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Uncertainty relations in simultaneous measurements for arbitrary observables

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ABSTRACT

We consider the simultaneous measurement for arbitrary observables and show its existence theorem and uncertainty relations.

1. Introduction.

In the measurement of the position q of a particle and the measurement of the momentum p of a particle with the same state, the following uncertainty relation holds:

$$\Delta q \cdot \Delta p \geq \hbar/2 \tag{1}$$

where $\hbar$ is " Plank's constant/2π" and Δq and Δp are respectively the errors in determining the particle position q and the momentum p.

On the other hand, in the simultaneous measurement of the position q and the momentum p of a particle, the following simultaneous uncertainty relation holds:

$$\Delta q \cdot \Delta p \geq \hbar. \tag{2}$$

A particular but very important simultaneous measurement of a pair of conjugate observables was first discussed in [2], where a simultaneous uncertainty relation (2) in the measurement was derived. For the precise arguments of the simultaneous measurement, also, see [10]. Also, a certain simultaneous measurement for a pair of non-conjugate observables and its simultaneous uncertainty relation were studied in [11]. In this paper, we shall give mathematical foundations to a simultaneous measurements and show the existence theorem of a simultaneous measurement of arbitrary observables. And we shall lastly derive, from the mathematical deduction, ordinary and simultaneous uncertainty relations.

2. Preliminaries.

In order to exhibit our purpose in this paper, we shall consider, according to Holevo's book ([6], p123), a certain simultaneous measurement of the position q and the momentum p of a particle A in one dimensional real line $\mathbb{R}$, which has a state function $u(x)$ $(\in L^2(\mathbb{R}),\ \|u\|_{L^2(\mathbb{R})}=1)$. If this simultaneous measurement is proper, we must get the following average results;

$$q=\int_{\mathbb{R}} x|u(x)|^2dx \qquad \text{and} \qquad p=\int_{\mathbb{R}} \overline{u}(x)[\frac{\hbar d}{idx}u(x)]dx. \tag{3}$$

after the measurement. Since the position observable x and the momentum observable $\frac{\hbar d}{idx}$ are not commutative, any simultaneous measurement (in the sense of [7]) for the position and the momentum of a particle A can not be realized in the $L^2(\mathbb{R})$. So we prepare another particle B with the state function $u_0(x)$ such that $\int_{\mathbb{R}} x|u_0(x)|^2dx =\int_{\mathbb{R}} \overline{u}_0(x)[\frac{\hbar d}{idx}u_0(x)]dx =0$, for example, $u_0(x)=\frac{1}{(\pi\hbar)^{1/4}}\exp(-\frac{x^2}{2\hbar})$. And we regard these two particles A and B as a "particle" C in two dimensional space $\mathbb{R}^2$ with the state function $u(x)u_0(y)$ $(\in L^2(\mathbb{R}^2),\ \|u(x)u_0(y)\|_{L^2(\mathbb{R}^2)}=1)$. Since the observables $(x-y)$ and $(\frac{\hbar\partial}{i\partial x}+\frac{\hbar\partial}{i\partial y})$ in $L^2(\mathbb{R}^2)$ are commutative, the simultaneous measurement (in the sense of [7]) of the observables $(x-y)$ and $(\frac{\hbar\partial}{i\partial x}+\frac{\hbar\partial}{i\partial y})$ of the "particle" C can be realized simply in the $L^2(\mathbb{R}^2)$. Moreover, we can easily calculate these expectations as follows;

$$\iint_{\mathbb{R}^2}\bar{u}(x)\bar{u}_0(y)[(x-y)u(x)u_0(y)]\,dxdy=\int_{\mathbb{R}}x|u(x)|^2dx \tag{4}$$

and

$$\iint_{\mathbb{R}^2}\bar{u}(x)\bar{u}_0(y)[(\frac{\hbar\partial}{i\partial x}+\frac{\hbar\partial}{i\partial y})u(x)u_0(y)]\,dxdy=\int_{\mathbb{R}}\bar{u}(x)[\frac{\hbar d}{idx}u(x)]\,dx. \tag{5}$$

For the reason that the equalities (4)=q and (5)=p in (3) hold, it is possible to consider that the simultaneous measurement of the position q and the momentum p of the particle A can be realized.

Also, the errors Δq and Δp in this simultaneous measurement of the position q and the momentum p of the particle A are given respectively by

$$\begin{aligned}\Delta q&=[\iint_{\mathbb{R}^2}|[(x-y)u(x)u_0(y)]|^2dxdy-|\iint_{\mathbb{R}^2}\bar{u}(x)\bar{u}_0(y)[(x-y)u(x)u_0(y)]\,dxdy|^2]^{1/2}\\&=[\int_{\mathbb{R}}|xu(x)|^2dx-|\int_{\mathbb{R}}x|u(x)|^2dx|^2+\int_{\mathbb{R}}|yu_0(y)|^2dy]^{1/2}\end{aligned} \tag{6}$$

and

$$\Delta p=[\int_{\mathbb{R}}|\frac{\hbar d}{idx}u(x)|^2dx-|\int_{\mathbb{R}}\bar{u}(x)[\frac{\hbar d}{idx}u(x)]\,dx|^2+\int_{\mathbb{R}}|\frac{\hbar d}{idy}u_0(y)|^2dy]^{1/2}. \tag{7}$$

Hence, we can get, by the arithmetic-geometric mean inequality and the well-known uncertainty principle, the following simultaneous uncertainty relation

$$\begin{aligned}\Delta q\cdot\Delta p\geq 2&[\int_{\mathbb{R}}|xu(x)|^2dx-|\int_{\mathbb{R}}x|u(x)|^2dx|^2]^{1/4}\cdot[\int_{\mathbb{R}}|yu_0(y)|^2dy]^{1/4}\\&\cdot[\int_{\mathbb{R}}|\frac{\hbar d}{idx}u(x)|^2dx-|\int_{\mathbb{R}}\bar{u}(x)[\frac{\hbar d}{idx}u(x)]\,dx|^2]^{1/4}\cdot[\int_{\mathbb{R}}|\frac{\hbar d}{idy}u_0(y)|^2dy]^{1/4}\geq\hbar.\end{aligned} \tag{8}$$

For further arguments about this simultaneous measurement, see [6].

