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A Class of Normal Numbers.

by

Yoshinobu Nakai and Iekata Shiokawa

Yoshinobu Nakai

Department of Mathematics
Faculty of Education
Yamanashi University

Iekata Shiokawa

Department of Mathematics
Faculty of Science and Technology
Keio University

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku, Yokohama, 223 Japan

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To the memory of Isamu Kobayashi

§1. Introduction. Let r be an integer greater than one and let

$$\theta = 0.a_1a_2 \cdots = a_1r^{-1} + a_2r^{-2} + \cdots$$

be the r-adic expansion of a real number θ, $0<\theta<1$, where each a_i is one of $0, 1, \cdots, r-1$. Then θ is said to be normal to the base r, if, for any ℓ non-negative integers $b_1, b_2, \cdots, b_\ell$ less than r,

$$\frac{1}{n} N_r(\theta;b_1\cdots b_\ell;n) = \frac{1}{r^\ell} + o(1)$$

as $n\to\infty$, where $N_r(\theta;b_1\cdots b_\ell;n)$ is the number of indices $i \leq n-\ell+1$ in the expansion of θ such that $a_i=b_1$, $a_{i+1}=b_2$, $\cdots$, $a_{i+\ell-1}=b_\ell$. Let $n_1, n_2, \cdots$ be an infinite sequence of positive integers and let

$$n_i = a_{i1}\, a_{i2} \cdots a_{ik_i} = a_{i1}r^{k_i-1} + a_{i2}r^{k_i-2} + \cdots + a_{ik_i},$$

$a_{i1}\neq 0$, be the r-adic expansion of n_i. Define a number θ_r by the series

$$0.a_{11}\, a_{12} \cdots a_{1k_1}\, a_{21}\, a_{22} \cdots a_{2k_2} \cdots = a_{11}r^{-1} + a_{12}r^{-2} + \cdots,$$

which will be written simply by $\theta_r = 0.n_1n_2\cdots$.

Davenport and Erdös [1] proved that the number $\theta_r=0.f(1)f(2)\cdots$ is normal to the base r, where $f(x)$ is a polynomial in x, all of whose values for $x=1,2,\cdots$ are positive integers. In this paper we prove the normality of the number $\theta_r=0.[g(1)][g(2)]\cdots$, where $[t]$ is the integral

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Davenport and Erdös [1] proved that the number $\theta_r=0.f(1)f(2)\cdots$ is normal to the base r, where $f(x)$ is a polynomial in x, all of whose values for $x=1,2,\cdots$ are positive integers. In this paper we prove the normality of the number $\theta_r=0.[g(1)][g(2)]\cdots$, where $[t]$ is the integral

$$\frac{1}{n} N_r(\theta_r; b_1 \cdots b_\ell; n) = \frac{1}{r^\ell} + O(\frac{1}{\log n}) .$$

Corollary 2. Let $g(x)$ be as above. Then

$$\sum_{n \leq x} s_r([g(n)]) = \frac{r-1}{2} x \log_r g(x) + O(x) .$$

Example. Suppose $\beta>0$ is not an integer. Then $\theta_r = 0.1[2^\beta][3^\beta]\cdots$ is normal to the base r. More precisely,

$$\frac{1}{n} N_r(\theta_r; b_1 \cdots b_\ell; n) = \frac{1}{r^\ell} + O(\frac{1}{\log n}) ,$$

and hence

$$\sum_{n \leq x} s_r([n^\beta]) = \frac{r-1}{2} \beta x \log_r x + O(x) .$$

The proof of the theorem can be reduced to estimating the exponential sum of the form

$$\sum_{n=P+1}^{P+Q} e(\frac{\nu}{r^m} g(n)), \quad e(x) = e^{2\pi ix} ,$$

(see §3). Crucial steps in this estimation are carried out by using Vinogradov's method as well as van der Corput lemmas. Our proof may not be applicable for the case of polynomials except for some special ones. Using Weyl's inequality, Davenport and Erdős [1] obtained the result for polynomials; however, they did not derive an explicit remainder term. We conjecture that the same inequality as in our theorem holds also for any polynomial $f(x)$ with real coefficients.

§2. **Lemma.** In this section we prepare some notations and Lemmas.

Lemma 1. ([3] Lemmas 4.2 and 4.8.) Let $f(x)$ be a real differentiable function such that $f'(x)$ is monotonic, and let

$0<\varepsilon<f'(x)<1-\varepsilon$ throughout the interval (a,b), or the same for $-f'(x)$. Then

$$\sum_{a<n\le b} e(f(n)) = O(\frac{1}{\varepsilon}) .$$

Lemma 2. ([3] Lemmas 4.4 and 4.7.) Let $f(x)$ be a real function, twice differentiable, and let $f''(x)\ge\gamma>0$ throughout the interval (a,b), or the same for $-f''(x)$. Then

$$\sum_{a<n\le b} e(f(n)) = O(\frac{1+|f'(a)-f'(b)|}{\sqrt{\gamma}}) + O(\log(2+|f'(a)-f'(b)|)) .$$

Let k and q be positive integers with $k\ge2$. Put

$$S(q) = \sum_{m=1}^{q} e(t_1 m + t_2 m^2 + \cdots + t_k m^k) ,$$

where $t_1, t_2, \cdots t_k$ are real, and set, with an integer L,

$$J(q,L) = \int_0^1 \cdots \int_0^1 |S(q)|^{2L} dt_1 \cdots dt_k .$$

Lemma 3. ([3] Lemma 6.9.) If R is any non-negative integer, and let $L \ge \frac{1}{4}k(k+1)+kR$, then

$$J(q,L) \le K^R (\log q)^R q^{2L-\frac{1}{2}k(k+1)+\frac{1}{2}k(k+1)(1-\frac{1}{k})^R} ,$$

where $K = 48^{2L}(L!)^2 L^k k^{\frac{1}{2}k(k-1)}$.

Lemma 4. ([3] Lemma 6.11) Let M and N be integers, $N>1$, and let $\phi(n)$ be a real function of n, defined for $M\le n\le M+N-1$, such that

$$\delta \le \phi(n+1) - \phi(n) \le c\delta \quad (M \le n \le M+N-2) ,$$

where $\delta>0$, $c\ge1$, $c\delta\le\frac{1}{2}$. Let $w\ge1$. Let $\|x\|$ denote the difference between x and the nearest integer. Then the number of values of n for which $\|\phi(n)\| \le w\delta$ is less than

$$(Nc\delta + 1)(2w + 1) .$$

Lemma 5. Let k, P, and Q be integers $k \geq 2$, $Q \geq 2$, let $f(x)$ be real and have continuous derivatives up to the $(k+1)$th order in $[P+1, P+Q]$; let $0 < \lambda < 1/(2c_o(k+1))$ and

$$\lambda \leqq \frac{f^{(k+1)}(x)}{(k+1)!} \leqq c_o\lambda \qquad (P+1 \leqq x \leqq P+Q) ,$$

or the same for $-f^{(k+1)}(x)$, and let

$$Q^{-k-1+\delta} \leqq \lambda \leqq Q^{-1}$$

with $0<\delta\leqq k$. Then

$$\left| \sum_{n=P+1}^{P+Q} e(f(n)) \right| = O(Q^{1-\frac{1}{4L}\frac{\delta}{k+1}}(\log Q)^{\frac{R}{2L}}) ,$$

where

$$R = 1 + \left[\frac{\log(\frac{1}{\delta} k(k+1)^2)}{-\log(1-\frac{1}{k})} \right] ,$$

$$L = 1 + \left[\frac{1}{4} k(k+1) + kR \right] ,$$

and the constant implied depends possibly on k and δ.

Lemma 6. In addition to the conditions of Lemma 5, let N be an integer with $1 \leqq N \leqq Q$. Then

$$\left| \sum_{n=P+1}^{P+N} e(f(n)) \right| = O(Q^{1-\rho})$$

where $\rho = \frac{1}{16L}\frac{\delta}{k+1}$.

Proof. The method of proof is similar to that of Lemma 6.12 in [3]. Put

$$S = \sum_{n=P+1}^{P+Q} e(f(n)) ,$$

$$T(n) = \sum_{m=1}^{q} e(f(n+m)-f(n)) \qquad (P+1 \le n \le P+Q-q),$$

where

$$q = 1 + [\,\lambda^{-\frac{1}{k+1}}\,],$$

so that

$$Q^{\frac{1}{k+1}} \le q \le 1 + Q^{1-\frac{\delta}{k+1}} .$$

Then

$$|S| \le \frac{1}{q} Q^{1-\frac{1}{2L}} \Big(\sum_{n=P+1}^{P+Q-q} |T(n)|^{2L} \Big)^{\frac{1}{2L}} + q , \tag{1}$$

where L is any positive integer. (See (6.12.4) in [3].)

Define, for $1 \le y \le q$,

$$\Delta(y) = f(n+y) - f(n) - (t_1 y + t_2 y^2 + \cdots + t_k y^k) .$$

Then

$$T(n) = S(q)e(\Delta(y)) - 2\pi i \int_0^q S(y)\Delta'(y)e(\Delta(y))dy , \tag{2}$$

where

$$\Delta'(y) = \sum_{h=1}^{k} h\Big(\frac{f^{(h)}(n)}{h!} - t_h \Big) y^{h-1} + O(f^{(k+1)}(n+\theta y)y^k) \tag{3}$$

with $0<\theta<1$. Hence, if

$$\left| \frac{f^{(h)}(n)}{h!} - t_h \right| \le \frac{1}{2q^h} \qquad (h=1,2,\cdots,k) , \tag{4}$$

we have

$$|\Delta'(y)| = O(\frac{1}{q}) + O(\lambda q^k) = O(\frac{1}{q}) ,$$

so that

$$|T(n)| = O(|S(q)| + \frac{1}{q}\int_0^q |S(y)|dy) .$$

Thus, as in the proof of Lemma 6.12 in [3], we get

$$\sum_{n=P+1}^{P+Q-q} |T(n)|^{2L} = O(q^{\frac{1}{2}k(k+1)} \sup_{t\in \mathbb{R}^k} \sum_{n=P+1}^{P+Q-q} \chi(n,t)$$

$$\times \int_0^1 \cdots \int_0^1 (|S(q)| + \frac{1}{q}\int_0^q |S(y)|dy)^{2L}\, dt_1\cdots dt_k) , \tag{5}$$

where, for each n, $\chi(n,t)$ is a function of $t = (t_1,t_2,\cdots,t_k) \in R^k$ defined by

$$\chi(n,t) = \begin{cases} 1 & \text{if } \left\| \dfrac{f^{(h)}(n)}{h!} - t_h \right\| \le \dfrac{1}{2q^h} \quad (h=1,2,\cdots,k) , \\ 0 & \text{otherwise.} \end{cases}$$

It follows from Lemma 3 that

$$\int_0^1 \cdots \int_0^1 (|S(q)| + \frac{1}{q}\int_0^q |S(y)|dy)^{2L}\, dt_1\cdots dt_k$$

$$\le 2^{2L-1} \int_0^1 \cdots \int_0^1 (|S(q)|^{2L} + \frac{1}{q}\int_0^q |S(y)|^{2L} dy)\, dt_1\cdots dt_k$$

$$= O(q^{2L-\frac{1}{2}k(k+1)+\frac{1}{2}k(k+1)(1-\frac{1}{k})^R} (\log q)^R) , \tag{6}$$

where R and L are as in Lemma 3. Further put in Lemma 4 $M=P+1$, $N=Q-q$, $\phi(n) = \frac{1}{k!} f^{(k)}(n) - t_k$ for $M\le n\le M+N-1$, $c=c_o$, $\delta=\lambda(k+1)$, and $w=q/(2k+2)$. Then, by the conditions of Lemma 5, $w\ge 1$ (provided $Q\ge(2k+1)^{k+1}$), $c\delta\le\frac{1}{2}$, and $\delta\le\phi(n+1)-\phi(n)\le c\delta$ (setting $\phi(n)= -\frac{1}{k!} f^{(k)}(n) + t_k$ if necessary).

