Research Report

KSTS/RR-90/005
September 14 1990

Szegö Operators and a Paley-Wiener Theorem on SU(1,1)

by

Takeshi Kawazoe

Takeshi Kawazoe

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku
Yokohama, 223 Japan

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku, Yokohama, 223 Japan

KSTS/RR90/005
September 14, 1990

Szegö Operators and a Paley-Wiener Theorem on SU(1,1)

By

Takeshi Kawazoe

§1. Introduction. In 1934 Paley and Wiener [PW] showed that the Fourier transform $f \to f^{\wedge}$ on R is a bijection of $C_c^{\infty}(R)$ onto the set of holomorphic functions of exponential type. Let G be a reductive Lie group with a maximal compact subgroup K of G. Then the analogous theorem to characterize the image of $C_c^{\infty}(G,K)$, K-finite functions in $C_c^{\infty}(G)$, under the Fourier transform was finally solved by Arthur [A] in 1983. During the these 50 years a number of authors had proved the Paley-Wiener theorem for particular classes of groups.

Some difficulties arise in the proof of the surjectivity, especially, of showing compactness of the support of a function whose Fourier transform is holomorphic of exponential type, and there are some directions to obtain the fact. The first one is, as in the case of R, the way of changing of contours of integration in the Fourier inversion formula. Ehrenpreis and Mautner [EM] solved the case of SU(1,1) and Johnson [J] rephrased the result in terms of Harish-Chandra's generalized c-functions. The main problems in this direction were (1) how to obtain a sharp estimate for Harish-Chandra expansion which allows us to change the contours of integration and (2) how to treat residues which appear during the contour change. For the K-biinvariant or right K invariant functions on general groups G the residues don't appear. Then the main problem (1) was solved by

Helgason [H1], Gangolli [G] for $C_c^\infty(K\backslash G/K)$ and by Helgason [H2] for $C_c^\infty(G/K)$. Roughly speaking in these cases the image is characterized by holomorphic functions of exponential type satisfying functional equations related with the small Weyl group of G.

When we treat K-finite functions on G, we encounter the residues during the contour change, so the problem (2) is essential. This was solved by noting a relation between the residues and matrix coefficients of nonunitary principal series of G, especially the discrete series of G. For the real rank one groups this was done by Campoli [C] and for arbitrary groups by Arthur [A]. In his proof W. Casselman's theory of a realization of $(\underline{g},K)$ modules played an important role to treat the residues. Then the image of $C_c^\infty(G,K)$ is characterized by holomorphic functions of exponential type satisfying functional equations that matrix coefficients of nonunitary principal series of G satisfy.

The second direction of proving the compactness is completely different from the first one and is algebraic in nature. For complex semisimple Lie groups the Paley-Wiener theorem was solved by Zelobenko [Z] and for any groups with one conjugacy classes of Cartan subgroups of G was done by Delorme [D].

The aim of this paper is to offer a third direction of proving the Paley-Wiener theorem. Especially, we shall give a new approach to obtain the theorem for right K-finite functions on G = SU(1,1). The Plancherel formula for $L^2(G)$ indicates that L^2 functions on G consist of wave packets and cusp forms on G. Although any functions in $C_c^\infty(G,K)$ are uniquely determined by the integral part - the sum of wave packets - in the Fourier inversion formula, the result stated above <the image satisfies functional equations that matrix

coefficients of nonunitary principal series of G satisfy> does not express clearly the relation between wave packets and cusp forms. So, we shall characterize simultaneouly the two parts of the right K-finite functions on G. As mentioned above, the residues, which appear in the contour change, are real obstacles in the proof of the surjectivity. Therefore, we want to avoid using the Harish-Chandra expansion from which the singularities arise. Actually, reducing the theorem to the one for right K-invariant functions, we won't use the theory of c-functions. In this approach the theory of Szegö operators will play an important role.

We shall treat right n type functions on G; $n \in \frac{1}{2}\mathbb{Z}$ and the left K-type is of free. In §3, as generalization of the classical Szegö projection defined on the unit circle (cf. [R], p.178), the Szegö operators $S_{\varepsilon,\nu,n}$ ($\varepsilon=0$, $\frac{1}{2}$ and $\nu \in \mathbb{R}$) will be defined (see (3.11)). They are deeply related with the principal series and the discrete sereis of G, and some properties will be investigated in §4 and §5. Then, in §6, we shall rephrase the Plancherel formula for $L^2_n(G)$, L^2 functions on G with right K-type n, by using the Szegö operators (see Theorem 6.10). Actually, wave packets can be written as an integral of $S_{\varepsilon,\nu,n}$ with respect to $\mu_\varepsilon(\lambda)d\lambda$, where μ_ε is the Plancherel measure and $\nu = \frac{1}{2}+i\lambda$, and the discrete part - L^2 sum of cusp forms - as a finite sum of $S_{\varepsilon,m,n}$ ($1 \leqq m \leqq n$, $m \in \frac{1}{2}\mathbb{Z}$ and $2m=2n$ mod(2)). This new phrase of the Plancherel formula is useful to express the relation between wave packets and the discrete part of compactly supported, C^∞ functions on G (see Lemma 7.1), and moreover, it makes easy to see the fact that the formula can be reduced to the one for right K-invariant functions on G by applying a suitable differential operator on G (see Corollary 6.4 and Remark 6.5). This indicates that the Paley-Wiener theorem for right n type functions will be reduced to the one for right K-invariant functions which has no discrete part (see Remark 6.5). In this direction the Paley-Wiener theorem will be proved in §7. Especially, we don't use the Harish-Chandra expansion for K-finite spherical functions and we don't need to treat singularities of generalized c-functions, only we pay attention to the ones of $P_n(\lambda)^{-1}$ (see Corollary 6.4 (1)). By the same way, this direction is also applicable to the characterization of L^p Schwartz space on G with right K-type n (see Theorem 7.4).

§2. Notation. Let G be SU(1,1), the group of all C-linear transformations of C^2 which are of determinant one, and G = KAN an Iwasawa decomposition of G, where K, A and N are, respectively the maximal compact, vector and unipotent subgroups of G consisting of all matrices in G of the form:

$$k_\theta = \begin{pmatrix} e^{i\theta/2} & \\ & e^{-i\theta/2} \end{pmatrix} \qquad (0 \leqq \theta < 4\pi),$$

$$a_t = \begin{pmatrix} \mathrm{ch}t/2 & \mathrm{sh}t/2 \\ \mathrm{sh}t/2 & \mathrm{ch}t/2 \end{pmatrix} \qquad (t \in R)$$

and

$$n_\xi = \begin{pmatrix} 1+i\xi/2 & -i\xi/2 \\ i\xi/2 & 1-i\xi/2 \end{pmatrix} \qquad (\xi \in R).$$

Let $\underline{g} = \underline{k} + \underline{a} + \underline{n}$ denote the corresponding Iwasawa decomposition of the Lie algebra $\underline{g}$ of G. Let $A^+ = \{a_t \ ; \ t>0\}$ and $M = \{\pm 1\}$, the centralizer of A in K. Then the Cartan decomposition of G is given by $G = KCL(A^+)K$. For $x \in G$ we define H(x) as the unique element in $\underline{a}$ such that $x \in K\exp H(x)N$ and $\sigma(x)$ as the unique positive number such that

$x \in Ka_{\sigma(x)}K$. Let $\underline{u}_c$ denote the complexification of an algebra $\underline{u}$ and $\underline{u}_c^*$ the dual space of $\underline{u}_c$. Then $\underline{a}_c = C\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\underline{h}_c = \underline{k}_c = C\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ are Cartan subalgebras of $\underline{g}_c$. We define $\rho_0 \in \underline{a}_c^*$ and $\rho \in \underline{h}_c^*$ as follows.

$$\rho_0(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) = 1 \quad \text{and} \quad \rho(\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}) = i.$$

Let D be the open unit disk $|z|<1$ in C and T the boundary of D. Then each element g in G acts transitively as analytic automorphism of D under

$$z \to g \cdot z = (\bar{\beta} z + \bar{\alpha})^{-1}(\alpha z + \beta) \ ; \ g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \text{ and } z \in D.$$

This action is naturally extended to the boundary T. Then K and M are respectively the subgroups of G fixing 0 in D and 1 in T, so we have the identifications:

$$D = G/K \quad \text{and} \quad T = K/M.$$

Let $dk=(4\pi)^{-1}d\theta$ denote the normalized Haar measure on K and dg the one on G normalized as the following integral formula holds:

$$\int_G f(g)dg = 2\pi(4\pi)^{-2}\int_0^{4\pi}\int_0^{\infty}\int_0^{4\pi} f(k_\theta a_t k_{\theta'}) \operatorname{sh} t \, d\theta \, dt \, d\theta' \quad (2.1)$$

whenever the integral exists. For each measurable space (X,dx) $L^p(X)$ $(1 \leq p < \infty)$ denotes the space consisting of all the functions f on X for which $\int_X |f(g)|^p dx < \infty$ with obvious norm.

