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Gauss maps of compact surfaces with constant mean curvature

by

Masaaki Umehara
Kotaro Yamada

Masaaki Umehara

Institute of Mathematics
University of Tsukuba

Tsukuba, Ibaraki, 305 Japan

Kotaro Yamada

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku
Yokohama, 223 Japan

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku, Yokohama, 223 Japan

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Dedicated to Professor Shingo Murakami on his 60th birthday

Masaaki Umehara
University of Tsukuba

Kotaro Yamada
Keio University

Abstract

Let x be an immersion of a compact surface Σ into $\mathbf{E}^3$ with constant mean curvature. Then the Gauss map Ψ of x is a harmonic map from Σ into the unit sphere S^2. Identifying S^2 with $\mathbf{C} \cup \{\infty\}$, the conjugation $\overline{\Psi}$ is also a harmonic map. Moreover, this map is shown to be realized as a Gauss map of a branched immersion $\tilde{x}$ of Σ with constant mean curvature. In particular, $\tilde{x}$ have no branched points if Σ is a torus. In this case, the relationship between x and $\tilde{x}$ is also discussed.

0 Introduction.

Recently, the study of compact surfaces with constant mean curvature has been developed remarkably. In 1984, H. C. Wente [13] discovered conformal immersions of rectangular tori in $\mathbf{E}^3$ with constant mean curvature. Later, U. Abresh [1] and R. Walter [12] explained such tori in terms of elliptic functions. Moreover, immersions of non-rectangular tori with constant mean curvature were constructed by Wente [14]. More recently, N. Kapouleas [7] [8] constructed compact surfaces with constant mean curvature of higher genus.

It is well-known that the Gauss maps of such surfaces are harmonic. Conversely, K. Kenmotsu [9] established the fundamental tools for reverse construction of an immersion from prescribed Gauss map and mean curvature. As a special case of his results, a harmonic map Ψ of a simply connected Riemann surface Σ into the unit sphere S^2 determines a conformal branched immersion of Σ into $\mathbf{E}^3$ whose Gauss map is Ψ. These results are reviewed in Section 1. Through these facts, it seems that the theory of harmonic maps between Riemann surfaces is very useful for the study of surfaces with constant mean curvature.

Unfortunately, Kenmotsu's result cannot apply for a compact surface unless it is simply connected. In fact, even if Ψ is a harmonic map of a compact Riemann surface

Σ into the sphere, it does not determine a branched immersion of Σ but that of the universal cover of Σ in general.

The purpose of this paper is to study the Gauss maps of *compact* surfaces with constant mean curvature in $\mathbf{E}^3$.

Let $\Psi : \Sigma \to S^2$ and $H : \Sigma \to \mathbf{R}^+$ be mappings defined on a compact Riemann surface Σ. In Section 2, a necessary and sufficient condition for an immersion determined by Ψ and H to give an immersion Σ itself is given.

The Gauss map Ψ of an immersion $x : \Sigma \to \mathbf{E}^3$ is considered as a map from Σ into $\mathbf{C} \cup \{\infty\}$ with the stereographic projection from the north pole. Assume the mean curvature of x is constant. Then under the above identification, the Gauss map Ψ of the immersion x satisfies

$$\Psi_{z\bar{z}} - \frac{2\overline{\Psi}}{1+|\Psi|^2}\Psi_z\Psi_{\bar{z}} = 0. \tag{0.1}$$

Evidently, the conjugation $\overline{\Psi}$ also satisfies (0.1). In Sections 3 and 4, properties of the immersion $\tilde{x}$ determined by $\overline{\Psi}$, namely $\overline{\Psi}$-*immersion* are discussed. It should be remarked that the definition of $\tilde{x}$ is independent of the stereographic projection, that is, by composing an isometry τ of $\mathbf{E}^3$, the Gauss maps of x and $\tau \circ \tilde{x}$ are mutually antipodal.

In Section 3, we shall prove that if x is an immersion of compact surface Σ, $\overline{\Psi}$ also determines a branched immersion $\tilde{x}$ of Σ not of the universal cover of it. Moreover, $\tilde{x}$ has no branched points if Σ is torus. This implies that for each immersion of a torus with constant mean curvature, there exists a corresponding immersion with such properties.

Immersions admitting immersions with the conjugate Gauss maps are not necessarily with constant mean curvature. A sufficient condition that such an immersion admits an immersion with the conjugate Gauss map will be given in Section 4.

In the last Section, the relationship between the immersions of tori constructed by Wente [13] [14] and the corresponding immersions with the conjugate Gauss maps of them is discussed.

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1 Weierstrass formula for surfaces of prescribed mean curvature.

In this section, we observe the representation formula and some related formulas for surfaces in the euclidean 3-space with prescribed mean curvature established by K. Kenmotsu [9].

First, we shall review the classical theory of surfaces. Let $x : U \to \mathbf{E}^3$ be an conformal immersion of a simply connected domain U in $\mathbf{C}$ into the euclidean 3-space.

Here, conformality of x says that the canonical coordinate system $z = u^1 + \sqrt{-1}\,u^2$ of $\mathbf{C}$ is isothermal, *i.e.* the first fundamental form g of x is written as

$$g = e^{2\sigma}\{(du^1)^2 + (du^2)^2\} = e^{2\sigma}|dz|^2,$$

where σ is a smooth function defined on U.