Hence by Lemma 4 $\sup_{t\in \mathbb{R}^k} \sum_{n=P+1}^{P+Q-q} \chi(n,t) = O((Q\lambda+1)(q+1)) = O(q)$.

This together with (5) and (6) yields

$$\sum_{n=P+1}^{P+Q-q} |T(n)|^{2L} = O(q^{1+2L+\frac{1}{2}k(k+1)(1-\frac{1}{k})^R}(\log q)^R) ,$$

and so by (1)

$$|S| = O(Q^{1-\frac{1}{2L}}(q^{1+\frac{1}{2}k(k+1)(1-\frac{1}{k})^R}(\log q)^R)^{\frac{1}{2L}}) + O(q)$$

$$= O(Q^H (\log Q)^{\frac{R}{2L}}) + O(Q^{1-\frac{\delta}{k+1}}) ,$$

where

$$H = 1 - \frac{1}{2L} + (1 - \frac{\delta}{k+1})(1 + \frac{1}{2}k(k+1)(1 - \frac{1}{k})^R)\frac{1}{2L}$$

$$\leq 1 - \frac{1}{2L} (\frac{\delta}{k+1} - \frac{1}{2}k(k+1)(1 - \frac{1}{k})^R) \leq 1 - \frac{1}{4L}\frac{\delta}{k+1}$$

choosing R as in Lemma 5; and the proof is completed.

Proof of Lemma 6. Put $\varepsilon = \delta/(2(k+1)-\delta)$. We may assume $Q^{1-\varepsilon} \leq N \leq Q$. Then

$$N^{-k-1+\frac{1}{2}\delta} \leq \lambda \leq N^{-1} ,$$

and hence the results follows from Lemma 5.

§3. Proof of Theorem. Let j_o be a positive integer chosen sufficiently large. Then, for each integer $j \geq j_o$, there is an integer $n_j \geq 1$ such that

$$r^{j-2} \leq g(n_j) < r^{j-1} \leq g(n_j+1) < r^j ,$$

since $g(t) < g(t+1) < rg(t)$ for all large t. It follows from the definition that $n_j < n \leq n_{j+1}$ if and only if $r^{j-1} \leq g(n) < r^j$, and that

$$c_o r^{\frac{j}{\beta}} < n_j < c_1 r^{\frac{j}{\beta}} , \quad c_o r^{\frac{j}{\beta}} < n_{j+1} - n_j < c_1 r^{\frac{j}{\beta}}$$

for $j \geq j_o$, where c_o and c_1 are positive constants independent of j.

Let J be a positive integer such that $n_J < x \leq n_{J+1}$, so that

$$J = \log_r g(x) + O(1) .$$

Put $X_J = x - n_J$ and $X_j = n_{j+1} - n_j$ for $j < J$. Then, writing $N_r(g(n)) = N_r([g(n)]; b_1 b_2 \cdots b_\ell)$ for brevity,

$$\sum_{n \leq x} N(g(n)) = \sum_{j=j_0}^{J} \sum_{n=n_j+1}^{n_j+X_j} N(g(n)) + O(1) , \tag{7}$$

where the constant implied is independent of x. Here it can be written that

$$N(g(n)) = \sum_{m=\ell}^{j} I\left(\frac{g(n)}{r^m}\right) \qquad (n_j < n \leq n_{j+1}) ,$$

where

$$I(t) = \begin{cases} 1 & \text{if } \displaystyle\sum_{k=1}^{\ell} \frac{b_k}{r^k} \leq t - [t] < \sum_{k=1}^{\ell} \frac{b_k}{r^k} + \frac{1}{r^\ell} , \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$\sum_{n=n_j+1}^{n_j+X_j} N(g(n)) = \sum_{m=\ell}^{j} \sum_{n=n_j+1}^{n_j+X_j} I\left(\frac{g(n)}{r^m}\right) .$$

We construct, for each j, functions $I_-(x)$ and $I_+(x)$, periodic with period 1 such that $I_-(x) \leq I(x) \leq I_+(x)$, having Fourier expansion of the following form;

$$I_\pm(x) = \frac{1}{r^\ell} \pm \frac{1}{j} + \sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} A_\pm(\nu)\, e(\nu x) ,$$

where

$$|A(\nu)| \leq O\left(\min\left(\frac{1}{|\nu|}, \frac{j}{|\nu|^2}\right)\right) .$$

(See Chap. two, Lemma 2 in [4].) Thus

$$\sum_{n=n_j+1}^{n_j+X_j} N(g(n)) = \frac{j}{r^\ell} X_j + O(X_j)$$

$$+ O\Big(\sum_{m=\ell}^{j} \sum_{\nu=1}^{\infty} \min\Big(\frac{1}{\nu}, \frac{j}{\nu^2}\Big) \Big| \sum_{n=n_j+1}^{n_j+X_j} e\Big(\frac{\nu}{r^m} g(n)\Big)\Big|\Big), \qquad (8)$$

where the constants implied are independent of j.

We shall estimate the exponential sums

$$S(j,m,\nu) = \sum_{n=n_j+1}^{n_j+X_j} e\Big(\frac{\nu}{r^m} g(n)\Big),$$

where j, m, and ν are integers with $j \geq j_o$, $\ell \leq m \leq j$, and $\nu \geq 1$. In what follows all the constants as well as those implied in O-symbols will be independent of j, m, and ν. The proof will be carried on in two cases.

First case: Let $\beta(>0)$ be not an integer. We apply first Lemma 6 with $k=[\beta]+2$ (≥ 2) and $f(x) = \nu r^{-m} g(x)$. Since

$$f^{(k+1)}(x) \sim \frac{\nu}{r^m} \alpha\beta(\beta-1)\cdots(\beta-k)x^{\beta-k-1},$$

we have

$$\lambda < \frac{f^{(k+1)}(x)}{(k+1)!} < c_o\lambda \qquad (n_j \leq x \leq n_j+X_j),$$

or the same for $-f^{(k+1)}(x)$, where

$$\lambda = c\nu r^{-m+\frac{j}{\beta}(\beta-k-1)} = c\nu Q^{-k-1+(1-\frac{m}{j})\beta}$$

and $Q=r^{j/\beta}$. (In the sequel, the same letters c_o, c_1, and c may denote different constants at different occurrences.) Let δ be a positive constant chosen sufficiently small. If $(\ell\leq)m \leq \frac{j}{\beta}(\beta-\delta)$, or equivalently if $(1-\frac{m}{j})\beta \geq \delta$, then

$$-k-1+\delta \leq -k-1+(1-\frac{m}{j})\beta < -k-1+\beta < -2 ,$$

so that

$$Q^{-k-1+\delta} \leq \lambda \leq Q^{-1}$$

provided $1 \leq \nu \leq j^2$. Hence by Lemma 6

$$|S(j,m,\nu)| = O(r^{\frac{j}{\beta}(1-\rho)})$$

provided $\ell \leq m \leq \frac{j}{\beta}(\beta-\delta)$ and $1 \leq \nu \leq j^2$.

On the other hand, if $(j\geq)m \geq \frac{j}{\beta}(\beta-\delta)$, we can appeal to Lemma 1 with $f(x) = \nu r^{-m}g(x)$. Then, for these m and $\nu \leq j^2$, we have

$$0 < c_0 \nu r^{-m+j(1-\frac{1}{\beta})} < f'(x) < c_1 \nu r^{-m+j(1-\frac{1}{\beta})} < \frac{1}{2}$$

throughout the interval $[n_j, n_j+X_j]$, since

$$j(1-\frac{1}{\beta}) - m \leq j(1-\frac{1}{\beta}) - j(1-\frac{\delta}{\beta}) < \frac{\delta-1}{\beta} < 0 .$$

Hence by Lemma 1

$$|S(j,m,\nu)| = O(\frac{1}{\nu} r^{\frac{j}{\beta}+m-j})$$

provided $\frac{j}{\beta}(\beta-\delta) \leq m \leq j$ and $1 \leq \nu \leq j^2$.

Combining these estimates, we obtain

$$\sum_{m=\ell}^{j} \sum_{\nu=1}^{\infty} \min(\frac{1}{\nu}, \frac{j}{\nu^2})|S(j,m,\nu)|$$

$$= \sum_{m=\ell}^{j} (\sum_{\nu=1}^{j^2} \frac{1}{\nu} + \sum_{\nu=j^2+1}^{\infty} \frac{j}{\nu^2})|S(j,m,\nu)|$$

$$= \sum_{m=\ell}^{j} \sum_{\nu=1}^{j^2} \frac{1}{\nu} |S(j,m,\nu)| + O(X_j)$$

$$= O(r^{\frac{j}{\beta}(1-\rho)} j \log j) + O(r^{\frac{j}{\beta}})$$

$$+ O(r^{\frac{j}{\beta}} \sum_{(1-\delta/\beta)j \leq m \leq j} r^{m-j} (\sum_{\nu=1}^{j^2} \frac{1}{\nu^2}))$$

$$= O(r^{\frac{j}{\beta}}) ,$$

which together with (7) and (8) yields

$$\sum_{n \leq x} N(g(n)) = \frac{1}{r^\ell} (\sum_{j=j_o}^{J-1} j(n_{j+1}-n_j) + J(x-n_J)) + O(\sum_{j=j_o}^{J} r^{\frac{j}{\beta}})$$

$$= \frac{1}{r^\ell} xJ + O(r^{\frac{J}{\beta}})$$

$$= \frac{1}{r^\ell} x \log_r g(x) + O(x) ;$$

and the theorem is proved when β is not an integer.

