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Padé Approximations to Certain Power Series.

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Let $f(z) = \sum_{k=0}^{\infty} f_k z^k$ be a formal power series with complex coefficients. Then, for each $n=1,2,\cdots$, there are polynomials $P_n(z)$, $Q_n(z)$ such that $\deg P_n(z) \le n$, $\deg Q_n(z) \le n$, $Q_n(z) \not\equiv 0$, and

$$Q_n(z)f(z) - P_n(z) = cz^{2n+1} + \cdots .$$

The rational function $P_n(z)/Q_n(z)$, which is determined uniquely for a given $f(z)$ and n, is called the *nth (diagonal) Padé approximation* for $f(z)$.

In this paper we study the Padé approximations for the entire function defined by the power series

$$f(z) = \sum_{k=0}^{\infty} f_k z^k , \quad f_k = q^{\frac{1}{2}k(k-1)} , \quad 0 < |q| < 1 . \qquad (1)$$

In §1 we prove some interesting identities satisfied by the denominators of the Padé approximants. In §2 we estimate, by using the Padé approximants, the irrationality measure of the number $\sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)} \alpha^k$, where α and $q = r/s$ are non-zero rationals with $|r|^3 < |s|$. Although our result is slightly weaker than that obtained by Bundschuh [2] and the first named author [6], the method might be of some interest.

§1. Padé approximations.

We denote by $P_n(z)/Q_n(z)$ the nth approximant for the function $f(z)$ defined by (1) and write

$$Q_n(z) = u_{n+1}^{(n)} + u_n^{(n)} z + \cdots + u_1^{(n)} z^n ,$$

where we may choose for simplicity $u_{n+1}^{(n)} = 1$. Then we have the following

<u>Lemma 1</u>. Let $L_i^{(n)}$, $R_j^{(n)}$ $(i,j = 1, 2, \cdots, n)$ be the $(n+1)\times(n+1)$ matrices written by

$$L_i^{(n)} = \left(\begin{array}{ccc|cccc} 1 & & & & & & \\ & \ddots & & & & 0 & \\ & & 1 & & & & \\ \hline & & & q^{-1} & & & \\ & & & & q^{-2} & & \\ & 0 & & & & \ddots & \\ & & & & & & q^{-i} \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\ \ddots\\ 1\end{array}}\right\} n-i+1 \\ \left.\vphantom{\begin{array}{c}1\\1\\ \ddots\\ 1\end{array}}\right\} i \end{array}$$

$$R_j^{(n)} = \left(\begin{array}{ccc|ccccc} 1 & & & & & & & \\ & \ddots & & & & 0 & & \\ & & 1 & & & & & \\ \hline & & & 1 & -1 & & & \\ & & & & 1 & -1 & & \\ & 0 & & & & \ddots & \ddots & \\ & & & & & & \ddots & -1 \\ & & & & & & & 1 \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\ \ddots\\ 1\end{array}}\right\} n-j \\ \left.\vphantom{\begin{array}{c}1\\1\\ \ddots\\ 1\\1\end{array}}\right\} j+1 \end{array}$$

and let $\underline{u}_n$, $\underline{a}_n$ be the column vectors of dimension $n+1$ written by

$$\underline{u}_n = {}^t(u_1^{(n)}, q u_2^{(n)}, q^3 u_3^{(n)}, \cdots, q^{n(n-1)/2} u_n^{(n)}, q^{n(n+1)/2} u_{n+1}^{(n)}) ,$$

$$\underline{a}_n = {}^t(\underbrace{0, 0, \cdots, 0}_{n-1}, -q^{n^2}, q^{n^2}) .$$

Then we have

$$\underline{u}_n = R_n^{(n)} L_{n-1}^{(n)} \cdots R_3^{(n)} L_2^{(n)} R_2^{(n)} L_1^{(n)} \underline{a}_n \ . \tag{2}$$

<u>Proof</u>. We have by definition

$$\begin{pmatrix} f_1 & f_2 & \cdots & f_n & f_{n+1} \\ f_2 & f_3 & \cdots & f_{n+1} & f_{n+2} \\ \vdots & \vdots & & \vdots & \vdots \\ f_n & f_{n+1} & \cdots & f_{2n-1} & f_{2n} \end{pmatrix} \begin{pmatrix} u_1^{(n)} \\ u_2^{(n)} \\ \vdots \\ u_{n+1}^{(n)} \end{pmatrix} = \underline{0} \ ,$$

where $\underline{0}$ is the zero vector of dimension n .