Let K^ and M^ denote the sets of equivalence classes of irreducible unitary representations of K and M respectively, which are parametrized as

$$K^\wedge = \{ \tau_n ; n \in \tfrac{1}{2}Z \} \text{ and } M^\wedge = \{ \sigma_\varepsilon ; \varepsilon = 0, \tfrac{1}{2} \}.$$

Actually, they are defined by $\tau_n(k_\theta) = e^{in\theta}$ and $\sigma_\varepsilon(\pm 1) = (\pm 1)^{2\varepsilon}$. Last for $\varepsilon=0, \frac{1}{2}$ we let

$$Z_\varepsilon = \{ n \in \tfrac{1}{2}Z ; 2n \equiv 2\varepsilon \mod(2) \}.$$

§3. <u>Szegö operators</u>. For $\sigma_\varepsilon \in M^\wedge$ and $\tau_n \in K^\wedge$ let

$$C^\infty(K, \varepsilon) = \{f \in C^\infty(K);\ f(mk) = \sigma_\varepsilon(m) f(k) \text{ for } m \in M,\ k \in K\}$$

and

$$C^\infty(G, \tau_n) = \{f \in C^\infty(G);\ f(gk_\theta) = \tau_n(k_\theta) f(g) \text{ for } k \in K,\ g \in G\}.$$

Obviously, if let

$$I_\varepsilon : C^\infty(T) \to C^\infty(K, \varepsilon)$$

denote the operator defined by $I_\varepsilon(F)(k_\theta) = e^{i\varepsilon\theta} F(e^{i\theta})$, we can identify $C^\infty(T)$ with $C^\infty(K, \varepsilon)$, especially, I_ε is an isometry between $L^2(T)$ and $L^2(K, \varepsilon)$, the L^2 completion of $C^\infty(K, \varepsilon)$.

For $\nu \in C$ the Szegö operator

$$S_{\varepsilon,\nu,n} : C^\infty(K, \varepsilon) \to C^\infty(G, \tau_n)$$

is defined by

$$S_{\varepsilon,\nu,n}(f)(x)=\int_K e^{\nu H(x^{-1}k)}\tau_n(\kappa(x^{-1}k))f(k)dk$$

$$= e^{in(\theta+\theta')}(1-|w|^2)^{-\nu} \qquad (3.1)$$

$$\times \int_0^{2\pi} \frac{|1-e^{-i\psi}w|^{2n}}{(1-e^{-i\psi}w)^{2n}} |1-e^{-i\psi}w|^{2\nu}e^{-in\psi}f(k_\psi)d\psi,$$

where $x=k_\theta a_t k_{\theta'}\in G$ and $w=x\cdot 0=\tanh t/2e^{i\theta}\in D$ (see [KW], p.178). Clearly, $S_{\varepsilon,\nu,n}(f)\equiv 0$ except $n\in Z_\varepsilon$, and when $\varepsilon=½$, $\nu=-½$ and $n=\pm½$, the integral of $S_{½,-½,\pm½}(f)(x)$ coincides with the classical Szegö projection operator on $L^2(T)$ (cf. [R], p.178). Actually, for $F\in L^2(T)$ with the Fourier series $\Sigma_{p\in Z}\, a_p e^{ip\theta}$, if we let

$$F_+(w) = \Sigma_{p=0}^{\infty}\, a_p w^p \quad (w\in D),$$

then

$$S_{½,-½,½}(I_\varepsilon(F))(x) = e^{½in(\theta+\theta')}(1-|w|^2)^{½}\, F_+(w)$$

and (3.2)

$$S_{½,-½,-½}(I_\varepsilon(F))(x) = e^{-½i(\theta+\theta')}(1-|w|^2)^{½}\, F_+(\bar{w}).$$

§4. Principal series and $S_{\varepsilon,\nu,n}$. For $\varepsilon\in\{0,½\}$ and $\nu\in C$ let $(\pi_{\varepsilon,\nu},L^2(T))$ denote the principal series representation of G, that is defined by, for $F\in L^2(T)$

$$\pi_{\varepsilon,\nu}(g)(F)(\xi)=|\bar{\beta}\xi+\bar{\alpha}|^{-2\nu}\left(\frac{\bar{\beta}\xi+\bar{\alpha}}{|\bar{\beta}\xi+\bar{\alpha}|}\right)^{2\varepsilon} F\left(\frac{\alpha\xi+\beta}{\bar{\beta}\xi+\bar{\alpha}}\right), \quad (4.1)$$

where $\zeta = e^{i\varphi} \in T$ and $g^{-1} = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \in G$ (cf. [Su], p.207). Let $\{e_p(\zeta)\ ;\ p \in Z\}$ denote the complete orthonormal system of $L^2(T)$ given by

$$e_p(\zeta) = \zeta^{-p} = e^{-ip\varphi}.$$

Then it follows from (4.1) that

$$\pi_{\varepsilon,\nu}(x)e_p(\zeta) = e^{i(p+\varepsilon)(\theta+\theta')} \frac{(1-|w|^2)^\nu}{|1-w\bar{\zeta}|^{2\nu}} \left(\frac{|1-w\bar{\zeta}|}{1-w\bar{\zeta}} \right)^{2(p+\varepsilon)} \zeta^{-p}, \quad (4.2)$$

where $x = k_\theta a_t k_{\theta'} \in G$ and $w = x \cdot 0 \in D$, so we see that

<u>Lemma 4.1</u>. There exists a positive constant C such that

$$|\pi_{\varepsilon,\nu}(x)e_p(\zeta)| \leqq Ce^{\sigma(x)|Re(\nu)|} \quad (x \in G).$$

Moreover, by comparing the definition (3.1) of $S_{\varepsilon,\nu,n}$ with (4.1), we can deduce that

<u>Proposition 4.2</u>. Let $\varepsilon \in \{0, 1/2\}$, $n \in Z_\varepsilon$ and $\nu \in C$. Then for $f \in L^2(K, \varepsilon)$

$$S_{\varepsilon,\nu,n}(f)(x) = \int_T \pi_{\varepsilon,-\nu}(x)e_{n-\varepsilon}(\zeta)I_\varepsilon^{-1}(f)(\zeta)d\zeta.$$

Let $\pi^{p,q}_{\varepsilon,\nu}(x)$ $(p, q \in Z)$ denote the matrix coefficient of $\pi_{\varepsilon,\nu}(x)$ $(x \in G)$ defined by

$$\pi_{\varepsilon,\nu}(x) = (\pi_{\varepsilon,\nu}(x)e_q, e_p). \quad (4.3)$$

Then, by substituting (4.2) for (4.3) the explicit form of $\pi^{p,q}_{\varepsilon,\nu}(a_t)$ is given by

$$(1-r^2)^{\nu} r^{p-q} \binom{-\nu-q-\varepsilon}{p-q} F(\nu-q-\varepsilon, \nu+p+\varepsilon; p-q+1; r^2) \quad (p \geqq q)$$

and (4.4)

$$(1-r^2)^{\nu} r^{q-p} \binom{-\nu+q+\varepsilon}{q-p} F(\nu+q+\varepsilon, \nu-p-\varepsilon; q-p+1; r^2) \quad (q \geqq p),$$

where r=tht/2 and F(a,b,c;z) is the hypergeometric function (cf. [Sa], p.74). Then, using this expression, we can easily deduce that the matrix coefficients satisfy the following relations (cf. [J], §4 and [B], p.26).

<u>Lemma 4.3</u>. Let $\varepsilon \in \{0, \frac{1}{2}\}$, $\nu \in C$ and p, $q \in Z$. Then for $x \in G$

(1) $\pi^{p,q}_{\varepsilon,\nu}(x) = \pi^{q,p}_{\varepsilon,1-\bar{\nu}}(x^{-1})$

(2) $\pi^{p,q}_{\varepsilon,\nu}(x) = \omega^{p,q}_{\varepsilon,\nu}\ \pi^{p,q}_{\varepsilon,1-\nu}(x)$,

where $\omega^{p,q}_{\varepsilon,\nu}$ is given by

$$\binom{-\nu-q-\varepsilon}{p-q} \Big/ \binom{-1+\nu-q-\varepsilon}{p-q} \ (p \geqq q) \text{ and } \binom{-\nu+q+\varepsilon}{q-p} \Big/ \binom{-1+\nu+q+\varepsilon}{q-p} \ (q \geqq p).$$

