Selberg zeta functions and Ruelle operators
for function fields.

by

Shin-ya Koyama

Shin-ya Koyama

Department of Mathematics
Faculty of Science and Technology
Keio University

Selberg zeta functions and Ruelle operators
for function fields

Shin-ya Koyama

§1. Introduction

Nowadays Selberg zeta functions are defined for semi-simple Lie groups G of real rank one and its discrete subgroups Γ, since Selberg defined it first in [S] in 1956. Two ways are known for studying Selberg zeta functions. One is by Selberg trace formulas and the other is by Ruelle operators. By the former method, Selberg zeta functions are finally expressed as the determinants of the Laplacians ([Sa],[E],[K],[Ko]). By the latter method, they are expressed as the determinants of $1-$(Ruelle operator) ([R],[F],[M]). Above all, the result of Mayer[M] is remarkable in number theoretic viewpoint. He treats the Selberg zeta function of $\Gamma = PSL(2,\mathbf{Z})$, which is the unique example of non-compact $\Gamma\backslash G$ for which the second method is applied.

In this paper we fulfill the second method in the case $\Gamma = PGL(2,\mathbf{F}[T])$, where $\mathbf{F}$ is a finite field of order q of odd characteristic. As is well be seen in number theory, Γ has similar properties to those of $PSL(2,\mathbf{Z})$. We can apply the second method by following Mayer[M]. For the present Γ, Akagawa[A] constructs Selberg trace formulas and proves that Selberg zeta functions are rational with respect to q^s. This paper will give another proof of Akagawa's result. It is much shorter than the original one, as is seen in the case of $PSL(2,\mathbf{Z})$ by Mayer[M]. In the next section, we will define continued fractions in function fields and deduce some properties. In the third section, we will classify conjugacy classes of Γ following Akagawa[A]. In the Last section, we will define Ruelle operators on function spaces over function fields, and establish the relation to Selberg zeta functions.

The author would like to express his profound gratitude to Professor N. Kurokawa for encouraging in the whole process of producing this paper. The author also would like to express his hearty thanks to Professor D. Fried for introducing the author to the beautiful theory of Ruelle operators.
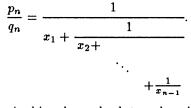
2   S. Koyama

§2. Continued fractions

Throughout this paper, we fix q a power of an odd prime and $\mathbf{F}$ the finite field of order q. We denote R, K, and F to be $\mathbf{F}[T]$, $\mathbf{F}(T)$, and $\mathbf{F}((T^{-1}))$, which are analogues of $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$. The field F is the completion of K with respect to the place T^{-1}. Elements in F are expressed as $x = \sum_{i=-\infty}^{k} a_i T^i$ $(a_i \in \mathbf{F}, k \in \mathbf{Z}, a_k \neq 0)$. We will write deg for the map from F to $\mathbf{Z}$ such that the element x corresponds to k. The map deg is a homomorphism from F^* onto $\mathbf{Z}$. When $\deg x \geq 0$, Gauss symbol [ ] is defined by $[x] := \sum_{i=0}^{k} a_i T^i \in R$. In the following we assume $\deg x < 0$. We define Gauss map T as $Tx := \frac{1}{x} - \left[\frac{1}{x}\right]$. Here $\frac{1}{x}$ is the inverse of x in F. It is easy to see that $\deg(Tx) < 0$. Letting x_i be the polynomial $\left[\frac{1}{T^i x}\right]$, we have an expansion

$$x = \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cdots}}}.$$

Let $p_n, q_n \in R$ be polynomials which express the finite part of the above as

$$\frac{p_n}{q_n} = \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{\ddots}{\quad + \frac{1}{x_{n-1}}}}}.$$

In what follows non-Archimedean absolute values in F are normalized as $|x| := q^{\deg(x)}$. The following lemma says that $\frac{p_n}{q_n}$ approximates x well.

Lemma.1.

$$\left| x - \frac{p_n}{q_n} \right| < \left| \frac{1}{q_n^2} \right| < \frac{1}{q^{2(n-1)}} \to 0 \ (n \to \infty)$$

As a corollary of the above, we get the following.

Proposition. *Let $x \in F$ be an algebraic element over K. The following (a) and (b) are equivalent.*
(a) x is quadratic over K.
(b) x has a cyclic expansion as continued fractions.