By the definition of f_k , we have $f_{k+1} = q^k f_k$ for $k \geq 1$ and so $f_i = q^{\frac{1}{2}(i+j-1)(i-j-2)} f_j$ for $i > j$. Hence we get

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 1 & q & \cdots & q^{j-1} & \cdots & q^n \\ \vdots & \vdots & & \vdots & & \vdots \\ 1 & q^{i-1} & \cdots & q^{(i-1)(j-1)} & \cdots & q^{(i-1)n} \\ \vdots & \vdots & & \vdots & & \vdots \\ 1 & q^{n-1} & \cdots & q^{(n-1)(j-1)} & \cdots & q^{(n-1)n} \end{pmatrix} \begin{pmatrix} f_1 u_1^{(n)} \\ f_2 u_2^{(n)} \\ \vdots \\ f_i u_i^{(n)} \\ \vdots \\ f_{n+1} u_{n+1}^{(n)} \end{pmatrix} = \underline{0} \ .$$

Remark. We have

$$H_n := \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_2 & f_3 & \cdots & f_{n+1} \\ \vdots & \vdots & & \vdots \\ f_n & f_{n+1} & \cdots & f_{2n-1} \end{vmatrix} = q^{\frac{1}{2}n^2(n-1)} \prod_{j=1}^{n-1} (q^j - 1)^{n-j} \neq 0 \ .$$

Note that $f_{i+j-1}/(f_i f_j) = q^{(i-1)(j-1)}$. Hence we have

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 0 & -1+q & \cdots & -1+q^{j-1} & \cdots & -1+q^{n} \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & -1+q^{i-1} & \cdots & -1+q^{(i-1)(j-1)} & \cdots & -1+q^{(i-1)n} \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & -1+q^{n-1} & \cdots & -1+q^{(n-1)(j-1)} & \cdots & -1+q^{(n-1)n} \end{pmatrix} \underline{u}_n = \underline{0} ,$$

which can be written as

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ 0 & 1 & 1+q & \cdots & 1+q+\cdots+q^{j-2} & \cdots & 1+q+\cdots+q^{n-1} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 1 & 1+q^{i-1} & \cdots & 1+q^{i-1}+\cdots+q^{(i-1)(j-2)} & \cdots & 1+q^{i-1}+\cdots+q^{(i-1)(n-1)} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 1 & 1+q^{n-1} & \cdots & 1+q^{n-1}+\cdots+q^{(n-1)(j-2)} & \cdots & 1+q^{n-1}+\cdots+q^{(n-1)(n-1)} \end{pmatrix} \underline{u}_n = \underline{0} .$$

Thus we have

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots & & \\ 0 & 1 & q^{i-2} & \cdots & q^{(i-2)(j-2)} & \cdots & q^{(i-2)(n-1)} \\ \vdots & \vdots & \vdots & & \vdots & & \\ 0 & 1 & q^{n-2} & \cdots & q^{(n-2)(j-2)} & \cdots & q^{(n-2)(n-1)} \end{pmatrix} \times$$

$$\times \begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & q & & & \\ & & & q^2 & & \\ & 0 & & & \ddots & \\ & & & & & q^{n-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & & 1 \\ & 1 & 1 & \cdots & 1 \\ & & \ddots & & \vdots \\ & 0 & & \ddots & \vdots \\ & & & & 1 \end{pmatrix} \underline{u}_n = \underline{0} .$$

Repeating this process, we get

$$\begin{pmatrix} 1 & & & & 0 \\ & 1 & & 0 & \vdots \\ & & \ddots & & \\ & & & & 0 \\ 0 & & & 1 & 1 \end{pmatrix} (L_1^{(n)})^{-1}(R_2^{(n)})^{-1} \cdots (L_{n-1}^{(n)})^{-1}(R_n^{(n)})^{-1} \underline{u}_n = \underline{0} .$$

Hence we obtain

$$(L_1^{(n)})^{-1}(R_2^{(n)})^{-1} \cdots (L_{n-1}^{(n)})^{-1}(R_n^{(n)})^{-1} \underline{u}_n = {}^t(0, \cdots, 0, -k, k)$$

with some k . Comparing the (n+1)th component in the both sides, we find $k = q^{n^2}$ (noticing that $u_{n+1}^{(n)} = 1$).

