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Limit Theorem and Large Deviation Principle for Voronoi tessellation generated by Gibbs point process

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Dedicated to Professor Joel L. Lebowitz on the occasion of his 60th birthday.

Abstract. The Voronoi tessellation generated by a Gibbs point process is considered. Using the algebraic formalism of polymer expansion, the limit theorem and the large deviation principle for the number of Voronoi vertices are proved.

1. Introduction

The Voronoi tessellation has been used in various fields such as mathematical statistics and statistical physics to study a topological structure of a system. It is defined for a random point process as an aggregate of cells which covers a space without overlapping. Let us consider a random point process $\xi = \{x_i\}_{i=1}^n$ in $V = (-\frac{1}{2}L, \frac{1}{2}L)^d$, $d = 2$ or 3. For each particle x_i, the set T_{x_i} of points in V having x_i as closest particle forms an open convex polygon (d=2) or polyhedron (d=3), and the aggregate $\mathcal{V} = \{T_{x_i}\}_{i=1}^n$ of such convex polygons or polyhedrons tessellates the space V. This random tessellation $\mathcal{V} = \{T_{x_i}\}$ constructed as above is called a Voronoi tessellation.

The mathematical investigation of the Voronoi tessellation has been made mainly for the Poisson point process ([1]∼[4]). For this process no interaction works on the system of particles. However, we consider a system in which a nonnegative(repulsive) interaction works between all pairs of particles. R.L.Dobrushin, O.Lanford and D.Ruelle [5∼ 9] introduced the notion of Gibbs point process or Gibbs measure to investigate an interacting particle system in an equilibrium state. In particular, it is used to investigate the phenomena of phase transitions mathematically. See [10] and [11] for detail.

The topological structure of Voronoi tessellation is obtained by studying the stochastic behavior of the number of particles N_V and the number of vertices D_V in the Voronoi tessellation. Other geometrical parameters such as the number of polygon edges B_V can be expressed in terms of N_V and D_V from the Euler identity.

For the Poisson point process Meijering [4] gave the expectations of the number of vertices D_V, the length of a polygon edge and the area of a polygon. R.E.Miles [2] defined the generalized Voronoi tessellations $\mathcal{V}_n$ $(n = 2, 3, ...)$ and determined the expectations of these geometrical parameters for $\mathcal{V}_n$. K.Tsuchikura [3] proved the central limit theorem for D_V in Voronoi tessellation generated by the Poisson point process.

In this paper we consider the Voronoi tessellation generated by the Gibbs point process with a nonnegative pair interaction. Using the algebraic formalism of the polymer(cluster) expansion we first express the expectation and the variance of D_V in terms of Ursell functions and derive its asymptotic estimates. Concerning the asymptotic behavior of N_V for the Gibbs point process, many results are obtained in various models. See [8] or [9] for references. We prove the existence of the limits

$$\frac{E_V[D_V]}{|V|} \to e_\beta(z) \qquad \text{and} \qquad \frac{Var_V(D_V)}{|V|} \to \tau_\beta^2(z) \text{ as } V \to \mathbf{R}^d,$$

where $E_V[D_V]$ and $Var_V(D_V)$ are the expectation and the variance of D_V with respect to the Gibbs measure in V with an activity z and the reciprocal temperature β, respectively(Theorem 1).

Secondly, we prove the central limit theorem for D_V. (Theorem 2)

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The central limit theorem for N_V for Gibbs point processes has already been investigated by several authors [12,13].

Finally, we prove the large deviation principle for D_V in the two dimensional system, that is, the probability distribution $p_V(K)$ of $\frac{D_V}{|V|}$ decays exponentially for all closed subsets $K \subset \mathbf{R}$ which do not contain $e_\beta(z)$. The decay rate is given by $-\inf_{x\in K} I(x)$, where $I(x)$ is some nonnegative function on $\mathbf{R}$ taking minimum value 0 at $x = e_\beta(z)$.(Theorem 3)

2. Gibbs measure and Voronoi tessellation

2.1 Gibbs measure

In this section we introduce necessary terminologies for the Gibbs measure and the Voronoi tessellation.

Let us consider a system of particles in a cube $V = (-\frac{L}{2}, \frac{L}{2})^d$, $d = 2$ or 3. We define a configuration space Ω_V of particles in V by

$$\Omega_V = \bigcup_{n=0}^{\infty} \Omega_{V,n} \quad , \text{ and } \quad \Omega_{V,n} = (V^n)'/S_n, \tag{2.1}$$

where $(V^n)' = \{(x_1, \cdots, x_n) \in V^n; x_i \neq x_j (i \neq j)\}$ and S_n is a symmetric group of order n.

For any $W \subset V$ we define a σ-algebra $\mathcal{B}_W$ as the smallest σ-algebra generated by subsets of Ω_V

$$\{\xi \in \Omega_V \; ; \; N_A(\xi) = k\} \quad , \quad A \subset W \quad \text{and} \quad k \geq 0,$$

where $N_A(\xi)$ is the number of particles of ξ in $A \in \mathcal{B}(\mathbf{R}^d)$.

Throughout this paper we use the following abbreviations

$$\underline{x}_n = (x_1, \cdots, x_n), \quad \{\underline{x}_n\} = \{x_1, \cdots, x_n\}, \quad \text{and} \quad d\underline{x}_n = dx_1 \cdots dx_n.$$

An interaction between two particles is defined by a $\mathbf{R}$ -valued measurable function on $\mathbf{R}^d$ satisfying the following conditions:

$$\Phi(x) \geq 0 \qquad \text{for all } x \in \mathbf{R}^d, \tag{2.2}$$

$$\Phi(-x) = \Phi(x) \qquad \text{for all } x \in \mathbf{R}^d, \tag{2.3}$$

(2.4) There exists a positive number r_0 such that $\Phi(x) = 0$ whenever $|x| > r_0$.(finite range)
For simplicity we assume that $r_0 = 1$.

To each configuration $\xi = \{\underline{x}_n\} \in \Omega_V$ we associate an interaction energy

$$U(\xi) = U(\underline{x}_n) = \sum_{1\leq i<j\leq n} \Phi(x_i - x_j).$$

Definition 2.1 *A probability measure $P_V(\cdot)$ on $(\Omega_V, \mathcal{B}_V)$ is called a Gibbs measure in V with an activity $z > 0$ and a reciprocal temperature $\beta > 0$ if the equality*

$$\int_{\Omega_V} f(\xi) P_V(d\xi) = \frac{1}{Z_V} \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{V^n} d\underline{x}_n f(\underline{x}_n) \exp\{-\beta U(\underline{x}_n)\}$$

holds for any $\mathcal{B}_V$ -measurable function $f \geq 0$.
The normalizing factor Z_V is called a partition function and is given explicitly as follows

$$Z_V = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{V^n} d\underline{x}_n \exp\{-\beta U(\underline{x}_n)\}.$$

Furthermore we define an interaction energy $U(\xi|\zeta)$ *with a boundary condition* $\zeta = \{\underline{v}_k\} \subset V^c$ *by*

$$U(\xi|\zeta) = U(\xi) + W(\xi|\zeta),$$

where $W(\xi|\zeta) = \sum_{1\le i\le n, 1\le j\le k} \Phi(x_i - v_j)$.

Changing the role of $U(\xi)$ *to* $U(\xi|\zeta)$ *we define a Gibbs measure* $P_{V,\zeta}(\cdot)$ *with a boundary condition* ζ *and a partition function* $Z_{V,\zeta}$ *with* ζ.

It is well known that the following limit exists

$$F_\beta(z) \;=\; \lim_{V\to\mathbf{R}^d} \frac{1}{|V|}\log Z_V \qquad \text{for all } z \ge 0.$$

In the above (and also in the sequel) $V \to \mathbf{R}^d$ means that $L \to \infty$. This function $F_\beta(z)$ is called a free energy, and is used for the study of the asymptotic behavior of $E_V(N_V)$ as $V \to \mathbf{R}^d$, where $E_V(\cdot)$ is the expectation with respect to $P_V(\cdot)$.

Using the algebraic method of cluster expansion we expand $F_\beta(z)$ in powers of z,

(2.6) $$F_\beta(z) \;=\; \sum_{n=0}^{\infty} b_n z^n,$$

where b_n is described in terms of Ursell functions.

The definition of Ursell functions and their properties will be stated in Section 4. Also we summarize the algebraic method of the cluster expansion in Section 4.

We denote the radius of convergence of the power series (2.6) by $\mathcal{R}_0$. For any $z \in (0, \mathcal{R}_0)$ the asymptotic behavior of $E_V(N_V)$ and the variance $\mathrm{var}_V(N_V)$ of N_V are obtained as follows,

(2.7) $$\lim_{V\to\mathbf{R}^d} \frac{1}{|V|} E_V(N_V) = \rho_\beta(z),$$

(2.8) $$\lim_{V\to\mathbf{R}^d} \frac{1}{|V|} \mathrm{var}_V(N_V) = \sigma_\beta^2(z),$$

where $\rho_\beta(z) = zF'_\beta(z)$ and $\sigma_\beta^2(z) = z\rho'_\beta(z)$.

2.2 Voronoi tessellation and Delaunay network

Let $\xi = \{x_1, \cdots, x_n\} \subset V$ be the point process which has the Gibbs measure as its distribution.

For any particle $x_i \in \xi$ we define a territory of x_i by

$$T_{x_i}(\xi) = \{p \in V \;;\; |p - x_i| < |p - x_j| \text{ for all } x_j \in \xi (j \ne i)\}.$$

Then $T_{x_i}(\xi)$ is an open convex polygon and $\mathcal{V} = \{T_{x_i}(\xi)\}_{i=1}^n$ is a convex polygonal random tessellation of V. Ignoring the null set of polygon boundaries, every point of V belongs to one and only one territory of $\mathcal{V}$. This random tessellation is called the *Voronoi tessellation*. Also we call the vertices in the tessellation *Voronoi vertices*.

Let us first consider the two dimensional system. Assume that $T_{x_i}(\xi)$ has m neighbouring territories $\{T_{x_{k_j}}(\xi)\}_{j=1}^m$. Each side of $T_{x_i}(\xi)$ is the portion of the perpendicular bisector between x_i and some x_{k_j}. A probability that more than four particles exist on the same circle is zero with respect to the Gibbs measure, so that each vertex of $T_{x_i}(\xi)$ is almost surely a circumcenter of x_i and an adjacent pair x_{k_j}, $x_{k_{j+1}}$. (See Fig.1.)