Motivated by this example, we shall consider the following two problems in this paper:

Problem 1. Does there exist a simultaneous measurement for a pair of arbitrary observables A_1 *and* A_2 *?*

Problem 2. Does the following simultaneous uncertainty relation hold

$$\Delta q \cdot \Delta p \geq \hbar$$

for any simultaneous measurement of the position q and the momentum p (there is a possibility that other precise simultaneous measurements exist) ?

In order to answer these problems, we shall prepare the following definitions.

Definition 1. *Let H be a Hilbert space with the inner product* $\langle \cdot , \cdot \rangle_H$ *(or simply* $\langle \cdot , \cdot \rangle$*). A triplet* $(\Omega, \mathbb{B}, E)$ *is called a positive operator valued probability space in H, if it satisfies the following conditions:*

(i) $(\Omega, \mathbb{B})$ *is a measurable space,*

(ii) *E is a mapping from* $\mathbb{B}$ *to* $B(H)$*, a space of bounded linear operators on H, such that*

(a) $E(B)$ *is a positive operator on H for any* $B \in \mathbb{B}$ *and particularly* $E(\varnothing)=0$, $E(\Omega)=I$ *, where 0 and I are respectively a 0-operator and an identity operator on H,*

(b) *for any countable decomposition* $\{B_j\}_{j=1}^{\infty}$ *of* $B \in \mathbb{B}$, $E(B) = \sum_{j=1}^{\infty} E(B_j)$ *holds where the series is weakly convergent.*

Positive operator valued probability spaces were introduced into quantum mechanics by Davies and Lewis [3]. A particular but very important class of positive operator valued probability spaces is that of orthogonal positive operator valued probability spaces, that is, they satisfy the additional requirement

$$E(B_1) \cdot E(B_2) = 0 \qquad \text{for} \quad B_1 \cap B_2 = \varnothing. \tag{9}$$

Note that, for any state u (*i.e.* $u \in H$, $\|u\|_H = 1$), $(\Omega, \mathbb{B}, \langle u, E(\cdot) u \rangle)$ is an ordinary probability space. Also, for any u, $v \in H$, $(\Omega, \mathbb{B}, \langle v, E(\cdot) u \rangle)$ is a signed measure space since $\langle v, E(B) u \rangle = \frac{1}{4} \{ \langle v+u, E(B)(v+u) \rangle - \langle v-u, E(B)(v-u) \rangle - i \langle v+iu, E(B)(v+iu) \rangle + i \langle v-iu, E(B)(v-iu) \rangle \}$.

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Definition 2. *A quartet* $\mathbb{M}=(\Omega,\mathbb{B},E,f)$ *(or* $(\Omega,\mathbb{B},E,f=(f_1,...,f_n))$ *is called a measurement (or n-measurement) if* $(\Omega,\mathbb{B},E)$ *is a positive operator valued probability space and* $f:\Omega\to\mathbb{R}^n$ *is a measurable mapping.*

Now, we postulate the following probabilistic interpretation of quantum mechanics: *when we take an n-measurement* $\mathbb{M}=(\Omega,\mathbb{B},E,f=(f_1,...,f_n))$ *on a system whose state is* u *(i.e.* $u\in H,\|u\|=1$*) before the measurement, the probability that the value* λ $(=(\lambda_1,...,\lambda_n)\in\mathbb{R}^n)$ *obtained in the measurement* $\mathbb{M}$ *belongs to a Borel set* G $(\subset\mathbb{R}^n)$ *is given by* $(u,E(f^{-1}(G))u)$.

Hence the average result $\bar{\lambda}$ $(=(\bar{\lambda}_1,...,\bar{\lambda}_n)\in\mathbb{R}^n)$ is equal to the expectation value $\overline{\mathbb{M}}(u)=(\overline{\mathbb{M}}(u)_1,...,\overline{\mathbb{M}}(u)_n)$:

$$\overline{\mathbb{M}}(u)=\int_\Omega f(\omega)(u,E(d\omega)u)=(\int_\Omega f_1(\omega)(u,E(d\omega)u),\ .\ .\ .\ ,\int_\Omega f_n(\omega)(u,E(d\omega)u)) \tag{10}$$

if it exists. Put

$$\begin{aligned}\Delta\mathbb{M}(u)_i&=[\int_\Omega|f_i(\omega)-\overline{\mathbb{M}}(u)_i|^2(u,E(d\omega)u)]^{1/2}\\&=[\int_\Omega|f_i(\omega)|^2(u,E(d\omega)u)-|\int_\Omega f_i(\omega)(u,E(d\omega)u)|^2]^{1/2}\quad(i=1,...,n).\end{aligned} \tag{11}$$

The value $\Delta\mathbb{M}(u)_i$ is called the i-th error (or i-th uncertainty) of $\mathbb{M}$ in the state u.

Definition 3. *Let* H *be a Hilbert space. And let* $(A_1,...,A_n)$ *be an n-tuple of self-adjoint operators (i.e. physical quantities or observables) on* H*. Let* $\int_{\mathbb{R}}\lambda F_i(d\lambda)$ *be the orthogonal spectral representation of* A_i $(i=1,...,n)$*. Let* G *be a subset of* $\mathbb{R}^+=\{\alpha\in\mathbb{R}:\alpha>0\}$ *and let* K_i *be a subspace of* $\{u\in H:\ \int_{\mathbb{R}}|\lambda|(u,F_i(d\lambda)u)<\infty\}$ *for each* $i=1,2,...,n$*. An n-measurement* $(\Omega,\mathbb{B},E,f=(f_1,...,f_n))$ *is called a* (G,K_i)*-simultaneous measurement for* $(A_1,...,A_n)$*, if it satisfies the following conditions :*

(i) *(moment conditions)*

$$\{u\in H:\ \int_{\mathbb{R}}|\lambda|^\alpha(u,F_i(d\lambda)u)<\infty\}=\{u\in H:\ \int_\Omega|f_i(\omega)|^\alpha(u,E(d\omega)u)<\infty\} \quad (\alpha\in G,\ i=1,...,n) \tag{12}$$

and

(ii) *(average value conditions)*

$$\int_{\mathbb{R}}\lambda\langle u,F_i(d\lambda)u\rangle=\int_{\Omega}f_i(\omega)\langle u,E(d\omega)u\rangle \qquad (u\in K_i,\ i=1,2,\ldots,n). \tag{13}$$