<u>Theorem 1</u>. We have for any $n \geq 1$

$$Q_n(z) = (-q^n z)Q_{n-1}(qz) + Q_{n-1}(q^2 z) . \tag{3}$$

<u>Proof</u>. It follows from (2) that

$$\underline{u}_{n-1} = R_{n-1}^{(n-1)}L_{n-2}^{(n-1)} \cdots R_2^{(n-1)}L_1^{(n-1)} \underline{a}_{n-1} .$$

Multiplying both sides by q^{2n-1}, we get

$$\begin{pmatrix} f_1 & q^{2n-1}u_1^{(n-1)} \\ f_2 & q^{2n-1}u_2^{(n-1)} \\ \vdots & \\ f_n & q^{2n-1}u_n^{(n-1)} \end{pmatrix} = R_{n-1}^{(n-1)}L_{n-2}^{(n-1)} \cdots R_2^{(n-1)}L_1^{(n-1)} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -q^{n^2} \\ q^{n^2} \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} n-2 \\ \\ \\ \end{matrix}$$

so that

$$\begin{pmatrix} & 0 \\ f_1 & q^{2n-1}u_1^{(n-1)} \\ f_2 & q^{2n-1}u_2^{(n-1)} \\ \vdots & \vdots \\ f_n & q^{2n-1}u_n^{(n-1)} \end{pmatrix} = R_{n-1}^{(n)}L_{n-2}^{(n)} \cdots R_2^{(n)}L_1^{(n)} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -q^{n^2} \\ q^{n^2} \end{pmatrix} \Bigg\} {n-1} =: \underline{v}_n \text{ , say.} \qquad (4)$$

From this we find

$$\begin{pmatrix} -f_1 & q^{2n-1}u_1^{(n-1)} \\ -f_2 & q^{2n-2}u_2^{(n-1)} \\ & \vdots \\ -f_n & q^{n}u_n^{(n-1)} \\ & 0 \end{pmatrix} = T_n \underline{v}_n , \quad T_n := \begin{pmatrix} 0 & -1 & & & \\ & 0 & -q^{-1} & & 0 \\ & & 0 & \ddots & \\ & 0 & & \ddots & -q^{-(n-1)} \\ & & & & 0 \end{pmatrix} . \qquad (5)$$

On the other hand, we have from (4)

$$\begin{pmatrix} & 0 \\ f_2 & q^{2n-2}u_1^{(n-1)} \\ f_3 & q^{2n-4}u_2^{(n-1)} \\ & \vdots \\ f_{n+1} & q^{0}u_n^{(n-1)} \end{pmatrix} = L_{n-1}^{(n)} \underline{v}_n , \qquad (6)$$

noticing that $f_i q^{2n-1} q^{-(i-1)} = f_{i+1} q^{2n-2i}$. It follows from (5), (6) and (4) that

$$\begin{pmatrix} -f_1\, q^{2n-1}\, u_1^{(n-1)} & \\ -f_2\, q^{2n-2}\, u_2^{(n-1)} & + f_2\, q^{2n-2}\, u_1^{(n-1)} \\ -f_3\, q^{2n-3}\, u_3^{(n-1)} & + f_3\, q^{2n-4}\, u_2^{(n-1)} \\ \vdots & \vdots \\ -f_n\, q^{n}\, u_n^{(n-1)} & + f_n\, q^{2}\, u_{n-1}^{(n-1)} \\ & f_{n+1}\, q^{0}\, u_n^{(n-1)} \end{pmatrix} = (T_n + L_{n-1}^{(n)})\, \underline{v}_n$$

$$= R_n^{(n)}\, L_{n-1}^{(n)}\, \underline{v}_n = R_n^{(n)}\, L_{n-1}^{(n)} \cdots R_2^{(n)}\, L_1^{(n)}\, \underline{a}_n$$

$$= \underline{u}_n = {}^t(f_1 u_1^{(n)},\ f_2 u_2^{(n)},\ \cdots,\ f_{n+1} u_{n+1}^{(n)})$$