When a circumcenter of the triangle with vertices x_{i_1}, x_{i_2} and x_{i_3} is a vertex of a polygon of $\mathcal{V}$, we call such a triangle "Delaunay triangle". The network consisting of Delaunay triangles is called the

Delaunay network. To each configuration ξ a Delaunay network is given almost surely with respect to the Gibbs measure. When a Delaunay network is given there is a one to one correspondence of the set of all vertices of Voronoi tessellation and the set of Delaunay triangles. The following property of Delaunay triangles plays an important role for the study of Voronoi tessellation.

Lemma 2.1 *When a set of Delaunay triangles is given there is no particle in any interior of the circumscribed circle of Delaunay triangle.*

In the case of d=3 a territory $T_{x_i}(\xi)$ is an open convex polyhedron and a vertex of a territory is a point with the same distance from four particles $\{x_{i_k}\}_{k=1}^4$ of four neighbouring territories $\{T_{x_{i_k}}(\xi)\}_{k=1}^4$ almost surely. We call the tetrahedron with vertices $\{x_{i_k}\}_{k=1}^4$ "Delaunay tetrahedron". In the same way as the case of d=2 a Delaunay tetrahedron is assigned to each vertex of Voronoi tessellation. Also the same property as Lemma 2.1 holds for three dimensional system.

Denote by $N_V(\xi)$ and $D_V(\xi)$ the number of particles in V and the number of Voronoi vertices in V respectively. For two dimensional system, if we denote by $B_V(\xi)$ the number of edges in the Voronoi tessellation, then $B_V(\xi)$ is expressed in terms of $N_V(\xi)$ and $D_V(\xi)$ from the Euler's identity;

$$N_V(\xi) + D_V(\xi) - B_V(\xi) = 1. \tag{2.9}$$

Let us note that $N_V(\xi)$ and $D_V(\xi)$ correspond to the number of vertices and Delaunay triangles(d=2) (or tetrahedrons(d=3)) in the Delaunay network, respectively. We use this fact in Section 5.

3. Statement of Results

Theorem 1

Assume the conditions (2.2)~(2.4) on Φ. Then there exist functions $e_\beta(z)$ and $\tau_\beta(z)$ defined on $[0,\mathcal{R}_0)$ for any $\beta > 0$, and the following estimates hold for any $z \in (0,\mathcal{R}_0)$

$$|E_V(D_V) - e_\beta(z)|V|| \quad < \quad C_\beta^1(z)|V|^{\frac{d-1}{d}}, \tag{3.1}$$

$$|\mathrm{var}_V(D_V) - \tau_\beta(z)^2|V|| < C_\beta^2(z)|V|^{\frac{d-1}{d}}, \tag{3.2}$$

where $C_\beta^1(z)$ and $C_\beta^2(z)$ are some positive constants.

Furthermore $e_\beta(z) = 2\rho_\beta(z)$ if d=2.

As a corollary of this theorem we have the following probabilistic estimate

$$P_V(|\frac{D_V(\cdot)}{|V|} - e_\beta(z)| \geq \frac{g(V)}{|V|^{\frac{1}{d}}}) \quad \leq \quad \frac{C_\beta^3(z)^2}{g(V)^2}, \tag{3.3}$$

for any function $g(V)$ of V satisfying

$$g(V) \to \infty \quad \text{and} \quad g(V)|V|^{-\frac{1}{d}} \to 0 \qquad \text{as } V \to \mathbf{R}^d,$$

where $C_\beta^3(z)$ is a positive constant.

The second result concerns a central limit theorem for $D_V(\cdot)$. We define a random variable $Y_V(\cdot)$ by

$$Y_V(\cdot) = \frac{1}{\sqrt{|V|}}\{D_V(\cdot) - E_V(D_V)\}.$$

Theorem 2

For any $z \in (0,\mathcal{R}_0)$

$$Y_V(\cdot) \to N(0,\tau_\beta(z)^2)$$

in law as $V \to \mathbf{R}^d$, *where* $N(0,\tau_\beta(z)^2)$ *is the normal distribution with mean* 0 *and variance* $\tau_\beta(z)^2$.

In particular, for a two dimensional system, we prove that

$$\tau_\beta^2(z) = 4z\rho_\beta'(z) \geq (1 - zC(\beta))\rho_\beta(z) > 0$$

for any $z \in (0, \mathcal{R}_0)$, where

$$C(\beta) = \int_{\mathbf{R}^d} dx(1 - \exp\{-\beta\Phi(x)\}).$$

Let us remind that

$$\frac{1}{|V|}\mathrm{var}_V(N_V) \to z\rho_\beta'(z) \qquad \text{as } V \to \mathbf{R}^2.$$

Also we prove that the limiting covariance matrix of N_V and D_V degenerates, more explicitly

$$\frac{1}{|V|}\begin{pmatrix} \mathrm{var}_V(N_V) & \mathrm{cov}_V(N_V, D_V) \\ \mathrm{cov}_V(D_V, N_V) & \mathrm{var}_V(D_V) \end{pmatrix} \longrightarrow \begin{pmatrix} z\rho_\beta'(z) & 2z\rho_\beta'(z) \\ 2z\rho_\beta'(z) & 4z\rho_\beta'(z) \end{pmatrix},$$

where $\mathrm{cov}_V(D_V, N_V) = E_V[(N_V - E_V(N_V))(D_V - E_V(D_V))]$.

Finally we shall state the result about the large deviation principle for the probability distribution of $D_V(\cdot)/|V|$.

We restrict our argument on a two dimensional system and assume that the hard core condition on $\Phi(\cdot)$, i.e. $\Phi(x) = \infty$ whenever $|x| < r_1$ for some $r_1 \leq r_0 = 1$.

In section 8 we prove that the limit

$$f_z(\theta) = \lim_{V \to \mathbf{R}^2} \frac{1}{|V|}\log E_V[\exp\{\theta D_V\}]$$

exists for any $\theta \in \mathbf{R}$ and $z \in (0, \infty)$ and satisfies

$$-z \leq f_z(\theta) = F_\beta(ze^{2\theta}) - F_\beta(z) \leq ze^{2\theta}.$$

Define the rate function $I_z(x)$ by

$$I_z(x) = \sup_{\theta \in \mathbf{R}}\{\theta x - f_z(\theta)\}.$$

Now we shall state our final result..

Theorem 3

(i) *For any closed set* $F \subset \mathbf{R}$

$$\overline{\lim_{V \to \mathbf{R}^2}} \frac{1}{|V|}\log P_V(\frac{D_V}{|V|} \in F) \leq -\inf_{x \in F} I_z(x).$$

(ii) *For any open set* $G \subset \mathbf{R}$

$$\underline{\lim_{V \to \mathbf{R}^2}} \frac{1}{|V|}\log P_V(\frac{D_V}{|V|} \in G) \geq -\inf_{x \in G} I_z(x).$$

4. cluster expansion

In this section we summarize the algebraic formalism of the cluster expansion in powers of an activity z. See [11] for detail.

Let $\mathcal{A}$ be a set of sequences ψ;

$$\psi = \{\psi(\underline{x}_n)\}_{n \geq 0},$$

where $\psi(\underline{x}_n)$ is a bounded complex valued Lebesgue measurable function on $\mathbf{R}^{nd}$. Notice that 0-th component $\psi(\emptyset)$ of ψ is a complex number. We denote by X a sequence $\underline{x}_n = (x_1, x_2, \cdots, x_n)$ of finite number of points in $\mathbf{R}^d$, and write

$$\psi(X) = \psi(\underline{x}_n).$$

We say (X_1, X_2) is a partition of X and write $X_1 + X_2 = X$, if X_1 is a subsequence of X and $X_2 = X \backslash X_1$. We define a product in $\mathcal{A}$ by

$$\psi_1 * \psi_2(X) = \sum_{X_1+X_2=X} \psi_1(X_1)\psi_2(X_2),$$

where the sum is taken over all partitions of X. With this product and a sum defined naturally $\mathcal{A}$ is a commutative algebra with a unit element $\mathbf{1}$ defined by

$$\mathbf{1}(X) = \begin{cases} 1, & \text{if } X = \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

where $\emptyset$ means the configuration of no particle.

We define subspaces $\mathcal{A}_0$ and $\mathcal{A}_1$ by

$$\mathcal{A}_0 = \{\psi \in \mathcal{A} : \psi(\emptyset) = 0\} \quad \text{and} \quad \mathcal{A}_1 = \{\psi \in \mathcal{A} : \psi(\emptyset) = 1\}, \text{ respectively.}$$

The power series expansion of the exponential yields a well-defined mapping from $\mathcal{A}_0$ to $\mathcal{A}_1$:

$$\text{Exp}\psi(X) = \mathbf{1}(X) + \sum_{n=1}^{\infty} \frac{\overbrace{\psi * \cdots * \psi}^{n \text{ times}}(X)}{n!}. \tag{4.1}$$

We remind that the above sum is a finite sum for any X, because some subsequence X_i of X must be $\emptyset$ for sufficiently large n.

As an inverse mapping of Exp we define a logarithm mapping from $\mathcal{A}_1$ to $\mathcal{A}_0$ by

$$\text{Log}\psi(X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \overbrace{\psi_0 * \cdots * \psi_0}^{n \text{ times}}(X), \qquad \psi \in \mathcal{A}_1, \tag{4.2}$$

where $\psi_0 = \psi - \mathbf{1} \in \mathcal{A}_0$. The above sum is also a finite sum for any X.

Now we define the Boltzmann's factor $\psi_\beta(X)$ and the Ursell function $\varphi_\beta(X)$ by

$$\psi_\beta(X) = \exp\{-\beta U(X)\} \quad \text{and} \quad \varphi_\beta(X) = \text{Log}\psi_\beta(X),$$

respectively. It is easily seen that $\varphi_\beta(X)$ is translation invariant.

The connection between the Boltzmann's factor and the Ursell function will be stated in the following lemma.

