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1. Introduction

The purpose of this paper is to study an asymptotic behavior for the gradient flow of the Yang-Mills functional near a stable or unstable Yang-Mills connection.

In differential geometry, many subjects are defined by variational problems on Riemannian manifolds. Most of them, however, do not satisfy the Palais-Smale condition. For the Palais-Smale condition, see R. Palais [**P**], R. Palais and S. Smale [**PS**], or J. Eells and J. H. Sampson [**ES2**]. If the variational problem defined by the functional $J(\cdot)$ on a function space X satisfies this condition, the equation of the gradient flow of J with initial value v:

$$\begin{cases} \dfrac{\partial u(t)}{\partial t} = -\operatorname{grad} J(u(t)) \\ u(0) = v \end{cases}$$

must have a unique time-global solution. If J does not satisfy this condition, we do not know, in general, whether the gradient flow exists globally in time or not.

There are some results on global existence of gradint flow. In 1964, J. Eells and J. H. Sampson proved the existence theorem of harmonic maps by means of the asymptotic behavior of the gradient flow when the target manifold had non-positive curvature [**ES1**].

In studying the existence of a time-global solution for the gradient flow, we need pay attention to the relation between the Morse theoretic stablity of a critical point and the asymptotic behavior of the solution of the gradient flow around the critical point. For harmonic maps, the third author studied the above relation in case of a stable harmonic map [**N1**]. Concerning more general variational problems, we can refer to L. Simon [**S**] and the third author [**N2**]. Recently, the third author proved the stable manifold theorem for quasi-linear parabolic equations, and showed, as an application, the asymptotic behavior of gradient flow even around an unsteble critical point assuming the ellipticity of the Euler-Lagrange operator [**N3**].

Since the Yang-Mills functional is invariant under the gauge transformation group,

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the equation governing the gradient flow of the Yang-Mills functional is not parabolic. To avoid this difficulty, the first and second authors consider a gauge condition and solve the gradient flow under the condition that the second variation is strictly positive [**MK**]. Similar idea is found in M. Yokotani [**Y**] and K. Kono and T. Nagasawa [**KN**]. These results, however, assume the Yang-Mills connection is strictly stable.

In this paper, we prove that the global existence of the grdient flow for the Yang-Mills functional with the initial value near a Yang-Mills connection at which the second variation is positive or not. Our basic set-up is the following. (This set-up is introduce by J. -P. Bourguignon and H. B. Lawson [**BL**].) Let (M,h) be a compact Riemannian manifold without boundary and P be a principal G-bundle over M where G is a compact Lie group.

We consider a G-vector bundle $E := P \underset{\rho}{\times} \mathbf{R}^N$, associated to P by a faithful orthogonal representation $\rho : G \to O(N)$. The group of all inner automorphisms is called the gauge group of P and will be denoted by $\mathcal{G}_P$. It can be easily identified with the group of smooth sections of the bundle of groups $G_P := P \underset{Ad}{\times} G$, i,e, $\mathcal{G}_P = \Gamma(M; G_P)$. Related to $\mathcal{G}_P$ is the infinitesimal gauge group (gauge algebra) which will be denoted $\mathbf{G}_P$. It is the Lie algebra of smooth sections of the bundle of Lie algebras $\mathbf{g}_P := P \underset{Ad}{\times} \mathbf{g}$, i,e, $\mathbf{G}_P := \Gamma(M; \mathbf{g}_P)$, where $\mathbf{g}$ expresses the Lie algebra of the Lie group G.

The gauge group can be easily re-expressed in terms of E. Let O_E be the orthogonal frame bundle of E over M, i,e, whose fiber at $x \in M$ is the group of orthogonal transformations in E_x. Let $\mathbf{so}_E$ be the bundle over M whose fiber at $x \in M$ is the Lie algebra of skew-symmetric transformations of E_x. Then the representation ρ gives embedding $G_P \hookrightarrow O_E$ and $\mathbf{g}_P \hookrightarrow \mathbf{so}_E$. We denote the images by G_E and $\mathbf{g}_E$ respectively. Clearly, $G_E \cong G_P$ and $\mathbf{g}_E \cong \mathbf{g}_P$.

We express the gauge group as the space $\mathcal{G}_E$ of smooth sections of G_E, i,e, $\mathcal{G}_E := \Gamma(M; G_E)$ and the gauge algebra as the space G_E of smooth sections of $\mathbf{g}_E$, i,e, $\mathbf{G}_E := \Gamma(M; \mathbf{g}_E)$.

Subsequently we introduce some notations. Given a smooth vector bundle F over M, let $\Omega^p(F) := \Gamma(\bigwedge^p T^*M \bigotimes F)$ be the space of exterior differential p-forms on M with values F. Note that $\Omega^0(F)$ is just the space of smooth sections of F and $\mathbf{G}_E = \Omega^0(\mathbf{g}_E)$. Now, we study the space $\mathcal{C}_P$ of connection on P, or equivalently $\mathcal{C}_E$ on E. (For the relation between a connection on P and a connection on E, we can refer to [**BL**].) It is easily shown that, for two connections ∇ and $\nabla' \in \mathcal{C}_E$, the difference $A = \nabla - \nabla'$ is an element of $\Omega^1(\mathbf{g}_E)$. In particular, if we fix $\nabla \in \mathcal{C}_E$ then there is a canonocal identification

$$T_\nabla(\mathcal{C}_E) \cong \Omega^1(\mathbf{g}_E). \tag{1.1}$$

To each connection $\nabla \in \mathcal{C}_E$, there is associated a curvature 2-form R^∇ in $\Omega^2(\mathbf{g}_E)$ given by

$$R^\nabla_{X,Y} := [\nabla_X, \nabla_Y] - R_{[X,Y]}$$

for tangent vectors X and Y.

The Yang-Mills functional $\mathcal{YM}$ on $\mathcal{C}_E$ is defined by

$$\mathcal{YM}(\nabla) = \frac{1}{2}\int_M \|R^\nabla\|^2, \tag{1.2}$$

where the norm is defined in terms of the Riemannian metric on M and a fixed Ad_G-invariant scalar product on Lie algebra $\mathbf{g}$ of G. Section 2 contains the precise definition of this norm.

Critical points of the smooth functional $\mathcal{YM} : \mathcal{C}_E \to \mathbf{R}$ are called Yang-Mills connections and their associated curvature tensors are called Yang-Mills fields. Clearly, a connection $\nabla \in \mathcal{C}_E$ is a Yang-Mills connection if and only if $\nabla \in \mathcal{C}_E$ satisfies the Euler-Lagrange equation of the Yang-Mills functional $\mathcal{YM}$. The Euler-Lagrange equation of $\mathcal{YM}$: $\mathrm{grad}(\mathcal{YM}(\nabla)) = 0$ is expressed by

$$\delta^\nabla R^\nabla = 0 \tag{1.3}$$

where δ^∇ is the formal adjoint operator of exterior derivative d^∇ associated with $\nabla \in \mathcal{C}_E$ with respect to the inner product induced by $\|\cdot\|$. In particular, we want to mention that the equation $\delta^\nabla R^\nabla = 0$ is the second ordered partial differential equation with respect to ∇.

We remark that the Yang-Mills functional is invariant under the gauge group $\mathcal{G}_E$ action on $\mathcal{C}_E$. Therefore if $\nabla \in \mathcal{C}_E$ is a Yang-Mills connection, then $\nabla^g := g \circ \nabla \circ g^{-1}$ is also a Yang-Mills connection, and in terms of the Euler-Lagrange equation, this fact implies $(\delta^\nabla R^\nabla)^g = \delta^{\nabla^g} R^{\nabla^g}$.

On the other hand, since $d^\nabla R^\nabla = 0$ (Bianchi identity), $\nabla \in \mathcal{C}_E$ is a Yang-Mills connection if and only if

$$\Delta^\nabla R^\nabla = 0 \tag{1.4}$$

where $\Delta^\nabla = d^\nabla \delta^\nabla + \delta^\nabla d^\nabla$ is the Hodge Laplacian on $\Omega^2(\mathbf{g}_E)$.

Concerning the above-mentioned, we want to construct a solution of the Yang-Mills gradient flow with initial value closed to a Yang-Mills connection. The Yang-Mills gradient flow with initial value ∇_1 on $\mathcal{C}_E$ is governed by the following equation:

$$\begin{cases} \dfrac{\partial \nabla}{\partial t} = -\delta^\nabla R^\nabla \\ \nabla(0) = \nabla_1. \end{cases} \tag{YMGF}$$

Unfortunately, the above (YMGF) is not a parabolic equation, since (YMGF) is invariant under the gauge group action on $\mathcal{C}_E$. In order to avoid this difficulty, we consider (YMGF) under a suitable gauge condition in [**MK**].

