $(b^{p,q}_{2,\pm\varepsilon})$ and $(b^{p,q}_{6,0})$, it follows from Lemma 3.2 and Lemma 4.1 that

$$\begin{aligned}
&\int_{\mathbf{R}} cx\tanh\pi x H(\nu,x)dx \\
&= \int_{\mathbf{R}+\sqrt{-1}\varepsilon} cx\tanh\pi x H(\nu,-x)dx \\
&= \int_{\mathbf{R}+\sqrt{-1}\varepsilon} (\eta_G(x)+\eta_G(-x))H(\nu,-x)dx \\
&= \int_{\mathbf{R}-\sqrt{-1}\varepsilon} \eta_G(x)H(\nu,x)dx + \int_{\mathbf{R}+\sqrt{-1}\varepsilon} \eta_G(x)H(\nu,-x)dx.
\end{aligned}$$

The second term is equal to

$$\begin{aligned}
&\int_{\mathbf{R}-\sqrt{-1}y} \eta_G(x)H(\nu,-x)dx - \sum_{j\in\mathbf{Z}} n_jH(\nu,\nu_j) - \sum_{1\le j\le M} H(\nu,-r_j) \\
&= -\sum_{j\in\mathbf{Z}} n_jH(\nu,\nu_j) - \sum_{1\le j\le M} H(\nu,-r_j)
\end{aligned}$$

by Proposition 3.3(ii). Therefore, it follows from Proposition 3.3 (i) that

$$\begin{aligned}
\eta(\nu) &= \int_{\mathbf{R}-\sqrt{-1}y} \eta_G(x)H(\nu,x)dx \\
&= \int_{\mathbf{R}-\sqrt{-1}\varepsilon} \eta_G(x)H(\nu,x)dx + \sum_{1\le j\le M} H(\nu,r_j) \\
&= \int_{\mathbf{R}} cx\tanh\pi x H(\nu,x)dx + \sum_{j\in\mathbf{Z}} n_jH(\nu,\nu_j) + \sum_{1\le j\le 2M} H(\nu,r_j).
\end{aligned}$$

Then, letting ε and δ (resp. E) sufficiently small (resp. large), we can obtain the following,

Proposition 4.2. *If ν satisfies*

$$\begin{cases} \Im(\nu) < \min(0, 1/2 - 5\alpha - 1/p) \\ 1 + 5\beta < -1/q, \end{cases}$$

where $1 \le p, q \le \infty$ and $1/p + 1/q = 1$, then

$$\eta(\nu) = c \int_{\mathbf{R}} x \tanh \pi x H(\nu, x) dx + \sum_{j \in \mathbf{Z}} n_j H(\nu, \nu_j) + \sum_{1 \le j \le 2M} H(\nu, r_j).$$

We put

$$P_G(x) = (\nu^2 - r_1^2)(\nu^2 - r_2^2) \dots (\nu^2 - r_M^2). \tag{17}$$

Then, replacing η_G with $P_G \eta_G$, we can obtain the following proposition by the quite same way.

Proposition 4.3. *If ν satisfies*

$$\begin{cases} \Im(\nu) < \min(0, 1/2 - (5 + 2M)\alpha - 1/p) \\ 1 + (5 + 2M)\beta < -1/q, \end{cases}$$

where $1 \le p, q \le \infty$ and $1/p + 1/q = 1$, then

$$\begin{aligned} P_G(\nu)\eta(\nu) &= \int_{\mathbf{R} - \sqrt{-1}\varepsilon} \eta_G(x) P_G(x) H(\nu, x) dx \\ &= c \int_{\mathbf{R}} x \tanh \pi x P_G(x) H(\nu, x) dx + \sum_{j \in \mathbf{Z}} n_j P_G(\nu_j) H(\nu, \nu_j). \end{aligned}$$

Remark 4.4. The same argument in Remark 3.5 is also applicable to the conditions of ν in Proposition 4.2 and Proposition 4.3.

5. A proof of the Riemann Hypothesis

We retain the notations in the previous sections. As said in §1, we here make an assumption of magnitude and distance of $u_i (i \in \mathrm{N})$, which is stronger than (**A**). Then, we shall prove the Riemann Hypothesis under the assumption.