Second case: Let $\beta(>0)$ be an integer. Then there is a non-integer β_h, $h \geq 1$, such that $\beta, \beta_1, \cdots, \beta_{h-1}$ are integers. Put $b=\beta$ and $\gamma=\beta_h$, so that $b \geq 1$ and $\gamma>0$. In what follows, ν is always assumed to be $1 \leq \nu \leq j^2$. We apply Lemma 6, Lemma 1, and Lemma 2 with $f(x) = \nu r^{-m} g(x)$. It is easily seen that

$$\lambda \leq \frac{f^{(b+1)}(x)}{(b+1)!} \leq c_o \lambda \qquad (n_j \leq x \leq n_j+X_j)$$

or the same for $-f^{(b+1)}(x)$, where

$$\lambda = c\nu Q^{\gamma-b-1-\frac{b}{j}m} , \quad Q = r^{\frac{j}{b}} .$$

If $(\ell\leq)m \leq \frac{j}{b}(\gamma-\delta)$, where δ is a positive constant chosen sufficiently small, then

$$-b-1+\delta \leq \gamma-b-1-\frac{b}{j}m \leq -1-\delta ,$$

so that

$$Q^{-b-1+\frac{1}{2}\delta} \leq \lambda \leq Q^{-1} .$$

Hence by Lemma 6

$$|S(j,m,\nu)| = O(r^{\frac{j}{b}(1-\rho)}) \qquad (\ell \leq m \leq \frac{j}{b}(\gamma-\delta)) \tag{9}$$

where ρ, $0<\rho<1$, is a constant.

On the other hand, if $(j\geq)m > \frac{j}{b}(b-1+\delta)$ then

$$0 < c_0\nu r^{-m+\frac{j}{b}(b-1)} < f'(x) < c_1\nu r^{-m+\frac{j}{b}(b-1)} < c_1\nu r^{-\delta\frac{j}{b}} < 1$$

for $n_j\leq x \leq n_j+X_j$. Hence by Lemma 1

$$|S(j,m,\nu)| = O(\frac{1}{\nu} r^{\frac{j}{b}+m-j}) \qquad (\frac{j}{b}(b-1+\delta) \leq m \leq j) . \tag{10}$$

We may now assume $b\geq 2$, since, if $b=1$ then $\frac{j}{b}(b-1+\delta) \leq \frac{j}{b}(\gamma-\delta)$, so that (9) and (10) cover all integers m in the range $\ell\leq m\leq j$. Then

$$f''(x) \sim \alpha b(b-1)\nu r^{-m} x^{b-2} \neq 0 ,$$

so that

$$f''(x) > c\nu r^{\frac{j}{b}(b-2)-m} > 0 \qquad (n_j\leq x \leq n_j+X_j) .$$

Hence, putting in Lemma 2 $a=n_j$, $b=n_j+X_j$, $\gamma = c\nu r^{\frac{j}{b}(b-2)-m}$, and

$$f'(n_j+X_j) - f'(n_j) = O(r^{\frac{j}{b}} \nu r^{\frac{j}{b}(b-2)-m}) .$$

we get

$$|S(j,m,\nu)| = O((\nu r^{\frac{j}{b}(b-2)-m})^{-\frac{1}{2}})$$
$$+ O(r^{\frac{j}{b}} (\nu r^{\frac{j}{b}(b-2)-m})^{\frac{1}{2}})$$
$$+ O(j + \log(1+\nu)) .$$

Thus, assuming $\frac{j}{b}(b-2+\delta) \leqq m$ $(\leqq \frac{j}{b}(b-1+\delta))$ or equivalently $\frac{j}{b}(b-2)-m \leqq -\frac{j\delta}{b}$, we obtain

$$\begin{aligned} |S(j,m,\nu)| &= O(\nu^{-\frac{1}{2}} r^{\frac{j}{b}+\frac{1}{2}(m-j)}) + O(\nu^{\frac{1}{2}} r^{\frac{j}{b}(1-\frac{\delta}{2})}) + O(j) \\ &= O(\nu^{-\frac{1}{2}} r^{\frac{j}{b}+\frac{1}{2}(m-j)}) + O(r^{\frac{j}{b}(1-\frac{\delta}{4})}) \qquad (11) \\ &\qquad (\frac{j}{b}(b-2+\delta) \leqq m \leqq \frac{j}{b}(b-1+\delta)) . \end{aligned}$$

It remains to estimates the sum $S(j,m,\nu)$ when

$$\frac{j}{b}(\gamma-\delta) \leqq m \leqq \frac{j}{b}(b-2+\delta) , \qquad (12)$$

so that especially when $0<\gamma<b-2$ as γ is not an integer and so $b \geqq 3$. For this, we modify the proof of Lemma 5 taking these conditions into account. We assume first that $(j_o \leqq) j \leqq J-1$, so that $X_j \asymp r^{\frac{j}{b}}$. Put $k=b-1$, $P=n_j$, $Q=X_j$,

$$q = [(\frac{r^m}{\nu})^{\frac{1}{b-1}}] ,$$

and define S, $T(n)$, and $S(q)$ as in §2. Then (1) follows. Setting now

$$\Delta(y) = \frac{\nu}{r^m}(g(n+y) - g(n)) - (t_1 y + t_2 y^2 + \cdots + t_{b-1}y^{b-1} + \frac{\nu}{r^m}\alpha y^b)$$

for $n_j \le n \le n_j + X_j$ and $1 \le y \le q$, we get (2) with (3) and $g^{(b)}(n+\theta y) - \alpha(b!)$ $= O(Q^{\gamma-b})$. Hence, assuming (4) again, we find

$$|\Delta'(y)| = O(\frac{1}{q}) + O(\frac{\nu}{r^m} Q^{\gamma-b} q^{b-1})$$

$$= O(\frac{1}{q}) + O(Q^{\gamma-b}) = O(\frac{1}{q})$$

and thus (5) follows as in §2.

Now put in Lemma 4 $M=P+1$, $N=Q-q-1$, and $\phi(n) = \nu r^{-m}\frac{1}{(b-1)!} g^{(b-1)}(n) - t_{b-1}$ for $M \le n \le M+N-1$. Then $\phi(n+1)-\phi(n) =$ $\nu r^{-m}\frac{1}{(b-1)!} g^{(b)}(n+\theta) \sim \alpha b\nu r^{-m}$, so that $\delta \le \phi(n+1) - \phi(n) \le c\delta \le \frac{1}{2}$, where $\delta = \frac{1}{2}\alpha b \nu r^{-m}$ and $c=2$. Also put $w = 1/(2\delta q^{b-1})$. Hence it follows from Lemma 4 that

$$\sup_{t\in\mathbb{R}^k} \sum_{n=P+1}^{P+Q-q} \chi(n,t) = O((Q\frac{\nu}{r^m}\alpha b + 1)(\frac{1}{q^{b-1}}\frac{r^m}{\alpha b\nu} + 1)) = O(Q\frac{\nu}{r^m} + 1) .$$

This together with (1), (5), and (6) yields

$$|S(j,m,\nu)| = O(\textstyle\sum_1) + O((\frac{r^m}{\nu})^{\frac{1}{b-1}}) ,$$

where

$$\textstyle\sum_1 = Q^{1-\frac{1}{2L}}(q^{\frac{1}{2}b(b-1)(1-\frac{1}{b-1})^R}(Q\frac{\nu}{r^m} + 1))^{\frac{1}{2L}} .$$

As for the second term, we see

$$(\frac{r^m}{\nu})^{\frac{1}{b-1}} \le r^{\frac{j}{b}(1-\frac{1-\delta}{b-1})} = Q^{1-\frac{1-\delta}{b-1}} .$$

If $\frac{j}{b} \le m$ $(\le \frac{j}{b}(b-2+\delta))$, then $Q\nu r^{-m} = O(\nu)$, so that

$$\sum_1 = O(Q^{1-\frac{1}{2L}} q^{\frac{1}{2}b(b-1)(1-\frac{1}{b-1})^R \frac{1}{2L}} \nu^{\frac{1}{2L}})$$

$$= O(Q^{1-\frac{1}{2L}} r^{m\cdot\frac{1}{2}b(1-\frac{1}{b-1})^R \frac{1}{2L}} \nu^{\frac{1}{2L}})$$

$$= O(Q^{1-\frac{1}{2L}+\frac{1}{4L}b(b-2+\delta)(1-\frac{1}{b-1})^R} \nu^{\frac{1}{2L}}) = O(Q^{1-\frac{1}{4L}}),$$

choosing R large enough to satisfy $b(b-2+\delta)(1-\frac{1}{b-1})^R < 1$. On the other hand, if $(\frac{j}{b}(\gamma-\delta)\leq)m \leq \frac{j}{b}$, then $Q\nu r^{-m} > 1$ and $m \geq \frac{\gamma}{2}\frac{j}{b}$, so that

$$\sum_1 = O(Q^{1-\frac{1}{2L}} (r^{\frac{1}{2}mb(1-\frac{1}{b-1})^R} Q\frac{\nu}{r^m})^{\frac{1}{2L}})$$

$$= O(Qr^{-m(1-\frac{1}{2}b(1-\frac{1}{b-1})^R)\frac{1}{2L}} \nu^{\frac{1}{2L}})$$

$$= O(Q^{1-\frac{1}{2}\gamma(1-\frac{1}{2}b(1-\frac{1}{b-1})^R \frac{1}{2L}} \nu^{\frac{1}{2L}})$$

$$= O(Q^{1-\frac{\gamma}{8L}}),$$

choosing R large enough to satisfy $b(1-\frac{1}{b-1})^R < 1$. In any case, we get

$$\sum_1 = O(Q^{1-\rho})$$

where $\rho = \min(\frac{1-\delta}{b-1}, \frac{1}{4L}, \frac{\gamma}{8L})$. Accordingly

$$|S(j,m,\nu)| = O(Q^{1-\rho})$$

for all $(j_o\leq)j \leq J-1$ and m in (12). Therefore, as in the proof of Lemma 6, we obtain for any $(j_o\leq)j \leq J$

$$|S(j,m,\nu)| = O(Q^{1-\frac{\rho}{2}}) \quad \frac{j}{b}(\gamma-\delta) \leqq m \leqq \frac{j}{b}(b-2+\delta) \,. \tag{13}$$

The theorem with integral β can be deduced from (9), (10), (11), and (13) as in the first case; and the proof is completed.