This idea is stated briefly as follows. Take a Yang-Mills connection $\nabla_0 \in \mathcal{C}_E$ and fix it. There is a natural splitting of the tangent space:

$$T_{\nabla_0}(\mathcal{C}_E) \cong \Omega^1(\mathbf{g}_E) = Z^1(\mathbf{g}_E) \bigoplus \Omega^1_*(\mathbf{g}_E), \tag{1.5}$$

where $Z^1(\mathbf{g}_E) := \{V = d^{\nabla_0}\phi;\ \phi \in \Omega^0(\mathbf{g}_E),\ \phi(x) = 0\} = \mathrm{Im}(d^{\nabla_0})$ and $\Omega^1_*(\mathbf{g}_E) := \{A \in \Omega^1_*(\mathbf{g}_E);\ \delta^{\nabla_0}A = 0\} = \ker(\delta^{\nabla_0})$, and there exists the exponential map

$$\exp : \mathbf{g}_E \to \mathcal{G}_E$$

induced form the exponential map of the Lie group: $\exp : \mathbf{g} \to G$. For smooth curves $d^{\nabla_0}S(t)$ and $A(t)$ in $Z^1(\mathbf{g}_E)$ and $\Omega^1_*(\mathbf{g}_E)$, respectively, we define the map $\sigma : Z^1(\mathbf{g}_E) \times \Omega^1_*(\mathbf{g}_E) \to \Omega^1(\mathbf{g}_E)$ by

$$\sigma(d^{\nabla_0}S(t), A(t)) := g(t) \circ (\nabla_0 + A(t)) \circ g(t)^{-1} - \nabla_0, \tag{1.6}$$

where $g(t) := \exp S(t)$.

Since $\sigma(0,0) = 0$ and the Fréchet derivative $D\sigma(0,0) = \mathit{Identity}$ on $\Omega^1(\mathbf{g}_E)$, σ is a local diffeomorphism near 0 in $\Omega^1(\mathbf{g}_E)$. Then, by (1.5), we may consider that σ gives a coordinate around ∇_0 in $\mathcal{C}_E$. Now, taking $\nabla(t) = \nabla_0 + \sigma(d^{\nabla_0}S(t), A(t))$ in (YMGF), we have the following equation (*) equivalent to (YMGF):

$$(*) \qquad \begin{cases} \dfrac{\partial A(t)}{\partial t} = -J^{\nabla_0}A(t) - Q(A(t)) - [A(t), S(t)] - d^{\nabla_0}S(t) \\ \delta^{\nabla_0}A(t) = 0 \\ A(0) = A_0. \end{cases}$$

For precise notation, see Section 2. Here we mention that the linear differential operator J^{∇_0} is the Jacobi operator (denoted in Section 2) of the Yang-Mills connection ∇_0.

For the evolution equation (*), our main result, briefly stated, is:

THEOREM. *Let $m > \frac{1}{2}\dim M + 2$. For the Yang-Mills connection ∇_0, there exists a finite codimensional stable manifold and a finite dimensional unstable manifold of (*) in $H^m(\Omega^1_*(\mathbf{g}_E))$: the m-th ordered Sobolev space on $\Omega^1_*(\mathbf{g}_E)$.*

Remark In [**MK**], they have shown an asymptotic stability of the Yang-Mills gradient flow near a strictly stable Yang-Mills connection. Our theorem asserts that we can get such behavior without assuming the stability of the critical point: the Yang-Mills connection.

In above theorem, a stable manifold is the submanifold in infinite dimensional manifold $H^m(\Omega^1_*(\mathbf{g}_E))$ containing zero such that the equation (*) with initial value in it has a unique time-global solution in some Banach space. Moreover the solution tends to the Yang-Mills connection in H^m-topology as $t \to \infty$. An unstable manifold implies a stable manifold with respect to the backward evolution equation.

COROLLARY. *Let $\nabla_0 \in \mathcal{C}_E$ be a Yang-Mills connection and $\nabla_1 = \nabla_0 + A_0$, where $A_0 \in \Omega^1(\mathbf{g}_E)$. If A_0 belongs to the stable manifold as in Theorem, there exists a unique solution $\nabla(t)$ of* (YMGF) *with initial value ∇_1. Moreover the solution tends to a Yang-Mills connection up to the gauge group action as $t \to \infty$.*

Here is an outline of the contains. In Section 2 we recall the set-up of the Yang-Mills theory and introduce the evolution equation (*). Section 3 discusses the evolution equation (*) and reduces it to an abstruct evolution equation. Section 4 is devoted to gives proofs of the existence of a solution for the abstruct evolution equation and the main result.

2. Preliminalies

As we mentioned in Section 1, we consider the Yang-Mills functional on $\mathcal{C}_E$.

Let (M, h) be a close Riemannian manifold and P be a principal G-bundle over M where G is a compact Lie group. Taking a faithful representation $\rho : G \to O(N)$, we can define a G-vector bundle $E := P \times_\rho \mathbf{R}^N$ associated to P. The Yang-Mills functional on the set of connections on E: $\mathcal{C}_E$ is

$$\mathcal{YM}(\nabla) = \frac{1}{2}\int_M \|R^\nabla\|^2. \tag{2.1}$$

The norm $\|\cdot\|$ is, in local coordinates, defined as follows. In a chart U of M, $\omega \in \Omega^k(\mathbf{g}_E)$ is expressed by

$$\omega = \omega_{j_1\cdots j_k}{}^b_a dx^{j_1} \wedge \cdots \wedge dx^{j_k} \bigotimes E^a_b \tag{2.2}$$

where $\{E^a_b\}$ is a basis of the fiber of $\mathbf{g}_E$ at $x \in M$ and $(x^1, \cdots, x^n)$ is a local coordinate on U $(n = \dim M)$.

In local coordinates neighborhooods, the norm of ω is defined by

$$\|\omega\|^2 := h^{j_1 i_1} \cdots h^{j_k i_k} \omega_{j_1\cdots j_k}{}^b_a \omega_{i_1\cdots i_k}{}^a_b. \tag{2.3}$$

That is to say, the fiber metric on E is defined by

$$\langle A, B\rangle := \frac{1}{2}\mathrm{trace}\,{}^t\!AB \tag{2.4}$$

for $A, B \in E_x$.

Now let us calculate the first and the second variation formulas. For a one-parameter family $\nabla_t \in \mathcal{C}_E$ of connections with ∇_0 at $t = 0$ and $A := \left.\frac{d}{dt}\right|_{t=0} \nabla_t$, the first variation formula is

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{YM}(\nabla_t) = \int_M \langle \delta^{\nabla_0} R^{\nabla_0}, A \rangle. \tag{2.5}$$

For a one-parameter family $\nabla_t \in \mathcal{C}_E$ of connections, with ∇_0 as a Yang-Mills connection and $A := \left.\frac{d}{dt}\right|_{t=0} \nabla_t$, the second variation formula is

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{YM}(\nabla_t) = \int_M \langle \delta^{\nabla_0} d^{\nabla_0} A + \left[R^{\nabla_0}, A\right], A \rangle. \tag{2.6}$$

In particular, if A satisfies $\delta^{\nabla_0} A = 0$ then

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{YM}(\nabla_0) = \int_M \langle \left(d^{\nabla_0}\delta^{\nabla_0} + \delta^{\nabla_0} d^{\nabla_0}\right) A + \left[R^{\nabla_0}, A\right], A \rangle. \tag{2.7}$$

Therefore, the Euler-Lagrange equation of $\mathcal{YM}$ is $\delta^{\nabla} R^{\nabla} = 0$. For $A \in \Omega^1(\mathbf{g}_E)$ and a Yang-Mills connection ∇_0, the operator $J^{\nabla_0} A := \left(d^{\nabla_0}\delta^{\nabla_0} + \delta^{\nabla_0} d^{\nabla_0}\right) A + \left[R^{\nabla_0}, A\right] = \Delta^{\nabla_0} A + \left[R^{\nabla_0}, A\right]$ is called *the Jacobi operator* of ∇_0.

Since there is a natural $\mathcal{G}_E$ action on $\mathcal{C}_E$, it is easily shown that $\mathcal{YM}(\nabla) = \mathcal{YM}(\nabla^g)$ and $\delta^{\nabla^g} R^{\nabla^g} = \left(\delta^{\nabla} R^{\nabla}\right)^g$, where $\nabla^g := g \circ \nabla \circ g^{-1}$.

Here we define the stability of a Yang-Mills connection. Since the restriction of the Jacobi operator J^{∇_0} to $\Omega^1_*(\mathbf{g}_E)$ has the discrete spectrum: $\{\lambda_1 \leq \lambda_2 \leq \cdots \nearrow +\infty\}$, we can define the *index*: $\mathrm{Index}(\nabla_0)$ of a Yang-Mills connection ∇_0 and the *nullity*: $\mathrm{Null}(\nabla_0)$ of ∇_0 as

$$\begin{aligned} \mathrm{Index}(\nabla_0) &:= \sharp\{\text{negative eigenvalues}\} \\ \mathrm{Null}(\nabla_0) &:= \sharp\{\text{zero eigenvalues}\}. \end{aligned} \tag{2.8}$$

A Yang-Mills connection ∇_0 is called *weakly stable* whenever $\mathrm{Index}(\nabla_0) = 0$. This definition is equivalent to

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{YM}(\nabla_t) \geq 0 \tag{2.9}$$

for any one-parameter family ∇_t of connections with ∇_0 as a Yang-Mills connection.

