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The TAO of TEX
Notes for Formulas
Part I

by

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The TAO of TEX

Notes for Formulas

Part I

Version 0

WITH NUMEROUS EXPLICIT ILLUSTRATIONS

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To TEXnician

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序文

"AMS-TeX" の入門書である "The TAO of TeX(version 0), Gourmet Guide ～" を発行してほぼ１ヶ月が過ぎ去り，早くもその続編"Notes for Formulas(Part I)"が完成することになった。今回のレポートには，特に関数近似に関連する数式の例題が豊富に含まれている。さらに，"AMS-TeX" を使って数式を記述する時に，多くの人が疑問を持つと思う" 数式の縦ぞろえ" に関して，詳細な説明を加えた。さらに， "参考文献の記述形式" に関しても，実例を中心に述べたつもりである。しかし，このレポートはジックリ読むようなものでなく，パラパラっと食後に寝っ転がって見る程度のものなのであしからず。

まだ，実際に，"AMS-TeX" を使っていない人は，早く使って見て欲しい。何故なら "TeX" の本当の良し悪しは，それを本当に使用してみて，初めて分かるからである。とにかく，"TeX" の好き嫌いを言う前に一度試してみてほしい。嫌いになるは，それからでも遅くない。

最後に，今回もいろいろなコメントや御意見をいただいた慶應義塾大学理工学部数理科学科の"AMS-TeX" 愛好者の皆さんどうも有り難う。そろそろ "ドラクエⅢ" に飽きた皆さん，次は，"TeX" が君を待っている。ただ，その前に， "AMS-TeX"で記述する内容を考え出さねばならないのだが。

著者

3.12.1988

目次

○ 数式の縦ぞろえ

一般形は次の通り。

```
\コマンド
    数　　式      \\
    数　　式      \\
    ・・・・・     ・・
    数　　式
\end コマンド
```

［記号の説明］

数式の種類には，下記の２つがある。

- 数　　式 ：数式のみ
- 数 ＆ 式 ：数式の中に＆を含む。＆は，数式の中で縦ぞろえしたい位置に入れる。

\コマンド ～ \end コマンド：数式の縦ぞろえに関するコマンドには，次のページの(1)～(8)がある。

● $$・・・$$の中でページがえを制御するコマンドとして次のものがある。

（イ）数式の途中の\\でページがえしてもよい場合には，\\の後に
　　\allowdisplaybreak を入れる。

（ロ）数式の途中の\\でページがえさせたい場合には，\\の後に，
　　\displaybreak　を入れる。

（ハ）数式の途中の何処の\\でページがえしてもよい場合には，
　　\コマンドの前に\allowdisplaybreaksを入れる。

［注］(1) \mathbreak,\nomathbreak,\allowmathbreakは，$$・・・・・$$の中では使用できない。これらのコマンドは，$・・・$の中でのみ使用可能。

● \コマンド ・・・ \end コマンド には次のものがある。

(1) \align ... \endalign
各数式に数式番号をつけることができる。

(2) \split.....\endsplit
$$ ....$$全体を１つにまとめて番号をつけたいときにこれを使う。

(3) \multiline ..... \endmultiline
縦ぞろえ不要，最上段の式は左ずめ，最下段の式は右ずめ，真ん中に位置する式は，センタリングされる。

(4) \gather .....\endgather
縦ぞろえ不要で，各々の式のセンタリングのみを行う。個々に $$ で囲むよりも行間がつまる。

(5) \aligned .....\endaligned
２列以上の同時縦ぞろえ。 \topaligned, \botaligned

(6) \gathered....\endgathered
\gatherの\aligned版であり，各数式はセンタリングされる。

(7) \alignat ......\endalignat
何列にもまたがる縦ぞろえに使用する。各列の幅を調節するコマンドとして
\xalignat, \xxalignat
がある。

(8) \cases ....\endcases
数式の場合分けに用いる。

［注］(1) 数式と数式の間のスペースをとるのに
\quad
及び
\qquad (=\quad\quad)
がある。

(1) $|\psi_{x_0}(f)| = |f(x_0)| < \epsilon \quad f \in U_{\Lambda,\epsilon}$

(2) $|\psi_{x_0}(f)| = |f(x_0)| < \epsilon \qquad f \in U_{\Lambda,\epsilon}$

```
$$|\psi_{x_0}(f)|=|f(x_0)|
  < \epsilon \quad f
     \in U_{\Lambda,\epsilon}\tag1$$

$$|\psi_{x_0}(f)|=|f(x_0)|
  < \epsilon \qquad f
     \in U_{\Lambda,\epsilon}\tag2$$
```

(1) \align ..... \endalign

$$
\begin{aligned}
\frac{1+t}{1+z} &= \frac{1+1/z-R/w}{1+z} = \frac{2}{1+w+R} \qquad (1.1)\\
\frac{1-t}{1-z} &= \frac{1-1/z+R/w}{1-z} = \frac{2}{1-w+R} \qquad (1.2)
\end{aligned}
$$

```
%------------------------
% An example for AMS-TeX
%------------------------
$$
\align
\frac{1+t}{1+z} & =
   \frac{1+1/z-R/w}{1+z} =
   \frac2{1+w+R}    \tag1.1 \\
\frac{1-t}{1-z} & =
   \frac{1-1/z+R/w}{1-z} =
   \frac2{1-w+R}    \tag1.2
\endalign
$$
```

[一般形]

```
\align
    数 & 式      \tag  数式番号  \\
    数 & 式      \tag  数式番号  \\
    .  .  .  .  .  .  .  .  .  .  .
    数 & 式      \tag  数式番号
\endalign
```

[注] (1) & 記号は，各数式を1列に合わせる所に入れる。
(2) 数式番号は，各数式に付けることができる。

(2) \split ......\endsplit

(1)

$$
\begin{aligned}
\frac{1+t}{1+z} &= \frac{1+1/z-R/w}{1+z} = \frac{2}{1+w+R} \\
\frac{1-t}{1-z} &= \frac{1-1/z+R/w}{1-z} = \frac{2}{1-w+R}
\end{aligned}
$$

```
% An example for AMS-TeX
%-----------------------
$$
\split
\frac{1+t}{1+z} & =
   \frac{1+1/z-R/w}{1+z} =
   \frac2{1+w+R}     \\
\frac{1-t}{1-z} & =
   \frac{1-1/z+R/w}{1-z} =
   \frac2{1-w+R}
\endsplit\tag1
$$
```

[一般形]

```
\split
    数 & 式    \\
    数 & 式    \\
    . . . . . .
    数 & 式
\endsplit\tag 数式番号
```

数式番号が右側に付く例

$$
\begin{aligned}
\frac{1+t}{1+z} &= \frac{1+1/z-R/w}{1+z} = \frac{2}{1+w+R} \\
\frac{1-t}{1-z} &= \frac{1-1/z+R/w}{1-z} = \frac{2}{1-w+R}
\end{aligned} \qquad (1)
$$

[注] (1) 数式番号は，全体の数式をまとめて1つだけ付く。左側に数式番号が付く場合いには先頭の数式に，右側に数式番号が付く場合には，最下段の数式に番号が付く。

● \align ... \split ... \endsplit .. \endalign の例

(1) 数式番号が左側に付く例

(1) $$L(\alpha x) = \Lambda(ax) - i\Lambda(iax)$$
$$= a\Lambda(x) - ia\Lambda(ix) = aL(x)$$

(2) $$L(ix) = \Lambda(ix) - i\Lambda(-x)$$
$$= i[\Lambda(x) - i\Lambda(ix)]$$
$$= iL(x).$$

```
%--------------------------------
% An example for  AMS-TeX
%--------------------------------
$$
\align
  L(\alpha x) & = \Lambda (ax)-i\Lambda (iax)\tag1 \\
              & = a \Lambda (x)-ia \Lambda(ix)=aL(x) \\
  \split
    L(ix) &= \Lambda (ix) -i\Lambda (-x)\\
          &= i[ \Lambda (x)- i\Lambda (ix)]\\
          &= iL(x).
  \endsplit\tag2
\endalign
$$
```

(2) 数式番号が右側に付く例

$$L(\alpha x) = \Lambda(ax) - i\Lambda(iax) \quad (1)$$
$$= a\Lambda(x) - ia\Lambda(ix) = aL(x)$$
$$L(ix) = \Lambda(ix) - i\Lambda(-x)$$
$$= i[\Lambda(x) - i\Lambda(ix)]$$
$$= iL(x). \quad (2)$$

```
% An example for  AMS-TeX
%--------------------------------
\TagsOnRight
$$
\align
  L(\alpha x) & = \Lambda (ax)-i\Lambda (iax)\tag1 \\
              & = a \Lambda (x)-ia \Lambda(ix)=aL(x) \\
  \split
    L(ix) &= \Lambda (ix) -i\Lambda (-x)\\
          &= i[ \Lambda (x)- i\Lambda (ix)]\\
          &= iL(x).
  \endsplit\tag2
\endalign
$$
```

［注］(1) と(2) では，数式番号の付く数式の位置が異なる。

(3) \multline ...\endmultline

$$(1)\quad A = B = C$$
$$d = c + e + f + g + i$$
$$j + k = m + n$$
$$O + P + U = Q + R + S + T$$

```
%-------------------------
% An example for AMS-TeX
%-------------------------
$$
\multline
A = B + C \\
d = c+ e + f + g + i \\
j + k = m + n \\
O + P +  U = Q + R + S + T
\endmultline\tag 1
$$
```

[一般形]

```
\multline
    数 式  \\
    数 式  \\
    . . . .
    数 式  \\
\endmultline \tag  数式番号
```

数式番号が右側に付く例

$$A = B = C$$
$$d = c + e + f + g + i$$
$$j + k = m + n$$
$$O + P + U = Q + R + S + T \quad (1)$$

[注] (1) 数式番号は，全体の数式をまとめて1つだけ付く。左側に数式番号が付く場合いには先頭の数式に，右側に数式番号が付く場合には，最下段の数式に番号が付く。

(3) \multline ...\endmultline

$$
\begin{aligned}
&(1)\quad A=B=C\\
&\qquad\qquad\qquad d=c+e+f+g+i\\
&\qquad\qquad\qquad\quad j+k=m+n\\
&\qquad\qquad\qquad\qquad\qquad O+P+U=Q+R+S+T
\end{aligned}
$$

```
%-------------------------
% An example for AMS-TeX
%-------------------------
$$
\multline
A = B + C \\
d = c+ e + f + g + i \\
j + k = m + n \\
O + P +  U = Q + R + S + T
\endmultline\tag 1
$$
```

[一般形]

```
\multline
    数 式  \\
    数 式  \\
    . . . .
    数 式  \\
\endmultline \tag 数式番号
```

数式番号が右側に付く例

$$
\begin{aligned}
&A=B=C\\
&\qquad\qquad d=c+e+f+g+i\\
&\qquad\qquad\quad j+k=m+n\\
&\qquad\qquad\qquad O+P+U=Q+R+S+T\quad(1)
\end{aligned}
$$

［注］(1) 数式番号は，全体の数式をまとめて1つだけ付く。左側に数式番号が付く場合いには先頭の数式に，右側に数式番号が付く場合には，最下段の数式に番号が付く。

(4) \gather .....\endgather

$$a = b = c \tag{1.1}$$

$$d = c + e + f + g + i \tag{1.2}$$

$$j + k = m + n \tag{1.3}$$

```
%-------------------------
% An example for AMS-TeX
%-------------------------
$$
\gather
a = b = c \tag 1.1 \\
d = c+ e + f + g + i \tag 1.2 \\
j + k = m + n \tag 1.3
\endgather
$$
```

［一般形］

```
\gather
    数　　式   \tag  数式番号  \\
    数　　式   \tag  数式番号  \\
    . . . . .        . . . . .
    数　　式   \tag  数式番号
\endgather
```

［注］(1) 数式番号は，各数式に付けることができる。
(2) 各数式は，各々センタリングされる。
(3) 各数式を$$...$$ で囲んで記述するよりも，式と式の間隔が狭まる。

(5) \aligned.....\endaligned

$$\left\{\begin{aligned} \alpha &= f(z)\\ \beta &= f(z^3)\\ \gamma &= f(z^2)\\ \delta &= f(z^4)\end{aligned}\right\}\qquad \left\{\begin{aligned} x &= \alpha^2-\beta\\ y &= 2\gamma-\delta\end{aligned}\right\}. \tag{20}$$

```
%------------------------
% An example for AMS-TeX
%------------------------
$$
\left\{
 \aligned
    \alpha & = f(z) \\
    \beta  & = f(z^3)\\
    \gamma & = f(z^2)\\
    \delta & = f(z^4)
 \endaligned
\right\}\qquad\left\{
 \aligned
    x & = \alpha^2-\beta \\
    y & = 2\gamma -\delta
 \endaligned\right\}.\tag 20
$$
```

[一般形]

```
\aligned
    数 & 式  \\
    数 & 式  \\
    . . . . . .
    数 & 式
\endaligned\tag 数式番号
```

[注] (1) 数式番号は，全体をまとめて1つしか付かないが，上の例のように中央に付く。

● \topaligned .... \endtopaligned, \botaligned .... \endbotaligned

(1) \topaligned ...\endtopaligned の例

$$\begin{aligned}[t] \alpha &= f(z) \\ \beta &= f(z^3) \\ \gamma &= f(z^2) \\ \delta &= f(z^4) \end{aligned} \qquad \begin{aligned}[t] x &= \alpha^2-\beta \\ y &= 2\gamma-\delta \end{aligned} \tag{21}$$

```
$$
 \topaligned
    \alpha & = f(z) \\
    \beta  & = f(z^3)\\
    \gamma & = f(z^2)\\
    \delta & = f(z^4)
 \endtopaligned
\qquad
 \topaligned
    x & = \alpha^2-\beta \\
    y & = 2\gamma -\delta
 \endtopaligned\tag 21
$$
```

(2) \topaligned ...\endtopaligned
\botaligned .... \endbotaligned の例

$$\begin{aligned}[t] \alpha &= f(z) \\ \beta &= f(z^3) \\ \gamma &= f(z^2) \\ \delta &= f(z^4) \end{aligned} \qquad \begin{aligned}[b] x &= \alpha^2-\beta \\ y &= 2\gamma-\delta \end{aligned} \tag{22}$$

```
$$
 \topaligned
    \alpha & = f(z) \\
    \beta  & = f(z^3)\\
    \gamma & = f(z^2)\\
    \delta & = f(z^4)
 \endtopaligned
\qquad
 \botaligned
    x & = \alpha^2-\beta \\
    y & = 2\gamma -\delta
 \endbotaligned\tag 22
$$
```