Lemma 4.1 *Let $\chi(\cdot)$ be a Lebesgue integrable function on $\mathbf{R}^d$. If the power series*

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\mathbf{R}^{nd}} d\underline{x}_n \psi_\beta(\underline{x}_n) \prod_{i=1}^{n} \chi(x_i) \quad \text{is absolutely convergent,}$$

then the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\mathbf{R}^{nd}} d\underline{x}_n \varphi_\beta(\underline{x}_n) \prod_{i=1}^{n} \chi(x_i)$ *is also absolutely convergent and the following relation holds*

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\mathbf{R}^{nd}} d\underline{x}_n \psi_\beta(\underline{x}_n) \prod_{i=1}^{n} \chi(x_i) = \exp\{\sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\mathbf{R}^{nd}} d\underline{x}_n \varphi_\beta(\underline{x}_n) \prod_{i=1}^{n} \chi(x_i)\}. \tag{4.3}$$

Let us introduce a power series of z,

(4.4) $$F_\beta(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where

$$b_n = \frac{1}{n!}\int_{\mathbf{R}^{nd}} dx_2 \cdots dx_n \varphi_\beta(o, x_2, \cdots, x_n).$$

Lemma 4.2 *If an interaction function* Φ *satisfies* (2.2) $\sim$(2.4), *then the following properties* (1)$\sim$ (3) *are obtained.*

(i) $$(-1)^{n-1}\varphi_\beta(\underline{x}_n) \geq 0 \qquad \textit{for all } n \textit{ and } \underline{x}_n,$$

(ii) $$(-1)^{n-1}b_n \geq 0 \quad \textit{for all} \quad n \quad \textit{and} \quad \frac{1}{n} \leq \frac{|b_n|}{C(\beta)^{n-1}} \leq \frac{n^{n-2}}{n!},$$

(iii) *the radius of convergence* $\mathcal{R}_0$ *of* $F_\beta(z)$ *satisfies*

$$\frac{1}{eC(\beta)} \leq \mathcal{R}_0 \leq \frac{1}{C(\beta)}.$$

We define the Boltzmann's factor $\psi_\beta(X|\zeta)$ with boundary condition $\zeta = \{\underline{y}_k\} \in \mathbf{R}^d$ by

$$\psi_\beta(X|\zeta) = \exp\{-\beta U(X|\zeta)\}.$$

In the same way as $\varphi_\beta(X)$ we define the Ursell function $\varphi_\beta(X|\zeta)$ with boundary condition ζ by

$$\varphi_\beta(X|\zeta) = \mathrm{Log}\psi_\beta(X|\zeta).$$

It follows from a simple calculation that

$$\varphi_\beta(X|\zeta) = \exp\{-\beta W(X|\zeta)\}\varphi_\beta(X).$$

Lemma 4.2 implies that the power series,

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{V^n} d\underline{x}_n \varphi_\beta(\underline{x}_n)$$

is absolutely convergent if $|z| < \mathcal{R}_0$. Hence, it follows from Lemma 4.1 that the partition function Z_V can be rewritten as

(4.5) $$Z_V = \exp\{\sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{V^n} d\underline{x}_n \varphi_\beta(\underline{x}_n)\} \qquad \text{if } |z| < \mathcal{R}_0.$$

From this formula we have

(4.6) $$\lim_{V\to\mathbf{R}^d}\frac{1}{|V|}\log Z_V = \sum_{n=0}^{\infty} b_n z^n \qquad \text{if } |z| < \mathcal{R}_0.$$

Let $N_V(\xi)$ be a number of particles in V of the configuration ξ. In the same way as (4.5) we have

(4.7) $$E_V(\exp\{\sqrt{-1}\,tN_V\}) = \exp\{\sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{V^n} d\underline{x}_n(\exp\{\sqrt{-1}\,tN_V(\underline{x}_n)\} - 1)\varphi_\beta(\underline{x}_n)\},$$

if $|z| < \mathcal{R}_0$. From this formula we can express the expectation and the variance of N_V as follows

(4.8) $$E_V(N_V) = \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!} \int_{V^n} d\underline{x}_n \varphi_\beta(\underline{x}_n),$$

(4.9) $$E_V(N_V^2) - E_V(N_V)^2 = \sum_{n=1}^{\infty} \frac{nz^n}{(n-1)!} \int_{V^n} d\underline{x}_n \varphi_\beta(\underline{x}_n).$$

Using the translation invariance of $\varphi_\beta(\underline{x}_n)$ we can prove the convergence,

(4.10) $$\lim_{V \to \mathbf{R}^d} \frac{1}{|V|} E_V(N_V) = \rho_\beta(z),$$

(4.11) $$\lim_{V \to \mathbf{R}^d} \frac{1}{|V|} \{E_V(N_V^2) - E_V(N_V)^2\} = \sigma_\beta^2(z),$$

uniformly on every compact set contained in $\{z \in \mathbf{R} : 0 \leq z < \mathcal{R}_0\}$, where $\rho_\beta(z) = zF_\beta'(z)$ and $\sigma_\beta^2 = z\rho_\beta'(z)$. The explicit form of $\rho_\beta(z)$ is given by,

$$\rho_\beta(z) = \sum_{n=1}^{\infty} nb_n z^n \qquad \text{for} \quad z \in (0, \mathcal{R}_0).$$

5. Asymptotic behavior of $E_V(D_V)$

In this section we study an asymptotic behavior of the expectation of the number of Voronoi vertices in V. Let us remind that the number of Voronoi vertices is the same as the number of Delaunay triangles (tetrahedrons) with their centers of the circumscribed circles (spheres) in V, and that there is no particle in the interior of the circumscribed circle(sphere) of each Delaunay triangle (tetrahedrons). If a configuration of particles in V is given by $\underline{x}_n = (x_1, \cdots, x_n) \in (V^n)'$, then the number of Voronoi vertices $D_V(\underline{x}_n)$ of $\underline{x}_n$ is written as

(5.1) $$D_V(\underline{x}_n) = D_V(\{\underline{x}_n\}) = \sum_{\{\underline{y}_{d+1}\} \subset \{\underline{x}_n\}} J_{S(\underline{y}_{d+1})}(\underline{x}_n) H_V(\underline{y}_{d+1}),$$

where $S(\underline{y}_{d+1})$ is the interior of the circumscribed circle (sphere) of the triangle (tetrahedron) with vertices $y_1, \cdots, y_{d+1}$,

$$J_A(\underline{x}_n) = J_A(\{\underline{x}_n\}) = \begin{cases} 1 & \text{if } \{\underline{x}_n\} \cap A = \emptyset \\ 0 & \text{otherwise} \end{cases}, \quad A \subset \mathbf{R^d},$$

and

$$H_V(\underline{y}_{d+1}) = \begin{cases} 1 & \text{if the center of } S(\underline{y}_{d+1}) \text{ is in } V \\ 0 & \text{otherwise.} \end{cases}$$

It follows from (5.1) and Lemma 4.1 that

(5.2) $$\begin{aligned} E_V(D_V) &= \frac{z^{d+1}}{(d+1)!} \frac{1}{Z_V} \int_{V^{d+1}} d\underline{x}_{d+1} \psi_\beta(\underline{x}_{d+1}) H_V(\underline{x}_{d+1}) \\ &\quad \cdot \sum_{m=0}^{\infty} \frac{z^m}{m!} \int_{V^m} d\underline{y}_m J_{S(\underline{x}_{d+1})}(\underline{y}_m) \exp\{-\beta W(\underline{x}_{d+1} | \underline{y}_m)\} \psi_\beta(\underline{y}_m) \\ &= \frac{z^{d+1}}{(d+1)!} \int_{V^{d+1}} d\underline{x}_{d+1} H_V(\underline{x}_{d+1}) \psi_\beta(\underline{x}_{d+1}) \exp\{-K_{V,z}(\underline{x}_{d+1})\}, \end{aligned}$$

where

$$K_{V,z}(\underline{x}_{d+1}) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \int_{V^m} d\underline{y}_m \varphi_\beta(\underline{y}_m)(1 - J_{S(\underline{x}_{d+1})}(\underline{y}_m) \exp\{-\beta W(\underline{x}_{d+1}|\underline{y}_m)\}). \tag{5.3}$$

Now we shall prove (3.1) for $e_\beta(z)$ given by

$$e_\beta(z) = \frac{z^{d+1}}{(d+1)!} \int_{\mathbf{R}^{d\times d}} d\underline{x}_d \psi_\beta(o, \underline{x}_d) \exp\{-K_z(o, \underline{x}_d)\}, \tag{5.4}$$

where

$$K_z(\underline{x}_{d+1}) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \int_{\mathbf{R}^{dm}} d\underline{y}_m \varphi_\beta(\underline{y}_m)(1 - J_{S(\underline{x}_{d+1})}(\underline{y}_m)\exp\{-\beta W(\underline{x}_{d+1}|\underline{y}_m)\}). \tag{5.5}$$

For simplicity we shall only prove (3.1) in the case of d=2. The extension of the proof to the case of d=3 is obtained similarly.

First we prepare several lemmas for the prof of (3.1).

Lemma 5.1 *For any z with $z \in (0, \mathcal{R}_0)$ there exist functions $c_1 = c_1(z) > 0$ and $L_0 = L_0(z) > 0$ such that*

$$K_{V,z}(\underline{x}_3) \geq c_1 |S(\underline{x}_3)| \tag{5.6}$$

for any $\underline{x}_3$ with $r(\underline{x}_3) \geq L_0$ and $H_V(\underline{x}_3) = 1$, where $r(\underline{x}_3)$ is a radius of $S(\underline{x}_3)$.

Furthermore we have the same estimate for $K_z(\underline{x}_3)$:

$$K_z(\underline{x}_3) > c_1 |S(\underline{x}_3)| \text{ for any } \underline{x}_3 \text{ with } r(\underline{x}_3) \geq L_0. \tag{5.7}$$

Proof. Since $W(\underline{x}_3|\underline{y}_m) \geq 0$, we have

$$K_{V,z}(\underline{x}_3) \geq \sum_{m=0}^{\infty} \frac{z^m}{m!} \{ \int_{V^m} d\underline{y}_m \varphi_\beta(\underline{y}_m) - \int_{(V \setminus S(\underline{x}_3))^m} d\underline{y}_m \varphi_\beta(\underline{y}_m) \}.$$

We decompose the integrals of the summands as follows,

$$\begin{aligned} \int_{V^m} d\underline{y}_m \varphi_\beta(\underline{y}_m) - \int_{(V \setminus S(\underline{x}_3))^m} d\underline{y}_m \varphi_\beta(\underline{y}_m) &= |S(\underline{x}_3) \cap V| \int_{\mathbf{R}^{2(m-1)}} d\underline{y}_{m-1} \varphi_\beta(o, \underline{y}_{m-1}) \\ &- \int_{V \cap S(\underline{x}_3)} dy_m \int_{\{\underline{y}_{m-1} \in \mathbf{R}^{2(m-1)}; \{\underline{y}_{m-1}\} \cap V^c \neq \emptyset\}} d\underline{y}_{m-1} \varphi_\beta(\underline{y}_m) \\ &+ \int_{V \setminus S(\underline{x}_3)} dy_m \int_{\{\underline{y}_{m-1} \in \mathbf{R}^{2(m-1)}; \{\underline{y}_{m-1}\} \cap S(\underline{x}_3) \neq \emptyset \text{ and } \{\underline{y}_{m-1}\} \subset V\}} d\underline{y}_{m-1} \varphi_\beta(\underline{y}_m). \end{aligned}$$

Here we have used the translation invariance of $\varphi_\beta(\cdot)$.