Furthermore, we define a strictly stable Yang-Mills connection. A Yang-Mills connection ∇_0 is called *strictly stable* whenever for any one-parameter family ∇_t of connections with ∇_0 as the Yang-Mills connection and $\left.\frac{d}{dt}\right|_{t=0} \nabla_t \in \Omega^1_*(\mathbf{g}_E)$,

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{YM}(\nabla_t) > 0. \tag{2.10}$$

(c.f. **[MK]**, **[BL]**.) In terms of the Jacobi operator, a Yang-Mills connection ∇_0 is strictly stable if and only if $\ker J^{\nabla_0} \subset Z^1(\mathbf{g}_E)$.

In this situation, let us avoid the difficulty that the Euler-Lagrange equation of $\mathcal{YM}$ is not elliptic.

In what follows, a Yang-Mills connection ∇_0 is fixed. As we mentioned in Section 1, there are a canonical identification of the tangent space $T_{\nabla_0}(\mathcal{C}_E)$ and $\Omega^1(\mathbf{g}_E)$, and the splitting of $\Omega^1(\mathbf{g}_E)$:

$$T_{\nabla_0} \cong \Omega^1(\mathbf{g}_E) = Z^1(\mathbf{g}_E) \bigoplus \Omega^1_*(\mathbf{g}_E). \tag{2.11}$$

Concerning (2.11), we define the map $\sigma : Z^1(\mathbf{g}_E) \times \Omega^1_*(\mathbf{g}_E) \to \Omega^1(\mathbf{g}_E)$ as follows:

$$\sigma(d^{\nabla_0}S, A) := g \circ (\nabla_0 + A) \circ g^{-1} - \nabla_0 \tag{2.12}$$

where $g := \exp S$ and $S \in \Omega^0(\mathbf{g}_E)$. It is easily shown that there exist three neighborhoods U_1, U_2 and U in $Z^1(\mathbf{g}_E)$, $\Omega^1_*(\mathbf{g}_E)$ and $\Omega^1(\mathbf{g}_E)$, respectively, such that the map σ induces a diffeomorphsim $U_1 \times U_2 \cong U$. In particular, taking $A = 0$ in (2.12), we get

$$\sigma(d^{\nabla_0}S, 0) = g \circ \nabla_0 \circ g^{-1} - \nabla_0.$$

Therefore the tangent space at ∇_0 of orbits of the action of the gauge group coincides with $Z^1(\mathbf{g}_E)$.

In the above formulation, the Yang-Mills gradient flow is rewritten to the evolution equation (*) (in Section 1).

For a smooth curve $\nabla(t)$ in $\mathcal{C}_E$ with $\nabla(0) = \nabla_0$: the Yang-Mills connection, we can find smooth curves $S(t)$ and $A(t)$ in $\Omega^0(\mathbf{g}_E)$ and $\Omega^1_*(\mathbf{g}_E)$, respectively, such that

$$\nabla(t) - \nabla(0) = \sigma(d^{\nabla_0}S(t), A(t)) \tag{2.13}$$

using diffeomorphsim $U_1 \times U_2 \overset{\sigma}{\cong} U$. These U, U_1 and U_2 give local coordinates neighborhoods around zero, respectively, therefore we call the system $(\sigma; U, U_1, U_2)$ an *addmisible coordinates system.* For the sake of simplicity, $\sigma(t)$ denotes $\sigma(d^{\nabla_0}S(t), A(t))$.

The differential of σ in t gives:

PROPOSITION 2.1 (**[MK]** (3.8)). *For the map σ defined by (2.12), we obtain*

$$\begin{aligned} \frac{d\sigma(t)}{dt} &= g(t) \circ \left(\frac{dA(t)}{dt} + [\nabla_0 + A(t), S(t)] \right) \circ g(t)^{-1} \\ &= g(t) \circ \left(\frac{dA(t)}{dt} + d^{\nabla_0}S(t) + [A(t), S(t)] \right) \circ g(t)^{-1}. \end{aligned} \tag{2.14}$$

On the other hand, the Euler-Lagrange operator at $\nabla(t) = \nabla_0 + \sigma(t)$ is $\delta^{\nabla_0+\sigma(t)} R^{\nabla_0+\sigma(t)}$. For $g \in \mathcal{G}_E$ and $\nabla \in \mathcal{C}_E$, the formulas

$$R^{\nabla^g} = g \circ R^{\nabla} \circ g^{-1}, \tag{2.15}$$

and

$$\delta^{\nabla^g} R^{\nabla^g} = \left(\delta^{\nabla} R^{\nabla}\right)^g = g \circ \delta^{\nabla} R^{\nabla} \circ g^{-1} \tag{2.16}$$

guarantee

$$\begin{aligned} &\delta^{\nabla_0+\sigma(t)} R^{\nabla_0+\sigma(t)} R^{\nabla_0+\sigma(t)} \\ =& g(t) \circ \left(\delta^{\nabla_0+A(t)} R^{\nabla_0+A(t)}\right) \circ g(t)^{-1} \\ =& g(t) \circ \left(\delta^{\nabla_0} R^{\nabla_0} + \delta^{\nabla_0} d^{\nabla_0} A(t) + \left[R^{\nabla_0}, A(t)\right] + Q(A(t))\right) \circ g(t)^{-1}, \end{aligned} \tag{2.17}$$

where

$$Q(A) = \delta^{\nabla_0}[A, A] + \left[d^{\nabla_0} A, A\right] + [[A, A], A]. \tag{2.18}$$

(c.f. [**MK**], Section 3.)

Therefore we obtain:

PROPOSITION 2.2. *Let ∇_0 be a Yang-Mills connection, and $(\sigma; U, U_1, U_2)$ be an addmisible coordinates system. If a smooth curve $\nabla(t)$ in U is the solution of*

$$\begin{cases} \dfrac{\partial \nabla}{\partial t} = -\delta^{\nabla} R^{\nabla} \\ \nabla(0) = \nabla_1 \end{cases}$$

then a pair $\{S(t), A(t)\}$ of smooth curve in $U_1 \times U_2$ satisfying $\nabla(t) - \nabla_0 = \sigma(d^{\nabla_0} S(t), A(t))$ is the solution of

$$\begin{cases} \dfrac{\partial A(t)}{\partial t} = -\left\{\left(d^{\nabla_0}\delta^{\nabla_0} + \delta^{\nabla_0} d^{\nabla_0}\right) A(t) + \left[R^{\nabla_0}, A(t)\right]\right. \\ \qquad \left. + Q(A(t)) + [A(t), S(t)] + d^{\nabla_0} S(t)\right\} \\ \delta^{\nabla_0} A(t) = 0 \\ A(0) = A_1, \end{cases} \tag{2.19}$$

where $A_1 = \nabla_1 - \nabla_0$ (note that $\sigma(0, A_1) = \nabla_1$).

The convese is also true.

The first equation of (2.19) is parabolic since $J^{\nabla_0} A = \left(d^{\nabla_0}\delta^{\nabla_0} + \delta^{\nabla_0} d^{\nabla_0}\right) A + \left[R^{\nabla_0}, A\right]$ is an elliptic operator on $\Omega^1_*(\mathbf{g}_E)$. These ideas are introduced by [**MK**]. In this section, we showed that the Yang-Mills gradient flow yields the evolution equation (2.19) by taking a gauge condition into account. Hence, our main purpose is to solve the evolution equation (2.19) when A_1 is small. If the Yang-Mills connection ∇_0 is strictly stable, J^{∇_0} is a strictly positive operator in $\Omega^1_*(\mathbf{g}_E)$. In this case, [**MK**] has shown an asymptotic stability of the solution of (2.19) using the method in Sections 2 and 3. We want to show such a result of (2.19), without assuming the (strictry) stability of ∇_0. In our case, J^{∇_0} restricted $\Omega^1_*(\mathbf{g}_E)$ may have finite dimensional non-positive eigenspaces. To solve (2.19) in such case, we need introduce a new evolution equation which is equivalent to (2.19), and some preliminaries of analysis: definitions of Banach spaces, some basic inequalities.

3. Reduction of (*) to a new evolution equation and some analysis.

In this section, the equation (*) will be reduced to a new evolution equation.

Let P be a projection from $\Omega^1(\mathbf{g}_E)$ to $\Omega^1_*(\mathbf{g}_E)$:

$$P : \Omega^1(\mathbf{g}_E) \to \Omega^1_*(\mathbf{g}_E). \tag{3.1}$$

The operator G^{∇_0} denotes the Green operator of the Hodge Laplacian $\Delta^{\nabla_0} = \delta^{\nabla_0} d^{\nabla_0}$ acting on $\Omega^0(\mathbf{g}_E)$. See [**MK**, Section 3.3].

Our main purpose of this section is to prove the following theorem.