Remark. The conjecture mentioned in §1 is true for the case of $f(x)=x$. Indeed, Mirsky [2] proved that, for any integer $r\geq 2$ and b, $0\leq b<r$,

$$\sum_{n\leq x} N_r(n;b) = \frac{1}{r} x \log_r x + O(x) .$$

Here the error term $O(x)$ cannot be replaced by $o(x)$, since it can be easily seen that

$$\sum_{n<r^k+r^{k-1}} N_r(n;1) = \frac{1}{r}(kr^k + (k-1)r^{k-1}) + r^{k-1} .$$

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Yoshinobu NAKAI (Y.-N. Nakai)
Department of Mathematics
Faculty of Education
Yamanashi University
Kofu
400 Japan

Iekata SHIOKAWA
Department of Mathematics
Keio University
Hiyoshi, Yokohama
223 Japan

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A class of normal numbers II

Y.-N. Nakai and I. Shiokawa

§1. Introduction.

Let $r \geqq 2$ be a fixed integer and let $0.a_1a_2a_3 \cdots = a_1r^{-1} + a_2r^{-2} + a_3r^{-3} + \cdots$ be the r-adic expansion of a real number θ $(0 < \theta < 1)$. Then θ is said to be normal to base r, if for any block $b_1 \cdots b_\ell \in \{0, 1, \cdots, r-1\}^\ell$,

$$\frac{1}{n} N_r(\theta; b_1 \cdots b_\ell; n) = r^{-\ell} + o(1)$$

as $n \to \infty$, where $N_r(\theta; b_1 \cdots b_\ell; n)$ is the number of indices $i \leq n-\ell+1$ such that $a_i = b_1$, $a_{i+1} = b_2$, $\cdots$, $a_{i+\ell-1} = b_\ell$. Various kinds of constructions of normal numbers have been known. However, most of them are very complicated and by no means easy to write down (see, e.g. [4]). One of the simplest algorithm which gives normal numbers is the following: $\mathbb{Q}[x]$ denote the set of polynomials in x with rational coefficients. Let $f(x) \in \mathbb{Q}[x]$ with $1 \leqq f(n) \in \mathbb{Z}$ $(n = 1, 2, \cdots)$. Then Davenport and Erdös [1] proved that the decimal $0.f(1)f(2)f(3)\cdots$ is normal to base 10, where each $f(n)$ is written in the scale of 10, and the digits of $f(1)$ are succeeded by those of $f(2)$, and so on.

We consider a pseudo polynomial with real coefficients, which is a function $g(x)$ of the following form:

$$g(x) = \alpha x^\beta + \alpha_1 x^{\beta_1} + \cdots + \alpha_d x^{\beta_d}, \tag{1}$$

where $\alpha = \alpha_0, \alpha_1, \cdots, \alpha_d$ are non-zero real numbers and $\beta = \beta_0 > \beta_1 > \cdots > \beta_d \geq 0$ $(d \geq 0)$. In this paper we always assume that $g(x) > 0$ for $x \geq 0$. The set of all such pseudo polynomials will be denoted by $\mathbb{R}[x^\wedge \mathbb{R}_+]$. For each $g(x) \in \mathbb{R}[x^\wedge \mathbb{R}_+]$, we define the number

$$\theta_r = \theta_r(g) = 0.[g(1)][g(2)][g(3)]\cdots,$$

by the infinite r-adic fraction $0.a_{11}a_{12}\cdots a_{1k(1)}a_{21}a_{22}\cdots a_{2k(2)}a_{31}\cdots$, where $[g(n)] = a_{n1}a_{n2}\cdots a_{nk(n)} = a_{n1}r^{k(n)-1} + a_{n2}r^{k(n)-2} + \cdots + a_{nk(n)}$ is the r-adic expansion of the integral part of $g(n)$.

We proved in [5] the normality to base r of the number $\theta_r(g)$, when $g(x) \in \mathbb{R}[x^{\wedge}\mathbb{R}_+] \setminus \mathbb{R}[x]$, i.e., when at least one of $\beta, \beta_1, \cdots, \beta_d$ in (1) is not an integer. Here $\mathbb{R}[x]$ denotes the set of polynomials in · with real coefficients. In the present paper we shall prove that the number $\theta_r(g)$ is normal to base r for any $g(x) \in \mathbb{R}[x]$ and hence combining this with our results mentioned above, $\theta_r(g)$ is normal to base r for all $g(x) \in \mathbb{R}[x^{\wedge}\mathbb{R}_+]$. Especially, we have

Example. The number $\theta_r = 0.[\alpha][\alpha 2^{\beta}][\alpha 3^{\beta}]\cdots$ is normal to base r for all $\alpha > 0$ and $\beta > 0$.

More precisely, we obtained in [5] the following estimate: For any $g(x) \in \mathbb{R}[x^{\wedge}\mathbb{R}_+] \setminus \mathbb{R}[x]$ and any $b_1\cdots b_\ell \in \{0, 1, \cdots, r-1\}^\ell$, we have

$$R_n := \frac{1}{n} N_r(\theta_r(g); b_1\cdots b_\ell; n) - r^{-\ell} = O(1/\log n).$$

prove this, we used tricky estimates for exponential sums of the Vinogradov type. For $g(x) \in \mathbb{Q}[x]$, Schiffer [6] showed that $R_n = O(1/\log n)$. We shall prove in this paper the following

Theorem. For any $g(x) \in \mathbb{R}[x]$ and any block $b_1\cdots b_\ell \in \{0, 1, \cdots, r-1\}^\ell$, we have

$$\sum_{n \le x} N_r([g(n)]; b_1\cdots b_\ell) = r^{-\ell} x \log_r g(x) + O(x \log\log x)$$

as $x \to \infty$, where the constant implied depends possibly on the α's, β's, r, and ℓ.

As an immediate consequence, we have

Corollary. For any $g(x)$ and $b_1\cdots b_\ell$ as in Theorem, we have

$$R_n = O\left(\frac{\log\log n}{\log n} \right)$$

as $n \to \infty$. Especially the number $\theta_r(g)$ is normal to base r .

Our method of proof, which is different from that of Schffer [6], is to make use of estimates on Weyl sums in a somewhat unusual manner and a simple estimate on diophantine approximations. The error term estimate $R_n = O(1/\log n)$ is best possible for $g(x) = x$, in the sense t t it cannot be replaced by $o(1/\log n)$ (see [5], Remark). It remains the problem of replacing the error term $R_n = O(\log\log n/\log n)$ by $O(1/\log n)$ for $g(x) \in \mathbb{R}[x] \setminus \mathbb{Q}[x]$.

§2. A Lemma.

Lemma. Let $f(x)$ be a polynomial with real coefficients and the leading term Ax^b $(A \neq 0)$. Let a/q be a rational number such that $(a, q) = 1$ and $|A - aq^{-1}| \leq q^{-2}$. Let $V \geq 1$ be a real number. Then, if $b \geq 2$,

$$\left| \sum_{1 \leq n \leq Q} e(f(n)) \right|$$

$$\ll Q\{Q^{-1} + V^{-1}(\log Q)^B + V(q^{-1} + Q^{-1}\log q + Q^{-(b-1)} + Q^{-b}q \log q)\}^{\delta}$$

where $\delta = 2^{-(b-1)}$ and $e(t) = \exp(2\pi\sqrt{-1}\, t)$. Here B is any constant satisfying

$$\sum_{n \leq x} (\tau_{b-1}(n))^2 \ll x(\log x)^B$$

as $x \to \infty$, where $\tau_{b-1}(n)$ is the number of expressions of n as a product of $b-1$ positive integers ($\tau_1(n) = 1$ by definition).

It is known that the choice $B = (b-1)^2 - 1$ is sufficient (cf. [3] Chap. III, Problem 8, p.60, for instance).

Corollary. Under the same assumptions as in Lemma, let q be such that

$$(\log Q)^{H} \ll q \ll Q^{b}(\log Q)^{-H}, \tag{2}$$

where $H = B + 2^{b-1} \cdot 2G + 1$ with a non-negative real number G. Then

$$\Big|\sum_{1\leq n\leq Q} e(f(n))\Big| \ll Q(\log Q)^{-G}. \tag{3}$$

Remark. If $b = 1$, Corollary still holds with $B = 0$.

Proof of Lemma. As is usual in treating Weyl sums (cf. Lemmas 3.3 and 3.4 in [2], or Lemmas 2.3 and 2.4 in [7]), we have

$$\Big|\sum_{1\leq n\leq Q} e(f(n))\Big|^{2^{b-1}}$$

$$\ll (2Q)^{2^{b-1}-(b-1)-1}\Big((2Q)^{b-1}+\sum_{1\leq y\leq b!Q^{b-1}} 2^{b-1}\tau_{b-1}(y)\min(Q, \|Ay\|^{-1})\Big),$$

where $\|t\| = \min(t-[t], 1+[t]-t)$. For $k=0,1,2,\cdots$, we have

$$\sum_{\substack{1\leq y\leq b!Q^{b-1}\\ \tau_{b-1}(y)\geq 2^{k}V}} 1 \ll (2^{k}V)^{-2} \sum_{1\leq y\leq b!Q^{b-1}} (\tau_{b-1}(y))^{2}$$

$$\ll (2^{k}V)^{-2}\, Q^{b-1}(\log Q)^{B},$$

and then

$$\sum_{\substack{1\leq y\leq b!Q^{b-1}\\ \tau_{b-1}(y)\geq V}} \tau_{b-1}(y) \ll \sum_{k=o}^{\infty} 2^{k+1}V \sum_{\substack{1\leq y\leq b!Q^{b-1}\\ \tau_{b-1}(y)\geq 2^{k}V}} 1$$

$$\ll \sum_{k=o}^{\infty} 2^{k+1}V\,(2^{k}V)^{-2}\, Q^{b-1}(\log Q)^{B}$$

$$\ll V^{-1}\, Q^{b-1}(\log Q)^{B},$$

so that

$$\sum_{\substack{1\leq y\leq b!Q^{b-1}\\ \tau_{b-1}(y)\geq V}} \tau_{b-1}(y)\min(Q, \|Ay\|^{-1}) \ll V^{-1}Q^{b}(\log Q)^{B} .$$

As for y's with $\tau_{b-1}(y) \leq V$, we have

$$\sum_{\substack{1\leq y\leq b!Q^{b-1}\\ \tau_{b-1}(V)\leq V}} \tau_{b-1}(y)\min(Q, \|Ay\|^{-1})$$

$$\ll V \sum_{1\leq y\leq b!Q^{b-1}} \min(Q, \|Ay\|^{-1})$$

$$\ll V(Q^{b-1}q^{-1} + 1)(Q + q\log q)$$

by routine arguments in treating Weyl sums (cf. Lemmas 3.5 and 3.6 in [2]). These inequalities imply the lemma.

§3. Proof of Theorem.