Since $|S(\underline{x}_3) \cap V| \geq \frac{1}{2}|S(\underline{x}_3)|$ for all $\underline{x}_3$ with $H_V(\underline{x}_3) = 1$, we have

$$\begin{aligned} K_{V,z}(\underline{x}_3) &\geq \frac{1}{2} F_\beta(z) |S(\underline{x}_3)| \\ &- \sum_{m=1}^{\infty} \frac{z^m}{(m-1)!} \int_{\{\underline{y}_m \in \mathbf{R}^{2m}; y_m \in V \cap S(\underline{x}_3) \text{ and } \{\underline{y}_{m-1}\} \cap (V \cap S(\underline{x}_3))^c \neq \emptyset\}} d\underline{y}_m |\varphi_\beta(\underline{y}_m)|. \end{aligned} \tag{5.8}$$

Since $\Phi(\cdot)$ is the interaction of finite range, $\varphi_\beta(\underline{y}_m) = 0$ if $m \le |y_i - y_j|$ for some i and $j \in \{1, \cdots, m\}$. Taking this property into account we have

$$\begin{aligned}
&\sum_{m=1}^{\infty} \frac{z^m}{(m-1)!} \int_{\{\underline{y}_m \in \mathbf{R}^{2m}; y_m \in V \cap S(\underline{x}_3) \text{ and } \{\underline{y}_{m-1}\} \cap (V \cap S(\underline{x}_3))^c \neq \emptyset\}} d\underline{y}_m |\varphi_\beta(\underline{y}_m)| \\
&\le \sum_{m=1}^{\infty} \frac{z^m}{(m-1)!} \int_{\{\underline{y}_m \in \mathbf{R}^2; d(y_m, \partial(V \cap S(\underline{x}_3))) \le m\}} dy_m \int_{\mathbf{R}^{2(m-1)}} d\underline{y}_{m-1} |\varphi_\beta(\underline{y}_m)| \\
&\le |\partial(V \cap S(\underline{x}_3))| \sum_{m=1}^{\infty} m^2 |b_m| z^m,
\end{aligned}$$

where $|\partial W|$ is the length of the boundary ∂W of $W \subset \mathbf{R}^2$.

Combining this estimate with (5.8), we obtain (5.6).

From Lemma 5.1 we see that $e_\beta(z)$ is finite.

Lemma 5.2 *For any $z \in (0, \mathcal{R}_0)$ there exist positive constants $c_2 = c_2(z)$ and $c_3 = c_3(z)$ such that*

$$|K_{V,z}(\underline{x}_3) - K_z(\underline{x}_3)| \le c_2 |\tilde{S}(\underline{x}_3)| \exp\{-c_3 d(\tilde{S}(\underline{x}_3), \partial V)\} \tag{5.9}$$

where $\tilde{S}(\underline{x}_3) = S(\underline{x}_3) \cup B_{r_0}(x_1) \cup B_{r_0}(x_2) \cup B_{r_0}(x_3)$. $\quad (B_r(x) = \{y \in \mathbf{R}^2; |y - x| < r\})$

Proof. From (5.3) and (5.5) we have

$$\begin{aligned}
&|K_{V,z}(\underline{x}_3) - K_z(\underline{x}_3)| \le \sum_{m=1}^{\infty} \frac{z^m}{m!} \int_{\{\underline{y}_m \in \mathbf{R}^{2m}; \{\underline{y}_m\} \cap \tilde{S}(\underline{x}_3) \neq \emptyset \text{ and } \{\underline{y}_m\} \cap V^c \neq \emptyset\}} d\underline{y}_m |\varphi_\beta(\underline{y}_m)| \\
&\le |\tilde{S}(\underline{x}_3) \cap V^c| \sum_{m=1}^{\infty} m |b_m| z^m + \sum_{m=2}^{\infty} \frac{z^m}{(m-2)!} \int_{\tilde{S}(\underline{x}_3)} dy_1 \int_{V^c} dy_2 \int_{\mathbf{R}^{2(m-2)}} dy_3 \cdots dy_m |\varphi_\beta(\underline{y}_m)|.
\end{aligned}$$

Since $\Phi(\cdot)$ is the interaction of finite range,

$$\varphi_\beta(\underline{y}_m) = 0 \text{ if } m \le d(\tilde{S}(\underline{x}_3), \partial V), \{\underline{y}_m\} \cap \tilde{S}(\underline{x}_3) \neq \emptyset \text{ and } \{\underline{y}_m\} \cap V^c \neq \emptyset.$$

From this property we have

$$\begin{aligned}
&|K_{V,z}(\underline{x}_3) - K_z(\underline{x}_3)| \\
&\le |\tilde{S}(\underline{x}_3) \cap V^c| \sum_{m=1}^{\infty} m |b_m| z^m + |\tilde{S}(\underline{x}_3)| \sum_{m=d(\tilde{S}(\underline{x}_3), \partial V)}^{\infty} m(m-1) |b_m| z^m.
\end{aligned} \tag{5.10}$$

The radius of convergence of the second sum in (5.10) is also $\mathcal{R}_0$. Hence, we obtain (5.9).

Now we come to the position to prove (3.1) in the case of d=2. We decompose the right hand side of (5.2) into several terms,

$$\begin{aligned}
E_V(D_V) &= \frac{z^3}{3!} \int_V dx_3 \int_{\mathbf{R}^4} d\underline{x}_2 \psi_\beta(\underline{x}_3) \exp\{-K_z(\underline{x}_3)\} \\
&- \frac{z^3}{3!} \int_V dx_3 \int_{\{\underline{x}_2 \in \mathbf{R}^4; \{\underline{x}_2\} \cap V^c \neq \emptyset\}} dx\underline{x}_2 \psi_\beta(\underline{x}_3) \exp\{-K_z(\underline{x}_3)\} \\
&+ \frac{z^3}{3!} \int_{V^3} d\underline{x}_3 \psi_\beta(\underline{x}_3)(H_V(\underline{x}_3) - 1) \exp\{-K_z(\underline{x}_3)\} \\
&+ \frac{z^3}{3!} \int_{V^3} d\underline{x}_3 H_V(\underline{x}_3) \psi_\beta(\underline{x}_3) [\exp\{-K_{V,z}(\underline{x}_3)\} - \exp\{-K_z(\underline{x}_3)\}] \\
&\equiv e_\beta(z)|V| - I_1(V) + I_2(V) + I_3(V).
\end{aligned} \tag{5.11}$$

Using Lemma 5.1 and the fact that

$$|S(\underline{x}_3)| \geq \frac{\pi}{12}\{|x_1 - x_2|^2 + |x_1 - x_3|^2 + |x_2 - x_3|^2\},$$

we have

$$\begin{aligned}|I_1(V)| \leq \text{constant} \cdot \int_V dx_1 \int_{V^c} dx_2 \exp\{-\frac{\pi c_1}{12}|x_1 - x_2|^2\} \\ + \frac{1}{3}z^3 \cdot (L_0|\partial V|) \cdot (4\pi {L_0}^2)^2.\end{aligned}$$

Hence

$$|I_1(V)| = O(|\partial V|), \text{ as } V \to \mathbf{R}^2. \tag{5.12}$$

The third term $I_2(V)$ is estimated similarly:

$$\begin{aligned}|I_2(V)| \leq \frac{z^3}{3!} \int_{\{\underline{x}_3 \in V^3; c(\underline{x}_3) \in V^c \text{ and } r(\underline{x}_3) \geq L_0\}} d\underline{x}_3 \exp\{-c_1|S(\underline{x}_3)|\} \\ + \frac{z^3}{3!}(L_0|\partial V|) \cdot (4\pi L_0^2)^2,\end{aligned}$$

where $c(\underline{x}_3)$ is the center of the circle $S(\underline{x}_3)$.

Since

$$|S(\underline{x}_3)| \geq \frac{\pi}{12}\{|x_1 - x_2|^2 + |x_1 - x_3|^2 + d(x_1, \partial V)^2\} \text{ if } c(\underline{x}_3) \in V^c$$

we have

$$|I_2(V)| = O(|\partial V|), \text{ as } V \to \mathbf{R}^2. \tag{5.13}$$

Finally we shall estimate $I_3(V)$. Remind that

$$|\exp\{x\} - \exp\{y\}| \leq |x - y| \exp\{x \vee y\} \text{ for all } x, y \in \mathbf{R}.$$

Using this inequality and Lemma 5.2 we have

$$\begin{aligned}|I_3(V)| \leq \frac{c_2 z^3}{3!} \int_{\{\underline{x}_3 \in V^3; r(\underline{x}_3) \geq L_0\}} d\underline{x}_3 H_V(\underline{x}_3)|\tilde{S}(\underline{x}_3)| \exp\{-c_3 d(\tilde{S}(\underline{x}_3), \partial V) - c_1|S(\underline{x}_3)|\} \\ + \frac{c_2 z^3}{3!} \int_{\{\underline{x}_3 \in V^3; r(\underline{x}_3) \leq L_0\}} d\underline{x}_3 H_V(\underline{x}_3)|\tilde{S}(\underline{x}_3)| \exp\{-c_3 d(\tilde{S}(\underline{x}_3), \partial V)\}.\end{aligned}$$

Using the standard argument on calculus we obtain

$$|I_3(V)| = O(|\partial V|), \text{ as } V \to \mathbf{R}^2. \tag{5.14}$$

Putting these estimates together we have (3.1).

6. Asymptotic behavior of $\mathbf{var}_V(D_V)$

6.1 variance of D_V

In this section we study an asymptotic behavior of $Var_V(D_V)$, the variance of the number of Voronoi vertices in V.