(3) \botaligned .... \endbotaligned

\topaligned ...\endtopaligned の例

$$\begin{aligned}[b] \alpha &= f(z) \\ \beta &= f(z^3) \\ \gamma &= f(z^2) \\ \delta &= f(z^4) \end{aligned} \qquad \begin{aligned}[t] x &= \alpha^2 - \beta \\ y &= 2\gamma - \delta \end{aligned} \tag{23}$$

```
$$
 \botaligned
    \alpha & = f(z) \\
    \beta  & = f(z^3)\\
    \gamma & = f(z^2)\\
    \delta & = f(z^4)
 \endbotaligned
\qquad
 \topaligned
    x & = \alpha^2-\beta \\
    y & = 2\gamma -\delta
 \endtopaligned\tag 23
$$
```

(4) \botaligned .... \endbotaligned

\botaligned .... \endbotaligned の例

$$\begin{aligned}[b] \alpha &= f(z) \\ \beta &= f(z^3) \\ \gamma &= f(z^2) \\ \delta &= f(z^4) \end{aligned} \qquad \begin{aligned}[b] x &= \alpha^2 - \beta \\ y &= 2\gamma - \delta \end{aligned} \tag{24}$$

```
$$
 \botaligned
    \alpha & = f(z) \\
    \beta  & = f(z^3)\\
    \gamma & = f(z^2)\\
    \delta & = f(z^4)
 \endbotaligned
\qquad
 \botaligned
    x & = \alpha^2-\beta \\
    y & = 2\gamma -\delta
 \endbotaligned\tag 24
$$
```

(6) \gathered....\endgathered

$$\left\{\begin{aligned}\alpha &= f(z)\\ \beta &= f(z^3)\\ \gamma &= f(z^2)\end{aligned}\right\}\qquad\left\{\begin{gathered}x = \alpha^2-\beta\\ y = 2\gamma\end{gathered}\right\}.$$

```
$$
\left\{
 \aligned
    \alpha & = f(z) \\
    \beta  & = f(z^3)\\
    \gamma & = f(z^2)
 \endaligned
\right\}\qquad\left\{
 \gathered
    x  = \alpha^2-\beta  \\
    y  = 2\gamma
 \endgathered\right\}.
$$
```

[一般形]

```
\gathered
    数    式   \\
    数    式   \\
    . . . . . .
    数    式
\endgathered \tag 数式番号
```

\aligned の例

$$\left\{\begin{aligned}\alpha &= f(z)\\ \beta &= f(z^3)\\ \gamma &= f(z^2)\end{aligned}\right\}\qquad\left\{\begin{aligned}x &= \alpha^2-\beta\\ y &= 2\gamma\end{aligned}\right\}.$$

```
$$
\left\{
 \aligned
    \alpha & = f(z) \\
    \beta  & = f(z^3)\\
    \gamma & = f(z^2)
 \endaligned
\right\}\qquad\left\{
 \aligned
    x & = \alpha^2-\beta \\
    y & = 2\gamma
 \endaligned\right\}.
$$
```

[注] (1) 数式番号は，全体をまとめて１つしか付かないが，中央に付く。

(7) \alignat ....\endalignat

$$
\begin{alignedat}{3}
&(23)\qquad V_i &= v_i - q_i v_j, &\qquad X_i &= x_i - q_i x_j, &\qquad U_i = u_i, \qquad \text{for } i \ne j;\\
&(24)\qquad V_j &= v_i, &\qquad X_j &= x_j, &\qquad U_j = u_j + \sum_{i\ne j} q_i u_i.
\end{alignedat}
$$

```
$$
\alignat 3
V_i & =v_i-q_iv_j, & \qquad X_i & = x_i - q_ix_j,
  & \qquad U_i & =u_i, \qquad\text{for $i\ne j$;}\tag 23\\
V_j & =v_i,          & \qquad X_j & =x_j,
  & \qquad U_j & =u_j +\sum_{i\ne j}q_iu_i.\tag 24
\endalignat
$$
```

[一般形]

```
\alignat 列の数
  数&式 & 数&式 & 数&式 ・・\tag 数式番号 \\
  数&式 & 数&式 & 数&式 ・・\tag 数式番号 \\
  ・ ・ ・ ・ ・ ・ ・ ・ ・・ ・ ・ ・ ・ ・
  数&式 & 数&式 & 数&式 ・・\tag 数式番号
\endalignat
```

● \xalignat ...\endxalignat,　\xxalignat...\endxxalignat

(1) \xalignat ...\endxalignat の例

$$
\begin{aligned}
&(23)\quad V_i = v_i - q_i v_j, & X_i &= x_i - q_i x_j, & U_i &= u_i, \quad \text{for } i \ne j;\\
&(24)\quad V_j = v_i, & X_j &= x_j, & U_j &= u_j + \sum_{i\ne j} q_i u_i.
\end{aligned}
$$

```
$$
\xalignat 3
V_i & =v_i-q_iv_j, & \qquad X_i & = x_i - q_ix_j,
  & \qquad U_i & =u_i, \qquad\text{for $i\ne j$;}\tag23\\
V_j & =v_i,         & \qquad X_j & =x_j,
  & \qquad U_j & =u_j +\sum_{i\ne j}q_iu_i.\tag 24
\endxalignat
$$
```

(2) \xxalignat...\endxxalignatの例

$$
\begin{aligned}
V_i &= v_i - q_i v_j, & X_i &= x_i - q_i x_j, & U_i &= u_i, \quad \text{for } i \ne j;\\
V_j &= v_i, & X_j &= x_j, & U_j &= u_j + \sum_{i\ne j} q_i u_i.
\end{aligned}
$$

```
$$
\xxalignat 3
V_i & =v_i-q_iv_j, & \qquad X_i & = x_i - q_ix_j,
  & \qquad U_i & =u_i, \qquad \text{for $i\ne j$;}\\
V_j & =v_i,         & \qquad X_j & =x_j,
  & \qquad U_j & =u_j +\sum_{i\ne j}q_iu_i.
\endxxalignat
$$
```

[注] (1) x が増えるに従い，各列の空白の幅が増える。ただし，xxが最大の横幅に広がるコマンドである。\xxalignat の場合には，\tag コマンドが使えないので自分で数式番号をコントロールする必要がある。

(8) \cases ......\endcases

$$
f(x,y) = \begin{cases} 2^{2n} & \text{if}\quad \dfrac{1}{2^n} \le x < \dfrac{1}{2^{n-1}},\quad \dfrac{1}{2^n} \le y < \dfrac{1}{2^{n-1}}, \\ -2^{2n+1} & \text{if}\quad \dfrac{1}{2^{n+1}} \le x < \dfrac{1}{2^n},\quad \dfrac{1}{2^n} \le y < \dfrac{1}{2^{n-1}}, \\ 0 & \text{otherwise}. \end{cases} \tag{3.1}
$$

```
$$
 f(x,y) =
   \cases    2^{2n} & \text{if}\quad
        \dfrac1{2^n}\le x<\dfrac1{2^{n-1}},\quad
        \dfrac1{2^n}\le y<\dfrac1{2^{n-1}}, \\
          -2^{2n+1} & \text{if}\quad
        \dfrac1{2^{n+1}}\le x<\dfrac1{2^n},\quad
        \dfrac1{2^n}\le y<\dfrac1{2^{n-1}},\\
            0       & \text{otherwise}.
   \endcases\tag 3.1
$$
```

[一般形]

```
\cases
    数 式 & 数 式   \\
    数 式 & 数 式   \\
    . . . . . . .
    数 式 & 数 式
\endcases\tag 数式番号
```

[注] (1) 数式番号は1つしか付かない。

○ 参考文献の書き方

参考文献の書き方は，大まかに述べると下記のようになる。

(1) 参考文献の参照：本文の中で　{By　\cite {10, Theorem 4 } ..... と書くと，
By　[10, Theorem 4] ...... の形式に変換される。

(2) 参考文献のコマンド ：各文献の順序は使用者が指定する。

\Refs　見出しの印刷をする。

\ref　1つの文献の始まり。以下の各項目は，オプショナルで，順序も自由である。

\no　参考文献番号(\key　キーワード)

\by　著者(あるいは，\manyby，\bysame)

\pages　ページ(1ページだけのときは\page)

\paper　論文のタイトル (本の中の場合には，\inbookで本の名前を指定する

\yr　年

\vol　巻

\jour　雑誌名(本の中の場合には，\inbook)

\toappear　to appear をプリントする

\issue　特集号のときにその標題を指定する

\endref　1つの文献の終わり

[注 1] \Refsを使うと，通常，センタリングされた，Referencesの文字の見出しを印刷するが，この見出しを別の文字に換えることもできる。例えば，見出しをBibliographiesに換えるには，

\Refs\nofrills {Bibliographies}

とすればよい。

[注 2] 本の場合には，\paper の代わりに\bookを使う。この場合には，

\publ　出版社

\publaddr　出版社の住所

が利用できる。

[注 3] 説明用の　\paperinfo，\bookinfo，\finalinfo，　また，2 つの文献を1つにまとめる　\moreref がある。

(3) Reference の実例

〔1〕一般的な雑誌の例

7. S. S. Chern, *Integral formulas for hypersurfaces in Euclidean space and their applications to uniqueness theorems*, J. Math. Mech. **8** (1959), 947–955.

```
\ref
  \no 7\by S. S. Chern
  \paper Integral formulas for hypersurfaces in Euclidean
         space and their applications to uniqueness theorems
  \jour J. Math. Mech. \vol 8 \yr 1959
  \pages 947--955
\endref
```

〔2〕\vol と\pages を省略した例

7. S. S. Chern, *Integral formulas for hypersurfaces in Euclidean space and their applications to uniqueness theorems*, J. Math. Mech. (1959).

```
\ref
  \no 7\by S. S. Chern
  \paper Integral formulas for hypersurfaces in Euclidean
         space and their applications to uniqueness theorems
  \jour J. Math. Mech.  \yr 1959

\endref
```

〔3〕本の中に論文が掲載されている場合（\inbookの例）

3. R. E. Borland, *The AQ Algorithm*, in "Sparse Matrices and their Uses," Academic Press, London, 1981, pp. 309–314.

```
\ref
\no 3
\by R. E. Borland
\paper The AQ Algorithm
\inbook Sparse Matrices and their Uses
\publ Academic Press
\publaddr London
\yr 1981
\pages 309--314
\endref
```

〔4〕同じ著者の論文が複数ある場合（\manyby と \bysame の例）

7. S. S. Chern, *Integral formulas for hypersurfaces in Euclidean space and their applications to uniqueness theorems*, J. Math. Mech. **8** (1959), 947–955.

8. ———, *On Riemannian manifolds of four dimensions*, J. Math. Mech. **8** (1959), 947–955.

```
\ref
  \no 7\manyby S. S. Chern
  \paper Integral formulas for hypersurfaces in Euclidean
         space and their applications to uniqueness theorems
  \jour J. Math. Mech. \vol 8 \yr 1959
  \pages 947--955
\endref
%
%
\ref
  \bysame
  \no 8
  \paper On Riemannian manifolds of four dimensions
  \jour J. Math. Mech. \vol 8 \yr 1959
  \pages 947--955
\endref
```

〔5〕本を参考文献にあげる場合（\bookの例）

5. H. Bass, "Algebraic K-theory," W.A. Benjamin, New York, 1968, pp. 15–19.

```
\ref
 \no 5
 \by H. Bass
 \book Algebraic $K$-theory
 \publ W.A. Benjamin
 \publaddr New York
 \yr 1968
 \pages 15--19
\endref
```

〔6〕文献番号の形式を変更したい場合（\key の例）

[C1] S. S. Chern, *Integral formulas for hypersurfaces in Euclidean space and their applications to uniqueness theorems*, J. Math. Mech. **8** (1959), 947–955.

[C2] ———, *On Riemannian manifolds of four dimensions*, J. Math. Mech. **8** (1959), 947–955.

```
\ref
  \key[{\bf C1}]
  \manyby S. S. Chern
  \paper Integral formulas for hypersurfaces in Euclidean
         space and their applications to uniqueness theorems
  \jour J. Math. Mech. \vol 8 \yr 1959
  \pages 947--955
\endref
%
\ref
  \bysame
  \key[{\bf C2}]
  \paper On Riemannian manifolds of four dimensions
  \jour J. Math. Mech. \vol 8 \yr 1959
  \pages 947--955
\endref
```

〔7〕\toappearの例

6. S. B. Alexander, *Reducibility of Euclidean immersions of low codimension*, Thesis, Univ. of Illinois, J. Differential Geometry (to appear).

```
\ref
 \toappear
 \no 6
 \by S. B. Alexander
 \paper Reducibility of Euclidean immersions of low
        codimension
 \paperinfo Thesis, Univ. of Illinois
 \year 1967
 \jour J. Differential Geometry
\endref
```

〔8〕＼toappearを使わずに to appearを出力する例

6. S. B. Alexander, *Reducibility of Euclidean immersions of low codimension*, Thesis, Univ. of Illinois. (J. Differential Geometry, to appear.)

```
\ref
 \no 6
 \by S. B. Alexander
 \paper Reducibility of Euclidean immersions of low
        codimension
 \paperinfo Thesis, Univ. of Illinois
 \year 1967
 \finalinfo  (J. Differential Geometry, to appear.)
\endref
```

〔9〕＼paperinfo の例

[3]. N. R. Wallach, Unpublished manuscript notes on the Borel-Weil theorem.

```
\ref
\no [3]
 \by N. R. Wallach
 \paperinfo Unpublished manuscript notes on the
            Borel-Weil theorem
\endref
```

〔10〕＼paperinfo の例

5. J. Bruna and B. Korenblum, *On Kolomogorv's theorem, the Hardy-Littlewood maximal function and radial maximal function*, preprint.

```
\ref
 \no 5
 \by J. Bruna and B. Korenblum
 \paper On Kolomogorv's theorem, the Hardy-\linebreak
        Littlewood maximal function and radial maximal function
 \paperinfo preprint
\endref
```

〔11〕\bookinfo と\finalinfo の例

[4]. N. Bourbaki, "Groupes et algèbres de Lie," Eléments de Mathématique, Hermann, Paris, 1968. Chap.IV–VI.

```
\ref
 \no [4]
 \by N. Bourbaki
 \book Groupes et alg\graveaccent ebres de Lie
 \bookinfo El\'ements de Math\'ematique
 \publ Hermann
 \publaddr Paris
 \yr 1968
 \finalinfo Chap.IV--VI.
\endref
```

〔12〕2つの文献を1つにする場合（\moreref の例）

4. L. Auslander, *On the Euler characteristic of compact locally affine spaces*, Comment. Math. Helv. **35** (1961), 25–27; *II*, Bull. Amr. Math. Soc. 67 (1961), 405–406.

```
\ref
 \no 4
 \by L. Auslander
 \paper On the Euler characteristic of compact locally affine spaces
 \jour Comment. Math. Helv.
 \vol 35
 \yr 1961
 \pages 25--27
%----------------
% more reference
%----------------
\moreref
 \paper II
 \jour Bull. Amr. Math. Soc.
 \vol 67
 \yr 1961
 \pages 405--406
\endref
```

REFERENCES FOR CHAPTER 10

[10.1] Athans, M., and P. L. Falb, "Optimal Control: Introduction to the Theory and Its Applications," MacGrraw-Hill, New York, 1966.