§3.   **Conjugacy classes**

The group $\Gamma = PGL(2, R)$ acts on the algebraic closure of K by the linear fractional transformation. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ be a stabilizer of an algebraic element z. Then $z = \frac{a-d \pm \sqrt{(a+d)^2 - 4\alpha}}{2c}$, where $\alpha = \det \gamma \in \mathbf{F}^*$. We call z *real* or *imaginary* when it belongs to F or not, respectively. We call γ *hyperbolic* when its fixed point z is real quadratic. We can see that γ being hyperbolic is equivalent to that $\sqrt{(\mathrm{tr}\gamma)^2 - 4\alpha}$ is real. For any $p \in R$, $\sqrt{p}$ is real if and only if the degree of p is even and the coefficient of the term of the highest degree belongs to $\mathbf{F}^2$. So we deduce that any element $\gamma \in \Gamma$ whose trace has a positive degree is hyperbolic. As trace depends only on the conjugacy class of γ, we get a correspondence between the set of hyperbolic conjugacy classes $[\gamma]$ and the set of real quadratic fields $K(z)$ over K. Let $\mathrm{Prim}(\Gamma)$ be the set of all the primitive hyperbolic conjugacy classes of Γ, where we call a class *primitive* when it cannot be expressed by a non-trivial power of any other conjugacy class. From the fact that γ^n ($n \in \mathbf{Z}$) has common fixed points, we establish a correspondence between $\mathrm{Prim}(\Gamma)$ and the set of quadratic fields;

$$\mathrm{Prim}(\Gamma) \in [\gamma] \longmapsto K(\sqrt{(\mathrm{tr}\gamma)^2 - 4\det(\gamma)}).$$

Putting λ to be the eigenvalue of γ which has the larger non-archimedean absolute value, we define the norm of γ by $N(\gamma) := |\lambda|^2$. It is equal to $q^{2\deg\lambda} = q^{2\deg(\mathrm{tr}\gamma)}$. Selberg zeta function of Ruelle type is defined by

$$\zeta_\Gamma(s) := \prod_{\gamma \in \mathrm{Prim}(\Gamma)} (1 - N(\gamma)^{-s})^{-1}.$$

Akagawa[A] constructs the Selberg trace formula and introduces the zeta function from the hyperbolic terms in the trace formula, which is

$$Z_\Gamma(s) := \prod_{\gamma \in \mathrm{Prim}(\Gamma)} (1 - N(\gamma)^{-s}) = \zeta_\Gamma(s)^{-1}.$$

Theorem.   ([A])*Selberg zeta function $Z_\Gamma(s)$ absolutely converges in $\Re s > 1$ and*

$$Z_\Gamma(s) = \frac{q^{2s} - q^2}{q^{2s} - q}.$$

4   S. Koyama
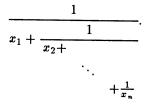
§4. Ruelle operators

We define Ruelle operators acting on the space $A(D)$ of analytic functions on $D = \{z \in F; \deg z < 0\}$ into D;

$$\mathcal{L}_s : f(z) \longmapsto \sum_{n \in R-\mathbf{F}} \left(\frac{1}{z+n}\right)^{2s} f\left(\frac{1}{z+n}\right),$$

where s is an integer. We can see that the above converges for sufficiently large s. By the theorem of Goss[G, 2.1.2], values at the large integers determine the function. By putting $\phi_n(z) := \left(\frac{1}{z+n}\right)^{2s}$, and $T_n(z) := \frac{1}{z+n}$, we decompose it into $\mathcal{L}_s f(z) = \sum_n \mathcal{L}_{s,n} f(z)$, where $\mathcal{L}_{s,n} f(z) = \phi_n(z) f(T_n(z))$. For a while we will fix s, and $\mathcal{L}_s$, $\mathcal{L}_{s,n}$ will be written as $\mathcal{L}$, $\mathcal{L}_n$. For given $x_i \in R$ $(i = 1,2,3,...)$, $[x_1, ..., x_n]$ will denote the cyclic continued fraction whose period is

$$\cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ddots + \frac{1}{x_n}}}}.$$

Lemma.2.

$$\mathrm{tr}(\mathcal{L}) = \sum_{n \in R-\mathbf{F}} \frac{[n]^{2s}}{1 + [n]^2}.$$

Proof. By the method of Kamowitz[Ka], we can deduce that the spectrum of $\mathcal{L}_n$ consists only of eigenvalues, which are $\lambda_l = [n]^{2s}(-[n])^l$ $(l = 0,1,2,...)$.   Q.E.D.

By the standard method of computing the trace of iterated Ruelle operators, the following is deduced.