In the same way as (5.2) we derive the expression of $E_V(D_V{}^2)$ in terms of Ursell functions:

$$\text{(6.1)}\qquad E_V(D_V{}^2) = \sum_{k=d+1}^{2(d+1)} \frac{z^k}{k!} \int_{V^k} d\underline{v}_k \psi_\beta(\underline{v}_k) \sum_{(\underline{x}_{d+1},\underline{y}_{d+1});\{\underline{x}_{d+1}\}\cup\{\underline{y}_{d+1}\}=\{\underline{v}_k\}} H_V(\underline{x}_{d+1}) H_V(\underline{y}_{d+1}) J_{S(\underline{x}_{d+1})\cup S(\underline{y}_{d+1})}(\underline{v}_k)\cdot\exp\{-K_{V,z}(\underline{x}_{d+1}|\underline{y}_{d+1})\},$$

where

$$\text{(6.2)}\qquad K_{V,z}(\underline{x}_{d+1}|\underline{y}_{d+1}) = \sum_{m=0}^{\infty} \frac{z^m}{m!}\int_{V^m} d\underline{w}_m \varphi_\beta(\underline{w}_m)\cdot \{1 - J_{S(\underline{x}_{d+1})\cup S(\underline{y}_{d+1})}(\underline{w}_m)\exp(-\beta W(\underline{x}_{d+1},\underline{y}_{d+1}|\underline{w}_m))\}.$$

Now we shall prove (3.2) for $\tau_\beta(z)$ given by

$$\text{(6.3)}\qquad \tau_\beta(z)^2 = \frac{z^{2(d+1)}}{((d+1)!)^2}\int_{\mathbf{R}^{d(2d+1)}} d\underline{x}_d d\underline{y}_{d+1} [\psi_\beta(o,\underline{x}_d,\underline{y}_{d+1}) J_{S(\underline{y}_{d+1})}(o,\underline{x}_d) J_{S(o,\underline{x}_d)}(\underline{y}_{d+1})\exp\{-K_z(o,\underline{x}_d|\underline{y}_{d+1})\} - \psi_\beta(o,\underline{x}_d)\psi_\beta(\underline{y}_{d+1})\exp\{-K_z(o,\underline{x}_d) - K_z(\underline{y}_{d+1})\}] + \sum_{k=d+1}^{2d+1}\frac{z^k}{k!}\int_{\mathbf{R}^{d(k-1)}} d\underline{v}_{k-1} \sum_{(\underline{x}_{d+1},\underline{y}_{d+1});\{\underline{x}_{d+1},\underline{y}_{d+1}\}=\{o,\underline{v}_{k-1}\}} \psi_\beta(o,\underline{v}_{k-1}) J_{S(\underline{y}_{d+1})}(\underline{x}_{d+1}) J_{S(\underline{x}_{d+1})}(\underline{y}_{d+1})\exp\{-K_z(\underline{x}_{d+1}|\underline{y}_{d+1})\},$$

where

$$\text{(6.4)}\qquad K_z(\underline{x}_{d+1}|\underline{y}_{d+1}) = \sum_{m=0}^{\infty}\frac{z^m}{m!}\int_{(\mathbf{R}^d)^m} d\underline{w}_m \varphi_\beta(\underline{w}_m) \{1 - J_{S(\underline{x}_{d+1})}(\underline{w}_m) J_{S(\underline{y}_{d+1})}(\underline{w}_m)\exp\{(-\beta W(\underline{x}_{d+1},\underline{y}_{d+1}|\underline{w}_m))\}.$$

For simplicity we shall only prove Proposition 6.1 in the case $d = 2$. The extension of the proof to the case $d = 3$ is obtained similarly. We prepare two lemmas similar to Lemma 5.1 and Lemma 5.2 for the proof of Proposition 6.1.

Lemma 6.1 *For any $z \in (0, \mathcal{R}_0)$ there exist positive constants c_4, L_1, M_1 such that*

(i) $K_{V,z}(\underline{x}_3|\underline{y}_3) \geq c_4|S(\underline{x}_3)\cup S(\underline{y}_3)|$ *if* $H_V(\underline{x}_3)\cdot H_V(\underline{y}_3) = 1$ *and* $|S(\underline{x}_3)\cup S(\underline{y}_3)| \geq L_1$,

(ii) $K_{V,z}(\underline{x}_3|\underline{y}_3) \geq -M_1$.

Lemma 6.2 *For any $z \in (0, \mathcal{R}_0)$ there exist positive constants c_5 and c_6 such that*

(i) $|K_z(\underline{x}_3|\underline{y}_3) - K_{V,z}(\underline{x}_3|\underline{y}_3)| \leq c_5|\tilde{S}(\underline{x}_3)\cup\tilde{S}(\underline{y}_3)|\exp\{-c_6 d(S(\underline{x}_3)\cup S(\underline{y}_3), V^c)\}$,

(ii) $|K_z(\underline{x}_3|\underline{y}_3) - K_z(\underline{x}_3) - K_z(\underline{y}_3)| \leq c_5|\tilde{S}(\underline{x}_3)\cup\tilde{S}(\underline{y}_3)|\exp\{-c_6 d(S(\underline{x}_3), S(\underline{y}_3))\}$

(iii) $|K_z(\underline{x}_3|\underline{y}_3) - K_{V,z}(\underline{x}_3|\underline{y}_3) - K_z(\underline{x}_3) - K_z(\underline{y}_3) + K_{V,z}(\underline{x}_3) + K_{V,z}(\underline{y}_3)|$
$\leq c_5|\tilde{S}(\underline{x}_3)\cup\tilde{S}(\underline{y}_3)|\exp\{-c_6(d(S(\underline{x}_3), S(\underline{y}_3))\vee d(S(\underline{x}_3)\cup S(\underline{y}_3), V^c)\}$.

The proof of these lemmas are obtained in a similar way to Lemma 5.1 and Lemma 5.2. Using these lemmas and modifying the argument in Section 5, we obtain the proof of (3.2).

In a similar way to the proof of (3.2) we obtain the following proposition.

Proposition 6.1 *For any z with $z \in (0, \mathcal{R}_0)$ the following limit exists,*

$$\lim_{V \to \mathbf{R}^2} \frac{1}{|V|^2} E_V[\{D_V - E_V(D_V)\}^4] = \gamma_\beta(z) < \infty,$$

6.2 Two Dimensional System

Now we shall restrict our argument on two dimensional systems. For two dimensional systems we have more detail results about $e_\beta(z)$ and $\tau_\beta(z)$.

Proposition 6.2 *When $d = 2$ the following relations hold for any $z \in (0, \mathcal{R}_0)$*

(i) $$e_\beta(z) = 2\rho_\beta(z)$$

(ii) $$\tau_\beta(z)^2 = 4\rho'_\beta(z).$$

To prove this proposition we prepare several lemmas. We take a boundary condition ζ as follows

$$\zeta = \partial \bar{V} \cap \mathbf{Z}^2,$$

where $\bar{V} = [-\frac{1}{2}L - 1, \frac{1}{2}L + 1]^2$. See Fig.2. We define a function $D_{V,\zeta}(\cdot)$ on Ω_V by

$$D_{V,\zeta}(\{\underline{x}_n\}) = \sum_{\{\underline{y}_3\} \subset \{\underline{x}_n\}} J_{S(\underline{y}_3)}(\{\underline{x}_n\} \cup \zeta).$$

For simplicity we abbreviate $D_{V,\zeta}(\{\underline{x}_n\})$ to $D_{V,\zeta}(\underline{x}_n)$ in the following.

Let us note that $D_{V,\zeta}(\underline{x}_n)$ is the number of Delaunay triangles whose vertices are taken from $\{\underline{x}_n\}$ under the configuration $\{\underline{x}_n\} \cup \zeta$.Denote a polygon consisting of all these Delaunay triangles by $G_{V,\zeta}(\underline{x}_n)$ and the number of vertices of this polygon by $N^b_{V,\zeta}(\underline{x}_n)$. Using the elementary argument on geometry we have

(6.5) $$D_{V,\zeta}(\underline{x}_n) = 2N_V(\underline{x}_n) - N^b_{V,\zeta}(\underline{x}_n) - 2.$$

Also we denote by $N^b_V(\underline{x}_n)$ the number of vertices of the polygon $G_V(\underline{x}_n)$ consisting of all Delaunay triangles for the configuration $\{\underline{x}_n\}$ with their centers of circumscribed circles in V. In the same way as we obtained (6.5) we have

(6.6) $$D_V(\underline{x}_n) = 2N_V(\underline{x}_n) - N^b_V(\underline{x}_n) - 2.$$

We remind that $N^b_V(\underline{x}_n)$ is rewritten as

$$N^b_V(\underline{x}_n) = \sharp\{i \in \{1, 2, \cdots, n\} : |x_i - y| = \min_{1 \le j \le n} |x_j - y|, \text{ for some } y \in \partial V\}.$$

We now present the analogue of Lemma 5.1.

Lemma 6.3. *For any z with $z \in (0, \mathcal{R}_0)$ there exist functions $c_1 = c_1(z) > 0$ and $L_0 = L_0(z) > 0$ such that*

$$K_{V,z}(\underline{x}_3) \ge c_1 |S(\underline{x}_3)|,$$

for any $\underline{x}_3$ *with* $r(\underline{x}_3) \geq L_0$ *and* $J_{S(\underline{x}_3)}(\zeta) = 1.$

The proof of this lemma is much the same as that of Lemma 5.1 and we omit it. We can show the following lemma on the same procedure as the proof of Theorem 1 by using Lemma 6.3 instead of Lemma 5.1.

Lemma 6.4. *For any* z *with* $z \in (0, \mathcal{R}_0)$

$$\text{(i)} \qquad \lim_{V \to \mathbf{R}^2} \frac{1}{|V|} E_V(D_{V,\zeta}) = e_\beta(z),$$

$$\text{(ii)} \qquad \lim_{V \to \mathbf{R}^2} \frac{1}{|V|} [E_V(D^2_{V,\zeta}) - E_V(D_{V,\zeta})^2] = \tau_\beta(z)^2.$$

From (6.5) and Lemma 6.4 it is enough to show the following lemma to prove Proposition 6.2.