We regard $X = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $Y = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ as left invariant differential operators on G and put $E_{\pm} = \pm X + Y$. Then, since $E_{\pm}^{\sim} = \pm d\pi_{\varepsilon,\nu}(X) + id\pi_{\varepsilon,\nu}(Y)$ make a shift of K-types according to

$$E_{+}^{\sim} e_p = (p+\varepsilon+\nu) e_{p+1}$$

and (4.5)

$$E_{-}^{\sim} e_p = (p+\varepsilon-\nu) e_{p-1}$$

(cf. [Su], p.216), it follows from Proposition 4.2 that

Lemma 4.4. Let the notation be as above. Then

$$E_{\pm}S_{\varepsilon,\nu,n}(f) = (n \mp \nu)S_{\varepsilon,\nu,n\pm 1}(f).$$

§5. Discrete series and $S_{\varepsilon,\nu,n}$. For $n \in \frac{1}{2}Z$ and $|n| \geqq 1$ let $(T_n, A_{2,n-1}(D))$ denote the discrete series representation of G, where $A_{2,n-1}(D)$ is the L^2 weighted Bergman space on the unit disc $D=\{z \in C; |z|<1\}$ defined by, for $n \geqq 1$

$$A_{2,n-1}(D)=\{F: D \to C;\ F \text{ is holomorphic on } D \text{ and}$$

$$\| F \|_{2,n-1}=[(2n-1)\pi^{-1} \int_D |F(z)|^2(1-|z|^2)^{2n-2}dz]^{1/2} < \infty\}$$

and for $n \leqq 1$, it is made up of conjugate holomorphic functions on D with the norm given by replacing n with $|n|$. Then $T_n(g)F$ ($g \in G$ and $F \in A_{2,n-1}(D)$) is defined by, for $n \geqq 1$

$$T_n(g)(F)(z)=(\bar{\beta}z+\bar{\alpha})^{-2n}F\left(\frac{\alpha z+\beta}{\bar{\beta}z+\bar{\alpha}}\right), \tag{5.1}$$

where $z \in D$ and $g^{-1}=\begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \in G$; for $n \leqq -1$, it is defined by $T_n(g)(F) = \mathrm{conj}(T_{|n|}(g)(\mathrm{conj}(F)))$, where conj is the operator taking the complex conjugation (cf. [Su], p.229). Let $\{e_p^n(z);\ p \in N\}$ denote the complete orthonormal system of $A_{2,n-1}(D)$ defined by

$$e_p^n(z) = \lambda_p^n z^p \ (n \geqq 1) \quad \text{and} \quad \lambda_p^n (z^-)^p \ (n \leqq -1),$$

where $(\lambda_p^n)^2 = \Gamma(p+2|n|)/\Gamma(p+1)\Gamma(2|n|)$. Let $T_n^{p\,q}(g)$ $(p,q \in N)$ denote the matrix coefficient of $T_n(g)$ $(g \in G)$ defined by

$$T_n^{p\,q}(g) = (T_n(g)e_q^n, e_p^n). \tag{5.2}$$

Then, $\| T_n^{p\,q} \|^2 = 4\pi(2|n|-1)^{-1}$ (cf. [Su], p.326), and comparing (4.1) with (5.1) and (4.3) with (5.2), we can deduce that

<u>Lemma 5.1</u>. Let $\varepsilon \in \{0, ½\}$, m, $n \in Z_\varepsilon$, $n \geqq m \geqq 1$ and p, $q \in Z$, $q \geqq m-\varepsilon$.

(1) $\pi_{\varepsilon,m}(g)e_{n-\varepsilon}(\zeta) = (\lambda_{n-m}^m)^{-1} T_{-m}(g) e_{n-m}^{-m}(z) \,|_{z=\zeta} e_{m-\varepsilon}(\zeta)$,

(2) $\pi_{\varepsilon,m}^{q,\,n-\varepsilon}(g) = (\lambda_q^m)^2 / \lambda_{n-m}^m \lambda_{q-m+\varepsilon}^m \, T_{-m}^{q-m+\varepsilon\ \ n-m}(g)$.

Let Ω be the Casimir operator in $U(\underline{g}_c)$ given by $-H^2 + ½(X^2+Y^2)$, where $-2H=[X,Y]$. Then it is well known (cf. [Su], p.288) that $Z(\underline{g}_c) = C\Omega$ and

<u>Lemma 5.2</u>. Let $\varepsilon \in \{0, ½\}$, $n \in Z_\varepsilon$ and $|n| \geqq 1$. Then

$$\Omega \pi_{\varepsilon,n}(x) = \Omega T_n(x) = n(1-n) \quad (x \in G).$$

In what follows we shall investigate the relation between the discrete sereis T_n and the Szegö operators $S_{\varepsilon,\nu,n}$.

Let V_ℓ $(\ell \in N)$ denote the set of all homogeneous polynomials of degree ℓ with variables z_1 and z_2. Then the finite dimensional representation (π_ℓ, V_ℓ) of $G_c = SL(2,C)$ is defined by

$$\pi_\ell(g)P(z) = P(z\cdot g) \quad \text{for } z = (z_1,z_2),$$

where $P\in V_\ell$ and $z\cdot g=(az_1+cz_2, bz_1+dz_2)$ if $g=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in G_c$. Especially, when $g=k_\theta a_t k_{\theta'}$, $z_1\cdot g$ and $z_2\cdot g$ are respectively given as follows.

$$e^{i(\theta+\theta')/2}\text{ch}t/2\ z_1 + e^{-i(\theta-\theta')/2}\text{sh}t/2\ z_2$$

and (5.3)

$$e^{i(\theta-\theta')/2}\text{sh}t/2\ z_1 + e^{-i(\theta+\theta')/2}\text{ch}t/2\ z_2.$$

Let $d=d_\ell=\dim V_\ell=\ell+1$ and $J_\ell=\{1,2,\cdots,d\}$. Then $v_i^\sim=[(i-1)!(\ell-i+1)!]^{-1/2} z_1^{i-1}z_2^{\ell-i+1}$ $(i\in J_\ell)$ is a $(2i-\ell-2)\rho$-weight vector with respect to $\underline{h}_c$ and $v_{d+1-i}=[(i-1)!(\ell-i+1)!/\ell!]^{-1/2}(z_1-z_2)^{i-1}(z_1+z_2)^{(\ell-i+1)}$ $(i\in J_\ell)$ is a $-(2i-\ell-2)\rho_0$-weight vector with respect to $\underline{a}_c$. Especially, we shall equip V_ℓ with the inner product for which $\{v_i; i\in J_\ell\}$ is an orthonormal system of V_ℓ. If we put $C_{ji}=(v_j^\sim,v_i)$ and $[D_{ij}]=[C_{ij}]^{-1}$, we see that $C_{id}=2^{-\ell}[(i-1)!(\ell-i+1)!]^{-1/2}$, $\|v_i^\sim\|^{-2}=\ell!2^\ell$ and

$$v_j^\sim = \sum_{i\in J_\ell} C_{ji}v_i \quad \text{and} \quad v_j = \sum_{i\in J_\ell} D_{ji}v_i^\sim. \tag{5.4}$$

Let $\pi_\ell^{ij}(x)$ $(i,j\in J_\ell)$ denote the matrix coefficient of $\pi_\ell(x)$ $(x\in G)$ defiend by

$$(\pi_\ell(x)v_j, v_i). \tag{5.5}$$

<u>Lemma 5.3</u>. Let a, $b\in Z_\varepsilon$ $(\varepsilon=0, 1/2)$ and $|b|\leqq a$. Then

$$e^{aH(x)}\tau_b(\kappa(x))^{-1} = C_{a-b+1\ d}^{-1}\sum_{i\in J_{2a}} C_{a-b+1\ i}\ \pi_{2a}^{id}(x),$$

where $d=2a+1$.

Proof. Noting the Iwasawa decomposition of $x \in G$, we easily see that the left hand side is equal to $C_{a-b+1\ d}{}^{-1}(\pi_{2a}(x)v_d, v_{a-b+1}{}^{\sim})$. Then, substituting with $v_{a-b+1}{}^{\sim} = \Sigma_i C_{a-b+1\ i} v_i$, we have the desired result.

Q.E.D.

Let $\varepsilon \in \{0, \frac{1}{2}\}$, $m \in Z_\varepsilon$ and $i \in J_{2m+1}$. Then for $f \in L^2(K, \varepsilon)$ we define

$$S^i_{\varepsilon,m}(f)(x) = \sum_{p \in J_{2m+1}} S_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}(f\,\pi^{pd}_{2m+1})(x)\,\pi^{ip}_{2m+1}(x^{-1}) \qquad (5.6)$$

where $d=2m+2$ and $f\pi^{pd}_{2m+1}$ is the function on K given by $f(k)\pi^{pd}_{2m+1}(k)$ $(k \in K)$. Then it follows that

<u>Proposition 5.4</u>. Let $\varepsilon \in \{0, \frac{1}{2}\}$ and $m, n \in Z_\varepsilon$. Suppose that $m \geqq -\frac{1}{2}$ and $-m \leqq n \leqq m+1$. Then for $f \in L^2(K, \varepsilon)$

$$S_{\varepsilon,m,n}(f)(x) = C_{m-n+2\ d}{}^{-1} \sum_{i \in J_{2m+1}} C_{m-n+2\ i}\, S^i_{\varepsilon,m}(f)(x),$$

where $x \in G$ and $d=2m+2$.