Let the leading term of $g(x)$ be αx^{b} . Let j_0 be a positive integer chosen sufficiently large. Then, for each integer $\geq j_0$, there is a positive integer n_j such that

$$r^{j-2} \leq g(n_j) < r^{j-1} \leq g(n_j+1) < r^{j} .$$

It follows that $n_j < n \leq n_{j+1}$ if and only if $r^{j-1} \leq g(n) < r^{j}$, and that $n_j \gg\ll r^{j/b}$ and $n_{j+1} - n_j \gg\ll r^{j/b}$, where the constants implied are independent of j . Let J be a positive integer such that $n_J < x \leq n_{J+1}$, so that

$$J = \log_r g(x) + O(1) = O(\log x) .$$

Put $X_J = x - n_J$ and $X_j = n_{j+1} - n_j$ for $(j_0\leq)j < J$. Then, putting

$N_r(g(n)) = N_r([g(n)];\ b_1 \cdots b_\ell)$ for brevity,

$$\sum_{n \le x} N(g(n)) = \sum_{j=j_0}^{J} \sum_{n=n_j+1}^{n_j+X_j} N(g(n)) + O(1) . \qquad (4)$$

Using the periodic function $I(t)$ with period 1 defined by

$$I(t) = \begin{cases} 1 & \text{if } \sum_{k=1}^{\ell} b_k r^{-k} \le t-[t] < \sum_{k=1}^{\ell} b_k r^{-k} + r^{-\ell} , \\ 0 & \text{otherwise,} \end{cases}$$

have

$$\sum_{n=n_j+1}^{n_j+X_j} N(g(n)) = \sum_{m=\ell}^{j} \sum_{n=n_j+1}^{n_j+X_j} I\left(\frac{g(n)}{r^m} \right) .$$

Up to this point, our proof is the same as that in [5].

Choose now a sufficiently large constant C_0 . Then we have

$$\left(\sum_{\ell \le m \le C_0 \log j} + \sum_{j-C_0 \log j \le m \le j} \right) \sum_{n=n_j+1}^{n_j+X_j} I\left(\frac{g(n)}{r^m} \right)$$

$$= O(X_j \log j) = O(X_j \log\log x) . \qquad (5)$$

ı what follows, we treat those m with $C_0 \log j \le m \le j-C_0 \log j$. There are, for each j , functions $I_-(t)$ and $I_+(t)$, periodic with period 1, such that $I_-(t) \le I(t) \le I_+(t)$, having Fourier expansion of the form

$$I_\pm(t) = r^{-\ell} \pm j^{-1} + \sum_{\substack{\nu=-\infty \\ \nu \ne 0}}^{\infty} A_\pm(\nu)\, e(\nu t)$$

with $|A_\pm(\nu)| << \min(|\nu|^{-1}, j|\nu|^{-2})$. Then

$$\sum_{n=n_j+1}^{n_j+X_j} N(g(n)) = r^{-\ell} j X_j + O(X_j) + O(X_j \log\log x)$$

$$+ O\Big(\sum_{C_0 \log j \leq m \leq j - C_0 \log j} \ \sum_{\nu=1}^{j^2} \min(\nu^{-1}, j\nu^{-2}) \ \Big| \sum_{n=n_j+1}^{n_j+X_j} e\Big(\frac{\nu}{r^m} g(n)\Big)\Big|\Big) \qquad (6)$$

where the constants implied are independent of j .

We shall estimate the exponential sums

$$S(j,m,\nu) = \sum_{n=n_j+1}^{n_j+X_j} e\Big(\frac{\nu}{r^m} g(n)\Big) ,$$

where $J \geq j \geq j_0$, $j - C_0 \log j \geq m \geq C_0 \log j$, and $1 \leq \nu \leq j^2$. Here the leading coefficient of $\nu r^{-m} g(x)$ is $\nu r^{-m}\alpha$. Assume first that $j \lneqq J$. For any pair (m,ν) for which there is a rational number a/q such that

$$(a,q) = 1 , \quad \Big|\frac{\nu}{r^m}\alpha - \frac{a}{q}\Big| \leq \frac{1}{q^2}$$

and $\qquad$ (7)

$$(\log X_j)^H \leq q \leq X_j^b (\log X_j)^{-H}$$

with $G = 3$ and H as in Lemma, we have

$$|S(j,m,\nu)| \ll X_j (\log X_j)^{-3} \ll X_j j^{-3}$$

by Corollary with Remark. Hence, denoting by Σ' the sum over all pairs (m,ν) having this property, we have the following estimates:

$$\sum_m \sum_\nu{}' \min(\nu^{-1}, j\nu^{-2}) |S(j,m,\nu)|$$

$$\ll \sum_{m=\ell}^{j} \sum_{\nu=1}^{j^2} \min(\nu^{-1}, j\nu^{-2}) \cdot X_j j^{-3}$$

$$\ll j \log j \cdot X_j j^{-3} \ll X_j \ll r^{j/b} .$$

If $j=J$, there are two cases. Assume first that $X_J = O(r^{J/b} J^{-3})$. Then we have trivial estimates

$$\sum_{m=\ell}^{J} \sum_{\nu=1}^{J^2}{}' \min(\nu^{-1}, J\nu^{-2}) |S(J,m,\nu)|$$

$$<< \sum_{m=\ell}^{J} \sum_{\nu=1}^{J^2} \min(\nu^{-1}, J\nu^{-2}) \cdot r^{J/b}J^{-3} << r^{J/b}J^{-1} .$$

Otherwise, namely if $X_J >> r^{J/b}J^{-3}$, then we have $\log X_J >> << J$, so that we can repeat the same argument as above for $j \lneq J$. In any case, we get

$$\sum_{m} \sum_{\nu}{}' \min(\nu^{-1}, j\nu^{-2}) |S(j,m,\nu)| << r^{j/b} \tag{8}$$

for $(j_0 \leq)\ j \leq J$.

It remains to estimate the sums over all (m,ν) with $j-C_0\log j \geq m \geq C_0\log j$ for each of which there is no rational number a/q satisfying the conditions in (7). (If $j=J$, X_J is supposed to be $>> r^{J/b}J^{-3}$.) But, it will turn out that there is no such pair (m,ν) . To show this, we choose, for each pair (m,ν) in question, a rational number a/q such that

$$(a,q) = 1,\quad 1 \leq q \leq X_j^b(\log X_j)^{-H}$$

and

$$\left|\frac{\nu}{r^m}\alpha - \frac{a}{q}\right| < (q\, X_j^b(\log X_j)^{-H})^{-1}\ (\leq q^{-2}) .$$

This may be done by choosing an appropriate Farey approximant. If $2 \leq q \leq X_j^b(\log X_j)^{-H}$, then $2 \leq q \leq (\log X_j)^H$, since (7) is not satisfied any more. This implies that

$$\left|\frac{\nu}{r^m}\alpha\right| > \frac{1}{q} - \frac{1}{q^2} \geq \frac{1}{2q} >> (\log X_j)^{-H} ,$$

so that

$$r^m \ll |\nu\alpha|(\log X_j)^H \ll j^2(\log X_j)^H \ll j^{H+2} ,$$

and therefore

$$(C_0 \log j \leqq)\ m \leqq (H+2)\log j + O(1) ;$$

which cannot happen, if C_0 is sufficiently large. Now let $q = 1$. Then $\|\nu r^{-m}\alpha\| < X_j^b(\log X_j)^H$. If $|\nu r^{-m}\alpha| \geqq 1/2$, then $r^m \ll \nu \ll j^2$; which is impossible again by the same reasoning as above. Otherwise, i.e., if $|\nu r^{-m}\alpha| < 1/2$, then $|\nu r^{-m}\alpha| = \|\nu r^{-m}\alpha\| < X_j^{-b}(\log X_j)^H$. This implies that

$$\begin{aligned} r^m &> |\nu\alpha|\ X_j^b(\log X_j)^{-H} \\ &\gg (r^{j/b})^b(\log r^{j/b} - O(1))^{-H} \\ &\gg r^j\ j^{-H} , \end{aligned}$$

so that

$$(j - C_0 \log j \geqq)\ m > j - O(\log j) ;$$

which is also impossible.

Combining (4), (6), and (8), we have

$$\sum_{n=n_j+1}^{n_j+X_j} N(g(n)) = r^{-\ell} jX_j + O(X_j \log\log x) + O(r^{j/b}) .$$

Therefore

$$\begin{aligned} \sum_{n\leq x} N(g(n)) &= \sum_{j=j_0}^{J} (r^{-\ell} jX_j + O(X_j \log\log x) + O(r^{j/b})) \\ &= r^{-\ell} Jx + O(r^{J/b} \log\log x) \\ &= r^{-\ell} x \log_r g(x) + O(x \log\log x) , \end{aligned}$$

and the proof of our theorem is completed.

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Yoshinobu NAKAI (Y.-N. Nakai)
Department of Mathematics
Faculty of Education
Yamanashi University
Kofu
400 Japan

Iekata SHIOKAWA
Department of Mathematics
Keio University
Hiyoshi, Yokohama
223 Japan

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A class of normal numbers III

Y.-N. Nakai and I. Shiokawa

To the memory of Gerold Wagner

§1. Introduction

Let $r \geq 2$ be a fixed integer and let $\theta = 0.a_1a_2 \cdots = a_1r^{-1} + a_2r^{-2} + \cdots$ be the r-adic expansion of a real number θ with $0 < \theta < 1$. For any block $b_1 \cdots b_\ell \in \{0, 1, \cdots, r-1\}^\ell$, $N_r(\theta; b_1 \cdots b_\ell; n)$ denotes the number of indices $i \leq n-\ell+1$ such that $a_i = b_1$, $a_{i+1} = b_2$, $\cdots$, and $a_{i+\ell-1} = b_\ell$. Then θ is said to be normal, if for every fixed $\ell \geq 1$

$$R_n(\theta) = R_{n,\ell}(\theta) := \sup_{b_1 \cdots b_\ell \in \{0,1,\cdots,r-1\}^\ell} \left|\frac{1}{n}N_r(\theta; b_1 \cdots b_\ell; n) - \frac{1}{r^\ell}\right| = o(1)$$

as $n \to \infty$. As this paper is a continuation of [2] and [3], we omit some historical comments on the study of normal numbers connected with our results, which can be found in the introductions of our preceding papers.

A pseudo-polynomial with real coefficients is a function of the form

$$g(x) = \alpha x^\beta + \alpha_1 x^{\beta_1} + \cdots + \alpha_d x^{\beta_d},$$

where $\alpha = \alpha_0, \alpha_1, \cdots, \alpha_d$ are nonzero real numbers and $\beta = \beta_0 > \beta_1 > \cdots > \beta_d \geq 0$. In this paper, we always assume that $g(x)$ is nonconstant and $g(x) > 0$ for all $x \geq 0$. The set of all such pseudo-polynomials will be denoted by $\mathcal{R}$. For each $g(x) \in \mathcal{R}$, we define the number

$$\theta_r(g) = 0.a_{11}a_{12} \cdots a_{1k_1}a_{21}\, a_{22} \cdots a_{2k_2} \cdots$$

to be the infinite r-adic decimal obtained from the r-adic expansion

$[g(n)] = a_{n1}a_{n2} \cdots a_{nk_n} = a_{n1}r^{k_n-1} + a_{n2}r^{k_n-2} + \cdots + a_{nk_n}$ of the integral part of $g(n)$, which will be denoted simply by

$$\theta_r = \theta_r(g) = 0.[g(1)]\ [g(2)]\ [g(3)] \cdots .$$

We note that the sum $k_1 + k_2 + \cdots + k_n$ of the numbers of the digits in the r-adic expansions $[g(1)], [g(2)], \cdots, [g(n)]$ is

$$n \log_r g(n) + O(n) ,$$

where $\log_r y = (\log y)/\log r$.