Lemma 6.5. For any z with $z \in (0, \mathcal{R}_0)$

$$\text{(i)} \qquad \lim_{V \to \mathbf{R}^2} \frac{1}{|V|} E_V(N^b_{V,\zeta}) = 0,$$

$$\text{(ii)} \qquad \lim_{V \to \mathbf{R}^2} \frac{1}{|V|} [E_V\{(N^b_{V,\zeta})^2\} - E_V(N^b_{V,\zeta})^2] = 0.$$

Proof. When $\{\underline{w}_2\} \subset \{\underline{x}_n\}$, $J_{S(\underline{w}_2, y)}(\{\underline{x}_n\} \cup \zeta) = 1$ for some $y \in \zeta$ if and only if the line segment $\overline{w_1 w_2}$ is a side of the polygon $G_{V,\zeta}(\underline{x}_n)$. Hence,

$$\text{(6.7)} \qquad N^b_{V,\zeta}(\underline{x}_n) = \sum_{y \in \zeta} \sum_{\{\underline{w}_2\} \subset \{\underline{x}_n\}} J_{S(\underline{w}_2 \cdot y)}(\{\underline{x}_n\} \cup \zeta).$$

We obtain the following formulas from the algebraic method of cluster expansion,

$$E_V(N^b_{V,\zeta}) = \sum_{y \in \zeta} \frac{z^2}{2} \int_{V^2} d\underline{w}_2 J_{S(\underline{w}_2 \cdot y)}(\zeta) \psi_\beta(\underline{w}_2) \exp\{-K_{V,z}(\underline{w}_2, y)\},$$

and
$$E_V\{(N^b_{V,\zeta})^2\} = \sum_{y_1, y_2 \in \zeta} \sum_{k=2}^{4} \frac{z^k}{k!} \int_{V^k} d\underline{v}_k \psi_\beta(\underline{v}_k) \sum_{(\underline{w}^1_2, \underline{w}^2_2) : \{\underline{w}^1_2\} \cup \{\underline{w}^2_2\} = \{\underline{v}_k\}}$$
$$J_{S(\underline{w}^1_2, y_1)}(\{\underline{v}_k\} \cup \zeta) J_{S(\underline{w}^2_2, y_2)}(\{\underline{v}_k\} \cup \zeta) \exp\{-K_{V,z}(\underline{w}^1_2, y_1 | \underline{w}^2_2, y_2)\}.$$

From the formulas and employing the same argument developed in Section 5 and Section 6.1 we have Lemma 6.5.

Hence, the proof of Proposition 6.2 is obtained .

7. Central Limit Theorem

To prove Theorem 3 it is enough to show

$$\text{(7.1)} \qquad \lim_{L \to \infty} \theta_L(s) = \exp\{-\frac{\tau^2_\beta}{2} s^2\}, \text{ for any } s \in \mathbf{R}\ ,$$

where $\theta_L(s)$ is the characteristic function of Y_V. For simplicity we shall only prove (7.1) in the case of $d = 2$. The extension of the proof to the case of $d = 3$ is obtained with minor modifications. Put $m = m(L) = [\frac{L}{L^{\frac{1}{2}} + L^{\frac{1}{3}}}]^2$, $a = a(L) = \frac{L}{\sqrt{m(L)}} - L^{\frac{1}{3}}$ and $b = b(L) = L^{\frac{1}{3}}$, where $[c]$ is the integer part of $c > 0$. We subdivide V into m squares $V_1, V_2, \cdots, V_m$ with side $a(L)$ and $B = V \backslash \sum_{i=1}^m V_i$, as in Fig.3. Define

$$\tilde{D}_{V_i}(\underline{x}_n) = \sum_{\underline{x}_3 \subset \underline{x}_n} \tilde{H}_{V_i}(\underline{x}_3) J_{S(\underline{x}_3)}(\underline{x}_n),$$

where

$$\tilde{H}_{V_i}(\underline{x}_3) = \begin{cases} 1, & \text{if } S(\underline{x}_3) \subset V_i, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\tilde{D}_{V_i}$ is the number of Delauny triangles whose circumscribed circles are contained in V_i. Employing the same argument developed for D_V in Section 5 and Section 6, we have

$$\lim_{L\to\infty} \frac{1}{|V_i|} E_{V_i,\zeta}[D_{V_i}] = e_\beta(z), \tag{7.2}$$

$$\lim_{L\to\infty} \frac{1}{|V_i|} E_{V_i,\zeta}[\{\tilde{D}_{V_i} - E_{V_i,\zeta}(\tilde{D}_{V_i})\}^2] = \tau_\beta(z)^2, \tag{7.3}$$

$$\lim_{L\to\infty} \frac{1}{|V_i|^2} E_{V_i,\zeta}[\{\tilde{D}_{V_i} - E_{V_i,\zeta}(\tilde{D}_{V_i})\}^4] = \gamma_\beta(z), \tag{7.4}$$

uniformly in $\zeta \in \Omega_B$, where $E_{V_i,\zeta}[\cdot]$ is the expectation with respect to the measure $P_{V_i,\zeta}$. Put

$$\tilde{\theta}_L(s) = \int_{\Omega_B} P_B(d\zeta) E_{V\setminus B,\zeta}[\exp\{\frac{\sqrt{-1}s}{L}\sum_{i=1}^m(\tilde{D}_{V_i} - E_{V\setminus B,\zeta}(\tilde{D}_{V_i}))\}].$$

The proof of (7.1) follows from the following two lemmas.

Lemma 7.1. *For each* $z \in (0, \mathcal{R}_0)$,

$$\lim_{L\to\infty} \tilde{\theta}_L(s) = \exp\{-\frac{\tau_\beta(z)^2}{2}s^2\}, \quad s \in \mathbf{R}. \tag{7.5}$$

Proof. Since $\tilde{D}_{V_i}, i = 1, 2, \cdots, m$ are independent under $P_{V\setminus B,\zeta}(\cdot)$, we have

$$\tilde{\theta}_L(s) = \int_{\Omega_B} P_B(d\zeta) \prod_{i=1}^m E_{V\setminus B,\zeta}[\exp\{\frac{\sqrt{-1}s}{L}(\tilde{D}_{V_i} - E_{V\setminus B,\zeta}(\tilde{D}_{V_i}))\}].$$

From (7.3) and (7.4) we obtain

$$\begin{aligned}
&\prod_{i=1}^m E_{V\setminus B,\zeta}[\exp\{\frac{\sqrt{-1}s}{L}(\tilde{D}_{V_i} - E_{V\setminus B,\zeta}(\tilde{D}_{V_i}))\}] \\
&\quad = \prod_{i=1}^m [1 - \frac{s^2}{2L^2} E_{V_i,\zeta}\{(\tilde{D}_{V_i} - E_{V_i,\zeta}(\tilde{D}_{V_i}))^2\}] + O(\frac{a^3}{L^3}) \\
&\quad = \exp\{-\frac{\tau_\beta^2}{2}s^2\} + o(1), \qquad \text{as } L \to \infty.
\end{aligned}$$

This completes the proof of the lemma.

Lemma 7.2. *For any* $s \in \mathbf{R}$,

$$\lim_{L\to\infty} |\theta_L(s) - \tilde{\theta}_L(s)| = 0. \tag{7.6}$$

Proof. From the definition of the Gibbs measure we have

$$\theta_L(s) = \int_{\Omega_B} P_B(d\zeta) E_{V\setminus B,\zeta}[\exp\{\frac{\sqrt{-1}s}{L}(D_V - E_V(D_V))\}].$$

Using the standard argument on calculus we have

(7.7)

$$\begin{aligned}
&|\theta_L(s) - \tilde{\theta}_L(s)| \\
&\le \int_{\Omega_B} P_B(d\zeta) E_{V\setminus B,\zeta}[|\exp\{\frac{\sqrt{-1}s}{L}(D_V - \sum_{i=1}^m \tilde{D}_{V_i} - E_V(D_V) + \sum_{i=1}^m E_{V\setminus B,\zeta}(\tilde{D}_{V_i}))\} - 1|] \\
&\le \frac{s}{L}\int_{\Omega_B} P_B(d\zeta) E_{V\setminus B,\zeta}[|D_V - \sum_{i=1}^m \tilde{D}_{V_i} - E_V(D_V) + \sum_{i=1}^m E_{V\setminus B,\zeta}(\tilde{D}_{V_i})|].
\end{aligned}$$

We estimate the last term in (7.7) as follows

$$
\begin{aligned}
(7.8)\qquad & \Big(\int_{\Omega_B} P_B(d\zeta) E_{V\setminus B,\zeta}[|D_V - \sum_{i=1}^{m} \tilde{D}_{V_i} - E_V(D_V) + \sum_{i=1}^{m} E_{V\setminus B,\zeta}(\tilde{D}_{V_i})|] \Big)^2 \\
& \le \int_{\Omega_B} P_B(d\zeta) E_{V\setminus B,\zeta}[|D_V - \sum_{i=1}^{m} \tilde{D}_{V_i} - E_V(D_V) + \sum_{i=1}^{m} E_{V\setminus B,\zeta}(\tilde{D}_{V_i})|^2] \\
& = 2E_V[\{(D_V - \sum_{i=1}^{m} \tilde{D}_{V_i}) - E_V(D_V - \sum_{i=1}^{m} \tilde{D}_{V_i})\}^2] \\
& + 2\int_{\Omega_B} P_B(d\zeta) E_{V\setminus B,\zeta}[E_{V\setminus B,\zeta}(\sum_{i=1}^{m} \tilde{D}_{V_i})^2 - E_V(\sum_{i=1}^{m} \tilde{D}_{V_i})^2]
\end{aligned}
$$

Employing the same argument as in Section 6, we have

$$
(7.9)\qquad \lim_{V\to\mathbf{R}^d} \frac{1}{|V|} E_V[\{(D_V - \sum_{i=1}^{m} \tilde{D}_{V_i}) - E_V(D_V - \sum_{i=1}^{m} \tilde{D}_{V_i})\}^2] = 0.
$$

From (7.3) we have

$$
(7.10)\qquad \lim_{V\to\mathbf{R}^d} \frac{1}{|V|} \int_{\Omega_B} P_B(d\zeta) E_{V\setminus B,\zeta}[E_{V\setminus B,\zeta}(\sum_{i=1}^{m} \tilde{D}_{V_i})^2 - E_V(\sum_{i=1}^{m} \tilde{D}_{V_i})^2] = 0.
$$

Putting these estimates together we obtain the proof of Lemma 7.2.

8. Large deviation

In this section we prove the large deviation principle for the probability distribution of $D_V(\cdot)/|V|$. Throughout this section, we restrict our argument on a two dimensional system. Furthermore, we assume that the potential function $\Phi(\cdot)$ satisfies the hard core condition, i.e. $\Phi(x) = \infty$ whenever $|x| < r_1$ for some $r_1 \le 1$.