In particular, a (G,K_i)*-simultaneous measurement is called a strongly simultaneous measurement when* $G=\mathbb{N}(=\{1,2,\ldots\})$ *and* $K_i=\{u\in H:\int_{\mathbb{R}}|\lambda|\langle u,F_i(d\lambda)u\rangle<\infty\}$ $(i=1,\ldots,n)$. *Also, it is called a weakly simultaneous measurement when* $G=\varnothing$ *(* $\varnothing$ *is an empty set) and* $K_i=\bigcap_{i=1}^{n}D(A_i)$ *where* $D(A)$ *is a domain of* A. *In the special case of* $n=1$, *a 1-measurement* $(\Omega,\mathcal{B},E,f_1)$ *satisfying (12) and (13) is called a* (G,K_1)*-measurement of* A_1.

Note that, for any n-tuple of commutative observables $(A_1,\ldots,A_n)$, there exists a unique common spectral representation (i.e., orthogonal simultaneous measurement on $\mathbb{R}^n$) $\mathbb{M}=(\mathbb{R}^n,\mathcal{B},E,f(\lambda_1,\ldots,\lambda_n)=(\lambda_1,\ldots,\lambda_n))$ with $E(d\lambda_1,\ldots,d\lambda_n)=F_1(d\lambda_1)F_2(d\lambda_2)\cdots F_n(d\lambda_n)$, which is not only a strongly simultaneous measurement for $(A_1,\ldots,A_n)$ but also a simultaneous measurement in the sense of [7], i.e., $F_i(d\lambda_i)=E(\mathbb{R}\times\cdots\times\mathbb{R}\times d\lambda_i\times\mathbb{R}\times\cdots\times\mathbb{R})$. The problem of the orthogonal simultaneous measurements for commutative observables is studied precisely in [7]. On the other hand, the physical interpretation of a simultaneous measurement for non-commutative observables is based on the following proposition.

Proposition 1 *(Holevo [5]). Let* H *be a Hilbert space, and let* $(\Omega,\mathcal{B},E)$ *be a positive operator valued probability space in* H. *Then there exist a Hilbert space* R, *an element* s_0 $(\|s_0\|_R=1)$ *in* R *and an orthogonal positive operator valued probability space* $(\Omega,\mathcal{B},\hat{E})$ *in the tensor Hilbert space* $H\otimes R$ *with the inner product* $\langle\cdot,\cdot\rangle_{H\otimes R}$ *(or simply* $\langle\cdot,\cdot\rangle$*) such that*

$$\langle u,E(B)u\rangle_H=\langle u\otimes s_0,\hat{E}(B)(u\otimes s_0)\rangle_{H\otimes R} \qquad (u\in H,\ B\in\mathcal{B}). \tag{14}$$

The triplet $(R,\hat{E},s_0)$ *is called a realization of* $(\Omega,\mathcal{B},E)$.

Conversely, any orthogonal positive operator valued probability space $(\Omega,\mathcal{B},\hat{E})$ *in* $H\otimes R$ *and* $s_0\in R$ *give rise to the unique positive operator valued probability space*

(Ω,B,E) in H satisfying (14).

For any measurement $\mathrm{M}=(\Omega,\mathrm{B},E,f=(f_1,\ldots,f_n))$, we can see, by the realization $(R,\hat{E},s_0)$, that

$$\overline{M}(u)_i=\int_\Omega f_i(\omega)(u,E(d\omega)u)=\int_\Omega f_i(\omega)(u\otimes s_0,\hat{E}(d\omega)(u\otimes s_0)), \tag{15}$$

which implies that the measurement M can be realized in the suitable extended Hilbert space $H\otimes R$, that is, the measurement M can be interpreted to be an orthogonal simultaneous measurement (i.e., simultaneous measurement in the sense of [7]) for the commutative observables $\hat{A}_i$ $(=\int_\Omega f_i(\omega)\hat{E}(d\omega))$ $(i=1,2,\ldots,n)$ of the state $u\otimes s_0$ in the Hilbert space $H\otimes R$. Therefore, the above proposition assures the possibility of the realization of the measurement $\mathrm{M}=(\Omega,\mathrm{B},E,f=(f_1,\ldots,f_n))$.

3. Existence theorems

In this section, we shall prove the existence theorem of the simultaneous measurement for arbitrary self-adjoint operators on H, which includes an answer to the problem 1.

Theorem 1. *Let $A_1,\ldots,A_n$ be any self-adjoint operators on a Hilbert space H. Then, there exists a strongly simultaneous measurement $\mathrm{M}=(\Omega,\mathrm{B},E,f=(f_1,\ldots,f_n))$ of $(A_1,\ldots,A_n)$.*

Proof. Let $\int_{\mathbb{R}}\lambda F_i(d\lambda)$ be the orthogonal spectral representation of A_i $(i=0,1,\ldots,n)$, where A_0 is a 0-operator on H. Let $\hat{H}$ be the Hilbert space which consists of the direct sum of n copies of H. That is, $\hat{H}=\{(x_1,\ldots,x_n): x_i\in H,\ i=1,\ldots,n\}$ with norm $\|(x_1,\ldots,x_n)\|_{\hat{H}}=(\sum_{i=1}^{n}\|x_i\|_H^2)^{1/2}$. Let H_i $(i=1,\ldots,n)$ be the subspace of $\hat{H}$ for which $x_j=0$ $(j\neq i)$. Define $\hat{T}_i:\hat{H}\to\hat{H}$, $(i=1,\ldots,n)$ such that $\hat{T}_ix=nA_ix\in H_i$ $(x\in D(A_i)\subset H_i)$, $=0$ $(x\in H_i^{\perp}$, orthogonal complement of $H_i)$, which are clearly commutative self-adjoint operators with the orthogonal spectral representations $\hat{T}_i = \int_{\mathbb{R}} \lambda \hat{F}_i(d\lambda)$ where $\hat{F}_i(d\lambda)x = F_i(d\lambda/n)x \in H_i$ $(x \in H_i)$, $= F_0(d\lambda/n)x$ $(= F_0(d\lambda)x) \in H_j$ $(x \in H_j\ (j \neq i))$. Let $P: \hat{H} \to \hat{H}$ be a projection on $H(=H_1)$. Define a unitary operator U on $\hat{H}$ such that $Ux = (x/\sqrt{n}, \ldots, x/\sqrt{n})$ for any $x \in H(=H_1)$. Now, we can easily see that