Denote $\mathbf{n}$ by the unit normal vector field of x:

$$\mathbf{n} = e^{-2\sigma}\left(\frac{\partial x}{\partial u^1} \times \frac{\partial x}{\partial u^2}\right),$$

where $\times$ is the vector product of $\mathbf{E}^3$.

Let $h = \sum_{i,j=1}^2 h_{ij}\, du^i du^j$ be the second fundamental form of x with respect to $\mathbf{n}$. The *mean curvature* H is defined by

$$H = \frac{1}{2}e^{-\sigma}(h_{11} + h_{22}).$$

Let ϕ be the (2,0)-component of $2h$ in the coordinate system z:

$$\phi = \frac{1}{2}(h_{11} - h_{22}) - \sqrt{-1}\,h_{12}. \tag{1.1}$$

Using these functions, the Gauss and the Codazzi equations are written as

$$|\phi|^2 = e^{4\sigma}(H^2 - K), \tag{1.2}$$

$$\frac{\partial \phi}{\partial \bar{z}} = e^{2\sigma}\frac{\partial H}{\partial z}, \tag{1.3}$$

where K is the Gaussian curvature of the surface, *i.e.* the half of the scalar curvature of (U, g). In particular, ϕ is a holomorphic function of z if H is constant.

Note that the definition of ϕ depends on the choice of a coordinate system. Kenmotsu used $e^{-2\sigma}\phi$ instead of our ϕ in [9], which is invariant under the change of a coordinate system.

The unit normal vector field $\mathbf{n}$ determines the *Gauss map*

$$\nu : U \to S^2 = \{x \in \mathbf{E}^3;\ |x| = 1\}.$$

Let Ψ be the composition of ν and the stereographic projection of S^2 from the north pole

$$p(x) = \frac{x^1 + \sqrt{-1}x^2}{1 - x^3} : S^2 \to \mathbf{C} \cup \{\infty\}. \tag{1.4}$$

We also call

$$\Psi = p \circ \nu : U \to \mathbf{C} \cup \{\infty\} = \text{Riemann sphere},$$

the Gauss map of the immersion x.

Before establishing the representation formula, we observe some relationships between H, ϕ and Ψ. The absolute values of H and ϕ are represented by Ψ as

$$|H| = 2e^{-\sigma}\frac{1}{1+|\Psi|^2}|\Psi_{\bar{z}}|, \tag{1.5}$$

$$|\phi| = 2e^{\sigma}\frac{1}{1+|\Psi|^2}|\Psi_z|, \tag{1.6}$$

where $\Psi_z = \partial\Psi/\partial z$ and $\Psi_{\bar{z}} = \partial\Psi/\partial\bar{z}$. And we have a Beltrami equation [9, Theorem 1]:

$$H\Psi_z = \phi e^{-2\sigma}\Psi_{\bar{z}}. \tag{1.7}$$

By these equation, ϕ is represented as

$$\phi = \frac{4\Psi_z\Psi_{\bar{z}}}{|H|(1+|\Psi|^2)^2}. \tag{1.8}$$

Using these equations, Kenmotsu showed that the coordinate functions of the immersion $x = (x^1, x^2, x^3)$ satisfy the following equations, namely a generalized Weierstrass formula,

$$\begin{cases} H\dfrac{\partial x^1}{\partial\bar{z}} = -\dfrac{1-\overline{\Psi}^2}{(1+|\Psi|^2)^2}\Psi_{\bar{z}}, \\ H\dfrac{\partial x^2}{\partial\bar{z}} = -\sqrt{-1}\dfrac{1+\overline{\Psi}^2}{(1+|\Psi|^2)^2}\Psi_{\bar{z}}, \\ H\dfrac{\partial x^3}{\partial\bar{z}} = -\dfrac{2\overline{\Psi}}{(1+|\Psi|^2)^2}\Psi_{\bar{z}}. \end{cases} \tag{1.9}$$

As a system of partial differential equations of x^1, x^2 and x^3, an integrability condition of (1.9) is given by

$$H(\Psi_{z\bar{z}} - \frac{2\overline{\Psi}}{1+|\Psi|^2}\Psi_z\Psi_{\bar{z}}) = H_z\Psi_{\bar{z}}. \tag{1.10}$$

Conversely, for given $H : U \to \mathbf{R}^+ = (0,\infty)$ and $\Psi : U \to \mathbf{C}\cup\{\infty\}$ satisfying (1.10), there exists a conformal immersion $x : U \to \mathbf{E}^3$ with the Gauss map Ψ and the mean curvature H. That is,

Propostion 1.1 [9, Theorem 4] *Let U be a simply connected domain in $\mathbf{C}$. Assume $H : U \to \mathbf{R}^+$ and $\Psi : U \to \mathbf{C}\cup\{\infty\}$ are smooth maps satisfying* (1.10). *Then there exists a conformal immersion $x : U \to \mathbf{E}^3$ with degenerate points in general, whose mean curvature and Gauss map are H and Ψ respectively. Moreover, such an immersion is determined uniquely up to translation of $\mathbf{E}^3$. More precisely, the immersion $x = (x^1, x^2, x^3)$ is represented by*

$$x^1(z) = \mathrm{Re}\int_{z_0}^{z}\frac{-1}{H}\frac{1-\Psi^2}{(1+|\Psi|^2)^2}\Psi_{\bar{z}}\,dz + c_1,$$

$$x^2(z) = \mathrm{Re}\int_{z_0}^{z} \frac{-\sqrt{-1}}{H}\frac{1+\Psi^2}{(1+|\Psi|^2)^2}\,\Psi_{\bar z}\,dz + c_2,$$
$$x^3(z) = \mathrm{Re}\int_{z_0}^{z} \frac{-1}{H}\frac{2\Psi}{(1+|\Psi|^2)^2}\,\Psi_{\bar z}\,dz + c_3,$$

where z_0 is a base point and $c_i\,(i = 1,2,3)$ are integral constants. The immersion x degenerates at points where $\Psi_{\bar z} = 0$.