5.1 We first modify the correspondence of p_i in Prim to δ_i in Prim_Γ. For an increasing map

$$\omega : \mathbf{N} \quad \rightarrow \quad \mathbf{N}$$

we put

$$\delta_{in} = \frac{1}{2} \inf_{\substack{(m,j)\in\mathbf{N}^2 \\ (m,j)\neq(n,\omega(i))}} |nu_{\omega(i)} - mu_j|, \tag{18}$$

$$\varepsilon_{in}^{\omega} = \varepsilon_{in}^{\omega}(\alpha,\beta,C) = Ce^{-\alpha n(\log p_i)}e^{-\beta n u_{\omega(i)}}, \tag{19}$$

$$h_{in}^{\omega} = \frac{a_{in}}{b_{\omega(i)n}}\phi(\frac{t-n(\log p_i)}{\varepsilon_{in}^{\omega}}) \quad (t\in\mathbf{R}), \tag{20}$$

$$H^{\omega}(\nu,x) = \sum_{i,n\geq 1} e^{\sqrt{-1}(nu_{\omega(i)}-n(\log p_i))x}\hat{h}_{in}^{\omega}(\nu-x) \tag{21}$$

(cf. (7), (8), (11) and (13)). Then it is easy to see that all results in the preceding sections are also valid when we replace $\delta_{in}, \varepsilon_{in}, h_{in}$ and $H(\nu,x)$ by $\delta_{in}^{\omega}, \varepsilon_{in}^{\omega}, h_{in}^{\omega}$ and $H^{\omega}(\nu,x)$ respectively and **(A)** by

(A)$^{\omega}$ There exist $\alpha,\beta\in\mathbf{R}$ and $C, A>0$ for which

$$\varepsilon_{in}^{\omega}(\alpha,\beta,C) \leq \min(A,\delta_{in}^{\omega}) \quad \text{for all } i \text{ and } n\geq 1.$$

We next modify the η functions. Let

$$\eta^{\circ}(r) = \sum_{i\geq 1} a_i e^{-\sqrt{-1}(\log p_i)r}, \tag{22}$$

where $a_i = (\log p_i)e^{-(\log p_i)/2}$, and let

$$\eta_G^{\circ}(r) = \sum_{i\geq 1} b_i e^{-\sqrt{-1}u_i r}, \tag{23}$$

where $b_i = u_i/2\sinh(u_i/2)$. Then, it is easy to see that $\eta(r)-\eta^{\circ}(r)$ and $\eta_G(r)-\eta_G^{\circ}(r)$ are holomorphic on $\Im(r)<0$ (cf. **[H]**, Proposition 3.5). Therefore, in order to prove the Riemann Hypothesis for η it is enough to prove it for η°. Since η° and η_G° inherit all singularities from η and η_G respectively, the whole arguments in the previous sections except one using the functional equation (16) are also applicable to η° and η_G°. Especially, if we define δ_i^{ω}, $\varepsilon_i^{\omega}(\alpha,\beta,C)$, h_i^{ω} and $H_{\circ}^{\omega}(\nu,x)$ by eliminating the sufix n in (18), (19), (20) and (21) respectively, we see that all the results except one containing $x\tanh\pi x$ are also valid when we replace η,η_G and H by $\eta^{\circ},\eta_G^{\circ}$ and $H_{\circ}^{\omega}$ respectively and **(A)**$^{\omega}$ by

(A)$_{\circ}^{\omega}$ There exist $\alpha,\beta\in\mathbf{R}$ and $C,A>0$ for which

$$\varepsilon_i^{\omega}(\alpha,\beta,C)\leq\min(A,\delta_i^{\omega}) \quad \text{for all } i\geq 1.$$

5.2 We now let

$$\omega : D \quad \rightarrow \quad \mathbf{N}, \quad D\subset\mathbf{N}$$

be an increasing map, and for each $i\in D$ we define $\delta_i^{\omega}, \varepsilon_i^{\omega}(\alpha,\beta,C)$ and h_i^{ω} as above. Especially, we put

$$H_{\circ}^{\omega}(\nu,x) = \sum_{i\in D} e^{\sqrt{-1}(nu_{\omega(i)}-n(\log p_i))x}\hat{h}_{in}^{\omega}(\nu-x) \tag{24}$$

and we define the corresponding assumption **(A)**$_{\circ}^{\omega}$, we denote by the same letter, by replacing $i\geq 1$ by $i\in D$. Then repeating the same arguments in the proof of Lemma 3.1 and Remark 3.5, we can deduce that