Theorem 3.1 ([**MK**] 3.3). *Let be ∇_0 a Yang-Mills connection. A pair $\{S(t), A(t)\}$ of smooth curve in $\Omega^0(\mathbf{g}_E) \times \Omega^1_*(\mathbf{g}_E)$ is a solution of* (2.19) *with $A(0) = A_1$ if and only if $\{S(t), A(t)\}$ is the solution of*

$$\begin{cases} \dfrac{\partial A(t)}{\partial t} + PJ^{\nabla_0}A(t) + P\left(Q(A(t)) + [A(t), S(t)]\right) = 0 \\ S(t) = G^{\nabla_0}\delta^{\nabla_0}\left(Q(A(t)) + [A(t), S(t)]\right) + G^{\nabla_0}\mathcal{R}A(t) \\ A(0) = A^1, \end{cases} \tag{3.2}$$

where $\mathcal{R}A := \delta^{\nabla_0}\left[R^{\nabla_0}, A\right] + \left(\delta^{\nabla_0}\right)^2 d^{\nabla_0}A.$

(*Proof*) The evolution equation (2.19) consists of

$$\frac{\partial A}{\partial t} = -\left(\Delta^{\nabla_0}A + \left[R^{\nabla_0}, A\right] + Q(A) + [A, S] + d^{\nabla_0}S\right) \tag{3.3}$$

and

$$\delta^{\nabla_0}A = 0. \tag{3.4}$$

Applying the projection P to both sides of (3.3), we get

$$\frac{\partial A}{\partial t} = -P\left(\Delta^{\nabla_0} A + [R^{\nabla_0}, A]\right) - P\left(Q(A) + [A, S]\right), \tag{3.5}$$

since $Pd^{\nabla_0} S = 0$. Note that $PA = A$ since $\delta^{\nabla_0} A = 0$. Furthermore applying δ^{∇_0} to both sides of (3.3), we have

$$\left(\delta^{\nabla_0}\right)^2 d^{\nabla_0} A + \delta^{\nabla_0}\left([R^{\nabla_0}, A] + Q(A) + [A, S]\right) + \delta^{\nabla_0} d^{\nabla_0} S = 0. \tag{3.6}$$

Since $\delta^{\nabla_0} d^{\nabla_0} S = \Delta^{\nabla_0} S$, we can rewrite (3.6) by using the Green operator G^{∇_0} as

$$S = -G^{\nabla_0}\delta^{\nabla_0}\left([R^{\nabla_0}, A] + Q(A) + [A, S]\right) + \left(\delta^{\nabla_0}\right)^2 d^{\nabla_0} A. \tag{3.7}$$

Therefore the assertion of this theorem follows form (3.5) and (3.7). ∎

The following proposition shows that the first equation of (3.2) is a parabolic equation.

PROPOSITION 3.1. *The operator $L^{\nabla_0} = PJ^{\nabla_0}$ is elliptic as an operator on $\Omega^1_*(\mathbf{g}_E)$.*

(*Proof*) It is sufficient to show that

$$\|u\|_{H^{m+2}} \leq C(\|f\|_{H^m} + \|u\|_{H^m}) \tag{3.8}$$

for $L^{\nabla_0} u = f$ and $u \in \ker \delta^{\nabla_0}$.

We express the projection P as follows:

$$Pu = u - d^{\nabla_0} G^{\nabla_0} \delta^{\nabla_0} u, \tag{3.9}$$

where G^{∇_0} is the Green operator. Form (3.9), the operator $L^{\nabla_0} = PJ^{\nabla_0}$ is denoted by

$$L^{\nabla_0} u = J^{\nabla_0} u - d^{\nabla_0} G^{\nabla_0} \delta^{\nabla_0} J^{\nabla_0} u. \tag{3.10}$$

Here note that $d^{\nabla_0} G^{\nabla_0} \delta^{\nabla_0}$ is a bounded operator from $H^m(\mathbf{g}_E)$ to $H^m(\mathbf{g}_E)$ (by Ricci formula). Since J^{∇_0} is an elliptic operator, we get the desired result. ∎

As we mentioned in Section2 $-L^{\nabla_0}$ has discrete spectra: $\{\lambda_1 \geq \lambda_2 \geq \cdots \searrow -\infty\}$. Now we re-number these spectra, and denote by $\{\lambda_N \geq \lambda_{N-1} \geq \cdots \geq \lambda_1\}$: positive spectra and $\{\lambda_{-1} \geq \lambda_{-2} \geq \cdots \searrow -\infty\}$: negative spectra. Moreover, π_+, π_0 and π_- denote projection operators onto positive, zero and negative eigenspaces of $-L^{\nabla_0}$ in $L^2(\Omega^1_*(\mathbf{g}_E))$, respectively. The space $L^2(\Omega^1_*(\mathbf{g}_E))$ is naturally defined by the norm on $\Omega^1_*(\mathbf{g}_E)$. We remark that the operator L^{∇_0} is self adjoint on $L^2(\Omega^1_*(\mathbf{g}_E))$.

Now we want to define some function spaces on $\Omega^1_*(\mathbf{g}_E)$ and $\Omega^0(\mathbf{g}_E)$. For $m > \frac{1}{2}\dim M + 2$, we define m-th ordered Sobolev spaces on $\Omega^1_*(\mathbf{g}_E)$ and $\Omega^0(\mathbf{g}_E)$.

The Sobolev space $H^m(\Omega^0(\mathbf{g}_E))$ with the norm $\|\cdot\|_{H^m(\Omega^0(\mathbf{g}_E))}$ is defined by

$$\|S\|^2_{H^m(\Omega^0(\mathbf{g}_E))} := \sum_{k=0}^{m}\sum_{|I|=k}\int_M \left(D_{i_1}\cdots D_{i_k}S^b_a(x)D_{i_1}\cdots D_{i_k}S^a_b(x)\right),$$

where $I = (i_1, \cdots, i_k)$ is a multi-index and $D_{i_1}\cdots D_{i_k} := \dfrac{\partial^{|I|}}{\partial x_{i_1}\cdots \partial x_{i_k}}$.

The Sobolev space $H^m(\Omega^1_*(\mathbf{g}_E))$ with the norm $\|\cdot\|_{H^m(\Omega^1_*(\mathbf{g}_E)))}$ is defined by

$$\|A\|^2_{H^m(\Omega^1_*(\mathbf{g}_E))} := \|(L^{\nabla_0})^{m/2}\pi_- A\|^2_{L^2} + \|\pi_0 A\|^2_{L^2} + \|\pi_+ A\|^2_{L^2}.$$

Since L^{∇_0} can be considered as a positive definite self adjoint operator on $\pi_-(L^2(\Omega^1_*(\mathbf{g}_E)))$, the first term of the right hand side is well-defined.

Banach spaces $L^2(\mathbf{R}_+; H^m(\Omega^1_*(\mathbf{g}_E))$ and $L^2(\mathbf{R}_+; H^m(\Omega^0(\mathbf{g}_E))$ with norms $|||\cdot|||_{m,1}$ and $|||\cdot|||_{m,0}$, respectively, are defined by

$$|||A|||^2_{m,1} := \int_0^\infty \|A(t)\|^2_{H^m(\Omega^1_*(\mathbf{g}_E))}\,dt$$

and

$$|||S|||^2_{m,0} := \int_0^\infty \|S(t)\|^2_{H^m(\Omega^0(\mathbf{g}_E))}\,dt.$$

Furthermore for $\mu > 0$, the Banach space $\mathcal{B}_{m,\mu} \subset L^2(\mathbf{R}_+; H^{m+1}(\Omega^1_*(\mathbf{g}_E))) \cap L^\infty(\mathbf{R}_+; H^m(\Omega^1_*(\mathbf{g}_E)))$ with the norm $|\cdot|_{\mu,m}$ is defined by

$$|A|^2_{\mu,m} := |||A|||^2_{m+1,1} + \sup_{t>0}\left[e^{2\mu t}\|A(t)\|^2_{H^m(\Omega^1_*(\mathbf{g}_E))}\right].$$

For the sake of simplicity, we abbreviate $|||S|||_{m,0}$, $|||A|||_{m,1}$, $\|S\|_{H^m(\Omega^0(\mathbf{g}_E))}$ and $\|A\|_{H^m(\Omega^1_*(\mathbf{g}_E))}$ to $|S|_m$, $|||A|||_m$, $\|S\|_{H^m}$ and $\|A\|_{H^m}$ for $S \in \Omega^0(\mathbf{g}_E)$ and $A \in \Omega^1_*(\mathbf{g}_E)$, respectively.

For the proof of the existence of a solution of (3.2), we prepare some inequalities.

PROPOSITION 3.2. *Let $m > \frac{1}{2}\dim M + 2$. For A_1, $A_2 \in \Omega^1_*(\mathbf{g}_E)$ and S_1, $S_2 \in \Omega^0(\mathbf{g}_E)$ satisfying $|A_i|_{\mu,m} < 1$, and $|S_i|_m < 1$, $(i = 1,2)$, the non-linear term $N(A,S) := -P\left(Q(A) + [A,S]\right)$ of (3.2) is estimated by*

$$\begin{aligned}&\|N(A_1,S_1) - N(A_2,S_2)\|_{H^{m-1}}\\ &\le C\left(\|A_1 - A_2\|_{H^{m+1}}\|A_1\|_{H^m} + \|A_1 - A_2\|_{H^m}\|A_2\|_{H^{m+1}}\right.\\ &\quad\left.+\|A_1 - A_2\|_{H^m}\|S_1\|_{H^m} + \|A_2\|_{H^m}\|S_1 - S_2\|_{H^m}\right).\end{aligned} \tag{3.11}$$

with $C > 0$ *independent of* A_i *and* S_i $(i = 1, 2)$.