Note that the integrability condition (1.10) is invariant under the change of a coordinate system. We call the function Ψ of a surface into the Riemann sphere satisfying (1.10) *H-harmonic*.

If H is non-zero constant, H-harmonicity means harmonicity. In this case, the immersion x with degenerate points becomes a branched immersion of U into $\mathbf{E}^3$ [6, Proposition 2.4].

Rewrite the integrability condition (1.10) as

$$\frac{\partial}{\partial z}(\log \Psi_{\bar z}) - \frac{2\bar\Psi}{1+|\Psi|^2}\,\Psi_z = \frac{\partial}{\partial z}(\log H). \tag{1.11}$$

By considering (1.11) as a differential equation of $\log H$, we have the integrability condition

$$\mathrm{Im}\left(\frac{\partial^2}{\partial z\partial\bar z}(\log\Psi_{\bar z}) + \frac{2\bar\Psi^2}{(1+|\Psi|^2)^2}\,\Psi_z\Psi_{\bar z} - \frac{2\bar\Psi}{(1+|\Psi|^2)^2}\,\Psi_{z\bar z}\right) = 0. \tag{1.12}$$

Note that $\Psi_z \neq 0$ is equivalent to $H \neq 0$.

By (1.11), H is determined up to homothety for given Ψ. Moreover, since the immersion x is determined by H and Ψ, x is completely determined only by Ψ if $H \neq 0$. This fact was pointed out by A. Hoffmann and R. Osserman [4] for surfaces in higher dimensional euclidean space.

From this observation, we call the immersion $x : \Sigma \to \mathbf{E}^3$ determined by $\Psi : \Sigma \to \mathbf{C}\cup\{\infty\}$ *Ψ-immersion.*

2 Gauss maps of compact surfaces.

Consider a compact Riemann surface Σ, and let $H : \Sigma \to \mathbf{R}^+$ and $\Psi : \Sigma \to \mathbf{C}\cup\{\infty\}$ be smooth maps satisfying the integrability condition (1.10). Applying Proposition 1.1 to a local isothermal coordinate system of Σ, we can construct *locally* an immersion with degenerate points of the surface into $\mathbf{E}^3$ whose mean curvature and Gauss map are H and Ψ respectively. In fact, the constructed immersion gives only an immersion of the universal cover $\widetilde{\Sigma}$ of Σ in general.

In this section, we shall give a condition for the immersion constructed like above to induce an immersion of the compact surface Σ itself.

For given maps $H : \Sigma \to \mathbf{R}^+$ and $\Psi : \Sigma \to \mathbf{C} \cup \{\infty\}$, we define

$$(2.1)\qquad \begin{aligned} \omega_1 &= \operatorname{Re} \frac{-1}{H} \frac{1-\overline{\Psi}^2}{(1+|\Psi|^2)^2} \Psi_{\bar z}\, d\bar z \\ \omega_2 &= \operatorname{Re} \frac{-\sqrt{-1}}{H} \frac{1+\overline{\Psi}^2}{(1+|\Psi|^2)^2} \Psi_{\bar z}\, d\bar z \\ \omega_3 &= \operatorname{Re} \frac{-2\overline{\Psi}}{(1+|\Psi|^2)^2} \Psi_{\bar z}\, d\bar z. \end{aligned}$$

Assume $\omega_i(z) = 0$ at the points $\Psi(z) = \infty$. Then ω_1, ω_2 and ω_3 are smooth 1-forms on Σ. Using these forms, we rewrite the integrability condition (1.10).

Propostion 2.1 *Let Σ be a Riemann surface and $\widetilde{\Sigma}$ the universal cover of it. Then for given smooth maps $H : \Sigma \to \mathbf{R}^+$ and $\Psi : \Sigma \to \mathbf{C} \cup \{\infty\}$, the following assertions are equivalent, where $\widetilde{H}$ and $\widetilde{\Psi}$ are the lifts of H and Ψ to $\widetilde{\Sigma}$ respectively.*

(1) *There exists a conformal immersion with degenerate points of $\widetilde{\Sigma}$ into $\mathbf{E}^3$ whose mean curvature and Gauss map are $\widetilde{H}$ and $\widetilde{\Psi}$ respectively.*