(using (2)). Hence we get

$$\begin{pmatrix} u_1^{(n)} \\ u_2^{(n)} \\ \vdots \\ u_n^{(n)} \\ u_{n+1}^{(n)} \end{pmatrix} = -q^n \begin{pmatrix} q^{n-1}\, u_1^{(n-1)} \\ q^{n-2}\, u_2^{(n-1)} \\ \vdots \\ q\, u_{n-1}^{(n-1)} \\ u_n^{(n-1)} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ q^{2(n-1)}\, u_1^{(n-1)} \\ q^{2(n-2)}\, u_2^{(n-1)} \\ \vdots \\ q^2\, u_{n-1}^{(n-1)} \\ u_n^{(n-1)} \end{pmatrix} .$$

The scalar product of the vector $(z^n, z^{n-1}, \cdots z, 1)$ and the vectors of the both sides of this equality yields (3).

Corollary 1.

$$Q_n(z) = 1 - q^n z \sum_{k=0}^{n-1} q^{n-k-1} Q_k(q^{2n-2k-1} z) .$$

Corollary 2.

$$Q_n(z) = (-1)^n q^{n^2} z^n + \sum_{k=0}^{n-1} (-1)^k (q^n z)^k \, Q_{n-k-1}(q^{k+2} z) .$$

Corollary 3. $Q_n(z)$ can be written as a polynomial in q and $q^n z$ with integral coefficients.

We next consider the continued fraction

$$\sum_{k=0}^{\infty} q^{k^2} z^k = 1 + \cfrac{qz}{1 + \cfrac{-q^3 z}{1 + \cfrac{q^3(1-q^2)z}{1 + \cdots + \cfrac{}{1 + \cfrac{-q^{4n-1}z}{1 + \cfrac{q^{2n+1}(1-q^{2n})z}{1 + \cdots}}}}}} . \tag{7}$$

(c.f. [5], §64, Formel (22), p.353). In view of the identity

$$\sum_{k=0}^{\infty} q^{k^2} z^k = \sum_{k=0}^{\infty} (q^2)^{\frac{1}{2}k(k-1)} (qz)^k , \tag{8}$$

the continued fraction (7) is transformed into the following one;

$$\sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)} z^k = 1 + \cfrac{z}{1 + \cfrac{-qz}{1 + \cfrac{q(1-q)z}{1 + \cdots + \cfrac{}{1 + \cfrac{-q^{2n-1}z}{1 + \cfrac{q^n(1-q^n)z}{1 + \cdots}}}}}} . \tag{9}$$

Let $\beta_0 + \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \cdots$ and $b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots$ denote continued fractions with nth approximants φ_n and f_n, resp. If $\varphi_{2n} = f_n$, then $b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots$ is called the even part of $\beta_0 + \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \cdots$. The even part, which exists if and only if $\beta_{2n} \neq 0$ $(n \geq 1)$, is given by the following formulas;

$$b_0 = \beta_0 \ , \ a_1 = \alpha_1\beta_2 \ , \ b_1 = \alpha_2 + \beta_1\beta_2 \ , \ a_2 = -\alpha_2\alpha_3\beta_4 \ ,$$

$$a_n = -\alpha_{2n-2}\,\alpha_{2n-1}\,\beta_{2n-4}\,\beta_{2n} \qquad (n \geq 3) \ ,$$

$$b_n = \alpha_{2n-1}\,\beta_{2n} + \beta_{2n-2}\,(\alpha_{2n} + \beta_{2n-1}\,\beta_{2n}) \qquad (n \geq 2)$$

(c.f. [4], §2.4.2). Taking now the even part of the continued fraction (9), we find

$$\sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)} z^k = 1 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots}} \ , \tag{10}$$

where $a_1 = z$, $b_1 = 1-qz$, and

$$a_n = q^{3n-4}(1-q^{n-1})z^2 \qquad (n \geq 2) \ ,$$

$$b_n = 1 + (q^{n-1} - q^{2n-2} - q^{2n-1})z \qquad (n \geq 2) \ .$$

Denoting by p_n/q_n the nth approximant of this continued fraction, we have

$$q_n = b_nq_{n-1} + a_nq_{n-2} \ , \quad p_n = b_np_{n-1} + a_np_{n-2} \qquad (n \geq 1)$$

with $p_0 = 1$, $q_0 = 1$, $p_{-1} = 1$, $q_{-1} = 0$.