[10.2] Bosarge, W. E., and O. G. Johnson, *Error bounds of high-order accuracy for the state regulator problem via piecewise polynomial approximations*, SIAM J. on Control 9 (1971), 15–28.

[10.3] ________________, *Direct method approximation to the state regulator control problem using a Ritz-Treffitz suboptimal control*, Proc. Joint Automatic Control Conference.

[10.4] ________________, *Numerical properties of the Ritz-Treffitz algorithm for optimal control*, Communications of the ACM 14 (1971), 402-406.

[10.5] Brauer, F., and J. Nohel, "Ordinary Differential Equations," W. A. Benjamin, Inc., New York, 1966.

[10.6] Schultz M. H., *A Ritz method for an optimal control problem*, Yale Computer Science Research Report 71–9.

```
%-------------------------------------
% A sample Document for AMS-TeX
%-------------------------------------
%   References
%---------------
\Refs\nofrills{REFERENCES FOR CHAPTER 10}
  \ref
    \key{{\bf 10.1}}
    \by Athans, M., and P. L. Falb
    \book Optimal Control: Introduction to the Theory and Its
           Applications
    \publ MacGrraw-Hill
    \publaddr New York
    \yr 1966
  \endref
%
  \ref
    \key{{\bf 10.2}}
    \manyby Bosarge, W. E., and O. G. Johnson
    \paper Error bounds of high-order accuracy for the state
           regulator problem via piecewise polynomial approximations
    \jour  SIAM J. on Control
    \vol 9
    \pages 15--28
    \yr 1971
  \endref
%
  \ref
    \bysame
    \key{{\bf 10.3}}
    \paper Direct method approximation to the state regulator control
           problem using a Ritz-Treffitz suboptimal control
    \paperinfo Proc. Joint Automatic Control Conference
    \yr 1970
  \endref
%
  \ref
    \bysame
    \key{{\bf 10.4}}
    \paper Numerical properties of the Ritz-Treffitz algorithm for
           optimal control
    \jour Communications of the ACM
    \vol 14
    \pages 402-406
    \yr 1971
  \endref
%
  \ref
    \key{{\bf 10.5}}
    \by Brauer, F., and J. Nohel
    \book Ordinary Differential Equations
    \publ W. A. Benjamin, Inc.
    \publaddr New York
    \yr 1966
  \endref
%
  \ref
    \key{{\bf 10.6}}
    \by Schultz M. H.
    \paper A Ritz method for an optimal control problem
    \paperinfo Yale Computer Science Research Report 71--9
    \yr 1971
  \endref
```

○ Poor Man's Bold

現在のAMS-TeX には，小文字のギリシャ文字やいくつかの数学記号の boldface の fontが存在していない（将来，提供される予定ではあるが?）。そこで，次のようなコマンドを定義することによって，文字のboldタイプを便宜的に印刷することができる。

```
\def\pmb#1 {\setbox0=\hbox {#1}
    \kern-.025em\copy0\kern-\wd0
    \kern.05em\copy0\kern-\wd0
    \kern-.025em\raise.0433em\box0 }
```

Test of **∞**.
Next is **abcde**.
［使用例］ Greek letters $\boldsymbol{\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\iota\kappa\lambda\mu\nu\xi o\pi\rho\sigma\tau\upsilon\phi\chi\psi\omega}$

$$-L[\mathbf{u},\mathbf{x};\boldsymbol{\lambda},\boldsymbol{\gamma}] \equiv J[\mathbf{u},\mathbf{x}] + \int_0^1 (\boldsymbol{\lambda}(t), -D_t\mathbf{x}(t) + A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t))\,dt + (\boldsymbol{\gamma}, \mathbf{x}(0) - \mathbf{x}_0). \tag{10.4}$$

```
\def\pmb#1{\setbox0=\hbox{#1}
    \kern-.025em\copy0\kern-\wd0
    \kern.05em\copy0\kern-\wd0
    \kern-.025em\raise.0433em\box0}

$$
\multline
 \qquad
  -L[{\bold u}, {\bold x};\pmb{$\lambda$},
    \pmb{$\gamma$}] \equiv J[{\bold u},{\bold x}]\\
  + \int_0^1(\pmb{$\lambda$}(t), -D_t{\bold x}(t)+
     A(t){\bold x}(t) + B(t){\bold u}(t))\,dt
     + (\pmb{$\gamma$}, {\bold x}(0)-{\bold x}_0).
 \qquad
\endmultline\tag 10.4
$$
```

○ TeXt Style($ ... $)で数字のboldfaceの出力の仕方

```
$ vbox {\bf 数字} $
```

〔参考文献〕

[1] D.E. Knuth, The TeXbook, Addison-Wesly, 1984.

[2] Michel D. Spivak, "The Joy of TeX-- A Gourmet Guide to Typesetting with the AMS-TeX Macro Package", Addison-Wesley, 1986.

[3] 大野義夫, AMS-TeX, bit Vol.20 No.3, pp.351-362 (1988).

[4] I. Aredon, The TAO of TeX, KSTS/RR-88/002, Keio Univ. (1988).

[5] M. H. Schultz, Spline Analysis, Prentice-Hall (1973).

[6] J. A. Shohat and J. D. Tamarkin, The Problem of Moments, AMS (1943).

KSTS/RR-88/003
March 12, 1988

KSTS/RR-88/003
March 12, 1988

数式に関する例題

<< 数式の縦ぞろえに関するインデックス >>

$$I \equiv [0,1] \equiv \{x|0 \le x \le 1\},$$
$$U \equiv [0,1] \times [0,1] \equiv \{(x,y)|0 \le x \le 1 and 0 \le y \ge 1\},$$

and for each positive integer t,

$$D^t\phi(x) \equiv \frac{d^t\phi}{dx^t}(x), \quad D_x^t\phi(x,y) \equiv \frac{\partial^t\phi}{\partial x^t}, \quad D_y^t\phi(x,y) \equiv \frac{\partial^t\phi}{\partial y^t}(x,y),$$

and

$$R^t \equiv \{x_1,\dots,x_t)|x_i \text{ is a real number, } 1 \le i \le t\},$$

i.e. R^t is Euclidean t-space.

```
%----------------------------------
%  A sample document for AMS-TeX
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%  No.1
%-------
%
$$
\gather
I\equiv [0,1]\equiv \{ x|0 \le x \le 1 \},\\
   U \equiv[0,1]\times [0,1] \equiv
      \{(x,y)|0\le x\le 1 and 0\le y\ge 1\},
\endgather
$$
and for each positive integer $t$,
$$
  D^t\phi(x)\equiv \frac {d^t\phi}{dx^t}(x), \quad
   D_x^t\phi(x,y) \equiv \frac {\partial^t\phi}{\partial x^t},
    \quad D_y^t\phi(x,y) \equiv
     \frac{\partial^t\phi}{\partial y^t}(x,y),
$$
and
$$
   R^t\equiv\{x_1,\dots,x_t)|\text{$x_i$ is a real number,
     $1\le i \le t$}\},
$$
i.e. $R^t$ is Euclidean t-space.\par
```

The L^p-norm of $D^t\phi$ is finite, i.e.

$$\|D^t\phi\| \equiv \left(\sum_{i=0}^s \int_{\gamma_i}^{\gamma_{i+1}} |D^t\phi(x)|^p\,dx\right)^{1/p} < \infty.$$

For the special case of $p=\infty$, we will demand that

$$\|D^t\phi\|_\infty \equiv \max_{0\le i\le s} \sup_{x\in(\gamma_i,\gamma_{i+1})} |D^t\phi(x)| < \infty$$

```
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%  No.2
%-------
%
\noindent
The $L^p$-norm of $D^t\phi$ is finite, i.e.
$$
\| D^t\phi \| \equiv
   \left(\sum_{i=0}^s \int_{\gamma_i}^{\gamma_{i+1}}
   |D^t\phi(x)|^p\,dx\right)^{1/p} < \infty.
$$
For the special case of $p=\infty$, we will demand that
$$
\|D^t\phi\|_\infty \equiv \max_{0\le i\le s}
     \sup_{x\in (\gamma_i, \gamma_{i+1})}
        |D^t\phi(x)| < \infty
$$
```

For all $0 \le l + k \le t$, the L^p-norm of $D_x^t D_y^k$ is finite, i.e.

$$\|D_x^l D_y^k \phi\|_\beta \equiv \left(\sum_{i=0}^{s} \sum_{j=0}^{r} \int_{\gamma_i}^{\gamma_{i+1}} \int_{\mu_j}^{\mu_{j+1}} |D_x^l D_y^k \phi|^p \, dy \, dx \right)^{1/p} < \infty$$

For the special case of $p = \infty$, we will demand that

$$\|D_x^l D_y^k \phi\|_\infty \equiv \max_{0 \le i \le s \atop 0 \le j \le r} \sup_{(x,y) \in (\gamma_i, \gamma_{i+1}) \times (\mu_j, \mu_{j+1})} |D_x^l D_y^k \phi(x, y)| < \infty.$$

```
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%-------
%
\noindent
For all $0 \le l+k \le t$, the $L^p$-norm
  of $D_x^tD_y^k$ is finite, i.e.
$$
\| D_x^lD_y^k\phi\|_\beta \equiv \left(\sum_{i=0}^s\sum_{j=0}^r
   \int_{\gamma_i}^{\gamma_{i+1}}\int_{\mu_j}^{\mu_{j+1}}
    |D_x^lD_y^k\phi|^p \, dy\, dx \right)^{1/p} < \infty
$$
For the special case of $p=\infty$, we will demand that
$$
\|D_x^lD_y^k\phi\|_\infty \equiv \max_{0\le i\le s\atop 0\le j \le r}
 \sup_{(x,y)\in (\gamma_i,\gamma_{i+1}) \times (\mu_j,\mu_{j+1})}
   |D_x^lD_y^k\phi(x,y)| < \infty.
$$
```

$$PC_0^{1,p}(a,b) \equiv \{\phi \in PC^{1,2}(a,b) | \phi(a) = \phi(b) = 0\}$$

and

$$\begin{split} PC_0^{1,p}(U) \equiv &\{\phi \in PC^{1,p}(U) | \phi(x,y) = 0 \text{ for all } (x,y) \text{ in the boundary} \\ &\qquad\text{of } U\text{, i.e. for } (x,y) \text{ with } x=0 \text{ or } 1\text{, or } y=0 \text{ or } 1.\} \end{split}$$

```
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%-------
%

$$
PC_0^{1,p}(a,b)\equiv \{\phi \in PC^{1,2}(a,b)|
   \phi(a)=\phi(b)=0\}
$$
and
$$
\split
PC_0^{1,p}(U)\equiv &\{\phi\in PC^{1,p}(U)|
   \phi(x,y)=0 \text{ for all $(x,y)$ in the boundary} \\
        &\qquad\text{of $U$, i.e. for $(x,y)$
          with $x=0$ or $1$, or $y=0$ or $1$}.\}
\endsplit
$$
```

THEOREM 1.2.

If $f \in PC_0^{1,2}(a,b)$, *then*

$$\pi^2 \int_a^b f^2(x)\,dx \le (b-a)^2 \int_a^b (Df(x))^2\,dx.$$

Moreover, we have equality if and only if

$$f(x) = a_1 \sin(\pi(b-a)^{-1}(x-a))$$

for some real number a_1.

```
%------------------------------------
%  A sample document for AMS-TeX
%------------------------------------
%  No.5
%-------
%
\proclaim{Theorem 1.2}\par
If $f\in PC_0^{1,2}(a,b)$, then
$$
\pi^2\int_a^bf^2(x)\,dx \le (b-a)^2\int_a^b(Df(x))^2\,dx.
$$
Moreover, we have equality if and only if
$$
f(x)=a_1\sin(\pi(b-a)^{-1}(x-a))
$$
for some real number $a_1$.
\endproclaim
```

$$E(f) = \frac{1}{n!} E_x \left[\int_a^b D^{n+1} f(t)(x-t)_+^n \, dt \right],$$

where

$$(x-t)_+^n \equiv \begin{cases} (x-t)^n, & x \ge t, \\ 0, & x < t, \end{cases}$$

```
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%  A sample document for AMS-TeX
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%-------
%
$$
E(f)=\frac1{n!}E_x\left[\int_a^b
   D^{n+1}f(t)(x-t)_+^n\,dt\right],
$$
where
$$
(x-t)_+^n\equiv
   \left\{ \matrix\format\c&\qquad\c\\
         (x-t)^n, &  x\ge t,\\
         0,       &  x < t,
          \endmatrix\right.
$$
```

$$f(x) = f(a) + Df(a)(x-a) + \cdots + \frac{1}{n!} D^n f(a)(x-a)^n \\ + \frac{1}{n!} \int_a^b D^{n+1} f(t)(x-t)_+^n \, dt$$

```
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%-------
%
$$
\multline
f(x)=f(a) +Df(a)(x-a)+\cdots
          +\frac 1{n!}D^nf(a)(x-a)^n \\
   + \frac 1{n!}\int_a^bD^{n+1}f(t)(x-t)_+^n\,dt
\endmultline
$$
```

Theorem 1.4

If f and $g \in PC^{0,2}(I)$ and

$$(f,g)_2 \equiv \int_0^1 f(x)g(x)\,dx = 0,$$

then

$$\|f\|_2^2 + \|g\|_2^2 = \|f+g\|_2^2.$$

```
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%-----------------------------------
%  No.8
%-------
%
\noindent
{\bf Theorem 1.4}\par
If $f$ and $g\in PC^{0,2}(I)$ and
$$
(f,g)_2 \equiv \int_0^1f(x)g(x)\,dx = 0,
$$
then
$$
\|f\|_2^2 +\|g\|_2^2 = \|f+g\|_2^2.
$$
```

$$|D^{k-1}g(x)| = |\int_\zeta^x |D^k g(s)|\,ds \le \int_0^1 |D^k g(s)|\,ds \le \|D^k g\|^2 = 0.$$

```
%-----------------------------------
%  A sample document for AMS-TeX
%-----------------------------------
%  No.9
%-------
%
$$
|D^{k-1}g(x)|=|\int_\zeta^x|D^kg(s)|\,ds
  \le \int_0^1|D^kg(s)|\,ds \le \|D^kg\|^2=0.
$$
```