(2) *The map Ψ satisfies (1.10), i.e. it is H-harmonic.*

(3) *The forms ω_1, ω_2 and ω_3 defined in (2.1) are closed 1-forms.*

PROOF. The equivalency between (1) and (2) is the conclusion of Proposition 1.1. It is easy to observe that (3) is equivalent to

$$\begin{cases} \operatorname{Im} \dfrac{\partial}{\partial z} \left(\dfrac{-1}{H} \dfrac{1-\overline{\Psi}^2}{(1+|\Psi|^2)^2} \Psi_{\bar z} \right) = 0, \\ \operatorname{Im} \dfrac{\partial}{\partial z} \left(\dfrac{-\sqrt{-1}}{H} \dfrac{1+\overline{\Psi}^2}{(1+|\Psi|^2)^2} \Psi_{\bar z} \right) = 0, \\ \operatorname{Im} \dfrac{\partial}{\partial z} \left(-\dfrac{2\overline{\Psi}}{(1+|\Psi|^2)^2} \Psi_{\bar z} \right) = 0. \end{cases}$$

This is an integrability condition of (1.9). Hence (2) and (3) are equivalent to each other. □

The immersion $x = (x^1, x^2, x^3)$ is written as the following:

$$(2.2)\qquad x^1 = \int_{z_0}^{z} \omega_1, \qquad x^2 = \int_{z_0}^{z} \omega_2, \qquad x^3 = \int_{z_0}^{z} \omega_3,$$

where z_0 is a base point. Thus, we have a condition for $x : \widetilde{\Sigma} \to \mathbf{E}^3$ to give an immersion of Σ.

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Theorem 2.2 *Let Σ, H and Ψ be as in Proposition 2.1. Then the immersion with degenerate points $x : \widetilde{\Sigma} \to \mathbf{E}^3$ constructed by Proposition 1.1 gives an immersion of $\Sigma \to \mathbf{E}^3$ if and only if the forms ω_1, ω_2 and ω_3 are exact.*

Assume that *the mean curvature H is non-zero constant.* In this case, Ψ satisfies (1.10) if and only if it is a harmonic map from Σ into S^2. Then, the degenerate points of x (points where $\Psi_{\bar{z}} = 0$) are isolated unless Σ is the sphere. The following existence theorem holds if H is constant.

Theorem 2.3 *Let Σ be a compact Riemann surface and $\Psi : \Sigma \to S^2 = \mathbf{C} \cup \{\infty\}$ a smooth mapping. Then the following assertions are equivalent:*

(1) *There exists a conformal immersion $x : \Sigma \to \mathbf{E}^3$ with constant mean curvature whose Gauss map is Ψ.*

(2) *The degree of Ψ is $-\chi(\Sigma)/2$ and 1-forms $\omega_i\,(i = 1, 2, 3)$ in (2.1) are exact.*

PROOF. It is well-known that if a harmonic map $\Psi : \Sigma \to S^2$ is the Gauss map of some surface, then the degree of Ψ is $-\chi(\Sigma)/2$, where $\chi(\Sigma)$ is the Euler number of Σ [5]. Hence (1) implies (2).

Conversely, by Theorem 2.2, (2.2) gives an immersion x of Σ into $\mathbf{E}^3$ with degenerate points. So, it is sufficient to show that x has no degenerate points.

By Eells-Wood [3], the following index formula holds:

$$\operatorname{Index} \bar{\partial}\Psi = -\chi(\Sigma) - 2 \deg \Psi.$$

Thus, if $\deg \Psi = -\chi(\Sigma)/2$, $\operatorname{Index} \bar{\partial}\Psi = 0$. On the other hand, the winding number of $\bar{\partial}\Psi$ around its zero must be positive [3]. Thus there is no zeros of $\Psi_{\bar{z}}$, which are degenerate points of x. Hence (2) implies (1). □

REMARK. Even if $\deg \Psi = -\chi(\Sigma)/2$, the forms ω_i are not necessarily exact. For example, a Gauss map of a cylinder in $\mathbf{E}^3$ gives a harmonic map from rectangular torus into S^2 with degree 0.

Recently, some examples of compact surfaces with constant mean curvature in $\mathbf{E}^3$ are constructed by H. C. Wente [13] [14], U. Abresh [1], R. Walter [12] and N. Kapouleas [7] [8]. The authors wish that such examples could be constructed from Gauss maps.

3 Conjugate of Gauss maps of surfaces with constant mean curvature.

If H is constant, the integrability condition (1.10) is invariant under the conjugation of Ψ. Then $\overline{\Psi}$ determines another immersion of the surface Σ based on the original immersion of Σ. In this section, we investigate such surfaces of constant mean curvature. Throughout this section, we assume $H \equiv 1$.

Let Σ be a compact Riemann surface and Ψ a harmonic map from Σ into $\mathbf{C}\cup\{\infty\}$, *i.e.* , Ψ satisfies

$$\Psi_{z\bar{z}} - \frac{2\overline{\Psi}}{1+|\Psi|^2}\Psi_z\Psi_{\bar{z}} = 0. \tag{3.1}$$

Clearly, if Ψ satisfies (3.1), so does $\overline{\Psi}$. Then $\overline{\Psi}$ is the Gauss map of some immersion of $\widetilde{\Sigma}$ with non-zero constant mean curvature. Consider the following 1-forms:

$$\begin{aligned} \tilde{\omega}_1 &= \operatorname{Re}\frac{-(1-\Psi^2)}{(1+|\Psi|^2)^2}\overline{\Psi}_{\bar{z}}\,d\bar{z}, \\ \tilde{\omega}_2 &= \operatorname{Re}\sqrt{-1}\frac{1+\Psi^2}{(1+|\Psi|^2)^2}\overline{\Psi}_{\bar{z}}\,d\bar{z}, \\ \tilde{\omega}_3 &= \operatorname{Re}\frac{-2\Psi}{(1+|\Psi|^2)^2}\overline{\Psi}_{\bar{z}}\,d\bar{z}. \end{aligned} \tag{3.2}$$

It is clear that $\tilde{\omega}_i\,(i=1,2,3)$ are closed if (3.1) holds. Moreover, exactness of ω_i implies that of $\tilde{\omega}_i$.