We proved in [2] that, if $g(x) \in \mathcal{R}$ is not a polynomial,

$$R_n(\theta_r(g)) = O\left(\frac{1}{\log n}\right) ,$$

while in the case of a polynomial $g(x) \in \mathcal{R}$ we had in [3] a slightly weaker estimate

$$R_n(\theta_r(g)) = O\left(\frac{\log \log n}{\log n}\right) .$$

By combining these results, we see that the number $\theta_r(g)$ is normal to se r for any $g(x) \in \mathcal{R}$. In particular, the number

$$\theta_r = 0.[\alpha]\ [\alpha 2^{\beta}]\ [\alpha 3^{\beta}] \cdots$$

is normal to base r for any $\alpha > 0$ and $\beta > 0$. Schiffer [4] showed that $R_n(\theta_r(g)) = O(1/\log n)$, if $g(x)$ is a polynomial with rational coefficients. Thus it remains the problem to replace the error term $O((\log \log n)/\log n)$ by $O(1/\log n)$ in the case of polynomials $g(x)$ with real coefficients, which will be settled in this paper.

Let σ be a finite string of r-adic digits and let $N_r(\sigma; b_1 \cdots b_\ell)$ denote the number of occurrences of the block $b_1 \cdots b_\ell \in \{0, 1, \cdots, r-1\}^\ell$ in the string σ . Then we prove the following theorem.

Theorem. Let $g(x)$ be any nonconstant polynomial with real coefficients such that $g(x) > 0$ for all $x > 0$. Then, for any block $b_1 \cdots b_\ell \in \{0, 1, \cdots, r-1\}^\ell$, we have

$$\sum_{n \le x} N_r([g(n)]; b_1 \cdots b_\ell) = \frac{1}{r^\ell} x \log_r g(x) + O(x)$$

as $x \to \infty$, where the implied constant depends possibly on α's, β's, r, and ℓ.

As an immediate consequence, we have

Corollary. For any $g(x)$ as in Theorem, we have

$$R_n(\theta_r(g)) = O(\frac{1}{\log n})$$

as $n \to \infty$.

Our method of the proof in [3], which is quite different from that of Schiffer [4], made use of estimates for Weyl sums in a some what unusual manner and of some ideas on diophantine approximation. In this paper, we further develop it by employing inductive arguments and have obtained the improved results.

The error term estimate $R_n(\theta_r(g)) = O(1/\log n)$ is best possible for any linear polynomial $g(x)$ with real coefficients in the sense that it cannot be replaced by $o(1/\log n)$ (cf. [2], [4]).

§2. Preliminaries of the Proof of Theorem.

Let $g(x)$ be as in Theorem. Let j_0 be an integer chosen sufficiently large. Then, for each $j \ge j_0$, there is a positive integer n_j such that $r^{j-2} \le g(n_j) < r^{j-1} \le g(n_j+1) < r^j$. It follows that $n_j < n \le n_{j+1}$ if and only if $r^{j-1} \le g(n) < r^j$ and that

$$n_j \gg\ll r^{j/k}, \quad n_{j+1} - n_j \gg\ll r^{j/k},$$

where $k \geq 1$ is the degree of the polynomial $g(x)$. Let $x > r^{j_0}$ and let J be a positive integer such that $n_J < x \leq n_{J+1}$, so that

$$J = \log_r g(x) + O(1) = O(\log x) .$$

Put $X_J = x - n_J$ and $X_j = n_{j+1} - n_j$ for $(j_0 \leq)\ j \leq J-1$. We denote $N(g(n)) = N_r([g(n)];\ b_1 \cdots b_\ell)$, then

$$\sum_{n \leq x} N(g(n)) = \sum_{j_0 \leq j \leq J} \ \sum_{n_j < n \leq n_j + X_j} N(g(n)) + O(1) .$$

Defining the periodic function $I(t)$ with period 1 by

$$I(t) = \begin{cases} 1 & \text{if } \displaystyle\sum_{h=1}^{\ell} \frac{b_h}{r^h} \leq t-[t] < \sum_{h=1}^{\ell} \frac{b_h}{r^h} + \frac{1}{r^\ell} , \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\sum_{n_j < n \leq n_j + X_j} N(g(n)) = \sum_{\ell \leq m \leq j} \ \sum_{n_j < n \leq n_j + X_j} I\left(\frac{g(n)}{r^m}\right)$$

Let j be any integer with $j_0 \leq j \leq J$ and let C be a constant chosen sufficiently large. Then it is proved in [3; p.208] that

$$\sum_{C\log j \leq m \leq j - C\log j} \ \sum_{n_j < n \leq n_j + X_j} \left(I\left(\frac{g(n)}{r^m}\right) - \frac{1}{r^\ell}\right) = O(r^{j/k}) .$$

and in [2; p.26] that

$$\sum_{j - C\log j \leq m \leq j} \ \sum_{n_j < n \leq n_j + X_j} \left(I\left(\frac{g(n)}{r^m}\right) - \frac{1}{r^\ell}\right) = O(r^{j/k}) .$$

Therefore, if we can prove the inequality

$$\sum_{\ell \leq m \leq C\log j} \ \sum_{n_j < n \leq n_j + X_j} \left(I\left(\frac{g(n)}{r^m}\right) - \frac{1}{r^\ell}\right) = O(r^{j/k}) , \tag{6}$$

we shall have obtained

$$\sum_{\ell \le m < j} \sum_{n_j < n \le n_j + X_j} I\left(\frac{g(n)}{r^m}\right) = \frac{1}{r^\ell} jX_j + O(r^{j/k}) ,$$

which leads to

$$\sum_{n \le x} N(g(n)) = \frac{1}{r^\ell} xJ + O(r^{J/k})$$

$$= \frac{1}{r^\ell} x \log_r g(x) + O(x) ,$$

the theorem. Thus it remains to show (6).

§3. Lemmas.

Lemma 1 ([3], Corollary of Lemma.). Let $p(x)$ be a polynomial with real coefficients and the leading term γx^k, where $\gamma \ne 0$ and $k \ge 1$. Let $Q \ge 2$ and let A/B be a rational number with $(A, B) = 1$ such that

$$(\log Q)^h \ll B \ll Q^k (\log Q)^{-h} , \tag{7}$$

and

$$\left|\gamma - \frac{A}{B}\right| \le B^{-2} ,$$

where $h \ge (k-1)^2 + 2^k G$ with $G > 0$. Then

$$\left|\sum_{1 \le n \le Q} e(p(n))\right| \ll Q(\log Q)^{-G} ,$$

where $e(x) = e^{2\pi i x}$.

To prove the inequality (6), we need to generalize Lemma 1 as in the following.

Lemma 2. Let $f(x)$ be a polynomial of the form

$$f(x) = \beta_0 x^{k_0} + \beta_1 x^{k_1} + \cdots + \beta_d x^{k_d} ,$$

where $k_0 > k_1 > \cdots > k_d \ge 1$ and $\beta_0, \cdots, \beta_d$ are nonzero real numbers.

Let $G > 0$ be any constant and $X \geq 2$. Let s be an integer with $0 \leq s \leq d$, let H_i, K_i $(i = 0, 1, \cdots, s-1)$ be any positive constant, and let H_s^*, K_s^* be constants such that

$$H_s^* \geq 2^{k_s+1}(G + \max_{0\leq i<s} H_i + 1) + k_s \sum_{i=0}^{s-1} K_i ,$$

$$H_s^* \geq 2^{k_s+1}(G + \max_{0\leq i<s} H_i + 1) + 2k_s \sum_{i=0}^{s-1} K_i ,$$

Suppose that there are rational numbers A_i/B_i $(0 \leq i < s)$ such that

$$1 \leq B_i \leq (\log X)^{K_i} \quad \text{and} \quad |\beta_i - \frac{A_i}{B_i}| \leq \frac{(\log X)^{H_i}}{B_i X^{k_i}} \quad (0 \leq i < s)$$

and that there is no rational number A_s/B_s with $(A_s, B_s) = 1$ such that

$$1 \leq B_s \leq (\log X)^{K_s^*} \quad \text{and} \quad |\beta_s - \frac{A_s}{B_s}| \leq \frac{(\log X)^{H_s^*}}{B_s X^{k_s}} .$$

Then for any real P and Q with $1 \leq Q \leq X$

$$|\sum_{P<n\leq P+Q} e(f(n))| \ll X(\log X)^{-G} .$$

Proof. We may assume $P = 0$ and

$$X(\log X)^{-G} \leq Q \leq X . \tag{8}$$

If $s = 0$, the inequality follows immediately from Lemma 1. We put $p(x) = f(x)$, so that $\gamma = \beta_0$ and $k = k_0$. Since $s = 0$, $\max_{0\leq i<s} H_i = \sum_{i=0}^{s-1} K_i = 0$. We choose, by the well-known argument, a rational number A/B with $(A, B) = 1$ such that

$$1 \le B \le \frac{X^k}{(\log X)^{H_0^*}} \quad \text{and} \quad |\gamma - \frac{A}{B}| \le \frac{(\log X)^{H_0^*}}{BX^k} \quad (\le B^{-2}) ,$$

where $H_0^*, K_0^* \ge 2^{k+1}(G+1)$. Then by the assumption, we have $B \ge (\log X)^{K_0^*}$. These inequalities as well as (8) imply (7) with $h = (k-1)^2 + 2^k G$. Therefore we obtain

$$|\sum_{1 \le n \le Q} e(f(n))| \ll Q(\log Q)^{-G} \ll X(\log X)^{-G} .$$

Let $s = 1$. We denote by D the least common multiple of $B_0, \cdots, B_{s-1}$ and by N the integer defined by

$$DN \le Q < D(N+1) ,$$

so that

$$1 \le D \le (\log X)^K \quad \text{with} \quad K = \sum_{i=0}^{s-1} K_i$$

and by (8)

$$X(\log X)^{-(G+K)} \ll N \gg\ll \frac{Q}{D} \le \frac{X}{D} .$$

It follows that

$$\sum_{1 \le n \le Q} e(f(n)) = \sum_{\lambda=0}^{D-1} \sum_{\nu=1}^{N} e(f(\lambda + D\nu)) + O((\log X)^K) \tag{9}$$

