Research Report

KSTS/RR-88/001
26 Jan. 1988

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A Pinching Theorem for Cusps of Negatively Curved Manifolds with Finite Volume

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Dedicated to Professor Shingo Murakami on his 60th birthday

Abstract. For a noncompact complete riemannian manifold M of negative curvature with finite volume, each cusp of M is shown to be diffeomorphic to $N \times [0, \infty)$ with N being a compact flat space form provided that the sectional curvature of M satisfies the pinching condition $-4 < -\Lambda^2 \le K \le -1$ and that $|\nabla R| \le$ const for the covariant derivative of the curvature tensor of M.

1. Introduction. One of main problems in differential geometry that has been attracting us is to reveal relationships between curvature and topology of riemannian manifolds. A celebrated success in this direction is the pinching theorem in positive curvature due to M. Berger and W. Klingenberg (cf. [**CE**], [**Sa**]), which claims that a closed simply connected riemannian manifold with sectional curvature $1 \le K < 4$ is homeomorphic to the sphere. It had lead us to the pinching problem in negative curvature as well — Is a closed riemannian manifold with sectional curvature $-4 < K \le -1$ homeomorphic to a space of constant negative curvature? However Gromov and Thurston [**GT**] gave counter-examples to the problem in any dimension greater than three. Though the original pinching problem fails in negative curvature as is demonstrated by the examples of Gromov–Thurston, it still remains possible that pinching curvature of negatively curved manifolds have topological obstructions. In fact, U. Hamenstädt recently announced a beautiful theorem which especially suggests that if M is a closed locally symmetric space of negative curvature other than the spaces of constant curvature then M never carries a metric of curvature $-4 < K \le -1$ (cf. Ville [**V**] and Pansu [**P**]). The purpose of the present paper is to give another theorem which indicates a topological obstruction to pinching curvature of an *open* negatively curved manifold with finite volume.

To be more precise, suppose that M is a noncompact complete riemannian manifold of finite volume whose sectional curvature K satisfies the pinching condition

$$-\Lambda^2 \le K \le -1 \qquad (1)$$

with a positive constant Λ. Then it is known (cf. [**Gro**], [**E**], [**Sc**]) that each end (or cusp) of M is topologically of the form

$$N \times [0, \infty), \qquad (2)$$

where N is a closed infranil-manifold. Note here that in case M is locally symmetric, N is homeomorphic to a flat space form if and only if M is of constant curvature. Our main result below roughly means that the pinching condition (1) with $\Lambda < 2$

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53C20; Secondary 53C22.

implies that the ends of M are topologically the same with those of a space of constant curvature, if we simultaneously impose a subordinate geometric uniformness condition on M.

THEOREM. *Suppose that M is a noncompact complete riemannian manifold of finite volume whose sectional curvature satisfies the pinching condition*

$$-4 < -\Lambda^2 \leq K \leq -1,$$

and assume further that the covariant derivative ∇R of the curvature tensor of M is uniformly bounded on M. Then each cusp of M is diffeomorphic to $N \times [0, \infty)$ with N being a closed flat space form.

The author wishes to thank Hajime Tsuji whose theorem in [**T**] partially initiated the present work.

2. Proof of The Theorem. We give the proof of the theorem in the rest of the paper, which proceeds as follows: First we show that the infranil-manifold N in (2) admits a torsion-free affine connection D (§2.3) which is shown to be flat (§2.4), and finally find a riemannian metric on N that is parallel with respect to D (§2.5). The construction of the torsion-free flat affine connection D is closely related to the preceding work [**K1**], [**K2**] of the author.

2.1. We begin the proof with reviewing the geometric description (2) of cusps of M (see Gromov [**Gro**], Eberlein [**E**] and Schroeder [**Sc**] for details). As before, let M be a noncompact complete riemannian manifold of finite volume whose sectional curvature satisfies the pinching condition (1) for some $\Lambda \geq 1$, and take an end, say e, of M. We can always find a length-minimizing geodesic ray $r = r(t)$ $(0 \leq t < \infty)$ of M diverging to e. For simplicity, assume that r is of unit speed. Then for the Busemann function b relative to r defined by

$$b(x) = \lim_{t \to \infty} \{t - d(x, r(t))\} \quad (x \in M),$$

there is a constant T satisfying the following conditions: (i) b is C^2-differentiable on $b^{-1}(T, \infty)$; (ii) $|\operatorname{grad} b| \equiv 1$ on $b^{-1}(T, \infty)$; (iii) $b^{-1}(t)$ is compact for any $t \in (T, \infty)$. By reparametrizing r if necessary, we may assume $T < 0$. Since b is a Morse function on $b^{-1}(T, \infty)$, the cusp $b^{-1}[0, \infty)$ is diffeomorphic to the product space $N \times [0, \infty)$ with $N = b^{-1}(0)$.