Lemma 3.1 *If $\omega_i\,(i=1,2,3)$ are exact, so are $\tilde{\omega}_i\,(i=1,2,3)$.*

PROOF. The forms ω_i and $\tilde{\omega}_i$ are related as

$$d\left(\frac{1}{1+|\Psi|^2}\right) = \omega_3 + \tilde{\omega}_3, \quad d\left(\frac{-\operatorname{Re}\Psi}{1+|\Psi|^2}\right) = \omega_1 + \tilde{\omega}_1, \quad d\left(\frac{\operatorname{Im}\Psi}{1+|\Psi|^2}\right) = \omega_2 - \tilde{\omega}_2.$$

The conclusion is immediate from these identities. □

Hence, if there exists an immersion with constant mean curvature, we can construct another immersion of Σ whose Gauss map is conjugate to that of the first one.

Theorem 3.2 *Let $x : \Sigma \to \mathbf{E}^3$ be a conformal immersion of a compact Riemann surface with non-zero constant mean curvature H whose Gauss map is Ψ. Then, there exists another branched immersion $\tilde{x} : \Sigma \to \mathbf{E}^3$ with constant mean curvature whose Gauss map is $\overline{\Psi}$. Moreover, each branched point of $\tilde{x}$ corresponds with an umbilic point of x.*

PROOF. Integrate $\tilde{\omega}_i$ in (3.2), we have an immersion $\tilde{x}$.

At a branched point of $\tilde{x}$, $\overline{\Psi}_{\bar{z}} = \Psi_z = 0$ by Proposition 1.1. This shows that a branched point of $\tilde{x}$ is an umbilic point of x, and vice-versa. □

REMARK. As a consequence of this, the immersion $\tilde{x}$ must have branched points if the genus of Σ is grater than 2.

Assume that Σ is of genus 1, *i.e.* a torus. In this case, the immersion $\tilde{x}$ in Theorem 3.2 have no degenerate points. Thus, for each immersion of a torus with constant mean curvature, there exists another immersion by Theorem 3.2.

4 Surfaces admitting the immersions with the conjugate Gauss maps.

In the previous section, we observed that for each immersion of a compact surface with constant mean curvature, there exists an immersion of the surface whose Gauss map is conjugate to the original one.

In this section, the condition of constant mean curvature is not necessarily assumed. Even though the mean curvature is not constant, some immersions may admit immersions whose Gauss maps are conjugate to those of the original ones. Hereafter, we call the immersion determined by the Gauss map $\overline{\Psi}$ the *$\overline{\Psi}$-immersion.*

The notion of $\overline{\Psi}$-immersion seems to depend on a choice of a stereographic projection $p : S^2 \to \mathbf{C} \cup \{\infty\}$. However, this concept is independent of the coordinate system by means of the following proposition.

Propostion 4.1 *Let $\Psi = p \circ \nu : \Sigma \to \mathbf{E}^3$ be the Gauss map of a conformal immersion x for which both Ψ_z and $\Psi_{\bar{z}}$ never vanish. Assume x admits the $\overline{\Psi}$-immersion $\tilde{x}$. Then there exists an isometry τ of $\mathbf{E}^3$ such that the Gauss map of $\tau \circ \tilde{x}$ is antipodal to Ψ in S^2.*

PROOF. Let $\nu = p^{-1} \circ \Psi$ and $\tilde{\nu} = p^{-1} \circ \overline{\Psi}$ be the "spherical" Gauss maps of x and $\tilde{x}$ respectively, where p is the stereographic projection defined in (1.4). Then $\nu = (\nu^1, \nu^2, \nu^3)$ and $\tilde{\nu} = (\tilde{\nu}^1, \tilde{\nu}^2, \tilde{\nu}^3)$ are related by

$$(\nu^1, -\nu^2, \nu^3) = (\tilde{\nu}^1, \tilde{\nu}^2, \tilde{\nu}^3).$$

Let $\tau : \mathbf{E}^3 \to \mathbf{E}^3$ be the isometry defined as

$$\tau(x^1, x^2, x^3) = (x^1, -x^2, x^3).$$

Then,

$$\tau(x \times y) = -\tau(x) \times \tau(y).$$

Hence, the Gauss map of $\tau \circ \tilde{x}$ equals to $-\nu$. □

In the rest of this section, we introduce a necessary and sufficient condition for an immersion with the Gauss map Ψ to admit the $\overline{\Psi}$-immersion.

Using (1.12), we give a condition that Σ admits an immersion with the conjugate Gauss map.