Theorem 1.5

If $p_n(x)$ is a polynomial of degree $n = 1, 2,,$ or 3, then

$$\int_a^b [Dp_n(x)]^2 \le 4k_n(b-a)^{-2}\int_a^b [p_n(x)]^2\,dx$$

where $k_1 \equiv 3, k_2 \equiv 15, k_3 \equiv \frac{1}{2}(45+\sqrt{1605}) \approx 42.6$.

```
%------------------------------------
%  A sample document for AMS-TeX
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%-------
%
\noindent
{\bf Theorem 1.5}\par
If $p_n(x)$ is a polynomial of degree
$n=1,2,$, or $3$, then
$$\int_a^b[Dp_n(x)]^2 \le 4 k_n (b-a)^{-2}
   \int_a^b[p_n(x)]^2\,dx
$$
where $k_1\equiv 3, k_2\equiv 15, k_3\equiv
 \frac 12(45+\sqrt{1605})\approx 42.6$.
```

$$\int_{-1}^1 L_i(x)L_j(x) = \delta_{ij} \equiv \begin{cases} 1, & i=j, \\ 0, & i \ne j \end{cases}$$

```
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%-------
%
$$
\int_{-1}^1L_i(x)L_j(x)=\delta_{ij}
 \equiv\cases 1, & i=j,\\
              0, & i\ne j
        \endcases
$$
```

Using the representation (1.7) of $P_n(x)$ in terms of Legendre polynomials, we have

$$\begin{aligned} k_n &= \sup_{\beta\ne0} \frac{\int_{-1}^1[Dp_n(x)]^2\,dx}{\int_{-1}^1[p_n(x)]^2\,dx} \\ &= \sup_{\beta\ne0} \frac{\beta^T A\beta}{\beta^T \beta} \equiv \sup_{\beta\ne0} R[\beta], \end{aligned}$$

where

$$A_n \equiv [a_{ij}]_{0\le i,j\le n} \equiv \left[\int_{-1}^1 DL_i(x)DL_j(x)\,dx\right]_{0\le i,j\le n}$$

is symmetric, nonnegative definite and $R[\beta]$ is the Rayleigh quotient of A_n.

```
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%
\noindent
Using the representation (1.7) of $P_n(x)$ in terms
of Legendre polynomials, we have
$$
\split
k_n &= \sup_{\beta\ne0}
          \frac {\int_{-1}^1[Dp_n(x)]^2\,dx}
                {\int_{-1}^1[p_n(x)]^2\,dx} \\
    &=\sup_{\beta\ne0}
          \frac {\beta^T A\beta}{\beta^T \beta}
          \equiv\sup_{\beta\ne0} R[\beta],
\endsplit
$$
where
$$
A_n \equiv [a_{ij}]_{0\le i,j\le n}
    \equiv \left[
              \int_{-1}^1DL_i(x)DL_j(x)\,dx
           \right]_{0\le i,j\le n}
$$
is symmetric, nonnegative definite and $R[\beta]$
is the Rayleigh quotient of $A_n$.
```

Furthermore we compute that

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & \sqrt{21} \\ 0 & 0 & 15 & 0 \\ 0 & \sqrt{21} & 0 & 42 \end{pmatrix}.$$

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%
\noindent
Furthermore we compute that
$$
\aligned
  A_1= \pmatrix
         0 & 0\\
         0 & 3
       \endpmatrix,
\endaligned\quad
\aligned
  A_2 = \pmatrix
         0 & 0 & 0 \\
         0 & 3 & 0 \\
         0 & 0 & 15
       \endpmatrix,
\endaligned\quad
\aligned
 A_3 = \pmatrix
         0 & 0 & 0 & 0 \\
         0 & 3 & 0 & \sqrt{21}\\
         0 & 0 & 15 & 0 \\
         0 & \sqrt{21} & 0 & 42
       \endpmatrix.
\endaligned
$$
```

$$\left(\sum_{i=1}^N a_i\right)^{1/p} \le \left(\sum_{i=1}^N b_i\right)^{1/q}$$

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$$
\left(\sum_{i=1}^Na_i\right)^{1/p}
 \le
\left(\sum_{i=1}^Nb_i\right)^{1/q}
$$
```

To prove the inequality (1.6) for arbitrary a and b, we use the change of independent variable

$$y \equiv 2(a-b)^{-1}(a-x)-1$$

and obtain

$$\begin{split}\int_a^b[D_xp_n(x)]^2 &= 2(b-a)^{-1}\int_{-1}^1[D_yp_n(a+\tfrac 12(y+1)(b-a))]^2\,dy \\ &\le \int_{-1}^1 k_n[D_yp_n(a+\tfrac 12(y+1)(b-a))]^2\,dy \\ &\le 4(b-a)^{-2}k_n\int_a^b[p_n(x)]^2\,dx.\end{split}$$

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\noindent
To prove the inequality (1.6) for arbitrary
$a$ and $b$, we use the change of independent
variable
$$ y\equiv 2(a-b)^{-1}(a-x)-1$$
and obtain
$$
\split
\int_a^b[D_xp_n(x)]^2 & = 2(b-a)^{-1}
   \int_{-1}^1[D_yp_n(a+\tfrac 12(y+1)(b-a))]^2\,dy \\
 & \le \int_{-1}^1k_n[D_yp_n(a+\tfrac 12(y+1)(b-a))]^2\,dy \\
 & \le 4(b-a)^{-2}k_n\int_a^b[p_n(x)]^2\,dx.
\endsplit
$$
```

Moreover,

$$
\begin{aligned}
l_0(x) &\equiv \begin{cases} (x_i - x)x_1^{-1}, & 0 \le x \le x_i, \text{and} \\ 0, & x_i \le x \le 1, \end{cases} \\
l_i(x) &\equiv \begin{cases} (x - x_{i-1})(x_i - x_{i-1})^{-1}, & x_{i-1} \le x \le x_i, \\ (x_{i+1} - x)(x_{i+1} - x_i)^{-1}, & x_i \le x \le x_{i+1}, \text{ and} \\ 0 & 0 \le x \le x_{i-1} \text{ or } x_{i+1} \le x \le 1, \end{cases}
\end{aligned}
$$

and

$$
l_{N+1}(x) \equiv \begin{cases} (x - x_N)(I - x_N)^{-1}, & x_N \le x \le 1, \text{ and} \\ 0 & 0 \le x \le x_N. \end{cases}
$$

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\noindent
Moreover,
$$
\align
 l_0(x) &\equiv
    \cases (x_i - x)x_1^{-1}, & 0\le x\le x_i, \text{and}\\
          0,                  & x_i \le x \le 1, \\
    \endcases \\
 l_i(x) &\equiv
    \cases
     (x-x_{i-1})(x_i-x_{i-1})^{-1}, & x_{i-1} \le x \le x_i,\\
     (x_{i+1}-x)(x_{i+1} -x_i)^{-1}, & x_i \le x \le x_{i+1}, \text{ and}\\
     0  & 0\le x\le x_{i-1} \text{  or  }  x_{i+1} \le x \le 1,\\
    \endcases\\
\intertext{and}
l_{N+1}(x) &\equiv
    \cases (x - x_N)(I - x_N)^{-1}, & x_N \le x \le 1, \text{ and}\\
              0    & 0 \le x \le x_N.\\
    \endcases
\endalign
$$
```

Theorem 2.1

If $f(x,y)$ is defined for all $(x,y) \in U$, then

$$\mathcal{G}_{L(p)}f = \mathcal{G}_{L(\Delta_y)}\mathcal{G}_{L(\Delta)}f = \mathcal{G}_{L(\Delta)}\mathcal{G}_{L(\Delta_y)}f.$$

Proof. We prove only the first equality, as the second is proved the same way. By definition,

$$\begin{aligned}
\mathcal{G}_{L(\Delta_y)}\mathcal{G}_{L(\Delta)}f &= \mathcal{G}_{L(\Delta_y)}\left[\sum_{i=0}^{N+1} f(x_i,y)l_i(x)\right] \\
&= \sum_{j=0}^{M+1}\left(\sum_{i=0}^{N+1} f(x_i,y_j)l_i(x)\right) l_j(y) \\
&= \sum_{i=0}^{M+1}\sum_{j=0}^{N+1} f(x_i,y_j)l_i(x)l_j(y) \\
&= \mathcal{G}_{L(p)}f.
\end{aligned}$$

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\noindent
{\bf Theorem 2.1}\par
If $f(x,y)$ is defined for all $(x,y)\in U$, then
$$
\Cal G_{L(p)}f = \Cal G_{L(\Delta_y)} \Cal G_{L(\Delta)}f
       = \Cal G_{L(\Delta)} \Cal G_{L(\Delta_y)}f.
$$
{\it Proof.}
We prove only the first equality, as the second is proved
the same way. By definition,
$$
\split
 \Cal G_{L(\Delta_y)} \Cal G_{L(\Delta)}f
     &  =\Cal G_{L(\Delta_y)}
          \left[\sum_{i=0}^{N+1}f(x_i,y)l_i(x)\right] \\
     &  =\sum_{j=0}^{M+1}
          \left(\sum_{i=0}^{N+1}f(x_i,y_j)l_i(x)\right) l_j(y) \\
     &  =\sum_{i=0}^{M+1}\sum_{j=0}^{N+1}f(x_i,y_j)l_i(x)l_j(y) \\
     &  =\Cal G_{L(p)}f.
\endsplit
$$
```

Applying this theorem to the functional $e(x)$ for fixed $x \in [x_i, x_{i+1}]$, we have

$$e(x) = \int_{x_i}^{x_{i+1}} K_x(t) D^2 f(t)\, dt,$$

where

$$K_x(t) \equiv \begin{cases} (x_{i+1}-x)(t-x_i)(x_{i+1}-x_i)^{-1}, & x_i \le t \le x \le x_{t+1}, \\ (x-x_i)(x_{i+1}-t)(x_{i+1}-x_i)^{-1}, & x_i \le x \le t \le x_{i+1}. \end{cases}$$

Thus,

$$\begin{aligned} |e(x)| &\le \|D^2\|_\infty (x_{i+1}-x_i)^{-1} \left[\int_{x_i}^x (x_{i+1}-x)(t-x_i)\,dt \right. \\ &\quad + \left. \int_x^{x_{i+1}} (x-x_i)(x_{i+1}-t)\,dt \right] \\ &= \tfrac12 \|D^2 f\|_\infty (x_{i+1}-x_i)^{-1} [(t-x_i)(x_{i+1})|_{x_i}^{x_{i+1}} \\ &\quad - (t-x_{i+1})^2 (x-x_i)|_x^{x_{i+1}}] \\ &= \tfrac12 \|D^2 f\|_\infty (x_{i+1}-x_i)^{-1} [(x-x_i)^2 (x_{i+1}-x) \\ &\quad + (x-x_{i+1})^2 (x-x_i)]. \end{aligned}$$

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\noindent
Applying this theorem to the functional $e(x)$ for fixed
$x\in[x_i, x_{i+1}]$, we have
$$
e(x) = \int_{x_i}^{x_{i+1}}K_x(t)D^2f(t)\, dt,
$$
where
$$
K_x(t)\equiv
      \cases (x_{i+1}-x)(t-x_i)(x_{i+1}-x_i)^{-1},
               &  x_i \le t \le x \le x_{t+1},\\
             (x - x_i)(x_{i+1}-t)(x_{i+1}-x_i)^{-1},
               &  x_i \le x\le t \le x_{i+1}.
      \endcases
$$
Thus,
$$
\split
|e(x)|  &\le \|D^2\|_\infty (x_{i+1}-x_i)^{-1}
           \left[ \int_{x_i}^x (x_{i+1}-x)(t-x_i)\,dt\right.\\
        &\quad +\left.\int_x^{x_{i+1}}(x-x_i)(x_{i+1}-t)\,dt\right] \\
        &=\tfrac 12 \|D^2f\|_\infty (x_{i+1} - x_i)^{-1}
           [(t-x_i)(x_{i+1})|_{x_i}^{x_{i+1}}\\
        &\quad -(t-x_{i+1})^2(x-x_i)|_x^{x_{i+1}}]\\
        &= \tfrac 12 \|D^2f\|_\infty (x_{i+1}-x_i)^{-1}[(x-x_i)^2
           (x_{i+1}-x)\\
        &\quad +(x-x_{i+1})^2(x-x_i)].
\endsplit
```

In fact, if

$$H(x) \equiv \begin{cases} (x+1)^2(1-2x), & -1 \le x \le 0 \\ 2x^3 - 3x^2 + 1, & 0 \le x \le 1 \\ 0, & x \in R - [-1,1], \end{cases}$$

and

$$H^1(x) \equiv \begin{cases} x(x+1)^2, & -1 \le x \le 0 \\ x(1-x)^2, & 0 \le x \le 1 \\ 0, & x \in R - [-1,1], \end{cases}$$

then

$$h_i(x) = H(h^{-1}x - i), \quad 0 \le i \le N+1,$$

and

$$h_i^1(x) = hH^1(h^{-1}x - i), \quad 0 \le i \le N+1.$$

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\noindent
In fact, if
$$
\align
H(x) &\equiv
 \cases
   (x+1)^2(1-2x), & -1\le x\le 0\\
   2x^3-3x^2+1, &   0 \le x\le 1 \\
   0,    &   x\in R-[-1,1],\\
 \endcases\\
\intertext{and}
H^1(x)&\equiv
 \cases
  x(x+1)^2, & -1\le x \le 0 \\
  x(1-x)^2, & 0 \le x \le 1 \\
  0,         &  x \in R - [-1, 1],\\
 \endcases\\
\intertext{then}
h_i(x)& =H(h^{-1}x -i),\quad 0\le i \le N+1, \\
\intertext{and}
h_i^1(x)&=hH^1(h^{-1}x-i),\quad 0\le i\le N+1.
\endalign
$$
```

$$\begin{aligned}\mathcal{G}_{H(\Delta)}f(x) = f_i&[\{(x_{i+1}-x_i)^{-1}(x-x_i)\}^2\{2(x_{i+1}-x_i)^{-1}(x-x_i)-3\}+1]\\ &+f_{i+1}[\{(x_{i+1}-x_i)^{-1}(x-x_i)\}^2\{-2(x_{i+1}-x_i)^{-1}(x-x_i)+3\}]\\ &+f_i^1[\{(x_{i+1}-x_i)^{-1}(x-x_i)\}(x-x_i)(x_{i+1}-x)^2]\\ &+f_{i+1}^1[\{(x_{i+1}-x_i)^{-1}(x-x_i)\}^2(x-x_{i+1})],\end{aligned}$$

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$$
\split
\Cal G_{H(\Delta)}f(x) & = f_i [\{(x_{i+1}-x_i)^{-1} (x-x_i)
     \}^2\{2(x_{i+1}-x_i)^{-1}(x-x_i)-3\}+1]\\
  &\quad +f_{i+1}[\{(x_{i+1}-x_i)^{-1}(x-x_i)
     \}^2\{-2(x_{i+1}- x_i)^{-1}(x-x_i)+3\}]\\
  &\quad +f_i^1[\{(x_{i+1}-x_i)^{-1}(x-x_i)\}
     (x-x_i)(x_{i+1}-x)^2]\\
  &\quad +f_{i+1}^1[\{(x_{i+1}-x_i)^{-1}(x-x_i)
     \}^{2}(x-x_{i+1})],
\endsplit
$$
```

$$\begin{multline}\mathcal{G}_{H(\Delta)}f(x) = f(x_i)[\theta^2(2\theta-3)+1]-f(x_{i+1})[\theta^2(2\theta-3)]\\ +Df(x_i)[\alpha(x-x_{i+1})(x_{i+1}-x_i)]+Df(x_{i+1})[\alpha]\end{multline}$$