Proof. We can rewrite the integral in the definition of $S_{\varepsilon,m,n}$ as

$$S_{\varepsilon,m,n}(f)(x) = \int_K e^{-\frac{1}{2}H(x^{-1}k)}\,\tau_{\frac{1}{2}}(\kappa(x^{-1}k))^{-1}$$

$$\times\ e^{(m+\frac{1}{2})H(x^{-1}k)}\,\tau_{n-\frac{1}{2}}(\kappa(x^{-1}k))^{-1} f(k)\,dk.$$

Then, noting the assumption on m and n, we can apply Lemma 5.3 for $a=m+\frac{1}{2}$ and $b=n-\frac{1}{2}$ to the right hand side. Then it follows that

KSTS/RR90/005
September 14, 1990

$$=C_{m-n+2}\ d^{-1}\sum_{i\in J_{2m+1}} C_{m-n+2\ i}\int_K e^{-\frac{1}{2}H(x^{-1}k)}\tau_{\frac{1}{2}}(\kappa(x^{-1}k)^{-1}$$

$$\times\ \pi^{id}_{2m+1}(x^{-1}k)f(k)dk.$$

Then, since $\pi^{id}_{2m+1}(x^{-1}k)=\Sigma_p\pi^{ip}_{2m+1}(x^{-1})\pi^{pd}_{2m+1}(k)$, the desired result follows from the definition of $S_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}$.

Q.E.D.

Theorem 5.5. Let $\varepsilon\in\{0,\frac{1}{2}\}$, $m\in Z_\varepsilon$ and $m\geqq-\frac{1}{2}$. Let $f(\zeta)=\sum_{p\in Z_\varepsilon} a_p\zeta^p$ be a function in $L^2(K,\varepsilon)$ satisfying $|a_p|=0$ for $|p|\leqq m$. Then

$$S_{\varepsilon,m,m+1}(f)(x)$$

$$=(1-|w|^2)^{m+\frac{1}{2}}\ e^{i(m+\frac{1}{2})(\theta+\theta')}\ S_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}(fe^{-i(m+\frac{1}{2})\theta})$$

$$=(1-|w|^2)^{m+1}\ e^{i(m+1)(\theta+\theta')}\ (I_\varepsilon^{-1}f)_+(w)w^{-(m+1)},$$

where $x=k_\theta a_t k_{\theta'}\in G$ and $w=x\cdot 0\in D$.

Proof. Since $v_s^{\sim}$ $(s\in J_{2m+1})$ are weight vectors with respect to $\underline{h}_c$, we see from (5,4) and (5.5) that

$$\pi^{pd}_{2m+1}(k_\theta)=\Sigma_s D_{ds}C_{sp}e^{(-(m+\frac{1}{2})+(s-1))i\theta}$$

Therefore, we can rewrite (5.6) as follows.

$$S^i_{\varepsilon,m}(f)(x)=\sum_{s,p\in J_{2m+1}} D_{ds}C_{sp}S_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}(fe^{-i(m+\frac{1}{2})\theta}e^{i(s-1)\theta})\,\pi^{ip}_{2m+1}(x^{-1})$$

$$= S_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}(fe^{-i(m+\frac{1}{2})\theta}) \ \Sigma_{s,p} D_{ds} C_{sp} w^{s-1} \pi_{2m+1}^{ip}(x^{-1}).$$

Here we used (3.2) and the assumption that $a_p = 0$ for $|p| \leqq m$ to obtain the last equation. We note that

$$\Sigma_{s,p} D_{ds} C_{sp} w^{s-1} \pi_{2m+1}^{ip}(x^{-1})$$

$$= \Sigma_s D_{ds} w^{s-1} (\pi_{2m+1}(x^{-1}) v_s^{\sim}, v_i)$$

$$= (\pi_{2m+1}(x^{-1})(wz_1+z_2)^{d-1}, v_i).$$

Then, it follows from Proposition 5.4 that

$$S_{\varepsilon,m,m+1}(f)(x) = C_{1d}^{-1} S_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}(fe^{-i(m+\frac{1}{2})\theta})$$

$$\times \sum_{i \in J_{2m+1}} C_{1i}(\pi_{2m+1}(x^{-1})(wz_1+z_2)^{d-1}, v_i).$$

$$= C_{1d}^{-1} S_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}(fe^{-i(m+\frac{1}{2})\theta})$$

$$\times \ (\pi_{2m+1}(x^{-1})(wz_1+z_2)^{d-1}, v_1^{\sim}).$$

We recall that $\pi_{2m+1}(x^{-1})$ transforms wz_1+z_2 to

$$(1-|w|^2)^{\frac{1}{2}} e^{i(\theta+\theta')/2} z_2 \qquad (5.7)$$

(see (5.3)). Therefore, since $v_1^{\sim} = (\ell\,!)^{-\frac{1}{2}} z_2^{d-1}$ and $C_{1d}^{-1} \| v_1^{\sim} \|^2 = (\ell\,!)^{-\frac{1}{2}}$, we can deduce that $S_{\varepsilon,m,m+1}(f)(x)$ must be equal to

$$S_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}(fe^{-i(m+\frac{1}{2})\theta})(1-|w|^2)^{m+\frac{1}{2}} e^{i(m+\frac{1}{2})(\theta+\theta')}.$$

The second equation in the statement easily follows from (3.2)

Q.E.D.

We retain the notation and the assumption in Theorem 5.5. Then the theorem and (2.1) implies that if $m \geqq 0$, $S_{\varepsilon,m,m+1}(f) \in L^2(G)$ and thus, by Lemma 4.4, $S_{\varepsilon,m,n}(f)$ $(0 \leqq m \leqq n-1)$ also belongs to $L^2(G)$. Then substituting the decomposition of f: $f(\zeta) = \Sigma a_p \zeta^p$, where $p \in Z_\varepsilon$ and $|p| > m$, with Proposition 4.2 and using (4.5), we see that $S_{\varepsilon,m,n}(f)$ can be written as an L^2 linear combination of the matrix coefficients $\pi_{\varepsilon,-m}^{q-\varepsilon\ n-\varepsilon}$, where $q \geqq m+1$ and so $T_{-(m+1)}^{q-m-1\ n-m-1}$ by Lemma 4.3 (2) and Lemma 5.1 (2). This fact also follows from the left K-type decomposition of $S_{\varepsilon,m,n}(f)$, say $\sum_{\ell \in Z_\varepsilon} {}_\ell S(f)$. In fact, each ${}_\ell S(f)$ is an L^2 function on G with K-type (ℓ, n) and, by Proposition 4.2 and Lemma 5.2, it is also a $Z(\underline{g}_c)$-eigenfunction wiht eigenvalue $-m(m+1)$. Therefore, ${}_\ell S(f)$ must be a cusp form on G, and thus a scalar multiplication of the matrix coefficient $T_{-(m+1)}^{\ell-m-1\ n-m-1}$ of $T_{-(m+1)}$. Clearly, $\ell \geqq m+1$. So, we obtained

<u>Proposition 5.6</u>. We keep the notation and the assumption in Theorem 5.5 and suppose that $n \in Z_\varepsilon$ and $0 \leqq m \leqq n-1$. Then $S_{\varepsilon,m,n}(f)$ can be written as an L^2 linear combination of $T_{-(m+1)}^{p\ n-m-1}$ $(p \geqq 0)$.

Next theorem will not be used in the argument below. However, it is an important and interesting property that expresses the relation among the Szegö operators $S_{\varepsilon,m,n}$ $(0 \leqq m \leqq n-1)$.

<u>Theorem 5.7</u>. Let $m \in Z_\varepsilon$ $(\varepsilon = 0, \frac{1}{2})$ and $m \geqq -\frac{1}{2}$. Then for f in $L^2(K, \varepsilon)$

$$\sum_{n \in J_{2m+1}} \binom{2m-1}{n-1} (e^{i(\theta+\theta')} W^-)^{n-1} S_{\varepsilon,m,m+2-n}(f)(x)$$

$$= (1-|w|^2)^{m+1/2}\, e^{i(m+1/2)(\theta+\theta')}\, S_{1/2,-1/2,1/2}(fe^{-i(m+1/2)\theta})$$

$$= (1-|w|^2)^{m+1}\, e^{i(m+1)(\theta+\theta')}\, (I_\varepsilon^{-1}f)_+(w)w^{-(m+1)},$$

where $x=k_\theta a_t k_{\theta'}\in G$ and $w=x\cdot 0\in D$.

Proof. We keep the notation in (3.1). Then we note that

$$\sum_{n\in J_{2m+1}} \binom{2m+1}{n-1} \left[\frac{(1-e^{-i\varphi}w)^2}{|1-e^{-i\varphi}w|^2}\, e^{i(\varphi-\theta)}\, |w| \right]^{n-1}$$

$$= \left(1 + \frac{(1-e^{-i\varphi}w)^2}{|1-e^{-i\varphi}w|^2}\, e^{i(\varphi-\theta)}\, |w|\right)^{2m+1}$$

$$= \left(\frac{1-|w|^2}{1-e^{i\varphi}\bar{w}} \right)^{2m+1}.$$

Therefore, the desired relation follows from (3.1) and (3.2).

Q.E.D.

§6. <u>Plancherel formula</u>. In this section we shall rewrite the Plancherel formula for $L^2(G)$ (cf. [Su], p.344 and p.346) by using the Szegö operators $S_{\varepsilon,\nu,n}$.