(*Proof*) In local coordinate neighborhoods, brackets $[A, A]$ and $[A, S]$ are expressed by

$$[A, A]_{ij}{}^{a}_{b} = A_{i}{}^{a}_{e} A_{j}{}^{e}_{b} - A_{j}{}^{a}_{e} A_{i}{}^{e}_{b}$$

and

$$[A, S]_{i}{}^{a}_{b} = A_{i}{}^{a}_{e} S^{e}_{b} - A_{i}{}^{e}_{b} S^{a}_{e}.$$

The $\mathbf{g}$-valued 1-form $Q(A) = \delta^{\nabla_0}[A, A] + [d^{\nabla_0} A, A] + [[A, A], A]$ is estimated by

$$\begin{aligned} &\|Q(A_1) - Q(A_2)\|_{H^{m-1}} \\ &\quad \leq C\left(\|A_1 - A_2\|_{H^{m+1}} \|A_1\|_{H^m} + \|A_1 - A_2\|_{H^m} \|A_2\|_{H^{m+1}}\right). \end{aligned} \tag{3.12}$$

Note that $\|A_i\|_{H^{m+1}} < 1$ $(i = 1, 2)$. Similarly, we have

$$\begin{aligned} &\|\,[A_1, S_1] - [A_2, S_2]\,\|_{H^{m-1}} \\ &\quad \leq C\left(\|A_1 - A_2\|_{H^{m-1}} \|S_1\|_{H^{m-1}} + \|A_2\|_{H^{m-1}} \|S_1 - S_2\|_{H^{m-1}}\right). \end{aligned} \tag{3.13}$$

Therefore (3.12) and (3.13) guarantee (3.11). ∎

Here note that assumptions $|A_i|_{\mu,m} < 1$ and $|S_i|_m < 1$ in Proposition 3.2 is used the proof in (3.12).

Since $(\delta^{\nabla_0})^2 d^{\nabla_0}$ is a differential operator of the first order by Ricci formula, we see $\mathcal{R}$ in (3.2) is the first ordered differential one. Therefore using a property of the Green operator, we obtain

$$\|G^{\nabla_0} \mathcal{R} A\|_{H^m} \leq C \|A\|_{H^{m-1}} \tag{3.14}$$

and

$$\|G^{\nabla_0} \delta^{\nabla_0} A\|_{H^m} \leq C \|A\|_{H^{m-1}}, \tag{3.15}$$

for all $A \in H^{m-1}(\Omega^1(\mathbf{g}_E))$ with $C > 0$ independent of A.
These estimates (3.11)–(3.15) play very important role in the proof of the main result.

The following lemma is a basic inequality for linear partial differential equations.

LEMMA 3.1. *For* $u \in L^2(\mathbf{R}_+; H^{m+1}(\Omega^1_*(\mathbf{g}_E)))$ *and* $v \in L^2(\mathbf{R}_+; H^{m-1}(\Omega^1_*(\mathbf{g}_E)))$, *we assume that*

$$\begin{cases} \dfrac{\partial u}{\partial t} = -L^{\nabla_0} u + \pi_- v \\ u(0) \in \mathrm{Im}(\pi_-). \end{cases}$$

Then we obtain

$$\int_0^\infty \|u(t)\|^2_{H^{m+1}}\,dt \le \|u(0)\|^2_{H^m} + \int_0^\infty \|v(t)\|^2_{H^{m-1}}\,dt \tag{3.16}$$

and

$$e^{2\mu t}\|u(t)\|^2_{H^{m+1}} \le \|u(0)\|^2_{H^m} + C\int_0^\infty \|v(t)\|^2_{H^{m-1}}\,dt, \quad \textit{for} \quad \textit{all} \quad t>0, \tag{3.17}$$

where $0<\mu<\min\{|\lambda_1|,|\lambda_{-1}|\}$.

For the proof see [**N3**].

In the next section, we will prove the main result and explain the meaning of stable manifolds and unstable manifolds clearly.

4. Proof of the main result.

In this section, we show our main result by solving (3.2), which is reduced to solve the following equation:

$$\begin{cases} \dfrac{\partial A(t)}{\partial t} = -PJ^{\nabla_0}A(t) + P\left(N(A(t),S(t)\right) \\ S(t) = G^{\nabla_0}\delta^{\nabla_0}\left(N(A(t),S(t))\right) + G^{\nabla_0}\mathcal{R}A(t) \\ A(0) = A^1. \end{cases} \tag{4.1}$$

For this purpose, we adopt the following iteration scheme:

$$\begin{cases} \dfrac{\partial A_n}{\partial t} = -L^{\nabla_0}A_n + PN(A_{n-1},S_{n-1}) \quad (n\ge 1) \\ S_n = G^{\nabla_0}\delta^{\nabla_0}\left(N(A_{n-1},S_{n-1})\right) + G^{\nabla_0}\mathcal{R}A_n \quad (n\ge 1) \\ A_0 = \pi_- A^1, \end{cases} \tag{4.2}$$

where A^1 is in $H^m(\Omega^1_*(\mathbf{g}_E))$ for $m>\frac{1}{2}\dim M+2$. For the sake of simplicity, we abbreviate $N(A(t),S(t))$ to $N(A,S)(t)$. This iteration scheme is re-expressed by the following integral equation system:

$$\begin{cases} A_n(t) = e^{-tL^{\nabla_0}}\pi_- A_0 + \displaystyle\int_0^t e^{-(t-s)L^{\nabla_0}}\pi_- PN(A_{n-1},S_{n-1})(s)\,ds \\ \qquad - \displaystyle\int_t^\infty \pi_0 PN(A_{n-1},S_{n-1})(s)\,ds \\ \qquad - \displaystyle\int_t^\infty e^{-(t-s)L^{\nabla_0}}\pi_+ PN(A_{n-1},S_{n-1})(s)\,ds \\ S_n(t) = G^{\nabla_0}\delta^{\nabla_0}N(A_{n-1},S_{n-1})(t) + G^{\nabla_0}\mathcal{R}A_n(t), \quad \text{for} \quad t>0. \end{cases} \tag{4.3}$$

If this iteration scheme converges, the initial value of a solution is expressed by

$$A(0) = \pi_- A_0 - \int_0^\infty \pi_0 PN(A,S)(s)\,ds - \int_0^\infty \pi_+ PN(A,S)(s)\,ds. \tag{4.4}$$

Therefore the initial value A^1 of (4.1) is expressed as (4.4). This implies that for a Yang-Mills connection ∇_0, if A^1 is expressed as (4.4) and satisfies a suitable condition (this condition will be mentioned in the proof of the result), then there exists a solution of (4.1) with the initial value A^1 which tends to zero as $t \to \infty$.

We assume $m > \frac{1}{2}\dim M + 2$. Then Sobolev spaces $H^m(\Omega^1_*(\mathbf{g}_E))$ and $H^m(\Omega^0(\mathbf{g}_E))$ is compactly embedded in $C^2(\Omega^1_*(\mathbf{g}_E))$ and $C^2(\Omega^0(\mathbf{g}_E))$, respectively. Moreover we choose a positive number μ satisfying $0 < \mu < \min\{|\lambda_1|, |\lambda_{-1}|\}$. We want to prove that the iteration scheme (4.3) converges in $\mathcal{B}_{\mu,m}$ and $L^2(\mathbf{R}_+; H^m(\Omega^0(\mathbf{g}_E)))$. Put $M_n := |A_n|^2_{\mu,m}$ and $K_n := |S_n|^2_m$.

Now we read:

THEOREM 4.1. *For the iteration scheme (4.3), there exist positive constants C_1, C_2 and C_3 depending only μ and m such that if $M_n < 1$ and $K_n < 1$ then*

$$\begin{cases} M_{n+1} \le \|\pi_- A_0\|^2_{H^m} + C_1\left(M_n^2 + M_n K_n\right) \\ K_{n+1} \le C_2\left(\|\pi_- A_0\|_{H^m} + M_n^2 + M_n K_n\right), \end{cases} \tag{4.5}$$

and

$$|A_{n+1} - e^{-tL^{\nabla_0}}\pi_- A_0|^2_{\mu,m} \le C_3\left(M_n^2 + M_n K_n\right). \tag{4.6}$$

(*Proof*) We will construct $\mathcal{B}_{\mu,m}$ and $L^2(\mathbf{R}_+; H^m(\Omega^0(\mathbf{g}_E)))$-estimate of (4.5-6) by separating (4.3) freely.

Step 1 The $H^m(\Omega^1_*(\mathbf{g}_E))$-estimate.