$$PU^*(y_1, \ldots, y_n) = (1/\sqrt{n}) \sum_{k=1}^{n} (y_k, 0, \ldots, 0) = (1/\sqrt{n}) \sum_{k=1}^{n} y_k \qquad ((y_1, \ldots, y_n) \in \hat{H}) \tag{16}$$

because $\langle x, PU^*(y_1, \ldots, y_n) \rangle_H = \langle Ux, (y_1, \ldots, y_n) \rangle_{\hat{H}} = \sum_{k=1}^{n} \langle x/\sqrt{n}, y_k \rangle_H = \langle x, (1/\sqrt{n}) \sum_{k=1}^{n} y_k \rangle_H$ for all $x \in H(=H_1)$. Since $\hat{F}_i(d\lambda)$ $(i=1,\ldots,n)$ are commutative orthogonal spectral measure on $\mathbb{R}$, we can define the orthogonal spectral measure $\hat{E}(d\lambda_1 \cdots d\lambda_n)$ on $\mathbb{R}^n$ such that

$$\hat{E}(d\lambda_1 \cdots d\lambda_n) = U^* \hat{F}_1(d\lambda_1) \cdots \hat{F}_n(d\lambda_n) U. \tag{17}$$

Define $E: \mathcal{B}_n \to B(H)$ (where $\mathcal{B}_n$ is a family of Borel sets in $\mathbb{R}^n$) such that $E(B)x = P\hat{E}(B)x$ for all $B \in \mathcal{B}_n$ and $x \in H$.

Now, we shall show that the n-measurement $\mathbb{M} = (\Omega, \mathcal{B}, E, f=(f_1, \ldots, f_n)) = (\mathbb{R}^n, \mathcal{B}_n, E, f=(\lambda_1, \ldots, \lambda_n))$ is a strongly simultaneous measurement for $(A_1, \ldots, A_n)$. We see by (16) that, for any $x \in H(=H_1)$,

$$\begin{aligned} E(d\lambda_1 \cdots d\lambda_n)x &= P\hat{E}(d\lambda_1 \cdots d\lambda_n)x = PU^* \hat{F}_1(d\lambda_1) \cdots \hat{F}_n(d\lambda_n) Ux \\ &= PU^* \hat{F}_1(d\lambda_1) \cdots \hat{F}_n(d\lambda_n)(x/\sqrt{n}, \ldots, x/\sqrt{n})) \\ &= \{\frac{1}{n} \sum_{k=1}^{n} F_0(d\lambda_1/n) \cdots F_0(d\lambda_{k-1}/n) F_k(d\lambda_k/n) F_0(d\lambda_{k+1}/n) \cdots F_0(d\lambda_n/n)\} x. \end{aligned} \tag{18}$$

Also, since $\int_{\mathbb{R}^n} |\lambda_i|^\alpha \langle u, F_0(d\lambda_1) \cdots F_0(d\lambda_{k-1}) F_k(d\lambda_k) F_0(d\lambda_{k+1}) \cdots F_0(d\lambda_n) u \rangle = \int_{\mathbb{R}} |\lambda|^\alpha \langle u, F_i(d\lambda) u \rangle$ $(k=i)$, $=0$ $(k \neq i)$, we see by(18) that, for any $u \in H$, $\alpha \in \mathbb{N}$ and $i=1,\ldots,n$,

$$\begin{aligned} \int_{\mathbb{R}} |\lambda|^\alpha \langle u, F_i(d\lambda) u \rangle &= \sum_{k=1}^{n} \int_{\mathbb{R}^n} |\lambda_i|^\alpha \langle u, F_0(d\lambda_1) \cdots F_0(d\lambda_{k-1}) F_k(d\lambda_k) F_0(d\lambda_{k+1}) \cdots F_0(d\lambda_n) u \rangle \\ &= \sum_{k=1}^{n} \int_{\mathbb{R}^n} |\lambda_i/n|^\alpha \langle u, F_0(d\lambda_1/n) \cdots F_0(d\lambda_{k-1}/n) F_k(d\lambda_k/n) F_0(d\lambda_{k+1}/n) \cdots F_0(d\lambda_n/n) u \rangle \\ &= n^{1-\alpha} \int_{\mathbb{R}^n} |\lambda_i|^\alpha \langle u, E(d\lambda_1 \cdots d\lambda_n) u \rangle \end{aligned}$$

$$=n^{1-\alpha}\int_\Omega |f_i(\omega)|^\alpha \langle u,E(d\omega)u\rangle. \qquad (19)$$

This implies that M satisfies (12) for G=N. Also, we can easily see that it also satisfies (13), because

$$\int_{\mathbb{R}}\lambda\langle u,F_i(d\lambda)u\rangle$$
$$=\sum_{k=1}^{n}\int_{\mathbb{R}^n}(\lambda_i/n)\langle u,F_0(d\lambda_1/n)\cdots F_0(d\lambda_{k-1}/n)F_k(d\lambda_k/n)F_0(d\lambda_{k+1}/n)\cdots F_0(d\lambda_n/n)u\rangle$$
$$=\int_\Omega f_i(\omega)\langle u,E(d\omega)u\rangle \qquad (20)$$

for any i=1,2,...,n and any $u\in H$ such that $\int_{\mathbb{R}}|\lambda|\langle u,F_i(d\lambda)u\rangle<\infty$. Hence, the proof is completed.

Remark 1. We shall mention some remarks in connection with Theorem 1.

(i). It is clear that the moment condition (19) holds for all $\alpha\in\mathbb{R}^+$. Hence, there exists a $(\mathbb{R}^+,\{u\in H:\int_{\mathbb{R}}|\lambda|\langle u,F_i(d\lambda)u\rangle<\infty\})$-simultaneous measurement for any $(A_1,...,A_n)$.