It is known in the theory of continued fractions (c.f. [5], §77, Satz 9, p.449) that the nth Padé approximant of a power series which is expanded in a continued fraction of the form $1 + \frac{c_1 z}{1} + \frac{c_2 z}{1} + \frac{c_3 z}{1} + \cdots$, where $c_1, c_2, c_3, \cdots$ are constants, coincides with the nth approximant of the even part of $1 + \frac{c_1 z}{1} + \frac{c_2 z}{1} + \frac{c_3 z}{1} + \cdots$. Applying this to (9) and (10), we find $p_n/q_n = P_n(z)/Q_n(z)$, where $P_n(z)/Q_n(z)$ is the nth Padé approximant for $f(z)$ defined by (1). Hence we have the following recurrence relations:

<u>Lemma 2</u>. We have for any $n \geq 2$

$$Q_n(z) = (1 + (q^{n-1} - q^{2n-2} - q^{2n-1})z)Q_{n-1}(z) + q^{3n-4}(1 - q^{n-1})z^2 Q_{n-2}(z) ,$$

and the same relation with $P_n(z)$ in place of $Q_n(z)$.

Combining Lemma 2 with Theorem 1, we find the following interesting formula:

<u>Theorem 2</u>. We have for any $n \geq 1$

$$ {}^t\begin{pmatrix} 1 \\ -q^{n+2}(1+q)z \\ q^{2n+4}z^2 - q^{n+1}(1 - q^{n+1} - q^{n+2})z - 1 \\ q^{2n+2}(1 - q^{n+1} - q^{n+2})z^2 + q^{n+1}z \\ -q^{3n+2}(1 - q^{n+1})z^2 \end{pmatrix} \begin{pmatrix} Q_n(q^4 z) \\ Q_n(q^3 z) \\ Q_n(q^2 z) \\ Q_n(qz) \\ Q_n(z) \end{pmatrix} = 0 .$$

§2 The irrationality measure.

In 1915 Bernstein and Szász [1] proved that the number $\sum_{k=0}^{\infty} q^{k^2}\alpha^n$ is irrational for any non-zero rationals α and q such that

$$\gamma(q) < 1/3 , \tag{10}$$

where, for any non-zero rational $q = r/s$ with coprime integers r and s , $\gamma(q)$ is defined by

$$\gamma(q) = \frac{\log|r|}{\log|s|} \ (\geq 0)$$

Note that $|q| < 1$ if and only if $\gamma(q) < 1$. Their method was applied an irrationality criterion for continued fractions to the continued fraction (7). Tschakaloff [9] showed that the number $\sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)}\alpha^k$ is irrational for all non-zero rational α and q such that

$$\gamma(q) < (3 - \sqrt{5})/2 \ (= 0.381 \cdots) . \tag{11}$$

Because of the relations (8) and $\gamma(q) = \gamma(q^2)$, this improves the result of Bernstein and Szász mentioned above. Under the same assumption (11), Bundschuh [2] and Shiokawa [6] proved the following theorem: For any $\varepsilon > 0$, there is a constant $C_0 = C_0(\alpha, q, \varepsilon) > 0$ such that

$$\left| \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)}\alpha^k - \frac{P}{Q} \right| > Q^{-\kappa-\varepsilon} \tag{12}$$

for all integers P , Q $(> C_0)$, where

$$\kappa = \kappa_0 = 1 + \frac{1 + \sqrt{5}}{2 - (3+\sqrt{5})\gamma} \ \left(\geq \frac{3 + \sqrt{5}}{2} = 2.6180 \cdots\right)$$

with $\gamma = \gamma(q)$. The infimum of such constants $\kappa(\geq 2)$ in (12) is called the irrationality measure of the number $\sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)}\alpha^n$. The proof used in [6] is quite different from that of [2]. Furthermore, the linear independence of these numbers were studied by Tschakaloff [10] and Bundschuh and Shiokawa [3].