Propostion 4.2 *Let $\widetilde{\Sigma}$ be a simply connected Riemann surface, and $\tilde{x} : \widetilde{\Sigma} \to \mathbf{E}^3$ a conformal immersion. Assume that the derivatives Ψ_z and $\Psi_{\bar{z}}$ of the Gauss map $\Psi : \widetilde{\Sigma} \to \mathbf{C} \cup \{\infty\}$ of x never vanish. Then, x admits the $\overline{\Psi}$-immersion if and only if*

$$\text{(4.1)} \qquad \operatorname{Im} \frac{\partial^2}{\partial z \partial \bar{z}} \log \phi = 0,$$

where ϕ is the function defined on each local isothermal coordinate system in (1.1).

Note that (4.1) is invariant under the changes of coordinate system.

PROOF. Since the Gauss map Ψ satisfies (1.12),

$$\text{Im}\left(\frac{\partial^2}{\partial z\partial\bar{z}}(\log\overline{\Psi}_z)+\frac{2\Psi^2}{(1+|\Psi|^2)^2}\Psi_z\Psi_{\bar{z}}-\frac{2\Psi}{(1+|\Psi|^2)^2}\Psi_{z\bar{z}}\right)=0. \tag{4.2}$$

And there exists the $\overline{\Psi}$-immersion if and only if $\overline{\Psi}$ also satisfies (1.12):

$$\text{Im}\left(\frac{\partial^2}{\partial z\partial\bar{z}}(\log\overline{\Psi}_{\bar{z}})+\frac{2\Psi^2}{(1+|\Psi|^2)^2}\Psi_z\Psi_{\bar{z}}-\frac{2\Psi}{(1+|\Psi|^2)^2}\Psi_{z\bar{z}}\right)=0. \tag{4.3}$$

Combining (4.2) and (4.3), we have

$$\text{Im}\left(\frac{\partial^2}{\partial z\partial\bar{z}}(\log\overline{\Psi}_z+\log\Psi_z)\right)=0. \tag{4.4}$$

Thus (1.8) is equivalent to (1.11). □

A criterion for admitting the $\overline{\Psi}$-immersion is related to a concept of H-deformability.

DEFINITION. Let $x:\Sigma\to\mathbf{E}^3$ be an immersion. Then x is *H-deformable* if and only if any point of Σ has a neighbourhood on which there exists a non-congruent deformation of x preserving its first fundamental form and mean curvature.

Note that constant mean curvature surfaces are H-deformable [10]. Recently, A. G. Corales and K. Kenmotsu [2] classified H-deformable surfaces with constant curvature.

As shown in [11], an immersion $x:\Sigma\to\mathbf{E}^3$ without umbilic points is H-deformable if and only if

$$\frac{1}{4}\frac{\partial^2}{\partial z\partial\bar{z}}\log\phi=|(\log\phi)_{\bar{z}}|^2. \tag{4.5}$$

As a corollary of this, we have the following.

Propostion 4.3 *Under the assumptions of Proposition* 4.2, *if x is H-deformable, then it admits the immersion with the conjugate Gauss map.*

This gives examples of immersions admitting the $\overline{\Psi}$-immersions even if H is not constant. Moreover, the following example shows that there exists an immersion of a surface admitting the $\overline{\Psi}$-immersion which is not H-deformable.

EXAMPLE. We use the notation in Section 1. Assume that ϕ is positive and real, and that the Gaussian curvature is identically 0. Then the Gauss and the Codazzi equations (1.2) (1.3) imply

$$\begin{cases} \phi = & e^{2\sigma} H, \\ \phi_{\bar{z}} = & e^{2\sigma} \dfrac{\partial H}{\partial z}, \\ K = & 0. \end{cases}$$

These are equivalent to

$$\begin{cases} 2\dfrac{\partial \sigma}{\partial \bar{z}} H + \dfrac{\partial H}{\partial \bar{z}} = & \dfrac{\partial H}{\partial z}, \\ \dfrac{\partial^2 \sigma}{\partial z \partial \bar{z}} = & 0. \end{cases}$$

For example, if we put

$$\sigma = u^2 = \text{Im } z, \qquad H = e^{-\sigma/2},$$

then these satisfy above conditions. Hence there exists a corresponding surface with σ, H and ϕ by the fundamental theorem of surfaces. Evidently, such an immersion admits the $\overline{\Psi}$-immersion. Nevertheless,

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \phi = \frac{\partial^2}{\partial z \partial \bar{z}} \log H = 0$$

shows that this surface is not H-deformable.

By Proposition 4.2, the function $(\log \phi)_{z\bar{z}}$ is real valued if x admits the $\overline{\Psi}$-immersion. Moreover, its value has the following property.

Propostion 4.4 *Let $x : \Sigma \to \mathbf{E}^3$ be a conformal immersion of a Riemann surface with non-vanishing mean curvature and Ψ the Gauss map of it. Assume that there exists the $\overline{\Psi}$-immersion $\tilde{x}$ of x. Then the function $\phi/\widetilde{H}$ is holomorphic, where ϕ is the function defined in* (1.1) *and $\widetilde{H}$ is the mean curvature of $\tilde{x}$. In particular,*

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \phi = \frac{\partial^2}{\partial z \partial \bar{z}} \log \widetilde{H}$$

holds.