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$$
\multline \Cal G_{H(\Delta)}f(x)=f(x_i)[\theta^2(2\theta-3)+1]-
    f(x_{i+1})[\theta^2(2\theta-3)]\\
     + Df(x_i)[\alpha(x-x_{i+1})(x_{i+1}-x_i)]
       + Df(x_{i+1})[\alpha]
\endmultline
$$
```

More precisely, given $\left\{f_{k+i}\right\}_{i=0}^{3},\ 0 \le k \le N-2$, we define

$$p_k(x) \equiv \sum_{i=0}^{3} \eta_{k,i}(x)f_{k+i},$$

where

$$\eta_{k,i}(x) \equiv \frac{\displaystyle\prod_{\substack{j=0\\ j\ne i}}^{3} (x-x_{k+j})}{\displaystyle\prod_{\substack{j=0\\ j\ne i}}^{3} (x_{k+i}-x_{k+j})}$$

which is the unique cubic polynomial interpolating $\left\{f_{k+i}\right\}_{i=0}^{3}$. If $N \ge 2$, i.e., if Δ has at least two interior points, we approximate the derivatives $f_l^1 \equiv Df(x_i), 0 \le i \le N+1$, in the following fashion:

$$f_l^1 = Df(x_i) \approx \begin{cases} Dp_i(x_i), & i=0, \\ Dp_{i-1}(x_i), & i=1, \\ \tfrac{1}{2}(Dp_{i-2}(x_i)+Dp_{i-1}(x_i)), & 2 \le i \le N-1, \\ Dp_{i-2}(x_i), & i=N, \text{ and} \\ Dp_{i-3}(x_i), & i=N+1. \end{cases}$$

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More precisely, given $\left\{f_{k+i}\right\}_{i=0}^3,\,
0 \le k \le N-2$, we define
$$
\align
p_k(x) & \equiv \sum_{i=0}^3 \eta_{k,i}(x)f_{k+i},\\
\intertext{where}
\eta_{k,i}(x) & \equiv
  \frac{\dsize{\prod_{j=0\atop j\ne i}^3 (x-x_{k+j})}}
       {\dsize{\prod_{j=0\atop j\ne i}^3 (x_{k+i}-x_{k+j})}}
\endalign
$$
which is the unique cubic polynomial interpolating
$\left\{f_{k+i}\right\}_{i=0}^3$. If $N\ge 2$, i.e., if
$\Delta$ has at least two interior points, we approximate the
derivatives $f_l^1\equiv Df(x_i), 0\le i\le N+1$, in the
following fashion:
$$
f_l^1= Df(x_i)\approx
   \cases Dp_i(x_i), & i =0, \\
          Dp_{i-1}(x_i), & i = 1, \\
          \tfrac 12(Dp_{i-2}(x_i)+Dp_{i-1}(x_i)),
              & 2\le i\le N-1,\\
          Dp_{i-2}(x_i), & i=N, \text{ and}\\
          Dp_{i-3}(x_i), & i =N+1.
  \endcases
$$
```

Moreover, the matrix of coefficients, $\Gamma_{ij} \equiv [\gamma_{mn}^{ij}]$, in (3.5) is given by

$$\Gamma_{ij} = A(\Delta x_{i-1}) K_{ij} A^T(\Delta y_{j-1}),$$

where

$$\begin{aligned} K_{ij} &\equiv \begin{pmatrix} B_{i-1,j-1} & B_{i-1,j} \\ B_{i,j-1} & B_{i.j} \end{pmatrix}, \\ B_{l,k} &\equiv \begin{pmatrix} f(x_l, y_k) & D_y f(x_l, y_k) \\ D_x f(x_l, y_k) & D_x D_y f(x_l, y_k) \end{pmatrix}, \text{ and} \\ A(h) &\equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3h^{-2} & -2h^{-1} & 3h^{-2} & -h^{-1} \\ 2h^{-3} & h^{-2} & -2h^{-3} & h^{-2} \end{pmatrix}. \end{aligned}$$

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\noindent
Moreover, the matrix of coefficients, $\Gamma_{ij}
\equiv [\gamma_{mn}^{ij}]$, in (3.5) is given by
$$\Gamma_{ij}= A(\Delta x_{i-1})K_{ij}A^T(\Delta y_{j-1}),$$
where
$$
\align
K_{ij}\equiv &
  \pmatrix \format\l&\l\\
     B_{i-1,j-1} & B_{i-1,j}\\
     B_{i,j-1}   & B_{i.j}\\
  \endpmatrix, \\
B_{l,k}\equiv &
  \pmatrix
     f(x_l,y_k)    & D_yf(x_l, y_k)\\
     D_xf(x_l,y_k) & D_xD_yf(x_l, y_k)
  \endpmatrix, \text{  and}\\
A(h)\equiv &
 \pmatrix
     1    &    0    &    0    &    0 \\
     0    &    1    &    0    &    0 \\
  -3h^{-2} & -2h^{-1} & 3h^{-2} & -h^{-1} \\
    2h^{-3} & h^{-2} & -2h^{-3} & h^{-2}
  \endpmatrix.
\endalign
$$
```

Given the vector

$$f \equiv \{f_{ij}, f_{i,j}^{1,0}, f_{i,j}^{0,1} f_{i,j}^{1,1}\}_{i=0,j=0}^{N+1,M+1}$$

we define

$$\mathcal{G}_{H(p)} f \equiv \sum_{i=0}^{N+1} \sum_{j=0}^{M+1} \{f_{ij} h_i(x) h_j(y) + f_{i,j}^{1,0} h_i^1(x) h_j(y) + f_{i,j}^{0,1} h_i(x) h_j^1(y) + f_{i,j}^{1,1} h_i^1(x) h_j^1(y)\} \tag{3.3}$$

as the interpolation mapping in $H(p)$.

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Given the vector
$$
f\equiv \{f_{ij}, f_{i,j}^{1,0}, f_{i,j}^{0,1}
  f_{i,j}^{1,1}\}_{i=0,j=0}^{N+1,M+1}
$$
we define
$$
\multline
\Cal G_{H(p)}f\equiv
 \sum_{i=0}^{N+1}\sum_{j=0}^{M+1} \{f_{ij}h_i(x)h_j(y)
   + f_{i,j}^{1,0}h_i^1(x)h_j(y) \\
   + f_{i,j}^{0,1}h_i(x)h_j^1(y)
   + f_{i,j}^{1,1}h_i^1(x)h_j^1(y)\}
\endmultline\tag 3.3
$$
as the interpolation mapping in $H(p)$.
```

Doing this, we find that for all $x \in [x_i, x_{i+1}]$

$$D^j(I-\mathcal{G}_H)f(x) = \int_{x_i}^{x_{i+1}} D_x^j K_x(t) D^4 f(t)\,dt, 0 \le i \le N,$$

where $K_x(t) \equiv (x-t)_+^4 - \mathcal{G}_H(x-t)_+^4$. As may be easily verified, $K_x(t) \ge 0$ so that

$$\begin{aligned}\|(I-\mathcal{G}_H)f(x)\|_\infty &\le \|D^4 f\|_\infty \max_{0\le i\le N} \int_{x_i}^{x{i+1}} -K_x(t)\,dt \\ &\le \max_{0\le i\le N} \|\frac{1}{4!}(x-x_i)^2(x-x_{i+1})^2\|_\infty \|D^4 f\|_\infty \\ &\le \frac{1}{384} h^4 \|D^4 f\|_\infty.\end{aligned}$$

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\noindent
Doing this, we find that for all $x\in [x_i, x_{i+1}]$
$$
\align
D^j(I-\Cal G_H)f(x) & =
  \int_{x_i}^{x_{i+1}}D_x^jK_x(t)D^4f(t)\,dt,
     0\le i\le N,\\
\intertext{where $K_x(t)\equiv (x-t)_+^4-\Cal G_H(x-t)_+^4$.
As may be easily verified, $K_x(t)\ge 0$ so that}
\|(I-\Cal G_H)f(x)\|_{\infty}
  &\le \|D^4f\|_{\infty}\max_{0\le i\le N}
        \int_{x_i}^{x{i+1}}-K_x(t)\,dt \\
  &\le \max_{0\le i \le N}\|\dfrac 1{4!}(x-x_i)^2
    (x-x_{i+1})^2\|_{\infty} \|D^4f\|_{\infty} \\
  &\le \frac 1{384}h^4\|D^4f\|_{\infty}.
\endalign
$$
```

$$
\begin{aligned}
& \left(\int_{x_i}^{x_{i+1}}(f(x)-\mathcal{G}_Hf(x))^2\,dx\right)^2\\
&\le \left(\int_{x_i}^{x_{i+1}}(f(x)-g(x))^2\,dx\right)^{1/2}+\left(\int_{x_i}^{x_{i+1}}(q(x)-\mathcal{G}_Hq(x))^2\,dx\right)^{1/2}\\
&\le \pi^{-3}(x_{i+1}-x_i)^3\left(\int_{x_i}^{x_{i+1}}(D^3f)^2\,dx\right)^{1/2}+\pi^{-4}(x_{i+1}-x_i)^4\left(\int_{x_i}^{x_{i+1}}(D^4q(x))^2\,dx\right)^{1/2}\\
&\le \pi^{-3}(x_{i+1}-x_i)^3(1+\pi^{-1}2\sqrt{15})\left(\int_{x_i}^{x_{i+1}}(D^3f(x))^2\,dx\right)^{1/2}.
\end{aligned}\tag{3.29}
$$

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$$
\aligned
& \left(\int_{x_i}^{x_{i+1}}(f(x)-\Cal G_Hf(x))^2\,dx\right)^2\\
&\le \left(\int_{x_i}^{x_{i+1}}(f(x)-g(x))^2\,dx\right)^{1/2}
     +\left(\int_{x_i}^{x_{i+1}}(q(x)-\Cal G_Hq(x))^2\,dx\right)^{1/2}\\
&\le \pi^{-3}(x_{i+1}-x_i)^3\left(\int_{x_i}^{x_{i+1}}
       (D^3f)^2\,dx\right)^{1/2}
     +\pi^{-4}(x_{i+1}-x_i)^4\left(\int_{x_i}^{x_{i+1}}
        (D^4q(x))^2\,dx\right)^{1/2}\\
&\le \pi^{-3}(x_{i+1}-x_i)^3(1+\pi^{-1}2\sqrt{15})
        \left(\int_{x_i}^{x_{i+1}}(D^3f(x))^2\,dx\right)^{1/2}.
\endaligned\tag3.29
$$
```

$$\|x^{\alpha}-\widehat{\mathcal{G}}_{H(\Delta_{\alpha,N}^{1})}x^{\alpha}\|_{\infty} \\ \le \max\left(1,\frac{1}{384}2^{4(g-1)}\alpha|\alpha-1\|\alpha-2\|\alpha-3|\right)(N+1)^{-4}$$

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$$
\multline
\|x^{\alpha}-\widehat{\Cal G}_{H(
   \Delta_{\alpha,N}^1)}x^{\alpha}\|_{\infty}\\
   \le \max \left( 1,\,\frac 1{384} 2^{4(g-1)}\,\alpha
      |\alpha - 1\| \alpha - 2 \| \alpha - 3|\right)
   (N+1)^{-4}
\endmultline
$$
```

We define their tensor or Kronecker product as the $mn \times mn$ matrix $B \otimes C$ given by

$$B \bigotimes C \equiv \begin{pmatrix} b_{11}C & \dots & b_{1n}C \\ \vdots & & \vdots \\ b_{n1}C & \dots & b_{nn}C \end{pmatrix}$$

and in particular $A_{L(p)} = A_{L(\Delta)} \otimes A_{L(\Delta_y)}$.

```
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\noindent
We define their tensor or Kronecker product as
the $mn \times mn$ matrix $B \otimes C$ given by
$$
B \bigotimes C \equiv
\pmatrix
 b_{11}C & \hdots & b_{1n}C \\
 \vdots  &        & \vdots  \\
 b_{n1}C & \hdots & b_{nn}C
\endpmatrix
$$
and in particular $A_{L(p)} = A_{L(\Delta)}\otimes
A_{L(\Delta_y)}$.
```

Prove analogues of the results of Section *.* for the nonlinear two-point boundary value problem of order 2n, i.e. find $u \in PC^{n,2}(I)$ such that

$$\sum_{i=0}^{n}(-1)^i D^i[p_i(x)D^i u(x)] = f(x,u), \quad 0 < x < 1,$$
$$D^i u(0) = D^i u(1) = 0, \quad 0 \le i \le n-1,$$

where the coefficients $p_i(x), 0 \le i \le n$, satisfy the hypotheses of Exercise (**.**),

$$f(x,u), \frac{\partial f}{\partial u}(x,u) \in C([0,1] \times (-\infty,\infty)),$$
$$\left|\frac{\partial f}{\partial u}(x,u)\right| \le B \text{ for all } (x,u) \in [0,1] \times (-\infty,\infty), \text{ and}$$
$$\frac{\partial f}{\partial u}(x,u) \le \lambda < \Lambda \equiv \inf_{w\in PC_0^{n,2}(I)} \int_0^1 \dfrac{\displaystyle\sum_{i=0}^{n} p_i(x)[D^i w(x)]^2\,dx}{\displaystyle\int_0^1 [w]^2\,dx}$$

Hint: Use the functional

$$F[w] \equiv \int_0^1 \left\{\sum_{i=0}^{n} p_i(x)[D^i w]^2\,dx - 2\int_0^{w(x)} f(x,t)\,dt\right\}\,dx.$$

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Prove analogues of the results of Section $*.*$ for the
nonlinear two-point boundary value problem of order 2n,
i.e. find $u\in PC^{n,2}(I)$ such that
$$
\gather
\sum_{i=0}^n(-1)^iD^i[p_i(x)D^iu(x)]=f(x,u),\quad
  0 < x < 1, \\
D^iu(0)=D^iu(1)=0,\quad 0 \le i \le n-1,
\endgather
$$
where the coefficients $p_i(x), 0\le i\le n$, satisfy
the hypotheses of Exercise ($**.**$),
$$
\gather
f(x,u), \frac{\partial f}{\partial u} (x,u)
 \in C([0,1] \times (-\infty, \infty)),\\
\left|\frac{\partial f}{\partial u}(x,u)\right| \le
  B \text{ for all $(x,u)\in [0,1] \times
    (-\infty, \infty)$, and}\\
\frac{\partial f}{\partial u}(x,u)\le \lambda
    < \Lambda \equiv \inf_{w\in PC_0^{n,2}(I)}
      \int_0^1\dfrac{\dsize{\sum_{i=0}^n
        p_i(x)[D^iw(x)]^2\,dx}}
      {\dsize{\int_0^1[w]^2\,dx}}
\endgather
$$
 Hint: Use the functional
$$
F[w]\equiv \int_0^1\left\{\sum_{i=0}^np_i(x)
  [D^iw]^2\,dx-2\int_0^{w(x)}f(x,t)\,dt\right\}
   \,dx.
$$
```

$$-D[p(x)Du_B] + q(x)u_B = f(x, \xi_B(u_B)), \quad 0 < x < 1,$$
$$u_B(0) = u_B(1) = 0,$$

where

$$\xi_B(u_B) \equiv \begin{cases} B + 1 - e^{B-u_B}, & B < u_B, \\ u_B, & |u_B| \le B, \\ -B - 1 + e^{B+u_B}, & u_B < -B \end{cases}$$