The Plancherel formula implies that each L^2 function f on G can be written as $f = {}^{p}f + {}^{\circ}f$, where ${}^{p}f$ is the sum of wave packets, the integral part of the formula, and ${}^{\circ}f$ is a linear combination of cusp forms, the discrete part of the formula, so $L^2(G)$ has a direct sum decomposition:

$$L^2(G) = {}^{P}L^2(G) \oplus {}^{\circ}L^2(G). \qquad (6.1)$$

For f in $L^2(G)$ we denote by $f = \sum {}_m f_n$ the K-type decomposition of f, where m, $n \in \frac{1}{2}Z$ and the K-type of ${}_m f_n$ is (m,n). When we restrict our attention to L^2 functions on G with right k-type n, we denote the decomposition (6.1) as

$$L^2{}_n(G) = {}^{P}L^2{}_n(G) \oplus {}^{\circ}L^2{}_n(G)$$

By the same way we denote the decomposition of compactly supported C^∞ functions on G with right K-type n as

$$C^\infty_{c.n}(G) = {}^{P}C^\infty_{c.n}(G) \oplus {}^{\circ}C^\infty_{c.n}(G).$$

For $R \geqq 0$ let G(R) denote the compact set in G defined by $\sigma(x) \leqq R$ when $x \in G(R)$. Then $C^\infty_c(G;R)$ denotes the set of all C^∞ functions on G whose supports are contained in G(R) and $C^\infty_{c.n}(G;R)$ the subspace with right K-type n. Let ${}^{\circ}C_n(G)$ denote the space of cusp forms on G with right K-type n. Then the following proposition will play an important role in §7.

<u>Proposition 6.1</u>. For each $R>0$

$${}^{\circ}C^\infty_{c.n}(G;R) = {}^{\circ}C_n(G).$$

Proof. By the definition it is clear that ${}^{\circ}C^\infty_{c.n}(G) \subset {}^{\circ}C_n(G)$, so we shall prove the reverse. Let f be in ${}^{\circ}C_n(G)$. First we assume that the left K-type of f is q ($q \geqq \varepsilon$). Then, as stated before Proposition 5.6, the discrete part of $L^2{}_n(G)$ with K-type (q,n) is an L^2 span of a finite number of cusp forms on G, say ϕ_s $(1\leqq s\leqq N)$ that are linearly independent and real analytic on G. Therefore, for an arbitrary open subset S in G(R) we can choose compactly supported, C^∞ functions h_t $(1\leqq t\leqq N)$ on G such that $(h_t, \phi_s) = \delta_{ts}$ $(1\leqq s,t\leqq N)$ and $\mathrm{supp}(h_t) \subset CL(S)$. Obviously, we may assume that the K-type of h_t is (q,n). Let

$$g = \sum_{1\leqq t\leqq N} (f, \phi_t)h_t.$$

Then, $g\in C^\infty_{c,n}(G)$, $\mathrm{supp}(g) \subset CL(S)$ and $^\circ g = f$, because

$$(g, \phi_s) = \sum_{1\leqq t\leqq N} (f, \phi_t)(h_t, \phi_s) = (f, \phi_s).$$

Therefore, we see that $f\in {}^\circ C^\infty_{c,n}(G)$.

Next we shall consider the case of an arbitrary f in $^\circ C_n(G)$. Let $f = \Sigma_m f$ denote the left K-type decomposition of f and S_m $(m\in Z_\varepsilon)$ the open subsets in G(R) such that $CL(S_p) \cap CL(S_q) = \phi$ if $p\neq q$. Then, as proved above, for each m there exists a compactly supported, C^∞ function $_m g$ with K-type (m,n) such that $^\circ {}_m g = {}_m f$ and $\mathrm{supp}({}_m g) \subset S_m$. Therefore, if we put $g = \Sigma_m g$, we see that $g\in C^\infty_{c,n}(G;R)$ and $^\circ g = f$, so $f\in {}^\circ C^\infty_{c,n}(G;R)$.

This completes the proof of the reverse: $^\circ C^\infty_{c,n}(G;R)\supset {}^\circ C_n(G)$.

Q.E.D.

Let $\varepsilon \in\{0,½\}$, $n\in Z_\varepsilon$ and $\nu=½+i\lambda \in C$. For f in $C^\infty_c(G)$ we define the Fourier transform $f^\wedge(\lambda,\zeta)$ $((\lambda,\zeta)\in R\times T)$ associated with the principal series $\pi_{\varepsilon,\nu}$ by

$$f^\wedge(\lambda,\zeta) = \int_G f(g)\mathrm{conj}(\pi_{\varepsilon,\nu}(g)e_{n-\varepsilon}(\zeta))dg \qquad (6.2)$$

and moreover, for $m \in Z_\varepsilon$ we define

$$ {}_{m}f^{\wedge}(\lambda) = \int_G f(g)\,\mathrm{conj}(\pi_{\varepsilon,\nu}^{m-\varepsilon\ n-\varepsilon}(g))\,dg. \qquad (6.3)$$

Let $\alpha(\lambda, \zeta)$ be a function on $R \times T$. Then we define $A_n(\alpha)(\lambda, x)$ on $R \times G$ by

$$A_n(\alpha)(\lambda, x) = S_{\varepsilon,\bar{\nu}-1,n}(I_\varepsilon \alpha(\lambda, \cdot))(x) \qquad (6.4)$$

whenever this integral exists. When $\alpha(\lambda, \zeta)$ is integrable in ζ for a fixed $\lambda \in R$, the integral exists for the λ (see Lemma 4.1 and Proposition 4.2). We call $\alpha(\lambda, \zeta)$ a holomorphic function of uniform exponential type R if it is holomorphic in λ and if there exists a constant $R \geqq 0$ such that for each $N \geqq 0$

$$\sup_{\lambda \in C,\ \zeta \in T} e^{-R|Im(\lambda)|}(1+|\lambda|)^N |\alpha(\lambda, \zeta)| < \infty.$$

Then, as noted above, it follows from Lemma 4.1 that, if $\alpha(\lambda, \zeta)$ is a holomorphic function of uniform exponential type, $A_n(\alpha)(\lambda, x)$ is well defined for $(\lambda, x) \in C \times G$ and holomorphic in λ. We also define antiholomorphic functions of uniform exponential type by the same way.

Lemma 6.2. Let f be in $C_c^\infty(G;R)$.

(1) ${}_{m}f^{\wedge}(\lambda)$ $(\lambda \in C)$ is an antiholomorphic function of exponential type R and ${}_{m}f^{\wedge}(\lambda) = ({}_{m}f_n)^{\wedge}(\lambda)$.

(2) $f^{\wedge}(\lambda, \zeta)$ $((\lambda, \zeta) \in C \times T)$ is an antiholomorphic

function of uniform exponential type R and

$$f^\wedge(\lambda, \zeta) = \sum_{m \in Z_\varepsilon} {}_m f^\wedge(\lambda) e_{\varepsilon - m}(\zeta).$$

(3) $$A_n(f^\wedge)(\lambda, x) = A_n(f^\wedge)(-\lambda, x) \quad ((\lambda, x) \in R \times G).$$

Proof. (1) and (2) are obvious from Lemma 4.1, (6.2) and (6.3), so we shall prove (3).

$$A_n(f^\wedge)(\lambda, x) = \int_T \pi_{\varepsilon, 1-\nu^-}(x) e_{n-\varepsilon} f^\wedge(\lambda, \zeta) d\zeta$$

$$= \sum_{m \in Z_\varepsilon} {}_m f^\wedge(\lambda) \pi_{\varepsilon, 1-\nu}^{m-\varepsilon\; n-\varepsilon}(x)$$

$$= \sum_{m \in Z_\varepsilon} \int_G f(g) \operatorname{conj}(\pi_{\varepsilon, \nu}^{m-\varepsilon\; n-\varepsilon}(g) \pi_{\varepsilon, \nu}^{n-\varepsilon\; m-\varepsilon}(x^{-1})) dg$$

$$= \int_G f(g) \operatorname{conj}(\pi_{\varepsilon, \nu}^{n-\varepsilon\; n-\varepsilon}(x^{-1} g)) dg.$$

Then, since $\omega_{\varepsilon, \nu}^{n-\varepsilon\; n-\varepsilon} \equiv 1$, (3) follows from Lemma 4.3 (2).

Q.E.D.