Lemma.3.

$$\mathrm{tr}(\mathcal{L}^n) = \sum_{i_1,...,i_n \in R-\mathbf{F}} \frac{\prod_{k=1}^{n}[i_k, ..., i_n, i_1, ..., i_{k-1}]^{2s}}{1 - (-1)^n \prod_{k=1}^{n}[i_k, ..., i_n, i_1, ..., i_{k-1}]^2}.$$

Proof. By the same method as that of Lemma.2, we can compute that the eigenvalues of $\mathcal{L}_{i_1}\mathcal{L}_{i_2}...\mathcal{L}_{i_n}$ are given by

$$\lambda_l = \prod_{k=1}^{n}[i_k, ..., i_n, i_1, ..., i_{k-1}]^{2s} \prod_{k=1}^{n}(-[i_k, ..., i_n, i_1, ..., i_{k-1}]^2)^l.$$

Q.E.D.

We call x a *primitive* fixed element of T^n if x is not fixed by T^k for any positive integer $k < n$. The set of all the primitive fixed element of T^n will be denoted by $\mathrm{Prim}(\mathrm{Fix}T^n)$.

Lemma.4. *Let* $x = [i_1, ...i_n] \in \mathrm{Prim}(\mathrm{Fix}T^n)$ *be a quadratic element over* K, *which corresponds to* $\gamma \in \Gamma$. *Then*

$$N(\gamma) = q^{2\sum_{k=1}^{n} \deg i_k}.$$

Let X be an operator having only eigenvalues as its spectrum. We write $\det(1 - |X|)$ and $\mathrm{tr}|X|$ for $\prod_\lambda (1 - |\lambda|)$ and $\sum_\lambda |\lambda|$, respectively, when they absolutely converges.

Theorem.

$$\zeta_\Gamma(s) = \frac{\det(1 - |\mathcal{L}_{s+1}|)}{\det(1 - |\mathcal{L}_s|)}.$$

Proof. Using the equality $\det(1 - |X|) = \exp - \sum_{m=1}^{\infty} \frac{\mathrm{tr}|X^m|}{m}$, we can compute the right hand side by substituting Lemma.3. It is related to $\zeta_\Gamma(s)$ by Lemma.4.   Q.E.D.

Corollary.

$$\zeta_\Gamma(s) = \frac{q^{2s} - q}{q^{2s} - q^2}.$$

Proof. From Lemma.3, $\mathrm{tr}|\mathcal{L}^n| = \sum_{d=n}^{\infty} \epsilon_n(d)\frac{q^{-2sd}}{1-q^{-2s}}$, where $\epsilon_n(d)$ $:= \#\{(i_1, ..., i_n) \in (R - \mathrm{F})^n | \sum_{l=1}^{n} \deg(i_l) = d\}$. It is computed that $\epsilon_n(d) = \frac{q^d(q-1)^n(d-1)!}{(d-k)!(k-1)!}$ and some technical calculations lead to the result.
                                         Q.E.D.

References

[A]   S. Akagawa, On Selberg zeta functions for modular groups over function fields, master thesis (1978), Tokyo University.

[E]   I. Efrat, Determinant of Laplacians on surfaces of finite volume, Commun. Math. Phys., **119** (1988), 443-451.

[F]   D. Fried, Zeta functions of Ruelle and Selberg I, Ann. Sci. Ec. Norm. Sup., **19** (1986), 491-517.

[G]   D. Goss, Von Staudt for $\mathrm{F}_q[T]$, Duke Math. J., **45** (1978), 885-910.

6   S. Koyama

[Ka] H. Kamowitz, The spectra of endomorphisms of the disk algebra, Pacific
      J. Math., **46** (1973), 433-440.

[Ko] S. Koyama, Determinant expression of Selberg zeta functions I, Trans.
      Amer. Math. Soc., **324** (1991), 149-168.

[K]   N. Kurokawa, Parabolic components of zeta functions, Proc. Japan
      Acad., **64A** (1988), 21-24.

[M]   D. H. Mayer, Selberg's zeta function for $PSL(2, \mathbf{Z})$ via the thermo-
      dynamic formalism for the continued fraction map, preprint (1990),
      Max-Planck-Institute.

[R]   D. Ruelle, Zeta functions for expanding maps and Anosov flows, Inv.
      Math., **34** (1976), 229–247.

[Sa] P. Sarnak, Determinants of Laplacians, Comm. Math. Phys., **110** (1987),
      113-120.

[S]   A. Selberg, Harmonic analysis and discontinuous groups on weakly sym-
      metric Riemannian surfaces with applications to Dirichlet series, J.
      Ind. Math. Soc., **20** (1956), 47–87.

Department of Mathematics
Keio University
Hiyoshi, Yokohama, 223 Japan