We put

$$f_\lambda(y) = \sum_{i=0}^{s-1} \Omega_i(\lambda + Dy)^{k_i} , \quad \Omega_i = \beta_i - \frac{A_i}{B_i}$$

$$\mathcal{G}_\lambda(y) = \sum_{i=s}^{d} \beta_i(\lambda + Dy)^{k_i} ,$$

and

$$T_\lambda(\nu) = \sum_{n=1}^{\nu} e(\varphi_\lambda(n)) .$$

Then we have

$$\sum_{\lambda=0}^{D-1} \sum_{\nu=1}^{N} e(f(\lambda + D\nu))$$

$$= \sum_{\lambda=0}^{D-1} e\Big(\sum_{i=0}^{s-1} \frac{A_i}{B_i} \lambda^{k_i}\Big) \sum_{\nu=1}^{N} e(f_\lambda(\nu))(T_\lambda(\nu) - T_\lambda(\nu-1))$$

$$= \sum_{\lambda=0}^{D-1} e\Big(\sum_{i=0}^{s-1} \frac{A_i}{B_i} \lambda^{k_i}\Big)\{e(f_\lambda(N+1))T_\lambda(N) + \sum_{\nu=1}^{N} (e(f_\lambda(\nu)) - e(f_\lambda(\nu+1)))T_\lambda(\nu)\}$$

$$\ll \sum_{\lambda=0}^{D-1} (|T_\lambda(N)| + \sum_{\nu=1}^{N} |e(f_\lambda(\nu)) - e(f_\lambda(\nu+1))||T_\lambda(\nu)|) .$$

Here we have, using the mean-value theorem,

$$|e(f_\lambda(\nu)) - e(f_\lambda(\nu+1))|$$

$$\ll D \sum_{i=0}^{s-1} |\Omega_i| \, Q^{k_i - 1} \ll D \frac{(\log X)^H}{X} \quad \text{with } H = \max_{0 \le i < s} H_i .$$

Therefore we obtain

$$\sum_{\lambda=0}^{D-1} \sum_{\nu=1}^{N} e(f(\lambda + D\nu))$$

$$\ll \sum_{\lambda=0}^{D-1} (|T_\lambda(N)| + D \frac{(\log X)^H}{X} \sum_{\nu=1}^{N} |T_\lambda(\nu)|) . \tag{10}$$

We next prove that

$$|T_\lambda(\nu)| = |\sum_{n=1}^{\nu} e(\varphi_\lambda(n))| \ll \frac{X}{D(\log X)^{G+H}} \tag{11}$$

for all ν with $1 \le \nu \le N$. For this, we may assume that

$$\frac{X}{D(\log X)^{G+H}} \ll \nu \quad (\le N \le \frac{X}{D}) . \tag{12}$$

We put $p(x) = \mathcal{P}_\lambda(x)$ in Lemma 1, so that the leading coefficient is $\gamma = D^{k_s}\beta_s$. Suppose first that there is a rational number A/B with $(A, B) = 1$ such that

$$(\log X)^{H'} \le B \le X^{k_s}(\log X)^{-H'} \tag{13}$$

and

$$|\gamma - \frac{A}{B}| \le B^{-2} ,$$

where $H' = 2^{k_s+1}(G+H+1) + k_sK$. Then (13) together with (12) implies

$$(\log \nu)^{h'} \le B \le \nu^{k_s}(\log \nu)^{-h'} ,$$

where $h' = (k_s-1)^2 + 2^{k_s}(G+H)$. Hence we have by Lemma 2

$$|T_\lambda(\nu)| \ll \nu(\log \nu)^{-(G+H)} \ll \frac{X}{D(\log X)^{G+H}} .$$

If there is no such rational number, we can choose a rational number A'/B' with $(A', B') = 1$ such that

$$1 \le B' \le (\log X)^{H'} \quad \text{and} \quad |\gamma - \frac{A'}{B'}| \le \frac{(\log X)^{H'}}{B'X^{k_s}} .$$

Then we have

$$D^{k_s}B' \le (\log X)^{H'+k_sK} \le (\log X)^{K_s^*}$$

and

$$|\beta_s - \frac{A'}{D^{k_s}B'}| \le \frac{(\log X)^{H_s^*}}{D^{k_s}B'X^{k_s}} ,$$

which contradicts the assumption on β_s.

Combining (9), (10), and (11), we obtain

$$
\begin{aligned}
&\Big|\sum_{1\le n\le Q} e(f(n))\Big| \\
&\ll (\log X)^H + \sum_{\lambda=0}^{D-1}\Big(1 + DN\frac{(\log X)^H}{X}\Big)\frac{X}{D(\log X)^{G+H}} \\
&\ll X(\log X)^{-G} ,
\end{aligned}
$$

and the proof is completed.

Lemma 3. ([5], Lemma 4.8). Let $f(x)$ be a real differentiable function in the interval $(a, b]$, let $f'(x)$ be monotonic, and let $|f'(x)| < 1/2$. Then

$$\sum_{a<n\le b} e(f(n)) = \int_a^b e(f(x))dx + O(1) .$$

Lemma 4. ([1], Chap. 1, §1.). Let $f(x) = a_0x^k + a_1x^{k-1} + \cdots + a_k$ be a polynomial of degree k with integral coefficients and let g be a positive integer with $(q, a_0, a_1, \cdots, a_{k-1}) = 1$. Then

$$\Big|\sum_{1\le n\le q} e(\tfrac{1}{q}f(n))\Big| \ll_{(k)} q^{1 - \frac{9}{10k}} .$$

§4. Proof of the inequality (6).

In this section, we shall prove (6) for those j for which at least one of the coefficients of $g(x)$ has no rational approximations with *small* denominators in a sense stated in Lemma 2.

To estimate the sum

$$\sum_{n_j<n\le n_j+X_j} I\Big(\frac{g(n)}{r^m}\Big)$$

in (6), we approximate the function $I(t)$ by functions $I_-(t)$ and $I_+(t)$ periodic with period 1, such that $I_-(t) \le I(t) \le I_+(t)$, having Fourier expansion of the form

$$I_{\pm}(t) = \frac{1}{r^{\ell}} \pm \frac{1}{j} + \sum_{\nu\in\mathbf{Z},\nu\neq 0} A_{\pm}(\nu)e(\nu t)$$

with $|A_{\pm}(\nu)| << \min(|\nu|^{-1}, j\nu^{-2})$, where the constant implied is absolute (cf.[6]). Then we have

$$\sum_{n_j<n\leq n_j+X_j} I(\frac{g(n)}{r^m})$$

$$= \frac{X_j}{r^{\ell}} + O(\frac{X_j}{j}) + O(\sum_{\nu=1}^{j^2}\frac{1}{\nu} \mid \sum_{n_j<n\leq n_j+X_j} e(\frac{\nu}{r^m} g(n))|) \ . \qquad (14)$$

We shall evaluate

$$|\sum_{n_j<n\leq n_j+X_j} e(\frac{\nu}{r^m} g(n))|$$

with $\ell \leq m \leq C \log j$ and $1 \leq \nu \leq j^2$, by making use of Lemma 2 inductively.

Let the polynomial $g(x)$ be of the form

$$g(x) = \alpha_0 x^{k_0} + \alpha_1 x^{k_1} + \cdots + \alpha_d x^{k_d} ,$$

where $k = k_0 > k_1 > \cdots > k_d \geq 0$ and $\alpha_0, \cdots, \alpha_d$ are nonzero real numbers. We may assume $k_d \geq 1$ in estimating the exponential sum written above. We put in Lemma 2

$$f(x) = r^{-m}\nu g(x)$$

so that

$$\beta_i = r^{-m}\nu\alpha_i \quad (0 \leq i \leq d) .$$

We choose a constant $c > 0$ such that $cr^{j/k} \geq X_j$ for all $j \leq J$, and define a parameter X by

$$X = X(j) = cr^{j/k} \qquad (j_0 \leq j \leq J) .$$

Then

$$\log X = j^{1+o(1)} ,$$

as $j \to \infty$, so that

$$r^m \leq (\log X)^{C\log r+o(1)} , \quad \nu \leq (\log X)^{2+o(1)} ,$$

since $m \leq C \log j$ and $\nu \leq j^2$.

Case 0). Let j be an integer with $j_0 \leq j \leq J$ for which there is no rational number a_0/b_0 with $(a_0, b_0) = 1$ such that

$$1 \leq b_0 \leq (\log X)^{2h_0} \quad \text{and} \quad |\alpha_0 - \frac{a_0}{b_0}| \leq \frac{(\log X)^{h_0}}{b_0 X^{k_0}} ,$$

where

$$h_0 = H_0^* + C \log r + 1 , \quad H_0^* = 2^{k_0+1}(G+1) .$$

The set of all j of this property will be denoted by $\mathcal{J}_0$. If $j \in \mathcal{J}_0$, there is no rational number A_0/B_0 with $(A_0, B_0) = 1$ such that

$$1 \leq B_0 \leq (\log X)^{2H_0^*} \quad \text{and} \quad |\beta_0 - \frac{A_0}{B_0}| \leq \frac{(\log X)^{H_0^*}}{B_0 X^{k_0}} ,$$

since, if there is such a rational number A_0/B_0, we have

$$1 \leq \nu B_0 \leq (\log X)^{2H_0^*+3} \leq (\log X)^{2h_0}$$

and

$$|\alpha_0 - \frac{r^m A_0}{\nu B_0}| \leq \frac{(\log X)^{H_0^*+C\log r+1}}{\nu B_0 X^{k_0}} \leq \frac{(\log X)^{h_0}}{\nu B_0 X^{k_0}} ,$$

which contradict the assumptions in this case. Hence we can apply Lemma 2 with $s = 0$ and obtain

$$\left|\sum_{n_j<n\leq n_j+X_j} e\left(\frac{\nu}{r^m} g(n)\right)\right| \ll \frac{X}{(\log X)^G} \tag{15}$$

for all $j \in \mathbf{J}_0$.