Now we shall consider the inversion formula of the Fourier transform defined by (6.2). Let

$$\mu_\varepsilon(\lambda) = \begin{cases} \lambda \pi \operatorname{th}(\pi \lambda) & (\varepsilon = 0) \\ \lambda \pi \operatorname{ch}(\pi \lambda) & (\varepsilon = \tfrac{1}{2}). \end{cases} \qquad (6.5)$$

Then it is well known (cf. [Su], Ch.V, §8 and [B], §10) that for $f \in {}^\varepsilon L^2{}_n(G)$

$${}_m f(x) = \int_R {}_m f^\wedge(\lambda)\, \pi_{\varepsilon,\nu}^{m-\varepsilon\ n-\varepsilon}(x)\, \mu_\varepsilon(\lambda) d\lambda$$

and (6.6)

$$\int_G |{}_m f(x)|^2 dx = \int_R |{}_m f^\wedge(\lambda)|^2 \mu_\varepsilon(\lambda) d\lambda .$$

Let

$$L^2_n(R\times T) = \{\alpha(\lambda,\zeta) \in L^2(R\times T, \mu_\varepsilon(\lambda) d\lambda d\zeta) ;$$

$$A_n(\alpha)(\lambda, x) = A_n(\alpha)(-\lambda, x) \text{ for } (\lambda, x) \in R\times G\}.$$

Then, for $\alpha \in L^2_n(R\times T)$, if we define

$$\alpha^\vee(x) = \int_R S_{\varepsilon,-\nu,n}(I_\varepsilon \alpha(\lambda,\cdot))(x)\, \mu_\varepsilon(\lambda) d\lambda \quad (x \in G), \quad (6.7)$$

we see the following

<u>Proposition 6.3</u>. The Fourier transform $f(x) \to f^\wedge(\lambda,\zeta)$ is an isometry of $L^2_n(G)$ onto $L^2_n(R\times T)$ and the inversion formula is given by

$$f(x) = \sum_{m \in Z_\varepsilon} \int_R {}_m f^\wedge(\lambda)\, \pi_{\varepsilon,\nu}^{m-\varepsilon\ n-\varepsilon}(x)\, \mu_\varepsilon(\lambda) d\lambda$$

$$= (f^\wedge)^\vee(x).$$

Proof. Except the last equation the assertions are obvious from Lemma 6.2 (2) and (6.6), so we shall prove the last equation. Clearly, it is enough to prove it for $f \in {}^\nu C_{c,n}^\infty(G)$. Then it follows from (4.3) and Lemma 6.2 (2) that

$$\sum_{m \in Z_\varepsilon} \int_R {}_m f^\wedge(\lambda)\, \pi_{\varepsilon,\nu}^{m-\varepsilon\ n-\varepsilon}(x)\, \mu_\varepsilon(\lambda) d\lambda$$

$$= \sum_{m \in Z_\varepsilon} \int_R {}_m f^\wedge(\lambda)(\pi_{\varepsilon,\nu}(x)e_{n-\varepsilon},\ e_{m-\varepsilon})\,\mu_\varepsilon(\lambda)d\lambda$$

$$= \int_R (\pi_{\varepsilon,\nu}(x)e_{n-\varepsilon},\ \mathrm{conj}(f^\wedge(\lambda,\cdot)))\,\mu_\varepsilon(\lambda)d\lambda.$$

This integral is nothing but $(f^\wedge)^\vee(x)$ by Proposition 4.2 and (6.7).

Q.E.D.

<u>Corollary 6.4</u>. Let f be in ${}^P C^\infty_{c,n}(G)$. Then

(1) $f(x) = (E_+)^{n-\varepsilon} \int_R S_{\varepsilon,-\nu,\varepsilon}(I_\varepsilon f^\wedge(\lambda,\cdot)P_n(\lambda)^{-1})\,\mu_\varepsilon(\lambda)d\lambda$,

where $P_n(\lambda) = (n-½+i\lambda)(n-3/2+i\lambda)\cdots(\varepsilon+½+i\lambda)$.

(2) $A_\varepsilon(f^\wedge P_n^{-1})(\lambda,x) = A_\varepsilon(f^\wedge P_n^{-1})(-\lambda,x) \quad ((\lambda,x)\in R\times G)$.

Proof. (1) follows from the inversion formula in Proposition 6.3 and Lemma 4.4. We shall prove (2). By the same argument in Lemma 6.1 (3) we see that

$$A_\varepsilon(f^\wedge P_n^{-1})(\lambda,x) = P_n(\lambda)^{-1} \int_G f(g)\,\mathrm{conj}(\pi^{0,n-\varepsilon}_{\varepsilon,\nu}(x^{-1}g))dg.$$

Then, since $\omega^{0,n-\varepsilon}_{\varepsilon,1-\nu} = P_n(\lambda)/P_n(-\lambda)$ and $\mathrm{conj}(P_n(\lambda)) = P_n(-\lambda)$ by the definition, it follows from Lemma 4.3 (2) that

$$P_n(\lambda)^{-1}\pi^{0,n-\varepsilon}_{\varepsilon,1-\nu} = P_n(-\lambda)^{-1}\pi^{0,n-\varepsilon}_{\varepsilon,\nu}.$$

Therefore, the desired relation is obtained.

Q.E.D.

KSTS/RR90/005
September 14, 1990

Remark 6.5. We note that the integral of the formula in Corollary 6.4 (1) is nothing but apply the inversion formula for $L^2_\varepsilon(G)$ to the function $f^\wedge(\lambda,\zeta)P_n(\lambda)^{-1}$ satisfying (2). The formula for $L^2_\varepsilon(G)$ is simpler than one for $L^2_n(G)$, because it is made up only of wave packets, that is, the discrete part does not appear. Actually, the following theorem is well known for $\varepsilon=0$ by [H2] and $\varepsilon=½$ by the same way.

Theorem. (1) $L^2_\varepsilon(G) = {}^P L^2_\varepsilon(G)$ and the Fourier transform $f \to f^\wedge$ is an isometry of $L^2_\varepsilon(G)$ onto $L^2_\varepsilon(R\times T)$.
(2) The Fourier transfrom $f \to f^\wedge$ is a bijection of $C^\infty_{c,\varepsilon}(G;R)$ onto the set of holomorphic functions $\alpha(\lambda,\zeta)$ of uniform exponential type R satisfying $A_\varepsilon(\alpha)(\lambda,x) = A_\varepsilon(\alpha)(-\lambda,x)$.

The reduction formula in Corollary 6.4 will play an important role in §7. In fact, it reduces the proof of the Paley-Wiener theorem for $C^\infty_{c,n}(G)$ to the one for $C^\infty_{c,\varepsilon}(G)$ stated in Theorem (2).

Next we shall consider the Fourier transform associated with the discrete series T_m ($m\in ½Z$ and $|m|\geqq 1$) and the inversion formula, which are investigated in [K].

Let $n\in Z_\varepsilon$ and $I_n=\{\ell\in Z_\varepsilon\ ;\ 1\leqq \ell\leqq n\}$. Then for $m\in I_n$ and $f\in C_c^\infty(G)$ we define the Fourier transform $F_{-m}(f)(z)$ ($z\in D$) associated with the discrete series T_{-m} by

$$F_{-m}(f)(z) = \int_G f(g)\,\mathrm{conj}(T_{-m}(g)e_{n-m}(z^m))dg \qquad (6.8)$$

(see §5 and [K]). When we express the dependence on n, we use the notation F_{-m} instead of F_{-m}. Let $f^\wedge(z)$ denote a vector of functions on D given by

$$f^\wedge(z) = (F_{-m}(f)(z)\ ;\ m \in I_n). \tag{6.9}$$

Then we see the following

Proposition 6.6. Let the notation be as above.

(1) $F^n_{-m}(L^2(G)) = F^m_{-m}(L^2(G)) = A_{2,m-1}(D)$.

(2) For each $\beta \in A_{2,m-1}(D)$ we define

$$\beta^\vee(x) = (4\pi)^{-1}(2m-1)(1-r^2)^m e^{im(\theta+\theta')}\beta(w),$$

where $x=k_\theta a_t k_{\theta'} \in G$ and $w=x \cdot 0=re^{i\theta} \in D$. Then

$$F^m_{-m}(\beta^\vee) = \beta.$$

(3) We keep the notation in (2). Then

$$F^n_{-m}((\Gamma(2m)/\Gamma(n-m+1)\Gamma(n+m))^{1/2} E_+^{n-m}\beta^\vee) = \beta.$$

Proof. See [K], Theorem 4.1 and Theorem 5.5. Here we shall give the proofs of (2) and (3). Obviously, it is enough to prove the assertion for each $\beta(z) = e_p^{-m}(z^-) = \lambda^m_p z^p$ $(p \in N)$. Then it easily follows from (5.1) and (5.2) that $\beta^\vee(x) = c_n^{-2}T^{p}_{-m}{}^{0}(x)$, where $c_n^2 = 4\pi(2m-1)^{-1}$, and moreover, since $T_{-m}(g)e_0^{-m} = \Sigma_q\ T^{q}_{-m}{}^{0}(g)e_q^{-m}$,

$$F^m_{-m}(c_n^{-2}T^{p}_{-m}{}^{0})(z) = \int_G c_n^{-2}T^{p}_{-m}{}^{0}(g)\,\mathrm{conj}(T_{-m}(g)e_0^{-m}(z))dg$$

$$= e_p\ (z^-).$$

Therefore, (2) is obtained. We recall that $T_{-m}^{p\ 0} = (\lambda_0^m \lambda_p^m / \lambda_{p+m-\varepsilon}^m) \times \pi_{\varepsilon,m}^{p+m-\varepsilon\ \ m-\varepsilon}$ (see Lemma 5.1 (2)). Then, applying E_+^{n-m} to the right hand side (see (4.5)) and using Lemma 5.1 (2) again, we see that

$$E_+^{n-m}\, T_{-m}^{p\ 0} = (\Gamma(n-m+1)\Gamma(n+m)/\Gamma(2m))^{1/2}\, T_{-m}^{p\ n-m}.$$

Then, repeating the argument in the proof of (2), we can obtain (3).