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$$
\gather
-D[p(x)Du_B] + q(x)u_B = f(x,\xi_B(u_B)), \quad 0 < x < 1,\\
u_B(0)=u_B(1)=0,
\endgather
$$
where
$$
\xi_B(u_B)\equiv
  \cases
     B + 1 - e^{B-u_B},  & B < u_B,\\
     u_B,                & |u_B|\le B,\\
    -B -1 + e^{B+u_B},   & u_B < -B
  \endcases
 $$
```

Let $\{B_i(x)\}_{i=1}^n$ denote n linearly independent basis functions in $PC^{0,2}(I)$ and $f \in PC^{0,2}(I)$. We consider the least squares variational problem of finding $\beta^* \in R^n$ such that

$$\phi(\beta^*) = \inf_{\beta\in R^n} \phi(\beta) \equiv \inf_{\beta\in R^n} \|f - \sum_{i=1}^n \beta_i B_i\|_2^2. \tag{6.1}$$

The function

$$\phi(\beta) = \|f\|_2^2 - 2(f, \sum_{i=1}^n \beta_i B_i)_2 + \|\sum_{i=1}^n \beta_i B_i\|_2^2$$

is clearly quadratic in $\beta \in R^n$ and hence β^* is a solution of (6.1) if and only if

$$D_i\phi(\beta^*) = 0, \quad 1 \le i \le n, \tag{6.2}$$

and the matrix $J[\beta^*] \equiv [D_iD_j\phi(\beta^*)]$ is positive definite.

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Let $\{B_i(x)\}_{i=1}^n$ denote n linearly independent
basis functions in $PC^{0,2}(I)$ and $f\in PC^{0,2}(I)$.
We consider the least squares variational problem of
finding $\beta^*\in R^n$ such that
$$
\phi (\beta^*) = \inf_{\beta\in R^n} \phi(\beta)\equiv
\inf_{\beta\in R^n}\|f - \sum_{i=1}^n\beta_iB_i\|_2^2.
\tag 6.1
$$
The function
$$
\phi(\beta)=\|f\|_2^2-2(f, \, \sum_{i=1}^n\beta_iB_i)_2
  + \|\sum_{i=1}^n\beta_iB_i\|_2^2
$$
is clearly quadratic in $\beta\in R^n$ and hence $\beta^*$
is a solution of (6.1) if and only if
$$ D_i\phi(\beta^*)=0,\quad 1\le i\le n,\tag6.2$$
and the matrix $J[\beta^*]\equiv [D_iD_j\phi(\beta^*)]$ is
positive definite.\par
```

$$
A_h \equiv \frac{6}{h}
\begin{pmatrix}
2 & 1 & & & \text{\Large 0} \\
1 & 4 & \ddots & & \\
 & \ddots & \ddots & \ddots & \\
 & & \ddots & 4 & 1 \\
\text{\Large 0} & & & 1 & 2
\end{pmatrix}
$$

```
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\font\b=cmr10 scaled\magstep4
\def\bigzerol{\smash{\hbox{\b 0}}}
\def\bigzerou{\smash{\lower1.7ex\hbox{\b 0}}}
$$
A_h \equiv \frac 6h
  \pmatrix
         2  &  1     &         &       &\bigzerou   \\
         1  &  4     & \ddots  &       &            \\
            &\ddots  & \ddots  &\ddots&             \\
            &        & \ddots  &  4    &  1         \\
   \bigzerol&        &         &  1    &  2         \\
  \endpmatrix
$$
```

$$
A_{L(p)} =
\begin{pmatrix}
\ddots & & \ddots & \text{\Large 0} \\
 & \ddots & \text{\Large 0} & \ddots \\
\ddots & \text{\Large 0} & \ddots & \\
\text{\Large 0} & \ddots & & \ddots
\end{pmatrix}
$$

```
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%  A sample document for AMS-TeX
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%  No.33
%-------
\font\b=cmr10 scaled\magstep4
\def\bigzerol{\smash{\hbox{\b 0}}}
\def\bigzerou{\smash{\lower1.7ex\hbox{\b 0}}}
$$
A_{L(p)}=
  \pmatrix
   \ddots    &           &\ddots      &\bigzerou    \\
             &\ddots     & \bigzerou  &\ddots       \\
   \ddots    &\bigzerol  & \ddots     &             \\
   \bigzerol& \ddots     &            & \ddots      \\
  \endpmatrix
$$
```

In the special case of a uniform partition with mesh length $h = (N+1)^{-1}$, the basis functions $s_i(x)$, $-3 \le i \le N$, can be expressed in terms of a "standard" basis function, $S(x)$. In fact, if

$$S(x) \equiv \left\{ \begin{matrix} (2-x)^3/24 - (1-x)^3/6 - x^3/4 + (1+x)^3/6, \qquad -2 \le x \le -1, \\ (2-x)^3/24 - (1-x)^3/6 - x^3/4, \qquad -1 \le x \le 0, \\ (2-x)^3/24 - (1-x)^3/6, \qquad 0 \le x \le 1, \\ (2-x)^3/24, \qquad 1 \le x \le 2, \\ 0, \qquad x \in R - [-2,2], \end{matrix} \right.$$

then $s_i(x) = S(h^{-1}x - i - 2),\; -3 \le i \le N$.

```
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In the special case of a uniform partition with mesh
length $h =(N+1)^{-1}$, the basis functions $s_i(x),
-3\le i\le N$, can be expressed in terms of a
{\lq\lq standard\rq\rq} basis function, $S(x)$. In fact, if
$$
S(x)\equiv
  \left\{
    \matrix\format\l\\
     (2-x)^3/24 -(1-x)^3/6-x^3/4+(1+x)^3/6,\qquad -2\le x\le -1,\\
     (2-x)^3/24-(1-x)^3/6-x^3/4, \qquad -1\le x\le 0,\\
     (2-x)^3/24-(1-x)^3/6,\qquad 0\le x\le 1,\\
     (2-x)^3/24, \qquad 1\le x\le 2, \\
     0, \qquad x\in R -[-2,2],
    \endmatrix
   \right.
$$
then $s_i(x)=S(h^{-1}x-i-2),\; -3\le i\le N$.\par
```

Theorem 6.11

If $f \in PC^{4,2}(I)$, then

$$\|f - P_{H(\Delta)}f\|_2 \le \pi^{-4}h^4\|D^4f\|_2, \tag{6.42}$$

$$\|D(f - P_{H(\Delta)}f)\|_2 \le \pi^{-3}(1 + 2\sqrt{90 + 2\sqrt{1605}}\pi^{-1}\underline{h}^{-1}h)h^3\|D^4f\|_2, \tag{6.43}$$

and

$$\|D^2(f - P_{H(\Delta)}f\|_2 \le \pi^{-2}(1 + 4\sqrt{1350 + 30\sqrt{1605}}\pi^{-2}\underline{h}^{-2}h^2)\|D^4f\|_2. \tag{6.44}$$

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\noindent
{\bf Theorem 6.11}\par
If $f\in PC^{4,2}(I)$, then
$$
\align
\|f-P_{H(\Delta)}f\|_2 & \le
  \pi^{-4}h^4 \|D^4f\|_2,\tag6.42 \\
\|D(f-P_{H(\Delta)}f)\|_2 &\le
  \pi^{-3}(1 + 2 \sqrt{90+2\sqrt{1605}}\pi^{-1}
  \underline{h}^{-1} h)h^3\|D^4f\|_2,\tag 6.43 \\
\intertext{and}
\|D^2(f-P_{H(\Delta)}f\|_2 &\le
   \pi^{-2}(1+4\sqrt{1350+30\sqrt{1605}}\pi^{-2}
   \underline{h}^{-2} h^2)\|D^4f\|_2.\tag6.44
\endalign
$$
```

Theorem 6.12

If $f \in PC^{4,\infty}(I)$, then

$$\|f - P_{H(\Delta)}f\|_2 \le \frac{11}{5760}h^4\|D^4f\|_2, \tag{6.45}$$

$$\|D(f - P_{H(\Delta)}f)\|_2 \le \frac{1}{120}(1 + 2\underline{h}^{-1}ih)h^3\|D^4f\|_2, \tag{6.46}$$

and

$$\|D^2(f - P_{H(\Delta)}f\|_2 \le \frac{1}{60}(6 + 5\underline{h}^{-2}h^2)\|D^4f\|_2. \tag{6.47}$$

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\noindent
{\bf Theorem 6.12}\par
If $f\in PC^{4,\infty}(I)$, then
$$
\align
\|f-P_{H(\Delta)}f\|_2 & \le
  \frac{11}{5760} h^4 \|D^4f\|_2,\tag6.45 \\
\|D(f-P_{H(\Delta)}f)\|_2 &\le
  \frac 1{120}(1+2\underline{h}^{-1}
       h)h^3\|D^4f\|_2,\tag 6.46 \\
\intertext{and}
\|D^2(f-P_{H(\Delta)}f\|_2 &\le
   \frac 1{60}(6+5
   \underline{h}^{-2} h^2)\|D^4f\|_2.\tag6.47
\endalign
$$
```

$$
A_h \equiv \begin{pmatrix}
\frac{2}{15} & -\frac{1}{10} & -\frac{1}{30} & 0 & 0 & & & & & & \\
-\frac{1}{10} & \frac{12}{5} & 0 & -\frac{6}{5} & \frac{1}{10} & & & & & & \\
-\frac{1}{30} & 0 & \frac{4}{15} & -\frac{1}{10} & -\frac{1}{30} & \ddots & & & \text{\Large 0} & & \\
0 & -\frac{6}{5} & -\frac{1}{10} & \ddots & \ddots & \ddots & & & & & \\
0 & \frac{1}{10} & -\frac{1}{30} & \ddots & & & & & & & \\
 & & \ddots & \ddots & & & & \ddots & \ddots & & \\
 & & & & & & & \ddots & -\frac{6}{5} & \frac{1}{10} & 0 \\
 & & & & & \ddots & \ddots & \ddots & -\frac{1}{10} & -\frac{1}{30} & 0 \\
 & & \text{\Large 0} & & & \ddots & -\frac{6}{5} & -\frac{1}{10} & \frac{12}{5} & 0 & \frac{1}{10} \\
 & & & & & & \frac{1}{10} & -\frac{1}{30} & 0 & \frac{4}{15} & -\frac{1}{30} \\
 & & & & & & 0 & 0 & \frac{1}{10} & -\frac{1}{30} & \frac{2}{15}
\end{pmatrix}
$$

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\font\b=cmr10 scaled\magstep4
\def\bigzerol{\smash{\hbox{\b 0}}}
\def\bigzerou{\smash{\lower1.7ex\hbox{\b 0}}}
$$
A_h \equiv
\pmatrix
\format\r&\quad\r&\quad\r&\quad\r&\quad\r&
\quad\r&\quad\r&\quad\r&\quad\r&\quad\r&\quad\r\\
 \frac2{15}& -\frac1{10}&-\frac1{30}&0 &0 & & & & & & \\
-\frac1{10}& \frac{12}5 & 0 & -\frac65 &
    \frac1{10}& & & & & & \\
-\frac1{30}& 0 & \frac4{15}&-\frac1{10} &
    -\frac1{30}& \ddots& & &\bigzerol & & \\
0 &-\frac65 & -\frac1{10} &\ddots &\ddots & \ddots& & & & & \\
0 & \frac1{10} &-\frac1{30} &\ddots & & & & & & & \\
  & &\ddots &\ddots & & & & \ddots& \ddots & & \\
  & & & & & & &\ddots &-\frac65 &\frac1{10}&0  \\
  & & & & &\ddots &\ddots &\ddots &-\frac1{10} &-\frac1{30}&0 \\
  & &\bigzerou & & &\ddots &-\frac65 &-\frac1{10} &
     \frac{12}5 &0 & -\frac1{10}\\
  & & & & & &\frac1{10}& \frac1{30}&0 &
      \frac4{15} &-\frac1{30}\\
  & & & & & & 0 & 0 &\frac1{10}&
     -\frac1{30}&\frac2{15}\\
\endpmatrix
$$
```

Theorem 7.24

If $u \in PC^{4,2}(U) \cap PC_0^{1,2}(U)$, then

$$
\|u-u_H\|_D \le \gamma^{-1/2}\mu^{1/2}\pi^{-3}(1 + \pi^{-1}2^{1/2}9)\overline{\rho}^3\|u\|_{4,2},
$$

where $\|u\|_{4,2}\equiv \sum_{k+j=4}\|D_x^kD_y^ju\|_2$ and u_H is the RRG approximation to u over

$$
H_0(\rho)\equiv\{h(x,y)\in H(\rho) | h(x,y)=0 \text{ for all } (x,y)\in \text{ boundary of } U\}.
$$

```
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\noindent
{\bf Theorem 7.24}\par
If $u\in PC^{4,2}(U) \cap PC_0^{1,2}(U)$, then
$$
\|u-u_H\|_D \le \gamma^{-1/2}\mu^{1/2}\pi^{-3}
  (1 + \pi^{-1}2^{1/2}9)\overline{\rho}^3\|u\|_{4,2},
$$
where $\|u\|_{4,2}\equiv \sum_{k+j=4}\|D_x^kD_y^ju\|_2$
and $u_H$ is the RRG approximation to $u$ over
$$
H_0(\rho)\equiv\{h(x,y)\in H(\rho) |
h(x,y)=0 \text{  for all } (x,y)\in \text{ boundary of $U$}\}.
$$
```

Moreover, each eigenvalue, $\lambda_j, j \ge 1$, can be characterized as

$$\lambda_j = \left\{ \begin{matrix} \inf\{R[w]|w \in PC_0^{1,2}(I) \text{ such that } b(w,u_k)=0, 1 \le k < j\}, \\ R[u_j], \\ \min\{\max_{c_1,\dots,c_j} R\left[\sum_{i=1}^j c_iv_i\right] |v_1(x),\dots,v_j(x) \in PC_0^{1,2}(I) \\ \qquad\qquad\qquad \text{linearly independent}\}. \end{matrix} \right.$$