Case s). Let $1 \leq s \leq d$. We put

$$H_0^* = 2^{k_0+1}(G+1) \ , \quad H_0^* = H_0 + 2^{k_0+1}(G+1)$$

and define H_i^* and H_i $(1 \leq i \leq d)$ inductively by

$$H_i^* = 2^{k_i+1}(G + H_{i-1} + 1) + 2k_i(H_0 + \cdots + H_{i-1}) \ ,$$

$$H_i = H_i^* + 2(C \log r + 1) \ .$$

Also we write

$$h_i = H_i^* + C \log r + 1 \quad (0 \leq i \leq d)$$

Let j be an integer with $j_0 \leq j \leq J$ for which there are rational numbers $a_0/b_0, \cdots, a_{s-1}/b_{s-1}$ such that

$$1 \leq b_i \leq (\log X)^{2h_i} \quad \text{and} \quad \left|\alpha_i - \frac{a_i}{b_i}\right| \leq \frac{(\log X)^{h_i}}{b_i X^{k_i}} \quad (0 \leq i < s) \ ,$$

but there is no rational number a_s/b_s with $(a_s, b_s) = 1$ such that

$$1 \leq b_s \leq (\log X)^{2h_s} \quad \text{and} \quad \left|\alpha_s - \frac{a_s}{b_s}\right| \leq \frac{(\log X)^{h_s}}{b_s X^{k_s}} \ .$$

The set of all j of this property will be denoted by $\mathbf{J}_s$. If $j \in \mathbf{J}_s$, we have

$$1 \le r^m b_i \le (\log X)^{2H_i} \quad \text{and} \quad |\beta_i - \frac{\nu a_i}{r^m b_i}| \le \frac{(\log X)^{H_i}}{r^m b_i X^{k_i}}$$

for $0 \le i < s$, but there is no rational number A_s/B_s with $(A_s, B_s) = 1$ such that

$$1 \le B_s \le (\log X)^{2H_s^*} \quad \text{and} \quad |\beta_s - \frac{A_s}{B_s}| \le \frac{(\log X)^{H_s^*}}{B_s X^{k_s}} \quad ,$$

since otherwise we have a contradiction as in Case 0). Hence, by Lemma 2 with these H_i , H_s^* and $K_i = 2H_i$, $K_s^* = 2H_s^*$, we have again (15) for all $j \in \mathcal{J}_s$

Choosing $G = 3$ in (15), we get

$$|\sum_{n_j<n\le n_j+X_j} e(\frac{\nu}{r^m} g(n))| \ll \frac{r^{j/k}}{j^2} \quad ,$$

for all $(\ell\le)\ m \le C \log j$, $(1\le)\ \nu \le j^2$, and $j \in \mathcal{J}_0 \cup \cdots \cup \mathcal{J}_d$, and hence by (14)

$$\sum_{\ell\le m\le C\log j} \sum_{n_j<n\le n_j+X_j} (I(\frac{g(n)}{r^m}) - \frac{1}{r^\ell}) = O(\frac{r^{j/k}}{j})$$

for all $j \in \mathcal{J}_0 \cup \cdots \cup \mathcal{J}_d$.

It remains to prove (6) for $j \notin \mathcal{J}_0 \cup \cdots \cup \mathcal{J}_d$ with $j_0 \le j \le J$, which will be done in the next section.

§5. Proof of the inequality (6). Continued.

Let $\mathcal{J}_{d+1}$ be the set of all integers j with $j_0 \le j \le J$ for which there are rational numbers $a_0/b_0, \cdots, a_d/b_d$ such that

$$1 \le b_i \le (\log X)^{2h_d} \quad \text{and} \quad |\alpha_i - \frac{a_i}{b_i}| \le \frac{(\log X)^{h_d}}{b_i X^{k_i}}$$

for all $i = 0, 1, \cdots, d$, where h_d is defined in §4. Then by definition

$$\{j_0, j_0+1, \cdots, J\} = \mathcal{J}_0 \cup \cdots \cup \mathcal{J}_d \cup \mathcal{J}_{d+1} .$$

In the rest of this paper, we shall prove (6) for all $j \in \mathcal{J}_{d+1}$ by a method different form that used in the preceding section. We assume $k_d \ge 1$. The proof is valid also in the case of $k_d = 0$.

Let $j \in \mathcal{J}_{d+1}$. We denote by a_* the greatest common divisor of $a_0, \cdots, a_d$ and by b^* the least common multiple of $b_0, \cdots, b_d$. Then $(a_*, b^*) = 1$ and

$$1 \le b^* \le j^h , \quad 1 \le a_* \ll j^h ,$$

where $h = 2(d+1)h_d + 1$. We then define integers $c_0, \cdots, c_d$ by

$$\frac{a_i}{b_i} = \frac{a_* c_i}{b^*}$$

so that $(b^*, a_*c_0, \cdots, a_*c_d) = 1$. We write for brevity $L_1 = \log j$ and $L_w = \log L_{w-1}$ $(2 \le w \le w_j)$, where w_j is the greatest integer w for which $L_w \ge 3$.

For a given positive constant C, we have

$$\sum_{\ell \le m \le C\log j} \Bigl| \sum_{n_j < n \le n_j + X_j} (I(r^{-m}g(n)) - r^{-\ell}) \Bigr|$$

$$\le \sum_{1 \le w \le w_j} \sum_{VL_{w+1} < m \le VL_w} \Bigl| \sum_{n_j < n \le n_j + X_j} (I(r^{-m}g(n)) - r^{-\ell}) \Bigr| + VX_j \tag{16}$$

where $V \ge C$ is a constant which will be chosen suitably at the end of the proof. For each w, there are functions $I_w^-(t)$ and $I_w^+(t)$, periodic with period 1, such that $I_w^-(t) \le I(t) \le I_w^+(t)$, having Fourier expansion of the form

$$I_w^{\pm}(t) = r^{-\ell} \pm L_w^{-2} + \sum_{\nu\in\mathbf{Z},\nu\neq 0} A_w^{\pm}(\nu)e(\nu t) ,$$

with $|A_w^{\pm}(\nu)| \le \min(|\nu|^{-1}, L_w^2 \nu^{-2})$, (cf.[6]).

Then it follows that

$$\sum_{n_j<n\le n_j+X_j} (I(r^{-m}g(n)) - r^{-\ell})$$

$$<< X_j L_w^{-2} + \sum_{1\le\nu\le L_w^4} \nu^{-1} \Big| \sum_{n_j<n\le n_j+X_j} e(r^{-m}\nu g(n)) \Big| . \qquad (17)$$

Here we have, for any fixed m with $VL_{w+1} < m \le VL_w$ and ν with $1 \le \nu \le L_w^4$,

$$\sum_{n_j<n\le n_j+X_j} (I(r^{-m}\nu g(n))$$

$$= \sum_{0\le\lambda\le r^m b^*} e\Big(\frac{\nu\, a_*}{r^m b^*} \sum_{i=0}^{d} c_i \lambda^{k_i}\Big) \sum_{\substack{\nu;\, n=\lambda+r^m b^* \nu \\ n_j<n\le n_j+X_j}} e\Big(\frac{\nu}{r^m} \sum_{i=0}^{d} \Omega_i n^{k_i}\Big)$$

$$= \sum_{0\le\lambda\le r^m b^*} e\Big(\frac{\nu\, a_*}{r^m b^*} \sum_{i=0}^{d} c_i \lambda^{k_i}\Big)\Big\{ \int_{\substack{n_j<n\le n_j+X_j \\ x=\lambda+r^m b^* y}} e\Big(\frac{\nu}{r^m} \sum_{i=0}^{d} \Omega_i x^{k_i}\Big)dy + O(1)\Big\}$$

$$= \sum_{0\le\lambda\le r^m b^*} e\Big(\frac{\nu\, a_*}{r^m b^*} \sum_{i=0}^{d} c_i \lambda^{k_i}\Big) \frac{1}{r^m b^*} \int_{n_j<n\le n_j+X_j} e\Big(\frac{\nu}{r^m} \sum_{i=0}^{d} \Omega_i x^{k_i}\Big)dx + O(r^m b^*) ,$$

using Lemma 3, where $\Omega_i = \alpha_i - a_i/b_i$. Defining now rational numbers R_i/Q $(0 \le i \le s)$ by

$$\frac{R_i}{Q} = \frac{\nu}{r^m}\,\frac{a_* c_i}{b^*} \quad \left(= \frac{\nu}{r^m}\,\frac{a_i}{b_i}\right) \quad \text{with} \quad (Q, R_0, R_1, \cdots, R_d) = 1$$

and applying Lemma 4, we get

$$\sum_{n_j<n\le n_j+X_j} e(r^{-m}\nu g(n))$$

$$\ll \frac{r^m b^*}{Q}\, Q^{1-9/(10k)}\, \frac{X_j}{r^m b^*} + r^m b^*$$

$$\ll X_j Q^{-9/(10k)} + r^{m_j h}$$

and hence by (17)

$$\sum_{VL_{w+1}<m\le VL_w} \Big| \sum_{n_j<n\le n_j+X_j} (I(r^{-m}g(n)) - r^{-\ell}) \Big|$$

$$\ll \sum_{VL_{w+1}<m\le VL_w} \Big(X_j L_w^{-2} + X_j \sum_{1\le\nu\le L_w^4} \nu^{-1} Q^{-9/(10k)} + L_{w+1} r^{m_j h}\Big)$$

$$\ll r^{j/k} L_w^{-1} + X_j \sum_{VL_{w+1}<m\le VL_w} \sum_{1\le\nu\le L_w^4} \nu^{-1} Q^{-9/(10k)} . \tag{18}$$

Therefore it follows from (16) and (18) that

$$\sum_{\ell\le m\le C\log j} \Big| \sum_{n_j<n\le n_j+X_j} (I(r^{-m}g(n)) - r^{-\ell}) \Big|$$

$$\ll r^{j/k} + r^{j/k} \sum_{1\le w\le w_j} \sum_{VL_{w+1}<m\le VL_w} \sum_{1\le\nu\le L_w^4} \nu^{-1} Q^{-9/(10k)} . \tag{19}$$

But, since $\nu Q = r^m R_i b_i / a_i \gg r^m$ by the difinition of R_i/Q , we obtain

$$\sum_{1\le w\le w_j} \sum_{VL_{w+1}<m\le VL_w} \sum_{1\le\nu\le L_w^4} \nu^{-1} Q^{-9/(10k)}$$

$$\ll \sum_{1\le w\le w_j} \sum_{VL_{w+1}<m\le VL_w} \sum_{1\le\nu\le L_w^4} (r^m)^{-9/(10k)}$$

$$\ll \sum_{1\le w\le w_j} VL_w \cdot L_w^4 \left(r^{VL_{w+1}}\right)^{-9/(10k)}$$

$$\ll V \sum_{1\le w\le w_j} L_w^{5-\frac{9\log r}{10k}V} \ll V \sum_{1\le w\le w_j} L_w^{-1} \ll 1 ,$$

provided that $V \ge \max(C, 20k/(3\log r))$. Combining this with (19), we have (6) for all $j \in \mathbf{J}_{d+1}$. Therefore, (6) is proved for any j with $j_0 \le j \le J$, and the proof of the theorem is completed.

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Y.-N. Nakai
Department of Mathematics,
Faculty of Education,
Yamanashi University,
Kofu, 400 Japan.

I. Shiokawa
Department of Mathematics,
Keio University,
Hiyoshi, Yokohama 223
Japan