We feel it is worthwhile using the Padé approximations, as discussed in §1, to estimate the irrationality measure of the number $\sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)} \alpha^k$. The constant $\kappa = \kappa_1$ thus obtained in Theorem 3 below, is slightly greater than κ_0 mentioned above; however, the method of proof is of some interest.

Theorem 3. Let α and q be non-zero rationals with $\gamma(q) < 1/3$. Then, for any $\varepsilon > 0$, there is a constant $C_1 = C_1(\alpha, q, \varepsilon) > 0$ such that the inequality (12) holds for all integers P, Q ($> C_1$), where

$$\kappa = \kappa_1 = 1 + \frac{2}{1 - 3\gamma} \ (\geq 3) , \quad \gamma = \gamma(q) .$$

Proof of Theorem 3. We transform the continued fraction (10) into the regular one. Then we find

$$\sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)} z^k = 1 + \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cdots}} ,$$

where $A_1 = b_1/a_1 = (1 - qz)/z$, and

$$A_{2n} = \frac{a_1 a_3 \cdots a_{2n-1}}{a_2 a_4 \cdots a_{2n}} b_{2n}$$

$$= \frac{(1 + (q^{2n-1} - q^{4n-2} - q^{4n-1})z)}{zq^{3n-1}} \cdot \frac{\prod_{k=1}^{n-1} (1 - q^{2k})}{\prod_{k=1}^{n} (1 - q^{2k-1})} ,$$

$$A_{2n+1} = \frac{a_2 a_4 \cdots a_{2n}}{a_1 a_3 \cdots a_{2n+1}} b_{2n+1}$$

$$= \frac{(1 + (q^{2n} - q^{4n} - q^{4n+1})z)}{zq^{3n}} \cdot \frac{\prod_{k=1}^{n} (1 - q^{2k-1})}{\prod_{k=1}^{n} (1 - q^{2k})} \quad (n \geq 1) .$$

Hence we have

$$\log |A_n| = -\frac{3}{2} n \log |q| + O(1) . \tag{13}$$

We need now the following

Lemma 3 ([7], c.f. [8]). Let

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \cdots$$

be a continued fraction with real partial denominators which represents an irrational number, and let P_n/Q_n denotes its nth approximant. Assume that

$$\sum_{n=1}^{\infty} |A_n A_{n+1}|^{-1} < \infty .$$

Then the ratios $P_n/(A_2 A_3 \cdots A_n)$ and $Q_n/(A_1 A_2 \cdots A_n)$ converge to finite non-zero limits as $n \to \infty$. Furthermore

$$\lim_{n\to\infty} A_{n+1}\theta_n = 1 ,$$

where θ_n is defined for $n \geq 0$ by the convergent continued fraction

$$\theta_n = \frac{1}{A_{n+1}} + \frac{1}{A_{n+2}} + \frac{1}{A_{n+3}} + \cdots .$$

Since, by (13)

$$\log |A_n A_{n+1}|^{-1} = 3n \log |q| + O(1)$$

the series $\sum_{n=1}^{\infty} |A_n A_{n+1}|^{-1}$ is convergent. Hence we may apply Lemma 3 with

(13) and obtain

$$\log |Q_n| = \log |A_1 A_2 \cdots A_n| + O(1)$$

$$= (1 + O(\frac{1}{n}))n^2 \frac{3}{4}(1 - \gamma) \log s \qquad (14)$$

and

$$\lim_{n\to\infty} \frac{\theta_n}{A_n} = \lim_{n\to\infty} \frac{\theta_n A_{n+1}}{A_n A_{n+1}} = 0 . \qquad (15)$$