```
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\noindent
Moreover, each eigenvalue, $\lambda_j, j\ge 1$,
can be characterized as
$$
\lambda_j =
\left\{\matrix\format\l\\
   \inf\{R[w]|w\in PC_0^{1,2}(I) \text{ such that }
      b(w,u_k)=0, 1\le k < j \}, \\
   R[u_j],\\
   \min\{\max_{c_1,\dots ,c_j} R\left[\sum_{i=1}^j
       c_iv_i\right]|v_1(x),\dots ,v_j(x)\in PC_0^{1,2}(I)  \\
       \qquad\qquad\qquad
         \text{linearly independent} \}.
       \endmatrix
 \right.
$$
```

The restriction of R can be viewed as

$$\begin{aligned} R\left[\sum_{i=1}^n \beta_iB_i(x)\right] &= \dfrac{a\left(\sum_{i=1}^n\beta_iB_i, \sum_{i=1}^n\beta_iB_i\right)}{b\left(\sum_{i=1}^n\beta_iB_i, \sum_{i=1}^n\beta_iB_i\right)} \\ &\equiv \left(a\left[\sum_{i=1}^n\beta_iB_i\right]\right)\left(b\left[\sum_{i=1}^n\beta_iB_i\right]\right)^{-1}. \end{aligned}$$

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\noindent
The restriction of $R$ can be viewed as
$$
\align
R\left[\sum_{i=1}^n\beta_iB_i(x)\right] &=
  \dfrac {a\left(\dsize{\sum_{i=1}^n}\beta_iB_i,
          \dsize{\sum_{i=1}^n}\beta_iB_i\right)}
         {b\left(\dsize{\sum_{i=1}^n}\beta_iB_i,
          \dsize{\sum_{i=1}^n}\beta_iB_i\right)}\\
    &\equiv \left(a\left[\dsize{\sum_{i=1}^n}
            \beta_iB_i\right]\right)
          \left(b\left[\dsize{\sum_{i=1}^n}
            \beta_iB_i\right]\right)^{-1}.
\endalign
$$
```

Theorem 9.4

If $a(u,v)$ is strongly coercive and $u \in PC^{3,\infty}(I \times (0,\infty))$, then

$$\begin{aligned}\|(u-u_L)(t)\|_2 &\le \gamma^{-1/2}\mu^{3/2}\pi^{-2}\Gamma h^2\|D_x^2u(t)\|_2 \\ &\quad +[(1+\gamma^{-1/2}\mu^{3/2}\Gamma)^2\|D_x^2u_0\|_2^2 + t(2\gamma\pi^2)^{-1}\gamma^{-1}\mu^3\Gamma^2 \\ &\quad \times \sup_{0\le s\le t}\|D_tD_x^2u(s)\|_2^2]^{1/2}\pi^{-2}h^2, \qquad t \ge 0,\end{aligned}\tag{9.18}$$

where u_L is the semi-descrete Galerkin approximation to u over $L_0(\Delta)$.

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\noindent
{\bf Theorem 9.4} \par
If $a(u,v)$ is strongly coercive and $u\in PC^{3,\infty}
  (I \times (0,\infty ))$, then
$$
\aligned
\|(u-u_L)(t)\|_2 &\le \gamma^{-1/2}
    \mu^{3/2}\pi^{-2}\Gamma h^2\|D_x^2u(t)\|_2 \\
 &\quad +[(1+\gamma^{-1/2}\mu^{3/2}\Gamma)^2
   \|D_x^2u_0\|_2^2 + t(2\gamma \pi^2)^{-1}
    \gamma^{-1}\mu^3\Gamma^2 \\
 &\qquad \times \sup_{0\le s\le t}
   \|D_tD_x^2u(s)\|_2^2]^{1/2}\pi^{-2}h^2, \qquad
    t \ge 0,
\endaligned\tag9.18
$$
where $u_L$ is the semi-descrete Galerkin approximation to
$u$ over $L_0(\Delta)$.
```

$$
\begin{aligned}
\|u_S^{p,q}(t_j)\|_2 &\le \Lambda^{1/2}|\boldsymbol{\beta}^{p,q}(j)|_2 \\
&\le \Lambda^{1/2}|B^{-1/2}|_2\,|\boldsymbol{\gamma}_j(p,q)|_2 \\
&\le \Lambda^{1/2}|B^{-1/2}|_2\{|E^{-1}\mathbf{c}|_2 + |R_{p,q}^j(\Delta tE)|_2\,|\mathbf{h}-E^{-1}\mathbf{c}|_2\} \\
&\le \Lambda^{1/2}|B^{-1/2}|_2\,\{|B^{1/2}A^{-1}\mathbf{k}|_2 \\
&\qquad + |R_{p,q}^j(\Delta tE)|_2\,|B^{-1/2}g - B^{1/2}A^{-1}\mathbf{k}|_2\} \\
&\le \Lambda^{1/2}|B^{-1/2}|_2|B^{1/2}|_2\{|A^{-1}\mathbf{k}|_2 \\
&\qquad + |R_{p,q}^j(\Delta tE)|_2\,(|B^{-1}\mathbf{g}|_2 + |A^{-1}\mathbf{k}|_2)\}.
\end{aligned}
\tag{9.58}
$$

```
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\def\pmb#1{\setbox0=\hbox{#1}
    \kern-.025em\copy0\kern-\wd0
    \kern.05em\copy0\kern-\wd0
    \kern-.025em\raise.0433em\box0}
$$
\aligned
\|u_S^{p,q}(t_j)\|_2
   &\le \Lambda^{1/2} |\pmb{$\beta$}^{p,q}(j)|_2\\
   &\le \Lambda^{1/2} |B^{-1/2}|_2\,|{\bold\gamma}_j(p,q)|_2\\
   &\le \Lambda^{1/2} |B^{-1/2}|_2\{|E^{-1}c|_2
       +|R_{p,q}^j(\Delta tE)|_2\,|{\bold h}-E^{-1}c|_2\}\\
   &\le \Lambda^{1/2} |B^{-1/2}|_2\,\{|B^{1/2}A^{-1}{\bold k}|_2\\
   &\qquad + |R_{p,q}^j(\Delta tE)|_2\,|B^{-1/2}g - B^{1/2}
       A^{-1}{\bold k}|_2\} \\
   &\le \Lambda^{1/2} |B^{-1/2}|_2|B^{1/2}|_2\{|A^{-1}{\bold k}|_2 \\
   &\qquad + |R_{p,q}^j(\Delta tE)|_2\,
       (|B^{-1}{\bold g}|_2 + |A^{-1}{\bold k}|_2)\}.
\endaligned\tag 9.58
$$
```

$\Delta(h)$		
h	$\dim S(\Delta(h))$	$\Vert e^x - \mathcal{G}_{S(\Delta(h))} e^x \Vert$
$\frac{1}{4}$	7	0.26×10^{-4}
$\frac{1}{5}$	8	0.11×10^{-4}
$\frac{1}{6}$	9	0.53×10^{-5}
$\frac{1}{7}$	10	0.29×10^{-5}
$\frac{1}{11}$	11	0.17×10^{-5}

```
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$$
\vbox{
\offinterlineskip
\def\tablerule{\noalign{\hrule}}
\halign{\strut#&\vrule#&
         \quad\hfil#\hfil\quad&\vrule#&
          \quad\hfil#\hfil\quad&\vrule#&
           \quad\hfil#\hfil\quad&\vrule#\cr\tablerule
&&\multispan5\hfil $\Delta(h)$\hfil&\cr\tablerule
 && $h$ &&
 $\dim S(\Delta(h))$ &&
 $\|e^x-{\Cal G}_{S(\Delta(h))}e^x\|$
 &\cr\tablerule
 &&$\frac14$ && $7$&& $0.26\times 10^{-4}$ & \cr\tablerule
 &&$\frac15$ && $8$&& $0.11\times 10^{-4}$ & \cr\tablerule
 &&$\frac16$ && $9$&& $0.53\times 10^{-5}$ & \cr\tablerule
 &&$\frac17$ && $10$&& $0.29\times 10^{-5}$ & \cr\tablerule
 &&$\frac1{11}$ && $11$&& $0.17\times 10^{-5}$ & \cr\tablerule
  \noalign{\smallskip} }
  }
$$
```

$$
\begin{aligned}
\tfrac12\int_0^1(\mathbf{u}, R\mathbf{u})\,dt &= \tfrac12\int_0^1(R^{-1}B^T\boldsymbol\lambda, RR^{-1}B^T\boldsymbol\lambda)\,dt \\
&= \tfrac12\int_0^1(BR^{-1}B^T\boldsymbol\lambda, \boldsymbol\lambda)\,dt,
\end{aligned}
$$

and

$$
\begin{aligned}
\tfrac12\int_0^1(\mathbf{x}, Q\mathbf{x})\,dt &= \tfrac12\int_0^1(Q^{-1}A^T\boldsymbol\lambda + Q^{-1}D_i\boldsymbol\lambda, A^T\boldsymbol\lambda + D_i\boldsymbol\lambda)\,dt \\
&= \tfrac12\int_0^1(Q^{-1}D_i\boldsymbol\lambda, D_i\boldsymbol\lambda)\,dt + \tfrac12\int_0^1(Q^{-1}A^T\boldsymbol\lambda, A^T\boldsymbol\lambda)\,dt \\
&\quad + \tfrac12\int_0^1(Q^{-1}A^T\boldsymbol\lambda, D_i\boldsymbol\lambda)\,dt + \tfrac12\int_0^1(Q^{-1}D_i\boldsymbol\lambda, A^T\boldsymbol\lambda)\,dt \\
&= \tfrac12\int_0^1(Q^{-1}D_i\boldsymbol\lambda, D_i\boldsymbol\lambda)\,dt + \tfrac12\int_0^1(AQ^{-1}A^T\boldsymbol\lambda, \boldsymbol\lambda)\,dt \\
&\quad + \int_0^1(AQ^{-1}D_i\boldsymbol\lambda, \boldsymbol\lambda)\,dt.
\end{aligned}
$$

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%  A sample document for AMS-TeX
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%  No.44
%-------

\def\pmb#1{\setbox0=\hbox{#1}
    \kern-.025em\copy0\kern-\wd0
    \kern.05em\copy0\kern-\wd0
    \kern-.025em\raise.0433em\box0}

\define\blambda{\pmb{$\lambda$}}

$$
\align
\tfrac12\int_0^1({\bold u}, R{\bold u})\,dt
 &= \tfrac12\int_0^1(R^{-1}B^T\blambda,
     RR^{-1}B^T\blambda)\,dt \\
 &= \tfrac12\int_0^1(BR^{-1}B^T\blambda,
     \blambda)\,dt,\\
\intertext{and}
\tfrac12\int_0^1({\bold x}, Q{\bold x})\,dt
 & =\tfrac12\int_0^1(Q^{-1}A^T\blambda
     +Q^{-1}D_i\blambda, A^T\blambda
     +D_i\blambda)\,dt\\
 & =\tfrac12\int_0^1(Q^{-1}D_i\blambda,
     D_i\blambda)\,dt + \tfrac12\int_0^1
      (Q^{-1}A^T\blambda, A^T\blambda)\,dt\\
 & \quad +\tfrac12\int_0^1(Q^{-1}A^T\blambda, D_i
    \blambda)\,dt + \tfrac12\int_0^1(Q^{-1}D_i
    \blambda, A^T\blambda)\,dt\\
 & =\tfrac12\int_0^1(Q^{-1}D_i\blambda,
    D_i\blambda)\,dt + \tfrac12\int_0^1(AQ^{-1}
    A^T \blambda,\blambda)\,dt\\
 & \quad + \int_0^1(AQ^{-1}D_i\blambda,
     \blambda)\,dt.
\endalign
$$
```

Applying the Gronwall Inequality (cf.[105]), to this last inequality, we obtain

$$|\boldsymbol{\epsilon}_s(t)|_2 \le \|B\|_2 \|\boldsymbol{\delta}_s\|_2 e^{\int_0^1 |A(z)|_2\,dz}$$

and

$$\begin{aligned} |\boldsymbol{\epsilon}_s\|_2^2 &\equiv \int_0^1 |\boldsymbol{\epsilon}_s(t)|_2^2\,dt \le \|B\|_2^2 \|\boldsymbol{\epsilon}_s\|_2^2 e^{2\int_0^1 |A(z)|_2\,dz} \\ &\equiv \Gamma^2 \|\mathbf{u}^* - \mathbf{u}_s\|_2^2. \end{aligned}$$

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%  A sample document for AMS-TeX
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%-------

\def\pmb#1{\setbox0=\hbox{#1}
    \kern-.025em\copy0\kern-\wd0
    \kern.05em\copy0\kern-\wd0
    \kern-.025em\raise.0433em\box0}

\define\bepsilon{\pmb{$\epsilon$}}

\noindent
Applying the Gronwall Inequality (cf.[105]), to
this last inequality, we obtain
$$
|\bepsilon_s(t)|_2\le\|B\|_2
  \|\pmb{$\delta$}_s\|_2 e^{\int_0^1|A(z)|_2\,dz}
$$
and
$$
\split
|\bepsilon_s\|_2^2
  &\equiv \int_0^1|\bepsilon_s(t)|_2^2\,dt
    \le \|B\|_2^2\|\bepsilon_s\|_2^2
     e^{2\int_0^1|A(z)|_2\,dz} \\
  & \equiv
   \Gamma^2\|{\bold u}^*-{\bold u}_s\|_2^2.
\endsplit
$$
```

Theorem 10.8

If $\lambda^* \in \overset{n}{\underset{i=1}{\times}} [PC^{2,2}(I)]_i$, then there exists a positive constant, K, such that

$$\|\lambda^* - \lambda_{\widetilde{H}_0^n}\|_2 \le Kh^3, \qquad \|\lambda^* - \lambda_{\widetilde{S}_0^n}\|_2 \le Kh^3 \tag{10.34}$$

$$\|\mathbf{u}^* - \mathbf{u}_{\widetilde{H}_0^n}\|_2 \le Kh^3, \qquad \|\mathbf{u}^* - \mathbf{u}_{\widetilde{S}_0^n}\|_2 \le Kh^3 \tag{10.35}$$

$$J[\mathbf{u}^*, \mathbf{x}^*] \le J[\mathbf{u}_{\widetilde{H}_0^{\hat{n}}} \mathbf{x}_{\widetilde{H}_0^{\hat{n}}}] \le J[\mathbf{u}^*, \mathbf{x}^*] + Kh^6, \tag{10.36}$$

$$J[\mathbf{u}^*, \mathbf{x}^*] \le J[\mathbf{u}_{\widetilde{S}_0^n} \mathbf{x}_{\widetilde{S}_0^n}] \le J[\mathbf{u}^*, \mathbf{x}^*] + Kh^6. \tag{10.37}$$

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\def\pmb#1{\setbox0=\hbox{#1}
    \kern-.025em\copy0\kern-\wd0
    \kern.05em\copy0\kern-\wd0
    \kern-.025em\raise.0433em\box0}

\define\blambda{\pmb{$\lambda$}}

\noindent
{\bf Theorem 10.8}\par
If $\blambda^*\in {\overset n \to{\underset{i=1}\to\times}}
[PC^{2,2}(I)]_i$,
then there exists a positive constant, $K$, such that
$$
\gather
\|\blambda^*-\blambda_{{\widetilde H}_0^n}\|_2 \le Kh^3,
 \qquad \|\blambda^*-\blambda_{{\widetilde S}_0^n}\|_2
\le Kh^3\tag10.34\\
\|{\bold u}^*-{\bold u}_{{\widetilde H}_0^n}\|_2 \le Kh^3,
 \qquad \|{\bold u}^*-{\bold u}_{{\widetilde S}_0^n}\|_2
\le Kh^3\tag10.35\\
J[{\bold u}^*,{\bold x}^*]\le
  J[{\bold u}_{{\widetilde H}_0^{\hat n}}
    {\bold x}_{{\widetilde H}_0^{\hat n}}]
     \le J[{\bold u}^*, {\bold x}^*] + Kh^6,\tag10.36\\
J[{\bold u}^*,{\bold x}^*]\le J[{\bold u}_{{\widetilde S}_0^n}
 {\bold x}_{{\widetilde S}_0^n} ]
 \le J[{\bold u}^*, {\bold x}^*] + Kh^6.\tag10.37
\endgather
$$
```