Q.E.D.

Let

$$A^2{}_n(D) = \bigoplus_{m \in I_n} A_{2,m-1}(D)$$

be the direct sum of the weighted Bergman spaces $A_{2,m-1}(D)$ $(m \in I_n)$ with the norm given by the sum of $\|\ \|_{2,m-1}$ $(m \in I_n)$. Then for each $\beta = (\beta_m; m \in I_n) \in A^2{}_n(D)$ we let

$$\beta^\vee(x) = \sum_{m \in I_n} (\Gamma(2m)/\Gamma(n-m+1)\Gamma(n+m))^{1/2}$$

$$\times\ E_+^{n-m}((4\pi)^{-1}(2m-1)(1-r^2)^m e^{im(\theta+\theta')} \beta_m(w)),$$

where $x = k_\theta a_t k_{\theta'}$ and $w = x \cdot 0 = re^{i\theta}$. Here we note the fact that the set of the discrete series T_ℓ that has an element with K-type n in the representation space $A_{2,\ell-1}(D)$ is just given by $\{T_{-m}\ ;\ m \in I_n\}$. Then, applying Proposition 6.6, we can deduce the following

<u>Proposition 6.7</u>. The Fourier transform $f(x) \rightarrow f^\wedge(z)$ is an isometry of ${}^\circ L^2{}_n(G)$ onto $A^2{}_n(D)$ and the inversion formula is given

$$f(x) = (f^\wedge)^\vee(x).$$

We say that $\beta=(\beta_m)\in A^2_n(D)$ has a bounded boundary value if each $\beta_m\in A_{2,m-1}(D)$ has a bounded boundary value function on T. Then we have the following

<u>Lemma 6.8</u>. Let $\beta_m\in A_{2,m-1}(D)$ and suppose that it has a bounded boundary value function on T. Then

$$\beta_m{}^{\vee}(x) = (4\pi)^{-1}(2m-1)S_{\varepsilon,m-1,n}(I_\varepsilon(\lambda^m_{n-m}/\lambda^m_{n-\varepsilon})^2\lambda^m_{n-m}{}^{-1}\beta_m e_{\varepsilon-m})(x).$$

roof. Since β_m is bounded on T, the right hand side is well defined (see Lemma 4.1 and Proposition 4.2), and so the equation holds if it holds to each $e_p^{-m}(z^-) = \lambda_p^m z^p$ $(p\in N)$, In fact, it follows from Proposition 4.2 and Lemma 5.1 (2) that

$$S_{\varepsilon,m-1,n}(I_\varepsilon(\lambda^m_{n-m}/\lambda^m_{n-\varepsilon})^2\lambda^m_{n-m}{}^{-1}(e_p^{-m})^- e_{\varepsilon-m})(x)$$

$$= \lambda^m_{n-m}\lambda^m_p\lambda^m_{n-\varepsilon}{}^{-2}\ \pi_{\varepsilon,1-m}^{p+m-\varepsilon\ \ n-\varepsilon}(x)$$

$$= \lambda^m_{n-m}\lambda^m_p\lambda^m_{n-\varepsilon}{}^{-2}\ \mathrm{conj}(\pi_{\varepsilon,m}^{n-\varepsilon\ \ p+m-\varepsilon}(x^{-1}))$$

$$= \mathrm{conj}(T^{n-\varepsilon\ \ p}_{-m}(x^{-1}))$$

$$= T^{p}_{-m}{}^{n-\varepsilon}(x).$$

Then, by the same argument in the proof of Proposition 6.6 (2) and (3), the desired equation for e_p^{-m-} follows.

Q.E.D.

Corollary 6.9. If $\beta=(\beta_m)\in A^2_n(D)$ has a bounded boundary value,

$$\beta^\vee(x)=\sum_{m\in I_n}(4\pi)^{-1}(2m-1)(\lambda_{n-m}/\lambda_{n-\epsilon})^2\, S_{\epsilon,m-1,n}(\lambda_{n-m}{}^{-1}I_\epsilon\,\beta_m e_{\epsilon-m})(x).$$

Last, for $f\in L^2_n(G)$, we let

$$f^\wedge=(f^\wedge(\lambda,\xi),\ f^\wedge(z))\qquad((\lambda,\xi,z)\in R\times T\times D).$$

see (6.2), (6.8) and (6.9)). Then Proposition 6.3 and Proposition 6.7 imply that

Theorem 6.10. The Fourier transform $f\to f^\wedge$ is an isometry of $L^2_n(G)$ onto $L^2_n(R\times T)\oplus A^2_n(D)$ and the inversion formula is given by

$$f(x)=f^\wedge(\cdot,\cdot)^\vee+f^\wedge(\cdot)^\vee$$

$$=\int_R S_{\epsilon,-\nu,n}(I_\epsilon f^\wedge(\lambda,\cdot))(x)\,\mu_\epsilon(\lambda)d\lambda$$

$$+\sum_{m\in I_n}(\Gamma(2m)/\Gamma(n-m+1)\Gamma(n+m))^{1/2}\,E_+^{n-m}\,F_{-m}(f)^\vee(x).$$

§7. Paley-Wiener theorem. We retain the notations in the previous sections. In this section we shall give a characterization of Fourier transforms $f^\wedge$ of compactly supported, C^∞ functions f on G.

Let f be in $C_c^\infty(G)$. Then, by Lemma 6.2 (2), $f^\wedge(\lambda,\xi)$ is an anti-holomorphic function of uniform exponential type, and by Lemma 5.1 (1) and Lemma 4.1, $F_{-m}(f)(z)$ $(m\in I_n)$ has a bounded boundary value on T. Especially, we can obtain the following relation.

<u>Lemma 7.1</u>.

$$f^\wedge(-(m-1/2)i, \zeta) = \lambda_{n-m}{}^{-1} F_{-m}(f)(\zeta) e_{\varepsilon-m}(\zeta).$$

Proof. It follows from Lemma 5.1 (1) that

$$f^\wedge(-(m-1/2)i, \zeta) = \int_G f(g) \mathrm{conj}(\pi_{\varepsilon,m}(g) e_{n-\varepsilon}(\zeta)) dg$$

$$= \lambda^{m}_{n-m}{}^{-1} \int_G f(g) \mathrm{conj}(T_{-m}(g) e^{-m}_{n-m}(z)) dg \,|_{z=\zeta}\, e_{\varepsilon-m}(\zeta)$$

$$= \lambda_{n-m}{}^{-1} F_{-m}(f)(\zeta) e_{\varepsilon-m}(\zeta).$$

Q.E.D.

Let PW be the subspace of $L^2_n(R\times T) \oplus A^2_n(D)$ defined by

$$PW = \{ \gamma = (\alpha(\lambda, \zeta), \beta(z)) \in L^2(R\times T, \mu_\varepsilon(\lambda) d\lambda d\zeta) \oplus A^2_n(D);$$

(1) $\alpha(\lambda, \zeta)$ is an antiholomorphic function of uniform exponential type,

(2) $A_n(\alpha)(\lambda, x) = A_n(\alpha)(-\lambda, x) \quad ((\lambda, x) \in R\times G)$,

(3) $\alpha(-(m-1/2)i, \zeta) = \lambda_{n-m}{}^{-1} \beta_m(\zeta) e_{\varepsilon-m}(\zeta) \quad (\zeta \in T)$,

where $\beta(z) = (\beta_m(z);\ m \in I_n)$. }

and PW(R) (R>0) the subspace of PW consisting of $\gamma = (\alpha, \beta)$ such that the exponential type of α is R. In particular, the condition (3) of PW implies that

β has a bounded boundary value, (7.1a)

and

$\alpha(-(m-\frac{1}{2})i, \zeta)$ has zero at $\zeta=0$ of order $m-\varepsilon$ (7.1b)

and has a holomorphic extension on D.

Then the main theorem can be stated as

Theorem 7.2. (Paley-Wiener Theorem on SU(1,1)) The Fourier transform $f \to f^\wedge$ is a bijection of $C_{c,n}^{\infty}(G;R)$ onto PW(R).

Proof. Except the surjectivity, the assertion follows from Theorem 6.10, Lemma 6.2 and Lemma 7.1, so we shall prove that if $\gamma \in PW(R)$, then $\gamma^\vee \in C_{c,n}^{\infty}(G;R)$. It follows from Theorem 6.10, (7.1a), Corollary 6.9 and (3) of PW that $\gamma^\vee$ can be written as

$$\gamma^\vee(x) = \alpha^\vee(x) + \beta^\vee(x)$$

$$= \int S_{\varepsilon,-\nu,n}(I_\varepsilon \alpha(\lambda, \cdot))(x)\mu_\varepsilon(\lambda)d\lambda$$

$$+ \sum_{m\in I_n} (\lambda_{n-m}/\lambda_{n-\varepsilon})^2 S_{\varepsilon,m-1,n}(I_\varepsilon \alpha(-(m-\tfrac{1}{2})i, \cdot))(x).$$

Lemma 7.3. If $\beta\equiv 0$, then $\gamma^\vee = \alpha^\vee \in C_{c,n}^{\infty}(G;R)$.