(i) To estimate π_--part of the first equation, we apply (3.17) in Lemma 3.1 to

$$f_-(t) := e^{-tL^{\nabla_0}}\pi_- A_0 + \int_0^t e^{-(t-s)L^{\nabla_0}}\pi_- PN(A_{n-1}, S_{n-1})(s)\,ds.$$

This function $f_-(t) = \pi_- A_n(t)$ satisfies

$$\begin{cases} \dfrac{\partial f_-}{\partial t} = -L^{\nabla_0} f_- + \pi_- PN(A_{n-1}, S_{n-1}) \\ f_-(0) = \pi_- A_0. \end{cases}$$

Lemma 3.1 and Proposition 3.2 imply that

$$e^{2\mu t}\|\pi_- A_n(t)\|^2_{H^m} \le \|\pi_- A_0\|^2_{H^m} + C\int_0^\infty \|\pi_- PN(A_{n-1}, S_{n-1})(s)\|^2_{H^{m-1}}\, ds$$

$$\le \|\pi_- A_0\|^2_{H^m} + C\int_0^\infty \left(\|A_{n-1}(s)\|^2_{H^{m+1}}\|A_{n-1}(s)\|^2_{H^m} + \|A_{n-1}(s)\|^2_{H^m}\|S_{n-1}(s)\|^2_{H^m}\right)\, ds$$

$$\le \|\pi_- A_0\|^2_{H^m} + C\left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|A_{n-1}(s)\|^2_{H^{m+1}}\, ds$$

$$+C\left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|S_{n-1}(s)\|^2_{H^m}\, ds.$$

By definitions of norms $|\cdot|_{\mu,m}$ and $|\cdot|_m$, we obtain

$$e^{2\mu t}\|\pi_- A_n(t)\|^2_{H^m} \le \|\pi_- A_0\|^2_{H^m} + C\left(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m\right). \tag{4.7}$$

(ii) We will estimate $\pi_+ A_n$ in H^m-norm. Put

$$f_+(t) := -\int_t^\infty e^{-(t-s)L^{\nabla_0}}\pi_+ PN(A_{n-1}, S_{n-1})(s)\, ds.$$

this function $f_+(t) = \pi_+ A_n(t)$ satisfies

$$\begin{cases} \dfrac{\partial f_+}{\partial t} = -L^{\nabla_0} f_+ + \pi_+ PN(A_{n-1}, S_{n-1}) \\ f_+(0) = 0. \end{cases}$$

Since $f_+(t) = \pi_+ A_n(t)$, Lemma 3.1 garantees as above the inequality:

$$e^{2\mu t}\|\pi_+ A_n(t)\|^2_{H^m} \le C\left(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m\right). \tag{4.8}$$

(iii) Concerning the estimate of π_0-part, one should note that all norms defined on $\mathrm{Im}(\pi_0)$ are equivalent, since $\dim(\mathrm{Im}(\pi_+)) < \infty$. Hence we obtain

$$e^{2\mu t}\|\pi_0 A_n(t)\|^2_{H^m} \le C\left(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m\right). \tag{4.9}$$

Combining (4.7)–(4.9), we conclude that

$$e^{2\mu t}\|A_n(t)\|^2_{H^m} \le \|\pi_- A_0\|^2_{H^m} + C\left(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m\right), \tag{4.10}$$

and

$$e^{2\mu t}\|A_n(t) - e^{-tL^{\nabla_0}}\pi_- A_0\|^2_{H^m} \le C\left(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m\right). \tag{4.11}$$

Step 2 The $L^2(\mathbf{R}_+; H^{m+1}(\Omega^1_*(\mathbf{g}_E)))$-estimate.

As is remarked in Step 1, since the dimension of $\mathrm{Im}(\pi_+ + \pi_0)$ is finite, there exists a constant C such that

$$\|\pi_+ A_n(t)\|^2_{H^{m+1}} + \|\pi_0 A_n(t)\|^2_{H^{m+1}} \le C\left(\|\pi_+ A_n(t)\|^2_{H^m} + \|\pi_0 A_n(t)\|^2_{H^m}\right). \tag{4.12}$$

Integrating (4.12) in t, and then applying (4.8-9), we get

$$|||\pi_+ A_n|||^2_{m+1} + |||\pi_0 A_n|||^2_{m+1} \le C\left(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m\right). \tag{4.13}$$

For $\pi_- A_n(t)$, we apply (3.16) in Lemma 3.1 and Proposition 3.2

$$f_-(t) := e^{-tL^{\nabla_0}}\pi_- A_0 + \int_0^t e^{-(t-s)L^{\nabla_0}}\pi_- PN(A_{n-1}, S_{n-1})(s)\,ds,$$

and get (note that $\pi_- A_n(t) = f_-(t)$)

$$\begin{aligned}
|||\pi_- A_n|||^2_{m+1} &\le \|\pi_- A_0\|^2_{H^m} + \int_0^\infty \|N(A_{n-1}, S_{n-1})(s)\|^2_{H^{m-1}}\,ds \\
&\le \|\pi_- A_0\|^2_{H^m} + \int_0^\infty \left(\|A_{n-1}(s)\|^2_{H^{m+1}}\|A_{n-1}(s)\|^2_{H^m} + \|A_{n-1}(s)\|^2_{H^m}\|S_{n-1}(s)\|^2_{H^m}\right) ds \\
&\le \|\pi_- A_0\|^2_{H^m} + C\left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|A_{n-1}(s)\|^2_{H^{m+1}}\,ds \\
&\qquad + C\left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|S_{n-1}(s)\|^2_{H^m}\,ds.
\end{aligned}$$

Hence it follows

$$|||\pi_- A_n|||^2_{m+1} \le \|\pi_- A_0\|^2_{H^m} + C\left(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m\right). \tag{4.14}$$

Combining (4.11) and (4.12), we obtain

$$|||A_n|||^2_{m+1} \le \|\pi_- A_0\|^2_{H^m} + C\left(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m\right), \tag{4.15}$$

and

$$|||A_n - e^{-tL^{\nabla_0}}\pi_- A_0|||^2_{m+1} \le C\left(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m\right). \tag{4.16}$$

Therefore (4.12-15) and (4.13-16) yield

$$|A_n|^2_{\mu,m} \le \|\pi_- A_0\|^2_{H^m} + C\left(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m\right). \tag{4.17}$$

and

$$|A_n - e^{-tL^{\nabla_0}}\pi_- A_0|^2_{\mu,m} \le C\left(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m\right), \tag{4.18}$$

respectively.

Step 3 The $L^2(\mathbf{R}_+; H^m(\Omega^0(\mathbf{g}_E)))$-estimate.

We estimate both sides of the second equality of (4.3) in $H^m(\Omega^0(\mathbf{g}_E))$-norm, we have

$$\|S_n(t)\|_{H^m} \leq \|G^{\nabla_0}\delta^{\nabla_0}\left(N(A_{n-1}, S_{n-1})(t)\right)\|_{H^m} + \|G^{\nabla_0}\mathcal{R}^{\nabla_0}A_n(t)\|_{H^m}.$$

Using (3.12-15), we get

$$\|S_n(t)\|^2_{H^m} \leq C\left(\|A_n(t)\|^2_{H^m} + \|A_{n-1}(t)\|^4_{H^m} + \|A_{n-1}(t)\|^2_{H^m}\|S_{n-1}(t)\|^2_{H^m}\right). \tag{4.19}$$

Integrating both sides of (4.19) in t, we see

$$\begin{aligned}
&\int_0^\infty \|S_n(t)\|^2_{H^m}\,dt \\
&\leq C\left[\int_0^\infty \|A_n(t)\|^2_{H^m}\,dt + \left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|A_{n-1}(s)\|^2_{H^m}\,ds\right. \\
&\qquad\left. + \left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|S_{n-1}(s)\|_{H^m}\,ds\right].
\end{aligned}$$

This yields

$$|S_n|^2_m \leq C\left(|A_n|^2_{\mu,m} + |A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m\right). \tag{4.20}$$

By (4.17) and (4.20), we obtain

$$|S_n|^2_m \leq C\left(\|\pi_- A_0\|^2_{H^m} + |A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m\right). \tag{4.21}$$

Then (4.5) follows from (4.17) and (4.21). Here one should note that the assumption $M_n < 1$ and $K_n < 1$ is needed for the estimate of non-linear terms. ∎

Theorem 4.1 leads apriori estimates.

COROLLARY 4.1. *Let $L_n := \max\{M_n, K_n\}$, $(n = 0, 1, \cdots)$. There exists $\epsilon_0 > 0$ and a monotone decreasing function $L(\epsilon)$ of $0 < \epsilon \leq \epsilon_0$ such that under the condition $L_0 < \epsilon_0$ we have*

$$L_n \leq L(\epsilon_0) \quad \textit{for all} \quad n \geq 0. \tag{4.22}$$

Moreover $L(\epsilon)$ satisfies

$$\lim_{\epsilon \downarrow 0} L(\epsilon) = 0. \tag{4.23}$$

(*Proof*) Theorem 4.1 yields

$$M_{n+1} \le \|\pi_- A_0\|^2_{H^m} + CL_n^2$$
$$K_{n+1} \le C\left(\|\pi_- A_0\|^2_{H^m} + L_n^2\right).$$

Taking $M_0 = \|\pi_- A_0\|^2_{H^m}$ and $K_0 = 0$, we have

$$L_{n+1} \le C\left(L_0 + L_n^2\right). \tag{4.24}$$

The elementary calculus yields the assertion of this corollary. ∎

COROLLARY 4.2. *There is $\epsilon > 0$ such that if $\|\pi_- A_0\|^2_{H^m} \le \epsilon$, $|A_n|_{\mu,m} < \epsilon$ and $|S_n|^2_m < \epsilon$ then A_{n+1} is contained in ϵ-ball in $\mathcal{B}_{\mu,m}$ whose center is $e^{-tL^{\nabla_0}}A_0$.*

Corollaries 4.1 and 4.2 are regered as apriori estimates.