(ii). From (19) and (20), we can easily calculate the i-th error of M in the state u, that is,

$$\Delta\mathrm{M}(u)_i=[\int_\Omega|f_i(\omega)|^2\langle u,E(d\omega)u\rangle-|\int_\Omega f_i(\omega)\langle u,E(d\omega)u\rangle|^2]^{1/2}$$
$$=[\,n\int_{\mathbb{R}}|\lambda|^2\langle u,F_i(d\omega)u\rangle-|\int_{\mathbb{R}}\lambda\langle u,F_i(d\omega)u\rangle|^2\,]^{1/2}$$
$$=[\,n\|A_iu\|^2-|\langle u,A_iu\rangle|^2\,]^{1/2} \qquad (21)$$

for any $u\in D(A_i)$ (i=1,...,n).

(iii). Let $(A_1,...,A_n)$ be an n-tuple of self-adjoint operators on H. Let $\{A_k\}_{k=n+1}^{\infty}$ be any sequence of self-adjoint operators on H and let M_k = $(\Omega_k,\mathrm{B}_k,E_k,f=(f_k^1,f_k^2,...,f_k^n,...,f_k^k))$ be a strongly simultaneous measurement for $(A_1,...,A_n,...,A_k)$ as defined in the proof of Theorem 1 for each k ($k\geq n$). Put M_k^n = $(\Omega_k,\mathrm{B}_k,E_k,f=(f_k^1,f_k^2,...,f_k^n))$. Then, it is clear that M_k^n is a strongly simultaneous measurement of $(A_1,...,A_n)$ for each k=n,n+1,n+2,.... Also, we see from (21) that

$$\Delta\mathrm{M}_k^n(u)_i=[\,k\|A_iu\|^2-|\langle u,A_iu\rangle|^2\,]^{1/2}\qquad u\in D(A_i),\quad i=1,...,n. \qquad (22)$$

Hence, we can conclude that there exist infinitely many strongly simultaneous measurements M_k^n $(k=n,n+1,n+2,...)$ of $(A_1,...,A_n)$ in general and if $k\leq l$, then M_k^n is more precise than M_l^n since $\Delta M_k^n(u)_i \leq \Delta M_l^n(u)_i$ $(i=1,...,n)$.

4. Uncertainty relations.

In this section, we shall discuss the uncertainty relations in the quantum measurement.

Lemma 1. *Let A_1 and A_2 be any symmetric operators on a Hilbert space H. Then, it holds that*

$$[\|A_1u\|^2-|\langle u,A_1u\rangle|^2]\cdot[\|A_2u\|^2-|\langle u,A_2u\rangle|^2]\geq\frac{1}{4}|\langle A_1u,A_2u\rangle-\langle A_2u,A_1u\rangle|^2 \tag{23}$$

for all $u\in D(A_1)\cap D(A_2)$.

Proof. Though this lemma is well-known, we give the proof for completeness. We see by Schwarz inequality that, for any $u\in D(A_1)\cap D(A_2)$,

$$\begin{aligned}
&[\|A_1u\|^2-|\langle u,A_1u\rangle|^2]\cdot[\|A_2u\|^2-|\langle u,A_2u\rangle|^2]\\
&=\|(A_1-\langle u,A_1u\rangle)u\|^2\cdot\|(A_2-\langle u,A_2u\rangle)u\|^2\\
&\geq|\langle(A_1-\langle u,A_1u\rangle)u,(A_2-\langle u,A_2u\rangle)u\rangle|^2\\
&\geq|\mathrm{Im}\{\langle(A_1-\langle u,A_1u\rangle)u,(A_2-\langle u,A_2u\rangle)u\rangle\}|^2\\
&=\frac{1}{4}|\langle(A_1-\langle u,A_1u\rangle)u,(A_2-\langle u,A_2u\rangle)u\rangle-\langle(A_2-\langle u,A_2u\rangle)u,(A_1-\langle u,A_1u\rangle)u\rangle|^2\\
&=\frac{1}{4}|\langle A_1u,A_2u\rangle-\langle A_2u,A_1u\rangle|^2,
\end{aligned} \tag{24}$$

where $\mathrm{Im}\{z\}$ is an imaginary part of a complex number z. This completes the proof.

Lemma 2. *Let A be any self-adjoint operator on a Hilbert space. H. Let K be a dense subspace of $D(A)$ and let (Ω,B,E,g) be a $(\varnothing,K)$-measurement for A. Then, it follows that*

$$\langle v,Au\rangle=\int_\Omega g(\omega)\langle v,E(d\omega)u\rangle \tag{25}$$

for all $u\in K\cap\{u\in H:\int_\Omega|g(\omega)|^2\langle u,E(d\omega)u\rangle<\infty\}$ and all $v\in H$.

Proof. Let u be any fixed element in $K \cap \{u\in H: \int_\Omega |g(\omega)|^2\langle u,E(d\omega)u\rangle<\infty\}$. We can see that, for any $v\in K$,

$$\begin{aligned}\langle v,Au\rangle &= \frac{1}{4}\{\langle v+u,A(v+u)\rangle-\langle v-u,A(v-u)\rangle-i\langle v+iu,A(v+iu)\rangle+i\langle v-iu,A(v-iu)\rangle\}\\ &= \frac{1}{4}\{\int_\Omega g(\omega)\langle v+u,E(d\omega)(v+u)\rangle-\int_\Omega g(\omega)\langle v-u,E(d\omega)(v-u)\rangle\\ &\qquad -i\int_\Omega g(\omega)\langle v+iu,E(d\omega)(v+iu)\rangle+i\int_\Omega g(\omega)\langle v-iu,E(d\omega)(v-iu)\rangle\}\\ &= \frac{1}{4}\int_\Omega g(\omega)\{\langle v+u,E(d\omega)(v+u)\rangle-\langle v-u,E(d\omega)(v-u)\rangle\\ &\qquad -i\langle v+iu,E(d\omega)(v+iu)\rangle+i\langle v-iu,E(d\omega)(v-iu)\rangle\}\\ &= \int_\Omega g(\omega)\langle v,E(d\omega)u\rangle. \end{aligned} \tag{26}$$

Also, using Proposition 1, we obtain that, for any $v\in H$,

$$\begin{aligned}|\int_\Omega g(\omega)\langle v,E(d\omega)u\rangle| &= |\int_\Omega g(\omega)\langle v\otimes s_0,\hat{E}(d\omega)(u\otimes s_0)\rangle| = |\langle v\otimes s_0,\hat{A}(u\otimes s_0)\rangle|\\ &\le \|\hat{A}(u\otimes s_0)\|_{\hat{H}}\cdot\|v\|_H,\end{aligned} \tag{27}$$

where $\hat{A}=\int_\Omega g(\omega)\hat{E}(d\omega)$. Since K is dense, we see, from (26) and (27), that

$$\langle v,Au\rangle=\int_\Omega g(\omega)\langle v,E(d\omega)u\rangle \tag{28}$$

for all $v\in H$. This completes the proof.