We define functions

$$
f_{V,z}(\theta) = \frac{1}{|V|} \log E_V[\exp\{\theta D_V\}].
$$

Proposition 8.1 For any $\theta \in \mathbf{R}$ and $z \in (0, \mathcal{R}_0)$ the limit

$$
f_z(\theta) = \lim_{V\to\mathbf{R}^2} f_{V,z}(\theta)
$$

exists and satisfies

$$
f_z(\theta) = F_\beta(ze^{2\theta}) - F_\beta(z) \le ze^{2\theta}.
$$

The following lemma plays a dominant role for the proof of this proposition.

Lemma 8.1 For almost all $\xi \in \Omega_V$ we have

$$
N_V^b(\xi) \le \frac{\pi}{2r_1} |\partial V|.
$$

Proof. Let us remind the definition of the polygon $G_V(\xi)$. To each edge A_iB_i of $G_V(\xi)$ we draw a perpendicular bisector ℓ_i. These perpendicular bisectors do not intersect with one another in $V\setminus G_V(\xi)$. If two of them intersect in $V\setminus G_V(\xi)$, they create a new Voronoi vertex. This leads to a contradiction.

Denote by $Q = \{q_1, q_2, \cdots, q_n\}$ a set of intersecting points of these perpendicular bisectors and ∂V. We draw an arc θ_i connecting A_i and B_i with center at q_i to each (q_i, ℓ_i, A_i, B_i), as shown in Fig.4.

From the hard core condition on $\Phi(\cdot)$ we have

$$r_1 N_V^b(\underline{x}_n) \le |\partial G_V(\underline{x}_n)| \le \sum_{i=1}^{k} |\theta_i| \quad \text{a.s.} \tag{8.1}$$

where $|\partial G_V(\underline{x}_n)|$ and $|\theta_i|$ means the length of $\partial G_V(\underline{x}_n)$ and θ_i, respectively.

Take one edge AB of the square V. Let $\{q_1, q_2, \cdots, q_{n_1}\}, n_1 \le n$, be the set of all points of Q on this edge. Denote by α a curve obtained jointing arcs $\theta_1, \theta_2, \cdots, \theta_{n_1}$. Take the x-axis along AB with the origin located at A. The $x-$coordinates of the points $q_1, q_2, \cdots, q_{n_1}$ are also denote by $q_1, q_2, \cdots, q_{n_1}$. Now we express the curve α by the following functions defined on $[0, L]$,

$$F(x) = \max_{1 \le i \le n_1} f_i(x),$$

where $\quad f_i(x) = \sqrt{\{a_i^2 - (x - q_i)^2\} \vee 0}, \quad$ and a_i is the distance between q_i and A_i (or B_i).

We call an arc θ_i type 1, if a line which is perpendicular to x-axis and passes through q_i intersects with θ_i. Other arcs are called type 2.

Let us consider a curve $\tilde{\alpha}$ consisting of arcs of type 1 and line segments on x-axis. It is explicitly given by the function

$$\tilde{F}(x) = \max_{1 \le i \le n_1; f_i(q_i) = F(q_i)} f_i(x). \quad \text{(See Fig.4.)}$$

Projecting all arcs of $\tilde{\alpha}$ upon x-axis we obtain the estimate of $|\tilde{\alpha}|$ as follows

$$|\tilde{\alpha}| \le \frac{\pi}{2} L. \tag{8.2}$$

Furthermore, comparing the lengths of arcs of α with $\tilde{\alpha}$ we have

$$|\alpha| \le |\tilde{\alpha}|. \tag{8.3}$$

Hence, from (8.1),(8.2) and (8.3) we have

$$N_V^b(\xi) \le \frac{\pi}{2r_1} |\partial V|.$$

This completes the proof of Lemma 8.1.

From Lemma 8.1 ,(6.6) and the definition of the free energy function $F_\beta(z)$ we obtain the proof of Proposition 8.1.

As a remark we state the following lemma.

Lemma 8.2 *For any $\theta \in \mathbf{R}$ and $z \in (0, \mathcal{R}_0)$ there exists a positive constant $c = c(z, \theta) > 0$ such that*

$$\exp\{-c|\partial V|\} \le \frac{E_{V,\zeta}[\exp\{\theta D_V\}]}{E_V[\exp\{\theta D_V\}]} \le \exp\{c|\partial V|\}$$

for any square $V \subset \mathbf{R}^2$ and $\zeta = \{\underline{v}_k\} \subset V^c$.

The proof of this lemma is also obtained from Lemma 8.1 and (6.6). Using this lemma we can prove that $f^\beta_{V,z,\zeta}(\theta)$ defined by

$$f^\beta_{V,z,\zeta}(\theta) = \frac{1}{|V|} \log E_{V,\zeta}[\exp\{\theta D_V\}]$$

converges to $f_z(\theta)$ independent of the boundary condition $\zeta = \{\underline{v}_k\} \subset V^c$.

Proof of Theorem 3

The proof of the first statement of the theorem follows from the exponential Tchebycheff's inequality,

$$P_V(\frac{D_V}{|V|} \in F) \le \exp\{-|\theta||V| \min_{x \in F} x\} E_V[\exp\{\theta D_V\}] \tag{8.4}$$

for any closed subset F of $\mathbf{R}$.

To prove the second statement we introduce a set function

$$Q(G) = - \varliminf_{V \to \mathbf{R}^2} \frac{1}{|V|} \log P_V(\frac{D_V}{|V|} \in G)$$

for a open subset G of $\mathbf{R}$, and a function

$$q(x) = \sup\{Q(G); G \ni x \text{ and } G: \text{ open }\},$$

where the supremum is taken for all open subsets G satisfying $G \ni x$.

It is easily seen that $Q(\cdot)$ satisfies

$$Q(E \cup F) = Q(E) \wedge Q(F) \tag{8.5}$$

for any open subsets E and F. It follows from this property that $q(x)$ is lower semicontinuous and

$$\varliminf_{V \to \mathbf{R}^2} \frac{1}{|V|} \log P_V(\frac{D_V}{|V|} \in G) \ge - \inf_{x \in G} q(x) \tag{8.6}$$

for any open subset $G \subset \mathbf{R}$. (See [17] for the proof of (8.6).)

For the proof of the second statement it is enough to prove that

$$I_z(x) = q(x). \tag{8.7}$$

The first step in proving (8.7) is to prove the following lemma.

Lemma 8.3 *For any $\theta \in \mathbf{R}$*

$$f_z(x) = \sup_{x \in \mathbf{R}} \{\theta x - q(x)\}.$$

Proof. First we shall show that

$$f_z(\theta) \ge \sup_{x \in \mathbf{R}} \{\theta x - q(x)\}. \tag{8.8}$$

Take any open subset $G \subset \mathbf{R}$ satisfying $G \ni x$. In the same way as (8.4) we have

$$E_V[\exp\{\theta D_V\}] \ge \exp\{|V| \inf_{y \in G} \theta y\} P_V(\frac{D_V}{|V|} \in G).$$

From this inequality we obtain

$$f_z(\theta) \ge \inf_{y \in G} \theta y - Q(G).$$

Taking the limit $G \to \{x\}$ we obtain the proof of (8.8).

Next we shall show the converse estimate of (8.8),

$$f_z(\theta) \le \sup_{x \in \mathbf{R}} \{\theta x - q(x)\} \tag{8.9}$$

for any $\theta \in \mathbf{R}$.

For any $M > 0$ we decompose $E_V[\exp\{\theta D_V\}]$ as follows

$$E_V[\exp\{\theta D_V\}] = E_V[\exp\{\theta D_V\}; \frac{D_V}{|V|} \leq M] + E_V[\exp\{\theta D_V\}; \frac{D_V}{|V|} > M].$$

Using the same argument as before we have

$$E_V[\exp\{\theta D_V\}; \frac{D_V}{|V|} > M] \leq \begin{cases} E_V[\exp\{2\theta D_V\}]\exp\{-\theta|V|M\} & \text{if } \theta > 0 \\ E_V[\exp\{D_V\}]\exp\{-|V|M\} & \text{if } \theta \leq 0, \end{cases}$$

consequently, the following estimate holds

$$\overline{\lim_{V\to\mathbf{R}^2}} \frac{1}{|V|}\log E_V[\exp\{\theta D_V\}; \frac{D_V}{|V|} > M] \leq \begin{cases} f_z(2\theta) - \theta M \text{ if } \theta > 0 \\ f_z(1) - M \text{ if } \theta \leq 0. \end{cases}$$

Since $f_z(\theta) \leq z\exp\{2\theta\}$, for any $\epsilon > 0$ there exists a number $M_0 = M_0(\epsilon) > 0$ such that

$$\overline{\lim_{V\to\mathbf{R}^2}} \frac{1}{|V|}\log E_V[\exp\{\theta D_V\}; \frac{D_V}{|V|} > M] < \epsilon$$

holds for any $M > M_0$.

Let $\{G_1, ..., G_k\}$ be a finite open covering of $[0, M]$. Taking into account the inequality

$$\underline{\lim_{n\to\infty}} \frac{1}{n}\log(a_n + b_n) \leq (\underline{\lim_{n\to\infty}} \frac{1}{n}\log a_n) \vee (\underline{\lim_{n\to\infty}} \frac{1}{n}\log b_n)$$

for sequences of positive real numbers, we have

$$\underline{\lim_{V\to\mathbf{R}^2}} \frac{1}{|V|}\log E_V[\exp\{\theta D_V\}; \frac{D_V}{|V|} \leq M] \leq \max_{1\leq i\leq k}\{\sup_{y\in G_i} \theta y - Q(G_i)\}.$$

From estimates obtained above it follows for any $\epsilon > 0$ that

$$f_z(\theta) \leq \max_{1\leq i\leq k}\{\sup_{y\in G_i} \theta y - Q(G_i)\} + \epsilon \qquad \text{for any } M > M_0(\epsilon). \tag{8.10}$$

Taking $\{G_1, ..., G_k\}$ such that $|G_i| \leq \frac{\epsilon}{|\theta|}$ for any i=1,...,k, we have

$$f_z(\theta) \leq \sup_{x\in\mathbf{R}}\{\theta x - q(x)\} + 2\epsilon.$$

This completes the proof of Lemma 8.3.

Next we shall prove the convexity of $q(x)$.

Lemma 8.4 $q(x)$ is a convex function..