Proof. Clearly, $\beta\equiv 0$ implies that each $S_{\varepsilon,m-1,n}(I_\varepsilon \alpha(-(m-\frac{1}{2})i, \cdot))$ $\equiv 0$ ($m\in I_n$) and thus, by applying E_+^{n-m}, $S_{\varepsilon,m-1,m}(I_\varepsilon \alpha(-(m-\frac{1}{2})i, \cdot))$ $\equiv 0$ (see Lemma 4.4). Then, since $I_\varepsilon \alpha(-(m-\frac{1}{2})i, \zeta)$ has zero at $\zeta=0$ of order m and has a holomorphic extension on D (see (7.1b)), it follows from Theorem 5.5 that

$$\alpha(-(m-\tfrac{1}{2})i, \xi) \equiv 0 \quad (m \in I_n),$$

that is, $\alpha(\lambda^-, \xi)$ is a holomorphic function of uniform exponential type R, that has zero at $\lambda=(m-\frac{1}{2})i$ $(m \in I_n)$. Then, comparing the zero points of $P_n(\lambda)$ (see Corollary 6.4), we see that $\alpha(\lambda^-, \xi)P_n(\lambda)^{-1}$ is a holomorphic function of uniform exponential type R. Then, noting Corollary 6.4 (2), we can apply Theorem (2) in Remark 6.5 to $\alpha(\lambda^-, \xi)$ $P_n(\lambda)^{-1}$ and thus, by Corollary 6.4 (1), we can conclude that $\alpha^\vee \in C^\infty_{c,n}(G;R)$. This completes the proof of Lemma.

Q.E.D.

Now we return to the proof of the theorem. Since $\beta^\vee \in {}^\circ C_n(G)$, Proposition 6.1 implies that there exists $g \in C^\infty_{c,n}(G;R)$ such that ${}^\circ g = \beta^\vee$, that is, $g\hat{}(z) \equiv \beta(z)$ $(z \in D)$. Therefore, if we let

$$h = \gamma^\vee - g,$$

we see that $h\hat{} \in PW(R)$ and $h\hat{}(z) \equiv 0$. Then, applying Lemma 7.1 to $h\hat{}$, we can deduce that $h=(h\hat{})^\vee \in C^\infty_{c,n}(G;R)$, and so $\gamma^\vee = h+g \in C^\infty_{c,n}(G;R)$.

This completes the proof of Theorem.

Q.E.D.

Let $C^p_n(G)$ $(0<p\leqq 2)$ denote the L^p Schwartz space with right K-type n, that is, the space of all C^∞ functions f in $L^2_n(G)$ such that for any $r \in N$ and $g, g' \in U(\mathfrak{g}_c)$

$$\sup_{x \in G} |f(g;x;g')|\, e^{\sigma(x)/p}(1+\sigma(x))^r < \infty$$

(cf. [EK], p.146). Let R[p] $(1<p\leqq 2)$ denote the strip in C defined by $\{z\in C; |Im(z)| \leqq (1/p-1/2)\}$ and R(p) the interior of R[p]. Then, we define

$$L^pS = \{\gamma=(\alpha(\lambda,\zeta),\ \beta(z))\in L^2(R\times T,\mu_\varepsilon(\lambda)d\lambda d\zeta)\oplus A^2{}_n(D)\ ;$$

(1) $\alpha(\lambda,\zeta)$ is, as a function of λ, an antiholomorphic function on R[p] and for any p, q, $r\in N$

$$\sup_{\lambda\in R(p),\ \zeta\in T} |(d/d\lambda)^p(d/d\zeta)^q\alpha(\lambda,\zeta)|(1+|\lambda|)^r<\infty,$$

(2) $$A_n(\alpha)(\lambda,x) = A_n(\alpha)(-\lambda,x)\quad ((\lambda,x)\in R\times G),$$

(3) If $m\in I_n$ satisfies $m\leqq 1/p$, then

$$\alpha(-(m-½)i,\zeta) = \lambda_{n-m}{}^{-1}\beta_m(\zeta)e_{\varepsilon-m}(\zeta),$$

where $\beta=(\beta_m(z);\ m\in I_n)\}$.

Then we can obtain the following

<u>Theorem 7.4</u>. Let $n\in Z_\varepsilon$ and $0<p\leqq 2$. The Fourier transform $f\to f^\wedge$ is a bijection of $C^p{}_n(G)$ onto L^pS.

Proof. When the right K-type n is trivial, we know that the discrete part $A^2{}_n(G)$ vanishes, and the theorem is obtained by [EK] for general groups; also when n=½, it can be obtained by the same way. So, we shall reduce the proof to the case of $n=\varepsilon$.

As in the case of n=0 (cf. [EK], §4), the image $f^\wedge$ of f in $C^p{}_n(G)$

satisfies (1) of L^pS, and moreover, as in Lemma 6.2 (3) and Lemma 7.1, $f^\wedge$ satisfies (2) and (3) of L^pS. Therefore, Theorem 6.10 implies that $f \to f^\wedge$ is an injection of $C^p_n(G)$ into L^pS.

Let $\gamma=(\alpha,\beta)\in L^pS$, and we shall show that $\gamma^\vee\in C^p_n(G)$. As in the proof of Lemma 7.3, we can find a compactly supported C^∞ function g on G such that the right K-type is n and $^\circ g = \beta^\vee$. Moreover, if we put $h = \gamma^\vee - g$, h has no discrete part and $h^\wedge(-(m-\frac{1}{2})i,\xi) \equiv 0$ for all $m\in I_n$. Then, since $h^\wedge = \gamma - g^\wedge$ is in L^pS, it satisfies (1) of L^pS and thus, $h^\wedge P_n^{-1}$ satisfies the conditions (1) and (2) of L^pS for $n=\varepsilon$ (see Corollary 6.4 (2)). Therefore, the result for $n=\varepsilon$ and Corollary 6.4 (1) deduce that $h\in C^p_n(G)$, so, $\gamma^\vee = h + g \in C^p_n(G)$.

Q.E.D.

Recently, Barker [B] removed completely the finite K-type restriction for $C^p(G)$ and gave a characterization of $C^p(G)$ under Fourier transform.

References

[A] Arthur,J.; A Paley-Wiener theorem for real reductive groups, Acta Math., 150 (1983), 1-89.

[B] Barker,W.H.; L^p harmonic analysis on SL(2,R), Memoirs of Amer. Math. Soc., 393 (1988).

[C] Campoli,O.A.; Paley-Wiener type theorems for rank-1 semisimple Lie groups, Revista de la Unión Matemática Argentina, 29 (1980), 197-221.

[D] Delorme,P.; Théorème de type Paley-Wiener pour les groupes de Lie semi-simples réels avec une seule classe de conjugaison de sous groupes de Cartan, J. Funct. Anal., 47 (1982),26-63.

[EK] Eguchi,M. and Kowata,A.; On the Fourier transform of rapidly dicreasing functions of L^p type on a symmetric space, Hiroshima J. Math., 6 (1976), 143-158.

[EM] Ehrenpreis,L. and Mautner,F.I.; Some properties of the Fourier transform on semisimple Lie groups I, Ann. of Math., 61 (1955), 406-439; II, Trans. Amer. Math. Soc., 84 (1957), 1-55; III, Trans. Amer. Math. Soc., 90 (1959), 431-484.

[G] Gangolli,R; On the Plancherel formula and Paley-Wiener theorem for spherical functions on semisimple Lie groups, Ann. of Math., 93(2) (1971), 150-165.

[H1] Helgason,S.; An analogue of the Paley-Wiener theorem for the Fourier trnasform on certain symmetric spaces, Math. Ann., 165 (1966), 297-308.

[H2] Helgason,S.; A duality for symmetric spaces with applications to group representations I, Advan. Math., 5 (1970), 1-154; II, Advan. Math., 22 (1976), 187-219.

[J] Johnson,K.D.; Functional analysis on SU(1,1), Advan. Math., 14 (1974), 346-364.

[K] Kawazoe,T; A transform on classical bounded symmetric domains associated with a holomorphic discrete series, to appear in Tokyo J. of Math.

[KW] Knapp,A.W. and Wallach,N.R.; Szegö kernels associated with discrete series, Invent. math., 34 (1976), 163-200.

[PW] Paley,R. and Wiener,N.; Fourier Transforms in the Complex Domain, Amer. Math. Soc., Providence, Rhode Island,1934.

[R] Range,R.M.; Holomorphic Functions and Integral Representations in Several Complex Variables, GTM 108, Springer-Verlag, New York, 1984.

[Sa] Sally,P.J.; Analytic continuation of irreducible unitary representations of the universal covering group of SL(2,R), Memoirs of Amer. Math. Soc., 69 (1967).

[Su] Sugiura,M.; Unitary Representations and Harmonic Analysis, Wiley, New York, 1975.

[Z] Zelobenko,D.P.; Harmonic Analysis on Complex Semisimple Lie Groups, WAUKA, Moscow, 1974 (in Russian).

Department of Mathematics
Faculty of Science and Techinology
Keio University
3-14-1, Hiyoshicho, Kohokuku,
Yokohama 223,
Japan