The final step of the proof of the main result is to show the convergence of the iteration scheme (4.3). For this purpose, we may prove that sequence $\{A_n\}$ and $\{S_n\}$ are Cauchy sequence in $\mathcal{B}_{\mu,m}$ and $L^2(\mathbf{R}_+; H^m(\Omega^0(\mathbf{g}_E))$, respectively.

THEOREM 4.2. *For the iteration scheme* (4.3)*, the following two inequalities hold:*

$$\begin{aligned}|A_{n-1} - A_n|^2_{\mu,m} \le & C\left(|A_n|^2_{\mu,m} + |A_{n-1}|^2_{\mu,m} + |S_n|^2_m + |S_{n-1}|^2_m\right)\\ & \times\left(|A_n - A_{n-1}|^2_{\mu,m} + |S_n - S_{n-1}|^2_m\right);\end{aligned} \tag{4.25}$$

$$\begin{aligned}|S_{n-1} - S_n|^2_m \le & C\left(|A_n|^2_{\mu,m} + |A_{n-1}|^2_{\mu,m} + |S_n|^2_m + |S_{n-1}|^2_m\right)\\ & \times\left(|A_n - A_{n-1}|^2_{\mu,m} + |S_n - S_{n-1}|^2_m\right).\end{aligned} \tag{4.26}$$

(*Proof*) The proof of this theorem is essentially the same as that of Theorem 4.1.

First we calculate successive differences:

$$\begin{aligned}&A_{n+1}(t) - A_n(t)\\ &\quad = \int_0^t e^{-(t-s)L^{\nabla_0}}\pi_- P\left[N(A_n, S_n)(s) - N(A_{n-1}, S_{n-1})(s)\right] ds\\ &\quad - \int_t^\infty \pi_0 P\left[N(A_n, S_n)(s) - N(A_{n-1}, S_{n-1})(s)\right] ds\\ &\quad - \int_t^\infty e^{-(t-s)L^{\nabla_0}}\pi_+ P\left[N(A_n, S_n)(s) - N(A_{n-1}, S_{n-1})(s)\right] ds\end{aligned} \tag{4.27}$$

and

$$S_{n+1}(t) - S_n(t) = -G^{\nabla_0}\delta^{\nabla_0}(N(A_n, S_n)(t) - N(A_{n-1}, S_{n-1})(t)) - G^{\nabla_0}\mathcal{R}(A_{n+1}(t) - A_n(t)). \tag{4.28}$$

As in the proof of Theorem 4.1, we will take three steps to show our assertion.

Step 1 The $H^m(\Omega^1_*(\mathbf{g}_E))$-estimate.

(i) Taking

$$f_-(t) := \int_0^t e^{-(t-s)L^{\nabla_0}}\pi_- P(N(A_n, S_n)(s) - N(A_{n-1}, S_{n-1})(s))\, ds, \tag{4.29}$$

we have $f_-(t) = \pi_-(A_n(t) - A_{n-1}(t))$ and

$$\begin{cases} \dfrac{\partial f_-}{\partial t} = -L^{\nabla_0} f_- + \pi_- P(N(A_n, S_n) - N(A_{n-1}, S_{n-1})) \\ f_-(0) = 0. \end{cases}$$

It follows from Lemma 3.1 and Proposition 3.2 that

$$\begin{aligned} e^{2\mu t}\|f_-(t)\|^2_{H^m} \le & C\int_0^t \|N(A_n, S_n)(s) - N(A_{n-1}, S_{n-1})(s)\|^2_{H^{m-1}}\, ds \\ \le & C\int_0^t (\|A_n(s) - A_{n-1}(s)\|^2_{H^{m+1}}\|A_{n-1}(s)\|^2_{H^m} \\ & + \|A_n(s)\|^2_{H^{m+1}}\|A_n(s) - A_{n-1}(s)\|^2_{H^m} \\ & + \|A_n(s) - A_{n-1}(s)\|^2_{H^m}\|S_{n-1}(s)\|^2_{H^m} \\ & + \|A_{n-1}(s)\|^2_{H^m}\|S_n(s) - S_{n-1}(s)\|^2_{H^m})\, ds \\ \le & C\left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|A_n(s) - A_{n-1}(s)\|^2_{H^{m+1}}\, ds \\ & + C\left[\sup_{t>0} e^{2\mu t}\|A_n(t) - A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|A_{n-1}(s)\|^2_{H^{m+1}}\, ds \\ & + C\left[\sup_{t>0} e^{2\mu t}\|A_n(t) - A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|S_{n-1}(s)\|^2_{H^m}\, ds \\ & + C\left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|S_n(s) - S_{n-1}(s)\|^2_{H^m}\, ds. \end{aligned}$$

The above inequality implies that

$$\begin{aligned} & e^{2\mu t}\|\pi_-(A_{n+1} - A_n)\|^2_{H^m} \\ & \le C\left[\left(|A_n|^2_{\mu,m} + |A_{n-1}|^2_{\mu,m}\right)|A_n - A_{n-1}|^2_{\mu,m} \right. \\ & \quad \left. + |A_n - A_{n-1}|^2_{\mu,m}|S_n|^2_m + |A_{n-1}|^2_{\mu,m}|S_n - S_{n-1}|^2_m\right] \\ & \le C\left[\left(|A_n|^2_{\mu,m} + |A_{n-1}|^2_{\mu,m} + |S_n|^2_m + |S_{n-1}|^2_m\right)\right. \\ & \quad \left.\left(|A_n - A_{n-1}|^2_{\mu,m} + |S_n - S_{n-1}|^2_m\right)\right]. \end{aligned} \tag{4.30}$$

(ii) In the similar manner of the proof of Lemma 3.1 and the above (i), we can estimate π_0 and π_+-parts of $A_{n+1}(t) - A_n(t)$, as follows:

$$\begin{aligned} e^{2\mu t}\|\pi_+(A_{n+1}(t)-A_n(t))\|^2_{H^m} & \\ \le C\,\big[\big(|A_n|^2_{\mu,m} + |A_{n-1}|^2_{\mu,m} + |S_n|^2_m + |S_{n-1}|^2_m\big) & \\ \times\,\big(|A_n - A_{n-1}|^2_{\mu,m} + |S_n - S_{n-1}|^2_m\big)\big]\,, & \end{aligned} \tag{4.31}$$

and

$$\begin{aligned} e^{2\mu t}\|\pi_0(A_{n+1}(t)-A_n(t))\|^2_{H^m} & \\ \le C\,\big[\big(|A_n|^2_{\mu,m} + |A_{n-1}|^2_{\mu,m} + |S_n|^2_m + |S_{n-1}|^2_m\big) & \\ \times\,\big(|A_n - A_{n-1}|^2_{\mu,m} + |S_n - S_{n-1}|^2_m\big)\big]\,. & \end{aligned} \tag{4.32}$$

Consequently, as for the $H^m(\Omega^1_*(\mathbf{g}_E))$-norm of the successive difference, we see that

$$\begin{aligned} e^{2\mu t}\|A_{n+1}(t)-A_n(t)\|^2_{H^m} & \\ \le C\,\big[\big(|A_n|^2_{\mu,m} + |A_{n-1}|^2_{\mu,m} + |S_n|^2_m + |S_{n-1}|^2_m\big) & \\ \times\,\big(|A_n - A_{n-1}|^2_{\mu,m} + |S_n - S_{n-1}|^2_m\big)\big]\,. & \end{aligned} \tag{4.33}$$

Step 2 The $L^2(\mathbf{R}_+; H^{m+1}(\Omega^1_*(\mathbf{g}_E)))$-estimate.

As we remarked in the proof of Theorem 4.1, $\mathrm{Im}(\pi_+ + \pi_0)$ is of finite dimension. In the similar manner of (4.31) and (4.32), we have

$$\begin{aligned} |||\pi_+(A_{n+1}-A_n)|||^2_{m+1} + |||\pi_0(A_{n+1} - A_n)|||^2_{m+1} & \\ \le C\,\big[\big(|A_n|^2_{\mu,m} + |A_{n-1}|^2_{\mu,m} + |S_n|^2_m + |S_{n-1}|^2_m\big) & \\ \times\,\big(|A_n - A_{n-1}|^2_{\mu,m} + |S_n - S_{n-1}|^2_m\big)\big]\,. & \end{aligned} \tag{4.34}$$

For the estimate of the π_--part of the successive difference, we apply (3.16) of Lemma 3.1 to (4.29). We thus obtain

$$\begin{aligned} |||f(t)|||^2_{m+1} \le & \int_0^t \|N(A_n, S_n)(s) - N(A_{n-1}, S_{n-1})(s)\|^2_{H^{m-1}}\, ds \\ \le & C\,\Bigg[\Big[\sup_{t>0} e^{2\mu t}\|A_n(t)\|^2_{H^m}\Big]\int_0^t e^{-2\mu s}\|A_n(s) - A_{n-1}(s)\|^2_{H^{m+1}}\, ds \\ & + \Big[\sup_{t>0} e^{2\mu t}\|A_n(t) - A_{n-1}(t)\|^2_{H^m}\Big]\int_0^t e^{-2\mu s}\|A_{n-1}(s)\|^2_{H^{m+1}}\, ds \\ & + \Big[\sup_{t>0} e^{2\mu t}\|A_n(t) - A_{n-1}(t)\|^2_{H^m}\Big]\int_0^t e^{-2\mu s}\|S_n(s)\|^2_{H^m}\, ds \\ & + \Big[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\Big]\int_0^t e^{-2\mu s}\|S_n(s) - S_{n-1}(s)\|^2_{H^m}\, ds\Bigg]\,. \end{aligned}$$