PROOF. Let Ψ be the Gauss map of x. Then $\overline{\Psi}$ is that of $\tilde{x}$. By (1.10),

$$\begin{aligned} (\log H)_z \Psi_{\bar{z}} = & \Psi_{z\bar{z}} - \frac{2\overline{\Psi}}{1+|\Psi|^2} \Psi_z \Psi_{\bar{z}} \\ = & \overline{\overline{\Psi}_{z\bar{z}} - \frac{2\Psi}{1+|\Psi|^2} \overline{\Psi}_z \overline{\Psi}_{\bar{z}}} \\ = & \overline{(\log \widetilde{H})_z \overline{\Psi}_{\bar{z}}} \\ = & (\log \widetilde{H})_{\bar{z}} \overline{\Psi}_z. \end{aligned} \tag{4.6}$$

Hence,

$$\begin{aligned}(\log\phi)_{\bar z} &= \frac{e^{2\sigma}}{\phi}H_z && \text{by (1.3)}\\ &= \frac{\Psi_{\bar z}}{\Psi_z}(\log H)_z && \text{by (1.7)}\\ &= (\log\widetilde{H})_{\bar z} && \text{by (4.6).}\end{aligned}$$

□

5 Tori with constant mean curvature.

Let T be a Riemann surface of genus 1, *i.e.* a torus. Then T is identified with $\mathbf{R}^2/\Gamma$, where Γ is a lattice of $\mathbf{R}^2$. Consider a conformal immersion $x : T \to \mathbf{E}^3$. Then the coordinate system (u^1, u^2) of $\mathbf{R}^2$ is an isothermal coordinate system of x under the above identification.

Assume that the mean curvature H of the immersion x is constant. In this case, the holomorphic function ϕ in (1.1) on $\mathbf{R}^2/\Gamma$ should be constant. Hence the lines of curvature of x are two families of straight lines foliating $\mathbf{R}^2$.

Since H is a non-zero constant, we can assume $H = 1$ without any loss of generality. And by a homothetic change of $\mathbf{R}^2/\Gamma$, we can also assume $|\phi| = 1$. From now on, immersions of tori are assumed to be constant mean curvature $H = 1$ and $|\phi| = 1$.

Let $x : T \to \mathbf{E}^3$ be a conformal immersion with constant mean curvature 1 and Gauss map Ψ. As we observed in Section 3, the conjugation $\overline{\Psi}$ determines a regular immersion $\tilde{x} : T \to \mathbf{E}^3$ with constant mean curvature 1.

Let $g = e^{\sigma}\,|dz|^2$ and $\tilde{g} = e^{\tilde\sigma}\,|dz|^2$ be the first fundamental forms of x and $\tilde{x}$ respectively. Then by (1.5),

$$\begin{aligned}e^{\sigma} &= \frac{2}{1+|\Psi|^2}|\Psi_{\bar z}|^2,\\ e^{\tilde\sigma} &= \frac{2}{1+|\Psi|^2}|\Psi_z|^2.\end{aligned}$$

Moreover,

(5.1) $$\sigma = -\tilde\sigma$$

because of (1.6). And by (1.8),

(5.2) $$\phi = \tilde\phi$$

holds. In particular,

Propostion 5.1 *Let $\tau : T \to T$ be a smooth mapping. Then*

(1) *$\tau^* g = g$ if and only if $\sigma \circ \tau = \sigma$.*

(2) $\tau^* g = \tilde{g}$ *if and only if* $\sigma \circ \tau = -\sigma$.

We call a map $\tau : T \to T$ an *odd isometry* of (T, g) when $\tau^* g = \tilde{g}$. Isometries and odd isometries are conformal transformations of (T, g). Then they are represented as isometries of the flat torus $\mathbf{R}^2/\Gamma$. Hence such transformations are compositions of the following mappings on $\mathbf{R}^2/\Gamma$.

(1) Parallel displacements by constant vectors of $\mathbf{R}^2$.

(2) Reflections by lines keeping the lattice Γ invariant.

(3) Rotations of angle 0, $\pm\pi/3$, $\pm\pi/2$ or π keeping Γ invariant.

Note that only special lattices admit the transformations (2) and (3).

Before considering explicit examples, we observe the rigidity of tori with constant mean curvature.

Propostion 5.2 *Let* τ *be an odd isometry (resp. isometry) of* (T, g). *Then* $\hat{x} = \tilde{x} \circ \tau$ *(resp.* $x \circ \tau$*) and* x *have the same first fundamental form and mean curvature. Moreover, the following assertions holds.*

(1) *If* τ *is a parallel displacement,* x *and* $\hat{x}$ *are congruent.*

(2) *If* τ *is a reflection by a line* ℓ, *then* x *and* $\hat{x}$ *are congruent if and only if one of the families of lines of curvature of* x *are parallel to* ℓ.

(3) *If* τ *is a rotation of angle* θ, x *and* $\hat{x}$ *are not congruent unless* $\theta \equiv 0$ *or* π (mod 2π).

PROOF. The assertion (1) is trivial.

We now prove (2). Assume τ is a reflection with respect to the real axis of $\mathbf{C} = \mathbf{R}^2$. Then the components of the second fundamental form $\hat{h}$ of $\hat{x}$ are related with those of $\tilde{x}$ as

$$\hat{h}_{11} = \tilde{h}_{11} \circ \tau, \quad \hat{h}_{12} = -\tilde{h}_{12} \circ \tau, \quad \hat{h}_{22} = \tilde{h}_{22} \circ \tau,$$

and the (2,0)-component $\hat{\phi}$ of $2\hat{h}$ satisfies

$$\hat{\phi} = \bar{\phi}.$$

Hence x and $\hat{x}$ are congruent if and only if ϕ is real since the induced metrics are coincide. This condition is equivalent with that a family of lines of curvature of x are parallel to the real axis.