Lemma 3. (i). *Let A_1 and A_2 be any self-adjoint operators on a Hilbert space H such that $D(A_1)\cap D(A_2)$ is a dense subspace in H. Assume that $(\Omega,\mathcal{B},E,f=(f_1,f_2))$ is any weakly simultaneous measurement of (A_1,A_2) and $(R,\hat{E},s_0)$ is a realization of $(\Omega,\mathcal{B},E)$. And put $\hat{A}_1=\int_\Omega f_1(\omega)\hat{E}(d\omega)$ and $\hat{A}_2=\int_\Omega f_2(\omega)\hat{E}(d\omega)$. Then the following equalities hold:*

$$\begin{aligned}&\int_\Omega f_i(\omega)f_j(\omega)\langle u,E(d\omega)u\rangle\\ &=\langle\hat{A}_i(u\otimes s_0),\hat{A}_j(u\otimes s_0)\rangle\\ &=\langle A_iu,A_ju\rangle+\langle(\hat{A}_i-A_i\otimes I)(u\otimes s_0),(\hat{A}_j-A_j\otimes I)(u\otimes s_0)\rangle \qquad (i,j\in\{1,2\})\end{aligned} \tag{29}$$

for all $u\in D(A_1)\cap D(A_2)\cap\{u\in H: \int_\Omega |f_i(\omega)|^2\langle u,E(d\omega)u\rangle<\infty\ (i=1,2)\}$,

(ii). *Let A be any self-adjoint operator on a Hilbert space H and let K be any dense subspace of $D(A)$. Let $(\Omega,\mathcal{B},E,f)$ be any $(\emptyset,K)$-measurement of A. Then the following inequality holds:*

$$\int_\Omega |f(\omega)|^2 \langle u, E(d\omega)u\rangle \geq \|Au\|^2 \tag{30}$$

for all $u \in K \bigcap \{u \in H: \int_\Omega |f(\omega)|^2 \langle u, E(d\omega)u\rangle < \infty\}$.

Proof. Let u be any element in $D(A_1) \bigcap D(A_2) \bigcap \{u \in H: \int_\Omega |f_i(\omega)|^2 \langle u, E(d\omega)u\rangle < \infty \ \ (i=1,2)\}$.

Then, we see that

$$\begin{aligned}
&\int_\Omega f_i(\omega) f_j(\omega) \langle u, E(d\omega)u\rangle \\
&= \int_\Omega f_i(\omega) f_j(\omega) \langle u\otimes s_0, \hat{E}(d\omega)(u\otimes s_0)\rangle \\
&= \langle \int_\Omega f_i(\omega)\hat{E}(d\omega)(u\otimes s_0), \int_\Omega f_j(\omega)\hat{E}(d\omega)(u\otimes s_0)\rangle \\
&= \langle \hat{A}_i(u\otimes s_0), \hat{A}_j(u\otimes s_0)\rangle \qquad\qquad (\textit{then, (29) follows}) \\
&= \langle (\hat{A}_i - A_i\otimes I)(u\otimes s_0) + A_i u\otimes s_0, (\hat{A}_j - A_j\otimes I)(u\otimes s_0) + A_j u\otimes s_0\rangle \\
&= \langle (\hat{A}_i - A_i\otimes I)(u\otimes s_0), (\hat{A}_j - A_j\otimes I)(u\otimes s_0)\rangle + \langle (\hat{A}_i - A_i\otimes I)(u\otimes s_0), A_j u\otimes s_0\rangle \\
&\qquad + \langle A_i u\otimes s_0, (\hat{A}_j - A_j\otimes I)(u\otimes s_0)\rangle + \langle A_i u\otimes s_0, A_j u\otimes s_0\rangle \\
&= \langle (\hat{A}_i - A_i\otimes I)(u\otimes s_0), (\hat{A}_j - A_j\otimes I)(u\otimes s_0)\rangle + \langle \hat{A}_i(u\otimes s_0), A_j u\otimes s_0\rangle - \langle A_i u, A_j u\rangle \\
&\qquad + \langle A_i u\otimes s_0, \hat{A}_j(u\otimes s_0)\rangle - \langle A_i u, A_j u\rangle + \langle A_i u, A_j u\rangle \\
&= \langle (\hat{A}_i - A_i\otimes I)(u\otimes s_0), (\hat{A}_j - A_j\otimes I)(u\otimes s_0)\rangle - \langle A_i u, A_j u\rangle \\
&\qquad + \int_\Omega f_j(\omega)\langle A_i u, E(d\omega)u\rangle + \int_\Omega f_i(\omega)\langle E(d\omega)u, A_j u\rangle \\
&= \langle A_i u, A_j u\rangle + \langle (\hat{A}_i - A_i\otimes I)(u\otimes s_0), (\hat{A}_j - A_j\otimes I)(u\otimes s_0)\rangle \qquad (\textit{by Lemma 2}).
\end{aligned} \tag{31}$$

Hence, the proof of (i) is completed. Also, the proof of (ii) is carried out just in a similar way, that is, we can easily see that $\int_\Omega |f(\omega)|^2 \langle u, E(d\omega)u\rangle = \|Au\|^2 + \|(\hat{A} - A\otimes I)(u\otimes s_0)\|^2 \geq \|Au\|^2$ for all $u \in K \bigcap \{u \in H: \int_\Omega |f(\omega)|^2 \langle u, E(d\omega)u\rangle < \infty\}$.

Now we can show the ordinary and the simultaneous uncertainty relations.