Hence we have

$$
(4.35)\qquad \begin{aligned}&|||\pi_-(A_{n+1}-A_n)|||^2_{m+1}\\ &\quad\le C\left[\left(|A_n|^2_{\mu,m}+|A_{n-1}|^2_{\mu,m}+|S_n|^2_m+|S_{n-1}|^2_m\right)\right.\\ &\qquad\left.\times\left(|A_n-A_{n-1}|^2_{\mu,m}+|S_n-S_{n-1}|^2_m\right)\right].\end{aligned}
$$

Therefore we get by (4.34-35)

$$
(4.36)\qquad \begin{aligned}&|||A_{n+1}-A_n|||^2_{m+1}\\ &\quad\le C\left[\left(|A_n|^2_{\mu,m}+|A_{n-1}|^2_{\mu,m}+|S_n|^2_m+|S_{n-1}|^2_m\right)\right.\\ &\qquad\left.\times\left(|A_n-A_{n-1}|^2_{\mu,m}+|S_n-S_{n-1}|^2_m\right)\right].\end{aligned}
$$

Consequently, it follows from (4.33) and (4.36) that

$$
(4.37)\qquad \begin{aligned}&|A_{n+1}-A_n|^2_{\mu,m}\\ &\quad\le C\left[\left(|A_n|^2_{\mu,m}+|A_{n-1}|^2_{\mu,m}+|S_n|^2_m+|S_{n-1}|^2_m\right)\right.\\ &\qquad\left.\times\left(|A_n-A_{n-1}|^2_{\mu,m}+|S_n-S_{n-1}|^2_m\right)\right].\end{aligned}
$$

Step 3 The $L^2(\mathbf{R}_+;H^m(\Omega^0(\mathbf{g}_E)))$-estimate.

We estimate (4.28) by $H^m(\Omega^0(\mathbf{g}_E))$-norm. We obtain

$$
(4.38)\qquad \begin{aligned}&\|S_n(t)-S_{n-1}(t)\|^2_{H^m}\\ &\quad\le\|G^{\nabla_0}\delta^{\nabla_0}\left(N(A_n,S_n)(t)-N(A_{n-1},S_{n-1}(t))\right)\|^2_{H^m}\\ &\qquad+\|G^{\nabla_0}\mathcal{R}\left(A_{n+1}(t)-A_n(t)\right)\|^2_{H^m}.\end{aligned}
$$

By using estimates (3.12), (3.13) and (3.15), the first term of the right hand side of (4.38) is estimated as

$$
(4.39)\qquad \begin{aligned}&\|G^{\nabla_0}\delta^{\nabla_0}(N(A_n,S_n)(t)-N(A_{n-1},S_{n-1})(t))\|^2_{H^m}\\ &\le C\left(\|A_n(t)-A_{n-1}(t)\|^2_{H^{m+1}}\|A_n(t)\|^2_{H^m}\right.\\ &\quad+\|A_{n-1}(t)\|^2_{H^{m+1}}\|A_n(t)-A_{n-1}(t)\|^2_{H^m}\\ &\quad+\|A_n(t)-A_{n-1}(t)\|^2_{H^m}\|S_n(t)\|^2_{H^m}\\ &\quad\left.+\|A_{n-1}(t)\|^2_{H^m}\|S_n(t)-S_{n-1}(t)\|^2_{H^m}\right).\end{aligned}
$$

By (3.14) we have

$$
(4.40)\qquad \|G^{\nabla_0}\mathcal{R}\left(A_{n+1}(t)-A_n(t)\right)\|^2_{H^m}\le C\|A_{n+1}(t)-A_n(t)\|^2_{H^{m-1}}.
$$

Therefore, we can estimate $|S_{n+1}-S_n|^2_m$ as follows:

$$
(4.41)\qquad \begin{aligned}|S_{n+1}-S_n|^2_m\le& C\left(|A_n|^2_{\mu,m}+|A_{n-1}|^2_{\mu,m}+|S_n|^2_m+|S_{n-1}|^2_m\right)\\ &\times\left(|A_{n+1}-A_n|^2_{\mu,m}+|A_n-A_{n-1}|^2_{\mu,m}+|S_n-S_{n-1}|^2_m\right).\end{aligned}
$$

Combining (4.37) and (4.41), we therefore obtain

$$\begin{aligned} |S_{n+1} - S_n|_m^2 \leq & C\left(|A_n|_{\mu,m}^2 + |A_{n-1}|_{\mu,m}^2 + |S_n|_m^2 + |S_{n-1}|_m^2\right) \\ & \times \left(|A_n - A_{n-1}|_{\mu,m}^2 + |S_n - S_{n-1}|_m^2\right). \blacksquare \end{aligned} \tag{4.42}$$

By virtue of Theorem 4.2, we can conclude the existence of a unique solution to (4.1), therefore (YMGF), with initial value deteminated by (4.4).

COROLLARY 4.3. *For $m > \frac{1}{2}\dim M + 2$ and μ satisfying $0 < \mu < \min\{|\lambda_1|, |\lambda_{-1}|\}$, there exists an $\epsilon > 0$ such that for every $A^1 \in \mathrm{Im}(\pi_-)$ with*

$$|e^{-tL^{\nabla_0}} A^1|_{\mu,m} < \epsilon$$

then there is a unique solution $\{S, A\}$ of

$$\begin{cases} \dfrac{\partial A(t)}{\partial t} = -PJ^{\nabla_0}A(t) + PN(A(t), S(t)) \\ S(t) = G^{\nabla_0}\delta^{\nabla_0}(N(A(t), S(t))) + G^{\nabla_0}\mathcal{R}A(t) \\ \pi_- A(0) = A^1 \end{cases}$$

in $A \in \mathcal{B}_{\mu,m}$ and $S \in L^2(\mathbf{R}_+; H^m(\Omega^0(\mathbf{g}_E)))$ with $|A|_{\mu,m} < \epsilon$. Moreover such solution A exponentially tends to zero in H^m-norm as $t \to \infty$.

This corollary is leaded by Corollaries 4.1 and 4.2 and Theorem 4.2.

Therefore the solution of (4.1) is expressed by

$$\begin{aligned} A(t) = & e^{-tL^{\nabla_0}}\pi_- A^1 + \int_0^t e^{-(t-s)L^{\nabla_0}}\pi_- N(A(s), S(s))\, ds \\ & - \int_t^\infty \pi_0 N(A(s), S(s))\, ds - \int_t^\infty e^{-(t-s)L^{\nabla_0}}\pi_+ N(A(s), S(s))\, ds, \end{aligned} \tag{4.43}$$

and

$$S(t) = G^{\nabla_0}\delta^{\nabla_0}(Q(A(t)) + [A(t), S(t)]) + G^{\nabla_0}\mathcal{R}A(t). \tag{4.44}$$

This expression (4.43)–(4.44) of a solution implies that one can show that the solution $A(t)$ and $S(t)$ depend smoothly on the initial value A^1 in the $H^m(\Omega^1_*(\mathbf{g}_E))$ and $H^m(\Omega^0(\mathbf{g}_E)))$-topology, respectively. The stable manifold of the Yang-Mills connection ∇_0 is the set of all initial values A^1 satisfying the condition in Corollary 4.3. That is, the stable manifold of ∇_0 is the set of the initial values of the Yang-Mills gradient flow with which the solution is tends to ∇_0 as $t \to \infty$. The stable manifold is clearly a submanifold of $H^m(\Omega^1_*(\mathbf{g}_E))$ with codimension: $\dim(\mathrm{Im}(\pi_+ + \pi_0))$.

To obtain the unstable manifold, we use the iteration scheme:

$$
(4.45)\quad \begin{cases}
A_n(t) = e^{-tL^{\nabla_0}}\pi_+ A^0 + \displaystyle\int_0^t e^{-(t-s)L^{\nabla_0}}\pi_+ PN(A_{n-1}(s), S_{n-1}(s))\, ds \\
\qquad - \displaystyle\int_t^{-\infty} \pi_0 PN(A_{n-1}(s), S_{n-1}(s))\, ds \\
\qquad - \displaystyle\int_t^{-\infty} e^{-(t-s)L^{\nabla_0}}\pi_- PN(A_{n-1}(s), S_{n-1}(s))\, ds \\
S_n(t) = G^{\nabla_0}\delta^{\nabla_0}(Q(A_{n-1}(t)) + [A_{n-1}(t), S_{n-1}(t)]) + G^{\nabla_0}\mathcal{R}A_n(t), \quad \text{for} \quad t < 0.
\end{cases}
$$

As similar to Corollary 4.3, one can show the existence of a set of the initial values with which solutions are asymptotically stable for the backward evolution equation. This set is clearly a submanifold of $H^m(\Omega^1_*(\mathbf{g}_E))$ and has dimension: $\dim(\mathrm{Im}(\pi_+))$.

On the existence theorem of the stable manifold, readers can refer to D. Henry [**H**] or C. L. Epstein and M. I. Weinstein [**EW**]. The method of the proof is also used in [**N3**].

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