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \hdotsfor{4} \\ \hdotsfor{4} \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} \equiv |\mu_{i+j}|_{i,j=0}^n; \quad n = 0, 1, 2, \dots, \tag{4}$$

$$\Delta_n^{(1)} = \begin{vmatrix} \mu_1 & \mu_2 & \cdots & \mu_n & \mu_{n+1} \\ \mu_2 & \mu_3 & \cdots & \mu_{n+1} & \mu_{n+2} \\ \hdotsfor{5} \\ \hdotsfor{5} \\ \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{2n} & \mu_{2n+1} \end{vmatrix} \equiv |\mu_{i+j+1}|_{i,j=0}^n; \quad n = 0, 1, 2, \dots. \tag{5}$$

```
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$$
\gather
 \Delta_n =
  \vmatrix
     \mu_0 & \mu_1 & \hdots & \mu_n \\
     \mu_1 & \mu_2 & \hdots & \mu_{n+1} \\
                \hdotsfor 4\\
                \hdotsfor 4\\
     \mu_n & \mu_{n+1} & \hdots &\mu_{2n} \\
  \endvmatrix
 \equiv |\mu_{i+j}|_{i,j=0}^n;\quad
     n = 0, 1, 2,\dots, \tag4\\
 \Delta_n^{(1)} =
  \vmatrix
    \mu_1 & \mu_2 & \hdots & \mu_n     & \mu_{n+1} \\
    \mu_2 & \mu_3 & \hdots & \mu_{n+1} & \mu_{n+2} \\
    \hdotsfor 5 \\
    \hdotsfor 5 \\
    \mu_{n+1} & \mu_{n+2} & \hdots & \mu_{2n} & \mu_{2n+1}\\
  \endvmatrix
 \equiv |\mu_{i+j+1}|_{i,j=0}^n;\quad
    n = 0, 1, 2, \dots. \tag5
\endgather
$$
```

Intordocing the difference

$$\begin{aligned}
&\Delta^0\mu_{\nu} = \mu_{\nu}\\
&\Delta^1\mu_{\nu} = \mu_{\nu} -\mu_{\nu +1},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\Delta^k\mu_{\nu} = \mu_{\nu}-\binom{k}{1}\mu_{\nu +1} +\binom{k}{2}\mu_{\nu +2} +\dots +(-1)^k\mu_{\nu +k} = \mu(t^{\nu}(1-t)^k),
\end{aligned}$$

```
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%-------
\noindent
Intorducing the difference
$$
\split
  &\Delta^0\mu_{\nu} = \mu_{\nu}\\
  &\Delta^1\mu_{\nu} = \mu_{\nu} -\mu_{\nu +1},\\
  &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
  &\Delta^k\mu_{\nu} = \mu_{\nu}-\binom kl\mu_{\nu +1}
    +\binom k2\mu_{\nu +2} +\dots +(-1)^k\mu_{\nu +k}
    = \mu(t^{\nu}(1-t)^k),
\endsplit
$$
```

Using the recurrence relations, it is readily found that

$$
\left\{
\begin{aligned}
A_{n+1}(z) &= z\sum_{\nu=0}^n P_{\nu}(0)P_{\nu}(z)(\beta_0,\dots,\beta_{\nu})^{-1} = z\sum_{\nu=0}^n\chi_{\nu}(0)\chi_{\nu}(z),\\
B_{n+1}(z) &= -1 + z\sum_{\nu=0}^n P_{\nu}(0)Q_{\nu}(z)(\beta_0,\dots,\beta_{\nu})^{-1} = -1 + \sum_{\nu=0}^n\chi_{\nu}(0)\omega_{\nu}(z),\\
C_{n+1}(z) &= 1 + z\sum_{\nu=0}^n Q_{\nu}(0)P_{\nu}(z)(\beta_0,\dots,\beta_{\nu})^{-1} = 1 + \sum_{\nu=0}^n\omega_{\nu}(0)\chi_{\nu}(z),\\
D_{n+1}(z) &= z\sum_{\nu=0}^n Q_{\nu}(0)Q_{\nu}(z)(\beta_0,\dots,\beta_{\nu})^{-1} = z\sum_{\nu=0}^n\omega_{\nu}(0)\omega_{\nu}(z).
\end{aligned}
\right.
$$

```
%-----------------------------------
%  A sample document for AMS-TeX
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%-------
\noindent
Using the recurrence relations, it is readily
found that
$$
\left\{
 \aligned
    A_{n+1}(z) & =
        z\sum_{\nu =0}^n P_{\nu}(0)P_{\nu}(z)
        (\beta_0,\dots,\beta_{\nu})^{-1} = z
        \sum_{\nu =0}^n\chi_{\nu}(0)\chi_{\nu}(z),\\
    B_{n+1}(z) & =
         -1 + z\sum_{\nu =0}^n P_{\nu}(0)Q_{\nu}(z)
         (\beta_0,\dots,\beta_{\nu})^{-1}
        = -1 + \sum_{\nu =0}^n\chi_{\nu}(0)\omega_{\nu}(z),\\
    C_{n+1}(z) & =
         1 + z\sum_{\nu =0}^n Q_{\nu}(0)P_{\nu}(z)
         (\beta_0,\dots,\beta_{\nu})^{-1}
        = 1 + \sum_{\nu =0}^n\omega_{\nu}(0)\chi_{\nu}(z),\\
    D_{n+1}(z) & =
        z\sum_{\nu =0}^n Q_{\nu}(0)Q_{\nu}(z)
        (\beta_0,\dots,\beta_{\nu})^{-1} = z
        \sum_{\nu =0}^n\omega_{\nu}(0)\omega_{\nu}(z).
 \endaligned
\right.
$$
```

We have two identities[Perron, 1]:

$$\omega'_{2n}(z)^2\int_0^{\infty}\frac{d\Psi}{z-t} -\omega'_{2n}(z)\int_0^{\infty}\frac{\omega'_{2n}(z)-\omega'_{2n}(t)}{z-t}\, d\Psi(t)\\ -\int_0^{\infty}\omega'_{2n}(t)\frac{\omega'_{2n}(z)-\omega'_{2n}(t)}{z-t}\, d\Psi(t) = \int_0^{\infty}\frac{\omega'_{2n}(t)^2}{z-t}\,d\Psi(t),$$

$$\omega'_{2n+1}(z)^2\int_0^{\infty}\frac{d\Psi}{z-t} -\omega'_{2n+1}(z)\int_0^{\infty}\frac{\omega'_{2n+1}(z)-\omega'_{2n+1}(t)}{z-t}\, d\Psi(t)\\ -\int_0^{\infty}\omega'_{2n+1}(t)\dfrac{\dfrac{\omega'_{2n+1}(z)}{z}-\dfrac{\omega'_{2n+1}(t)}{z}}{z-t}\,d\Psi(t)\\ =z\int_0^{\infty}\frac{\omega'_{2n+1}(t)^2}{t}\frac{d\Psi(t)}{z-t}.$$

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\noindent
We have two identities[Perron, 1]:
$$
\multline
 \omega'_{2n}(z)^2\int_0^{\infty}\frac{d\Psi}
  {z-t} -\omega'_{2n}(z)\int_0^{\infty}
    \frac{\omega'_{2n}(z)-\omega'_{2n}(t)}
          {z-t}\, d\Psi(t)\\
 -\int_0^{\infty}\omega'_{2n}(t)
    \frac{\omega'_{2n}(z)-\omega'_{2n}(t)}
          {z-t}\, d\Psi(t) = \int_0^{\infty}
            \frac{\omega'_{2n}(t)^2}{z-t}\,d\Psi(t),
\endmultline
$$
$$
\multline
 \omega'_{2n+1}(z)^2\int_0^{\infty}\frac{d\Psi}
  {z-t} -\omega'_{2n+1}(z)\int_0^{\infty}
    \frac{\omega'_{2n+1}(z)-\omega'_{2n+1}(t)}
          {z-t}\, d\Psi(t)\\
 -\int_0^{\infty}\omega'_{2n+1}(t)
    \dfrac{\dfrac{\omega'_{2n+1}(z)}z
      -\dfrac{\omega'_{2n+1}(t)}z}{z-t}\,d\Psi(t)\\
    =z\int_0^{\infty}\frac{\omega'_{2n+1}(t)^2}t
       \frac{d\Psi(t)}{z-t}.
\endmultline
$$
```

$$
\left\{
\begin{aligned}
&\text{1. } \rho_{0,n}+\rho_{1,n}+\dots + \rho_{j-1,n}\\
&\qquad\qquad \le \int^{x_{jn}-0}d\Psi\,
  \int_0^{x_{jn}+0}d\Psi \le
  \rho_{0,n}+\rho_{1,n}+\dots + \rho_{j,n}, \quad
   j=1,2,\dots,n\\
&\text{2. }\int_{x_{j+1,n}+0}^{x_{k+1,n}-0}d\Psi\ge
   \rho_{j,n}+\rho_{j+1,n}+\dots + \rho_{k,n} \ge
   \int_{x_{jn}-0}^{x_{kn}+0}d\Psi, \quad k>j.\\
&\text{3. } \int_a^{x_{0n}+0}d\Psi \le \rho_{0n};
 \quad \int_{x_{nn}}^b d\Psi \le \rho_{nn}.\\
&\text{4. } \int_{x_{j-1,n}+0}^{x_{j+1,n}-0}d\Psi
 > \rho_{jn}\ge \int_{x_{jn}-0}^{x_{jn}+0} d\Psi \\
&\qquad\qquad\qquad\qquad\qquad
 j=0,1,\dots,n,\quad x_{-1,n}\equiv \alpha, \quad
 x_{n+1,n}\equiv b.\\
&\text{5. } \int_{x_{j-1,n}+0}^{x_{j+2,n}-0}d\Psi \;>
  \;\rho_{jn} + \rho_{j+1,n} \;\ge\; \int_{x_{jn}-0}^{x_{j+1,n}+0}
   d\Psi. \\
&\text{6. } \Psi(x_{jn}-0)\ge \sigma_{jn}\equiv
   \rho_{1,n}+\rho_{2,n}+\dots + \rho_{j,n} \ge
   \Psi(x_{j+1,n}+0),\\
&\qquad\qquad\qquad\qquad\qquad j=0,1,\dots, n-1.
\end{aligned}
\right.
$$

```
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$$
\left\{
 \aligned
  &\text{1. } \rho_{0,n}+\rho_{1,n}+\dots + \rho_{j-1,n}\\
  &\qquad\qquad \le \int^{x_{jn}-0}d\Psi\,
    \int_0^{x_{jn}+0}d\Psi \le
     \rho_{0,n}+\rho_{1,n}+\dots + \rho_{j,n}, \quad
      j=1,2,\dots,n\\
  &\text{2. }\int_{x_{j+1,n}+0}^{x_{k+1,n}-0}d\Psi\ge
     \rho_{j,n}+\rho_{j+1,n}+\dots + \rho_{k,n} \ge
     \int_{x_{jn}-0}^{x_{kn}+0}d\Psi, \quad k>j.\\
  &\text{3. } \int_a^{x_{0n}+0}d\Psi \le \rho_{0n};
   \quad \int_{x_{nn}}^b  d\Psi \le \rho_{nn}.\\
  &\text{4. } \int_{x_{j-1,n}+0}^{x_{j+1,n}-0}d\Psi
   > \rho_{jn}\ge \int_{x_{jn}-0}^{x_{jn}+0} d\Psi \\
  &\qquad\qquad\qquad\qquad\qquad
   j=0,1,\dots,n,\quad x_{-1,n}\equiv \alpha, \quad
   x_{n+1,n}\equiv b.\\
  &\text{5. } \int_{x_{j-1,n}+0}^{x_{j+2,n}-0}d\Psi \;>
    \;\rho_{jn} + \rho_{j+1,n} \;\ge\; \int_{x_{jn}-0}^{x_{j+1,n}+0}
     d\Psi. \\
  &\text{6. } \Psi(x_{jn}-0)\ge \sigma_{jn}\equiv
     \rho_{1,n}+\rho_{2,n}+\dots + \rho_{j,n} \ge
     \Psi( x_{j+1,n}+0),\\
  &\qquad\qquad\qquad\qquad\qquad j=0,1,\dots, n-1.
 \endaligned
\right.
$$
```

APPENDIX

The TaO of TeX (version 0) 〔Gourmet Guide ～〕訂正表

3.10.1988

ページ	誤(error)	正(correct)
A－1	［注］（1）～よな形式	～ような形式
A－3	(3) ＼proclaim イタリック体	斜体(Slanted)
A－5	<ページがえ> ＼nonpagebreak	＼nopagebreak
A－9	(1) 分数，連分数 or $$z= { {x +y↑2 } ＼choose {x -y↑2 } }	＼over {x -y↑2 } }
A－12	(ハ) ＼align の前に ＼allowdisplaybreak	＼allowdisplaybreaks
A－12	＼multiline .... ＼endmultiline	＼multline .... ＼endmultline
A－13	各列の幅を調節するコマンドとして. . . . . ＼xaligned, ＼xxaligned	＼xalignat, ＼xxalignat
A－１６	［注２］＼pageinfo	＼paperinfo
インデックス	○ multiline	○ multline