Theorem 2. *Let* A_1 *and* A_2 *be any self-adjoint operators on a Hilbert space* H *such that* $D(A_1) \bigcap D(A_2)$ *is a dense subspace in* H*. Then, for any* $(\varnothing, D(A_1) \bigcap D(A_2))$*-measurement* $\mathrm{M}_1 = (\Omega_1, \mathrm{B}_1, E_1, f_1)$ *of* A_1 *and any* $(\varnothing, D(A_1) \bigcap D(A_2))$*-measurement* $\mathrm{M}_2 = (\Omega_2, \mathrm{B}_2, E_2, f_2)$ *of* A_2*, the following inequality holds:*

$$(\Delta M_1(u)_1)\cdot(\Delta M_2(u)_1)\geq\frac{1}{2}|\langle A_1u,A_2u\rangle-\langle A_2u,A_1u\rangle| \tag{32}$$

for all $u\in D(A_1)\cap D(A_2)$.

Proof. Putting $K=D(A_1)\cap D(A_2)$, $A=A_i$ and $f=f_i$ in the part of (ii) of Lemma 3, we see that, for any $u\in D(A_1)\cap D(A_2)\cap\{u\in H:\int_\Omega|f_i(\omega)|^2\langle u,E_i(d\omega)u\rangle<\infty\}$,

$$\begin{aligned}\Delta M_i(u)_1&=[\int_{\Omega_i}|f_i(\omega)|^2\langle u,E_i(d\omega)u\rangle-|\int_{\Omega_i}f_i(\omega)\langle u,E_i(d\omega)u\rangle|^2]^{1/2}\\&\geq[\|A_iu\|^2-|\langle u,A_iu\rangle|^2]^{1/2} \qquad (i=1,2).\end{aligned} \tag{33}$$

Also, note that (33) holds for all $u\in D(A_1)\cap D(A_2)$ though the left hand side of (33) may be infinite for some u. Hence, by Lemma 1, we can easily obtain (32). This completes the proof.

Note that, if M_i is a weakly simultaneous measurement for A_i, then it is also a $(\varnothing,D(A_1)\cap D(A_2))$-simultaneous measurement for A_i.

Now, we have the following main theorem which includes an answer to the problem 2 as the special case.

Theorem 3. *Let A_1 and A_2 be any self-adjoint operators on a Hilbert space H such that $D(A_1)\cap D(A_2)$ is a dense subspace in H. Then, for any weakly simultaneous measurement $M=(\Omega,B,E;f=(f_1,f_2))$ of (A_1,A_2), the following inequality holds:*

$$(\Delta M(u)_1)\cdot(\Delta M(u)_2)\geq|\langle A_1u,A_2u\rangle-\langle A_2u,A_1u\rangle| \tag{34}$$

for all $u\in D(A_1)\cap D(A_2)$.

Proof. Let $(R,\hat{E},s_0)$ be a realization of the positive operator valued probability space (Ω,B,E). Put $\hat{A}_1=\int_\Omega f_1(\omega)\hat{E}(d\omega)$ and $\hat{A}_2=\int_\Omega f_2(\omega)\hat{E}(d\omega)$.

Let u be any element in $D(A_1)\cap D(A_2)\cap\{u\in H:\int_\Omega|f_i(\omega)|^2\langle u,E(d\omega)u\rangle<\infty\ (i=1,2)\}$. We see, by the part (i) of Lemma 3 for $i=1$ and $j=2$, that

$$\begin{aligned}&\langle A_1u,A_2u\rangle+\langle(\hat{A}_1-A_1\otimes I)(u\otimes s_o),(\hat{A}_2-A_2\otimes I)(u\otimes s_o)\rangle\\&=\int_\Omega f_1(\omega)f_2(\omega)\langle u,E(d\omega)u\rangle\end{aligned}$$

$$=\langle A_2u,A_1u\rangle+\langle(\hat{A}_2-A_2\otimes I)(u\otimes s_o),(\hat{A}_1-A_1\otimes I)(u\otimes s_o)\rangle \tag{35}$$

from which, we get that

$$\langle(\hat{A}_1-A_1\otimes I)(u\otimes s_o),(\hat{A}_2-A_2\otimes I)(u\otimes s_o)\rangle$$
$$=\frac{1}{2}\{\langle(\hat{A}_1-A_1\otimes I)(u\otimes s_o),(\hat{A}_2-A_2\otimes I)(u\otimes s_o)\rangle$$
$$+\langle(\hat{A}_2-A_2\otimes I)(u\otimes s_o),(\hat{A}_1-A_1\otimes I)(u\otimes s_o)\rangle\}-\frac{1}{2}\{\langle A_1u,A_2u\rangle-\langle A_2u,A_1u\rangle\}. \tag{36}$$

Since the above $\{\langle..\rangle+\langle..\rangle\}$ is real and $\{\langle..\rangle-\langle..\rangle\}$ is imaginary, we see by Schwarz inequality that

$$\frac{1}{2}|\langle A_1u,A_2u\rangle-\langle A_2u,A_1u\rangle|\leq|\langle(\hat{A}_1-A_1\otimes I)(u\otimes s_o),(\hat{A}_2-A_2\otimes I)(u\otimes s_o)\rangle|$$
$$\leq\|(\hat{A}_1-A_1\otimes I)(u\otimes s_o)\|\cdot\|(\hat{A}_2-A_2\otimes I)(u\otimes s_o)\|. \tag{37}$$

Also, we obtain, by the part (i) of Lemma 3 for $i=i$ and $j=i$ and the arithmetic-geometric mean inequality, that, for any $u\in D(A_1)\cap D(A_2)\cap\{u\in H: \int_\Omega|f_i(\omega)|^2\langle u,E(d\omega)u\rangle<\infty\ \ (i=1,2)\}$,

$$[\int_\Omega|f_i(\omega)|^2\langle u,E(d\omega)u\rangle-|\int_\Omega f_i(\omega)\langle u,E(d\omega)u\rangle|^2]=[\|A_iu\|^2+\|(\hat{A}_i-A_i\otimes I)(u\otimes s_0)\|^2-|\langle u,A_iu\rangle|^2]$$
$$\geq2(\|A_iu\|^2-|\langle u,A_iu\rangle|^2)^{1/2}\cdot\|(\hat{A}_i-A_i\otimes I)(u\otimes s_0)\|\quad(i=1,2). \tag{38}$$