Lemma 5.1. *Let us suppose that* $(\mathbf{A})_\circ^\omega$ *holds. (1) If* ν *and* x *satisfy* (a_E) *and* $(b_0^{p,q})$, *then the series* $H_\circ^\omega(\nu,x)$ *converges absolutely and uniformly, and is holomorphic of* ν *and* x. *Moreover, if* $(b_\theta^{p,q})(\theta \geq 0)$ *is satisfied, there exists a positive constant* $C = C_{E,A,\theta}$ *such that* $|H_\circ^\omega(\nu,x)| \leq C|\nu - x|^{-\theta}$. *(2) If there exists a positive constant* B *for which* $u_{\omega(i)} \leq B\log p_i$ *(resp.* $u_{\omega(i)} \geq B\log p_i$*) for all* $i \in D$, *then the same inequality holds for* ν *and* x *satisfying* (a_E) *and* $(b_{\theta,B,\gamma}^{p,q})$ *for* $\gamma \geq 0$ *(resp.* $\gamma \leq 0$*).*

5.3 We here suppose the following condition.

(B) There exist a sufficiently small positive number τ, positive numbers L, N and C and increasing maps

$$\omega_\ell : D_\ell \quad \to \quad \mathbf{N} \quad D_\ell \subset \mathbf{N} \quad (\ell = 1,2),$$

for whhich

$$(B0) \quad D_1 \cup D_2 = \mathbf{N} \quad (disjoint),$$

$$(B1) \quad u_{\omega_1(i)} \leq (\frac{1}{4} - \tau)\log p_i \quad \text{and} \quad \varepsilon_i^{\omega_1}(N,0,C) \leq \delta_i^{\omega_1} \quad \text{for all } i \in D_1,$$

$$(B2) \quad u_{\omega_2(i)} \leq L\log p_i \quad \text{and} \quad C \leq \delta_i^{\omega_2} \quad \text{for all } i \in D_2.$$

Without loss of generality we may suppose that fixed sufficiently small (resp. large) positive numbers ε and δ (resp. E) satisfy

$$\frac{1}{4} - \tau < \frac{1/2 - \delta + \varepsilon}{2 + \delta + \varepsilon}, \tag{25}$$

$$\frac{3}{2}\tau - L(\delta + \varepsilon) - \delta > 0, \tag{26}$$

$$E > \frac{1}{2} + \frac{3}{2}(L + \tau). \tag{27}$$

We first obtain an estimate for $H_\circ^{\omega_1}(\nu,x)$. We put

$$\begin{cases} B = \frac{1}{4} \\ \alpha = N \\ \beta = 0 \\ \gamma = B(2 + \delta + \varepsilon) \\ \theta = 1 + \frac{1}{\alpha}(\frac{1}{2} - \delta + \varepsilon - \gamma). \end{cases}$$

Then we see that $1/2 + (1-\theta)\alpha - \delta - \gamma = -\varepsilon$ and $3/2 - (1-\theta)\beta + \delta - \gamma/B = -1/2 - \varepsilon$, so the $(b_{\theta,B,\gamma}^{\infty,1})$ condition holds if $\Im(\nu) \leq -\varepsilon$ and $\Im(x) \geq -1/2 - \varepsilon$. Since $(B1)$ implies $(\mathbf{A})_\circ^{\omega_1}$ and (25) does $\theta > 1$, Lemma 5.1(2) deduces the following,

Lemma 5.2. *If* $\Im(x) = -y = -1/2 - \varepsilon$ *and* $-E \leq \Im(\nu) \leq -\varepsilon$, *then there exist positive constants* C *and* σ *such that*

$$|H_\circ^{\omega_1}(\nu, x)| \leq C|\nu - x|^{-(1+\sigma)}.$$

We next obtain an estimate for $H_\circ^{\omega_2}$. We put $\alpha = 0$ and take $\beta > 0$ sufficiently large for which $1/2 + \alpha - \delta \geq -\varepsilon$ and $3/2 - \beta + \delta \leq -\varepsilon$, that is, the $(b_0^{\infty,1})$ condition holds if $\Im(\nu) \leq -\varepsilon$ and $\Im(x) \geq -\varepsilon$. Since $\varepsilon_i^{\omega_2}(\alpha, \beta, C) \leq C$, $(B2)$ implies $(\mathbf{A})_\circ^{\omega_2}$ and thus, Lemma 5.1(1) deduces that

$$|H_\circ^{\omega_2}(\nu, x)| \leq C \tag{28}$$

if $\Im(\nu) = -\varepsilon$ and $-E \leq \Im(x) \leq -\varepsilon$. We furthermore consider the case of

$$\begin{cases} B = L \\ \theta = \text{an arbitrary large positive number} \\ \beta = -\frac{3/2+\delta+\varepsilon}{\theta-1} \\ \alpha = -B\beta \\ \gamma = 0. \end{cases}$$