Proof. Since $q(x)$ is lower semicontinuous, it is enough to show the following inequality for the proof of Lemma 8.4,

$$q(\frac{1}{4}\sum_{i=1}^{4} x_i) \leq \frac{1}{4}\sum_{i=1}^{4} q(x_i) \tag{8.11}$$

We partition the square $V = (-\frac{L}{2}, \frac{L}{2}) \times (-\frac{L}{2}, \frac{L}{2})$ into four congruent squares

V_1, V_2, V_3, V_4 and a corridor part B:

$$V_1 = (\frac{1}{2}, \frac{L}{2}) \times (\frac{1}{2}, \frac{L}{2}) \qquad V_2 = (-\frac{L}{2}, -\frac{1}{2}) \times (\frac{1}{2}, \frac{1}{2})$$
$$V_3 = (-\frac{1}{L}, -\frac{1}{2}) \times (-\frac{L}{2}, -\frac{1}{2}) \qquad V_4 = (\frac{1}{2}, \frac{L}{2}) \times (-\frac{L}{2}, -\frac{1}{2})$$

and $B = V \setminus \bigcup_{i=1}^4 V_i$.

From Lemma 8.1, (6.6) and the hard core condition of Φ we obtain

$$|D_V(\xi) - \sum_{i=1}^{4} D_{V_i}(\xi)| \le 4L(\frac{2}{\pi r_1^2} + \frac{3\pi}{2r_1}) + 6 \tag{8.12}$$

for almost all $\xi \in \Omega_V$. Using this estimate we have

$$|\frac{D_V(\xi)}{|V|} - \frac{1}{4}\sum_{i=1}^{4} x_i| \le \frac{1}{4}\sum_{i=1}^{4} |\frac{D_{V_i}(\xi)}{|V_i|} - x_i| + O(\frac{1}{L}), \qquad \text{as } L \to \infty \tag{8.13}$$

for almost all $\xi \in \Omega_V$. Hence, for any $\epsilon > 0$ there exists a positive constant L' such that

$$P_V(|\frac{D_V(\xi)}{|V|} - \frac{1}{4}\sum_{i=1}^{4} x_i| < \epsilon) \ge \int P_B(d\zeta) \prod_{i=1}^{4} P_{V_i,\zeta}(|\frac{D_{V_i}(\xi)}{|V_i|} - x_i| < \frac{1}{2}\epsilon) \tag{8.14}$$

for all $L > L'$.

Put $\tilde{V}_i = \{t \in V_i; d(t, \partial V_i) > 1\}$ and $\tilde{\tilde{V}}_i = V_i \setminus \tilde{V}_i$. For any configuration $\xi \in \Omega_{V_i}$ we denote by $\tilde{\xi}$ and $\tilde{\tilde{\xi}}$ the restriction of ξ on $\tilde{V}_i$ and $\tilde{\tilde{V}}_i$ respectively, i.e. $\tilde{\xi} = \{t \in \xi; t \in \tilde{V}_i\}$ and $\tilde{\tilde{\xi}} = \{t \in \xi; t \in \tilde{\tilde{V}}_i\}$.

Using Lemma 8.1, (6.6) and hard core condition on Φ we have

$$|D_{V_i}(\xi) - D_{V_i}(\tilde{\xi})| \le 2N_{\tilde{\tilde{V}}_i}(\tilde{\tilde{\xi}}) + N^b_{V_i}(\xi) + N^b_{V_i}(\tilde{\xi}) = O(L), \qquad \text{as } L \to \infty$$

for almost all $\xi \in \Omega_{V_i}$.

Hence, for any $\epsilon > 0$ there exists a constant $L" > L'$ such that the following estimate holds for all $L > L"$ and any $\zeta \in \Omega_B$

$$\begin{aligned}
\frac{P_{V_i,\zeta}(|\frac{D_{V_i}(\xi)}{|V_i|} - x_i| < \frac{1}{2}\epsilon)}{P_{V_i}(|\frac{D_{V_i}(\xi)}{|V_i|} - x_i| < \frac{1}{4}\epsilon)} &\ge \frac{P_{V_i,\zeta}(|\frac{D_{V_i}(\tilde{\xi})}{|V_i|} - x_i| < \frac{1}{2}\epsilon \text{ and } \tilde{\tilde{\xi}} = \emptyset)}{P_{V_i}(|\frac{D_{V_i}(\tilde{\xi})}{|V_i|} - x_i| < \frac{1}{2}\epsilon)} \\
&= \frac{Z_{V_i}}{Z_{V_i,\zeta}} \cdot \frac{\sum_{m=0}^{\infty} \frac{z^m}{m!} \int_{\tilde{V}_i{}^m} d\underline{w}_m \psi_\beta(\underline{w}_m) \mathbf{1}(|\frac{D_{V_i}(\underline{w}_m)}{|V_i|} - x_i| < \frac{1}{2}\epsilon)}{\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{z^{m+k}}{m!k!} \int_{\tilde{V}_i{}^m} d\underline{w}_m \int_{\tilde{\tilde{V}}_i{}^k} d\underline{y}_k \psi_\beta(\underline{w}_m \cdot \underline{y}_k) \mathbf{1}(|\frac{D_{V_i}(\underline{w}_m)}{|V_i|} - x_i| < \frac{1}{2}\epsilon)} \\
&\ge \frac{1}{\sum_{m=0}^{\infty} \frac{z^k}{k!} |\tilde{\tilde{V}}_i|^k} \ge \exp\{-2zL\}.
\end{aligned}$$

Putting these estimate together we have

$$P_V(|\frac{D_V(\xi)}{|V|} - \frac{1}{4}\sum_{i=1}^{4} x_i| < \epsilon) \ge \exp\{-8zL\} \prod_{i=1}^{4} P_{V_i}(|\frac{D_{V_i}(\xi)}{|V_i|} - x_i| < \frac{1}{4}\epsilon).$$

This implies that

$$Q(B_\epsilon(\frac{1}{4}\sum_{i=1}^{4} x_i)) \le \frac{1}{4}\sum_{i=1}^{4} Q(B_{\frac{1}{4}\epsilon}(x_i)),$$

where $B_\epsilon(x) = \{y \in \mathbf{R}; |y - x| < \epsilon\}$.

From this estimate we immediately obtain (8.11).

From Lemma 8.3 and Lemma 8.4 we have (8.7) and this completes the proof of Theorem 3.

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Keywords. Voronoi tessellation, Gibbs measure, Cluster expansion, Central limit theorem, Large deviation

Fig. 1 Voronoi tessellation and Delaunay network. The edges of the Voronoi tessellation and the Delaunay network are indicated by solid lines and dotted lines, respectively.

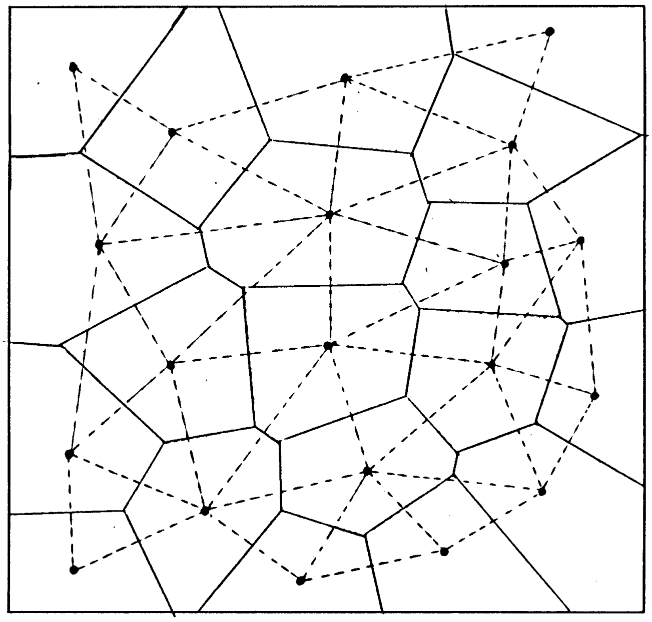

Fig. 2 The boundary condition ζ.

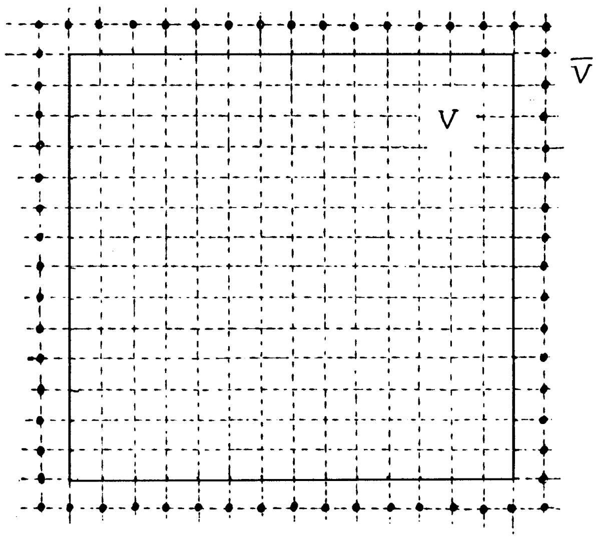

Fig. 3 Decomposition of V into $V_1,\ldots,V_m$ and B.

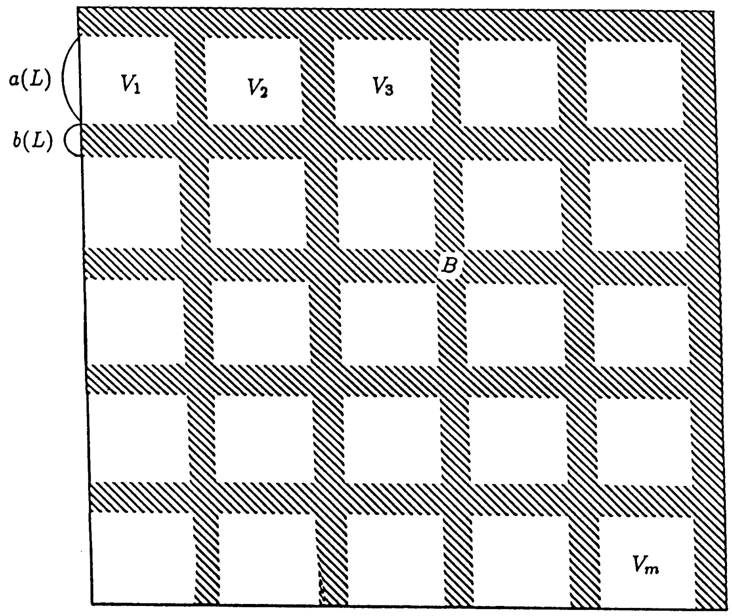

Fig. 4 Typical realization of a set of arcs $\{\theta_1,\ldots,\theta_{n_1}\}$.