From (37), (38) and Lemma 1, we see that

$$(\Delta M(u)_1)\cdot(\Delta M(u)_2)\geq|\langle A_1u,A_2u\rangle-\langle A_2u,A_1u\rangle| \tag{39}$$

for all $u\in D(A_1)\cap D(A_2)\cap\{u\in H:\int_\Omega|f_i(\omega)|^2\langle u,E(d\omega)u\rangle<\infty\ \ (i=1,2)\}$. Note that, if $u\in D(A_1)\cap D(A_2)$ and $\Delta M(u)_i=0$ for some i, then the equality $|\langle A_1u,A_2u\rangle-\langle A_2u,A_1u\rangle|=0$ holds because $0=(\Delta M(u)_i)^2\geq\|(A_i-\langle u,A_iu\rangle)u\|^2$ by (30). Therefore, (34) holds for all $u\in D(A_1)\cap D(A_2)$ in the sense of $\infty\cdot\infty=\infty$ and $\infty\cdot0=0\cdot\infty=0$ though the left hand side of (34) may be infinite for some $u\in D(A_1)\cap D(A_2)$. Hence, the proof is completed.

In the following corollary of Theorems 2 and 3, we give the mathematical formulation of the uncertainty relations in the ordinary and simultaneous measurements of the position q and the momentum p of a particle.

Corollary 1. *Let H be a Hilbert space. Let (P,Q) be a canonical conjugate pair on H. Then, the following uncertainty relations hold:*

(i) *(ordinary uncertainty relation)*

For any $(\emptyset,D(P)\cap D(Q))$*-measurement* $M_P=(\Omega_P,\mathcal{B}_P,E_P,f_P)$ *of P and any* $(\emptyset,D(P)\cap D(Q))$*-measurement* $M_Q=(\Omega_Q,\mathcal{B}_Q,E_Q,f_Q)$ *of Q, the following inequality holds:*

$$\Delta q_u\cdot\Delta p_u\geq\hbar/2 \tag{40}$$

for all $u\in D(P)\cap D(Q)$ $(\|u\|=1)$, *where*

$$\Delta p_u=\Delta M_P(u)_1=[\int_{\Omega_P}|f_P(\omega)|^2\langle u,E_P(d\omega)u\rangle-|\int_{\Omega_P}f_P(\omega)\langle u,E_P(d\omega)u\rangle|^2]^{1/2} \tag{41}$$

and

$$\Delta q_u=\Delta M_Q(u)_1=[\int_{\Omega_Q}|f_Q(\omega)|^2\langle u,E_Q(d\omega)u\rangle-|\int_{\Omega_Q}f_Q(\omega)\langle u,E_Q(d\omega)u\rangle|^2]^{1/2}. \tag{42}$$

(ii) *(simultaneous uncertainty relation)*

For any weakly simultaneous measurement $M=(\Omega,\mathcal{B},E,f=(f_P,f_Q))$ *of (P,Q), the following inequality holds:*

$$\Delta q_u\cdot\Delta p_u\geq\hbar \tag{43}$$

for all $u\in D(P)\cap D(Q)$ $(\|u\|=1)$, *where*

$$\Delta p_u=\Delta M(u)_P=[\int_\Omega|f_P(\omega)|^2\langle u,E(d\omega)u\rangle-|\int_\Omega f_P(\omega)\langle u,E(d\omega)u\rangle|^2]^{1/2} \tag{44}$$

and

$$\Delta q_u=\Delta M(u)_Q=[\int_\Omega|f_Q(\omega)|^2\langle u,E(d\omega)u\rangle-|\int_\Omega f_Q(\omega)\langle u,E(d\omega)u\rangle|^2]^{1/2}. \tag{45}$$

This proof immediately follows from Theorems 2, 3 and the fact that $|\langle Pu,Qu\rangle-\langle Qu,Pu\rangle|=\hbar$ for all $u\in D(P)\cap D(Q)$ $(\|u\|=1)$.

Remark 2. The above inequalities (40) and (43) are both best possible. For (40), see [7]. Also, we can easily see by (21) for $n=2$ that $\Delta q_u\cdot\Delta p_u=2\|Pu\|\cdot\|Qu\|=\hbar$ if we take a state $u\in H$ such that $\langle u,Qu\rangle=\langle u,Pu\rangle=0$ and $\|Pu\|\cdot\|Qu\|=\hbar/2$. Hence, the inequality (43) is also best possible.

References

[1]. Ahiezer, N.L., and Glazman, I.M., *The theory of linear operators in Hilbert space.* Akademie-Verlag, Berlin, 1954.

[2]. Arthurs, E., and Kelly, J.L.,Jr., *On the simultaneous measurement of a pair of conjugate observables.* Bell System Tech. J., 44, 725-729 (April 1965).

[3]. Davies, E.B., and Lewis, J.T., *An operational approach to quantum probability.* Commun. Math. Phys., 17, 239-260 (1970).

[4]. Helstrom, C.W., *Quantum detection and estimation theory.* Academic press, 1976.

[5]. Holevo, A.S., *Statistical decision theory for quantum systems.* J. Multivar. Anal., 3(4), 337-394 (1973).

[6]. Holevo, A.S., *Probabilistic and statistical aspects of quantum theory.* North-Holland publishing company, 1982.

[7]. von Neumann, J., *Die Mathematische Grundlagen Der Quantenmechanik.* Springer Verlag. Berlin, 1932.

[8]. Prugovečki, E., *Quantum mechanics in Hilbert space.* Academic Press, New York, 1981.

[9]. Riesz, F., and Nagy, B., *Leçons d'Analyse Fonctionnelle.* Akadémiai Kiadó, Budapest, 1952.

[10]. She, C.Y., and Heffner, H., *Simultaneous measurement of noncommuting observables.* Phys. Rev. 152, 1103-1110 (Dec. 23, 1966).

[11]. Yuen, H.P., *Generalized quantum measurements and approximate simultaneous measurements of noncommuting observables.* Physics letters, 91A, 101-104 (1982).

[12]. Hirota, O., *Optical communication theory. (in Japanese).* Morikita Press. (1985).