Put $z = \alpha = a/b$ and $q = r/s$, where a, b, r, s are integers with $s > 0$, $(a, b) = (r, s) = 1$, so that $\log |r| = \gamma \log s$ and $\log |q| = (\gamma-1)\log s$. We need to estimate the common denominators of the rationals P_n and Q_n . Noticing that, by definition,

$$A_1 A_2 \cdots A_{2n} = \frac{b_1 b_2 \cdots b_{2n}}{a_2 a_4 \cdots a_{2n}}$$

$$= \frac{\prod_{k=1}^{2n} (1 + (q^{k-1} - q^{2k-2} - q^{2k-1})z)}{z^{2n} q^{3n^2-n} \prod_{k=1}^{n} (1 - q^{2k-1})} ,$$

$$A_1 A_2 \cdots A_{2n+1} = \frac{b_1 b_2 \cdots b_{2n+1}}{a_1 a_3 \cdots a_{2n+1}}$$

$$= \frac{\prod_{k=1}^{2n+1} (1 + (q^{k-1} - q^{2k-2} - q^{2k-1})z)}{z^{2n+1} q^{3n^2+2n} \prod_{k=1}^{n} (1 - q^{2k})} ,$$

and

$$Q_n = A_n Q_{n-1} + Q_{n-2} , \quad P_n = A_n P_{n-1} + P_{n-2} ,$$

we put

$$D_{2n} = |a^{2n} r^{3n^2-n} s^n \prod_{k=1}^{n} (s^{2k-1} - r^{2k-2})| ,$$

$$D_{2n+1} = |a^{2n+1} r^{3n^2+2n} s^{n+1} \prod_{k=1}^{n} (s^{2k} - r^{2k})| .$$

Then $D_n P_n$ and $D_n Q_n$ are integers with

$$\log D_n = (1 + O(\frac{1}{n}))n^2 \frac{1 + 3\gamma}{4} \log s . \tag{16}$$

It follows from (13), (14) and (16) that

$$\log | \frac{A_{n+1}Q_n}{D_n} | = (1 + O(\frac{1}{n}))n^2 \frac{1 - 3\gamma}{2} \log s . \tag{17}$$

Hence, since $\gamma < 1/3$ by assumption, the sequence $|A_{n+1}Q_n/D_n|$ tends to infinity as $n \to \infty$.

Let P , Q be given integers. We may assume Q is sufficiently large. Then there is an integer $n = n(Q)$ such that

$$|A_n Q_{n-1}/D_{n-1}| \leqq 4Q < |A_{n+1}Q_n/D_n| . \tag{18}$$

Since $P_n Q_{n-1} - P_{n-1}Q_n \neq 0$, at least one of $P_nQ - Q_nP$, $P_{n-1}Q - Q_{n-1}P$ is different from zero. We assume first that $P_nQ - Q_nP \neq 0$. Putting

$\theta = \sum_{n=0}^{\infty} (r/s)^{n(n-1)/2}(a/b)^n$ for brevity, we have

$$D_n Q_n(\theta - \frac{P}{Q}) = \frac{D_n(P_nQ - Q_nP)}{Q} + D_n(Q_n\theta - P_n) .$$

Here $|D_n(P_nQ - Q_nP)| \geqq 1$, since $D_n(P_nQ - Q_nP)$ is a non-zero integer, and

$$|D_n(Q_n\theta - P_n)| = \frac{D_n}{|A_{n+1}Q_n||1 + \frac{\theta_{n+1}}{A_{n+1}} + \frac{Q_{n-1}}{A_{n+1}Q_n}|}$$

$$\leqq \frac{2}{|A_{n+1}Q_n/D_n|} \leqq \frac{1}{2Q} ,$$

noticing (15) and (18). Thus we get

$$|\theta - \frac{P}{Q}| \geq \frac{1}{2} Q^{-1-(\log|D_n Q_n|)/\log Q} . \tag{19}$$

In the case of $P_{n-1}Q - Q_{n-1}P \neq 0$, we have the same inequality as (19) with 1/2 and n^{θ} on the right-hand side replaced by a constant smaller than 1/2 and n-1 , respectively, since by (17)

$$\left| \frac{A_{n+1}Q_n}{D_n} \right| \cdot \left| \frac{D_{n-1}}{A_n Q_{n-1}} \right| = 1 + o(1) .$$

It follows from (18), (14), (16) and (17) that

$$\log |D_n Q_n| = (1 + O(\frac{1}{n}))n^2 \log s = \log |D_{n-1}Q_{n-1}| ,$$

and

$$\log Q = (1 + O(\frac{1}{n}))n^2 \frac{1 - 3\gamma}{2} \log s ,$$

Therefore we get

$$(\log |D_n Q_n|)/\log Q = \frac{2}{1 - 3\gamma} + o(1) ;$$

which together with (19) yields the theorem.

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