The rest of Proposition is proved in the same way. □

Unfortunately, we have no examples for which x and $\hat{x}$ are not congruent.

As examples of conformal immersions of tori with constant mean curvature, H. C. Wente [13] constructed immersions of rectangular tori into $\mathbf{E}^3$. Successively, U. Abresh [1] explained examples of Wente in terms of elliptic functions. We observe a relationship between these immersions and immersions with conjugate Gauss maps of them.

First, we review the construction in [13].

Assume a function σ on $\mathbf{R}^2$ satisfies the equation

$$\Delta\sigma = -4\sinh\sigma\cosh\sigma, \tag{5.3}$$

where Δ is the coordinate laplacian $\Delta = \partial^2/(\partial u^1)^2 + \partial^2/(\partial u^2)^2$. Then there exists an immersion $x : \mathbf{R}^2 \to \mathbf{E}^3$ with constant mean curvature 1 whose first and second fundamental form g and h are written as

$$\begin{aligned} g &= e^{2\sigma}\{(du^1)^2 + (du^2)^2\}, \\ h &= (e^{2\sigma}+1)(du^1)^2 + (e^{2\sigma}-1)(du^2)^2. \end{aligned} \tag{5.4}$$

Clearly, ϕ, the (2,0)-component of $2h$ is equal to 1.

To construct an immersion x of $\mathbf{R}^2$ invariant under an action of a lattice Γ from solutions of (5.3), we assume σ is a solution of the following Dirichlet problem:

$$\begin{cases} \Delta\sigma = -4\sinh\sigma\cosh\sigma & \text{in } \Omega_{AB}, \\ \sigma > 0 & \text{in } \Omega_{AB}, \\ \sigma = 0 & \text{on } \partial\Omega_{AB}, \end{cases} \tag{5.5}$$

where Ω_{AB} is a rectangle $(0,A)\times(0,B)$ on $\mathbf{R}^2$ with positive numbers A and B. If A and B are sufficiently large, there exists the unique solution of (5.5) [14]. Moreover, the solution σ is extended to the solution of (5.3) on $\mathbf{R}^2$ by odd reflections by $\partial\Omega_{AB}$. Such a solution σ is endowed with the symmetry properties:

$$\begin{aligned} \sigma(u^1+2A, u^2) &= \sigma(u^1,u^2), \\ \sigma(u^1, u^2+2B) &= \sigma(u^1,u^2), \\ \sigma(-u^1, u^2) &= -\sigma(u^1,u^2), \\ \sigma(u^1, -u^2) &= -\sigma(u^1,u^2), \\ \sigma(A-u^1, u^2) &= \sigma(u^1,u^2), \\ \sigma(u^1, B-u^2) &= \sigma(u^1,u^2), \\ \sigma(A/2-u^1, u^2) &= \sigma(A/2+u^1,u^2), \\ \sigma(u^1, B/2-u^2) &= \sigma(u^1,B/2+u^2), \end{aligned} \tag{5.6}$$

and so on.

The lines of curvature of the immersion x determined by (5.4) and (5.3) are lines parallel to the axes on $\mathbf{R}^2$. One can choose A and B for which the lines of curvature close up in the interval $[0, mA]$ and $[0, nB]$ with some integers m and n. Thus the immersion of the torus $\mathbf{R}^2/\Gamma$ is constructed, where Γ is the lattice generated by $(mA, 0)$ and $(0, nB)$.

Let $x : \mathbf{R}^2/\Gamma \to \mathbf{E}^3$ be an immersion of a torus constructed above with the Gauss map Ψ. Assume that $\tilde{x}$ is the $\overline{\Psi}$-immersion of x with the first fundamental form $\tilde{g} = e^{\tilde{\sigma}}|dz|^2$. Take a map $\tau : \mathbf{R}^2/\Gamma \to \mathbf{R}^2/\Gamma$ defined as

$$\tau(u^1, u^2) = (-u^1, u^2).$$

Since τ is an odd isometry of $\mathbf{R}^2/\Gamma$, the immersion $\hat{x} = \tilde{x} \circ \tau$ is congruent to x by Proposition 5.2. Hence, on the examples in [13] and explicit examples by Abresh [1], the immersion of a torus and the $\overline{\Psi}$-immersion of it are congruent with each other.

Later, Wente constructed immersions of tori of *skew* lattice with constant mean curvature [14]. Though his new examples have less symmetry than previous ones, they admits a parallel displacement τ as an odd isometry. Thus, such an immersion x is congruent to $\tilde{x} \circ \tau$. Then, the $\overline{\Psi}$-immersions of these examples also congruent to the original immersions.

Although these examples show that x and the $\overline{\Psi}$-immersion $\tilde{x}$ are congruent to each other, the authors wish to construct an immersion of tori which give an example that the $\overline{\Psi}$-immersion of it is really distinguished from the original one.

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Present Address:

Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki 305, JAPAN

Department of Mathematics
Faculty of Science and Technology
Keio University
Yokohama 223, JAPAN