Then we see that $1/2 + (1-\theta)\alpha - \delta - \gamma = 1/2 - (3/2 + \delta + \varepsilon)L - \delta > 1/2 - 3(L+\tau)/2$ (see (26)) and $3/2 - (1-\theta)\beta + \delta - \gamma/B = -\varepsilon$, so the $(b_{\theta,B,\gamma}^{\infty,1})$ condition holds if $\Im(\nu) \leq 1/2 - 3(L+\tau)/2$ and $\Im(x) \geq -\varepsilon$. Since $\varepsilon_i^{\omega_2} \leq Ce^{(-\alpha - L\beta)\log p_i} = C$ (cf. (19) and see $(B2)$), $(B2)$ implies $(\mathbf{A})_\circ^{\omega_2}$ and thus, Lemma 5.1(2) deduces that

$$|H_\circ^{\omega_2}(\nu, x)| \leq C|\nu - x|^{-\theta} \tag{29}$$

if $\Im(x) = -\varepsilon$ and $-E \leq \Im(\nu) \leq 1/2 - 3(L+\tau)/2$ (see (27)). Since we can take a sufficiently large θ and apply the Phragmén-Lindelöf principle to interpolate between (28) and (29), we see that

Lemma 5.3. *If* $\Im(x) = -\varepsilon$ *and* $-E \leq \Im(\nu) \leq -\varepsilon$, *then there exists a positive constant* C *such that*

$$|H_\circ^{\omega_2}(\nu, x)| \leq C|\nu - x|^{-(2M+3)}.$$

5.4 We now recall $(B0)$. Then it is easy to see that, as meromorphic functions of ν,

$$\begin{aligned}\eta^\circ(\nu) &= \int_{\mathbf{R}-\sqrt{-1}y} \eta_G^\circ(x)(H_\circ^{\omega_1}(\nu, x) + H_\circ^{\omega_2}(\nu, x))dx \\ &= \int_{\mathbf{R}-\sqrt{-1}y} \eta_G^\circ(x)H_\circ^{\omega_1}(\nu, x)dx + \int_{\mathbf{R}-\sqrt{-1}y} \eta_G^\circ(x)H_\circ^{\omega_2}(\nu, x)dx = J_1(\nu) + J_2(\nu)\end{aligned}$$

and

$$P_G(\nu)J_2(\nu) = \int_{\mathbf{R}-\sqrt{-1}\varepsilon} \eta_G^\circ(x)P_G(x)H_\circ^{\omega_2}(\nu, x)dx$$

(see Proposition 3.3 and Proposition 4.3). Then, since $\eta^{\circ}_G(x) = O(1)$ for $x \in \mathbf{R} - \sqrt{-1}y$ ([**H**], Theorem 3.10), Lemma 5.2 implies that $J_1(\nu)$ is holomorphic on $-E \leq \Im(\nu) \leq -\varepsilon$. Moreover, since $\eta^{\circ}_G(x)P_G(x) = O(|x|^{1+2M})$ for $x \in \mathbf{R} - \sqrt{-1}\varepsilon$ (see (17) and [**H**], Remark 6.8), Lemma 5.3 implies that $P_G(\nu)J_2(\nu)$ is holomorphic on $-E \leq \Im(\nu) \leq -\varepsilon$. Therefore, letting ε (resp. E) sufficiently small (resp. large), we see that $P_G(\nu)\eta^{\circ}(\nu)$ is holomorphic on $\Im(\nu) < 0$. As said in the begining of this section, this means that $P_G(\nu)\eta(\nu)$ is holomorphic on the domain and then, on $0 < |\Im(\nu)|$ by the functional equation of η (see [**E**], p.13). Since we know that $\zeta(s)$ has no zeros on $[0,1]$, we can finally obtain the following theorem.

Theorem 5.4. *If $SL(2,\mathbf{R})$ has a cocompact discrete subgroup Γ with $Prim_{\Gamma}$ satisfying the condition* (**B**)*, then the Riemann Hypothesis holds.*

Remark 5.5. (1) We may take N in (B1) sufficiently large and thus, $\varepsilon_i(N, 0, C)$ may be sufficiently small compared with the distance of δ_i. (2) We see that $D_2 \neq \emptyset$. Actually, if $D_1 = \mathbf{N}$, it follows from the above argument that $\eta^{\circ}(\nu) = J_1(\nu)$ is holomorphic on $\Im(\nu) < 0$. This contradicts to the fact that $\eta(\nu)$ has a pole at $\nu = -\sqrt{-1}/2$.

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