2.2. Next we give a few estimates for the gradient flow of the Busemann function b. Put $N_t = b^{-1}(t)$ for $t \geq 0$. Then the gradient flow of b gives rise to a C^1-diffeomorphism $\varphi_a : N_t \to N_{t+a}$ for any $t, a \geq 0$, and under the assumption (1), it obeys the following C^1-estimate which is an immediate consequence of the standard comparison argument (cf. Heintze–Im Hof [**HI**]):

$$e^{-\Lambda a} \cdot |X| \leq |d\varphi_a X| \leq e^{-a} \cdot |X|, \qquad X \in TN_t. \tag{3}$$

To obtain a C^2-estimate, assume further that $\Lambda < 2$ in (1) and that $|\nabla R| \leq$ const for the covariant derivative of the curvature tensor of M. Under these conditions, Green [**Gre**] proved that the Busemann function b is C^3 on $b^{-1}[0,\infty)$. In particular the diffeomorphism $\varphi_a : N_t \to N_{t+a}$ is C^2-differentiable. Now denote the riemannian connection of the hypersurface N_t of M by ∇_t. Then the pull-back $\varphi_a^*\nabla_{t+a}$ is a torsion-free C^0 affine connection of N_t, and $\nabla_t^2\varphi_a = \varphi_a^*\nabla_{t+a} - \nabla_t$ is a continuous (1,2)-tensor field on N_t. It is easy to see that $(d/da)\nabla_t^2\varphi_a = \varphi_a^*((d/d\alpha)|_{\alpha=0}\nabla_t^2\varphi_\alpha)$. In the right hand side, $\varphi_a^* : T^*N_{t+a}\otimes T^*N_{t+a}\otimes TN_{t+a} \to T^*N_t\otimes T^*N_t\otimes TN_t$ is induced by $d\varphi_a : TN_t \to TN_{t+a}$ that satisfies the inequalities (3), and therefore we have $|\varphi_a^*| \le e^{-(2-\Lambda)a}$. In addition we can follow arguments of Green [**Gre**] once again to show that $|(d/d\alpha)\nabla_t^2\varphi_\alpha|_{\alpha=0} \le$ const on N_t, where the constant is independent of $t \ge 0$. Hence we obtain

$$\left|\frac{d}{da}\nabla_t^2\varphi_a\right| \le \text{const}\cdot e^{-(2-\Lambda)a}.$$

Since, $\Lambda < 2$, this immediately implies a uniform estimate

$$|\nabla_t^2\varphi_a| \le \text{const} \tag{4}$$

on N_t with a constant independent of $t, a \ge 0$.

2.3. It is now possible to show that the family of the continuous affine connections $\varphi_t^*\nabla_t$ on $N = N_0$ converges to an affine connection D of N in the C^0-topology as t goes to infinity. In fact, for $a, t \ge 0$, we have

$$\left|\varphi_{t+a}^*\nabla_{t+a} - \varphi_t^*\nabla_t\right| = \left|\varphi_t^*(\nabla_t^2\varphi_a)\right| \le \text{const}\cdot e^{-(2-\Lambda)t}$$

by (4) and the fact that $|\varphi_t^*| \le e^{-(2-\Lambda)t}$. Since $\Lambda < 2$, this inequality actually guarantees that $\varphi_t^*\nabla_t$ converges uniformly to a C^0 affine connection D of N as $t \to \infty$. Of course D is torsion-free since so are $\varphi_t^*\nabla_t$'s.

2.4. The next purpose is to exhibit the flatness of the affine connection D on $N = N_0$ we have just obtained. Note here that D is only known to be continuous, and the curvature tensor of D is not defined in general. However the D-parallel translation along a smooth curve in N is always well defined, since the equation of the parallel translation is, in this case, a first order linear ordinary differential equation with *continuous* coefficients. What we are going to see here is that the D-parallel translation depends only on the homotopy class of the curve. Namely, let $c = c(s)$ $(0 \le s \le 1)$ be a homotopically trivial loop in $N = N_0$ with base point $x \in N$, and for each tangent vector $X \in T_xN$, denote the D-parallel translation of X along c by $X(s) \in T_{c(s)}N$ $(0 \le s \le 1)$. Then we should prove that $X(0) = X(1)$. Let $X_t(s)$ be the $\varphi_t^*\nabla_t$-parallel translation of X along c. Then $X_t(s)$ converges to $X(s)$ as t goes to ∞, since $\varphi_t^*\nabla_t$ tends to D uniformly. Thus it is enough to show

$$|\,X_t(0) - X_t(1)\,| \to 0 \quad \text{as} \quad t \to \infty. \tag{5}$$

To prove this, let $c_t = \varphi_t \circ c$ be the translation of the loop c in N by the diffeomorphism $\varphi_t : N \to N_t$, and $Y_t(s)$ the ∇_t-parallel translation of $Y_t = d\varphi_t X$ along c_t. Then it is obvious that

$$Y_t(s) = d\varphi_t X_t(s). \tag{6}$$

Now we can show that

$$|\,Y_t(1) - Y_t(0)\,| \le \mathrm{const} \cdot e^{-t} \cdot |Y_t(0)| \tag{7}$$

for sufficiently large t in the following way. First lift both the loop c_t in $N_t \subset M$ and the parallel translation $Y_t(s)$ along c_t to the universal covering $\widetilde{M}$ of M. Then we obtain a loop $\tilde{c}_t = \tilde{c}_t(s)$ in a lift $\widetilde{N}_t \subset \widetilde{M}$ of N_t, and a "parallel translation" $\widetilde{Y}_t(s)$ along $\tilde{c}_t$. Notice here that the equation of the parallel translation along $\tilde{c}_t(s)$ to which $\widetilde{Y}_t(s)$ is a solution is a first order linear ordinary differential equation with continuous coefficients, and the coefficients involve only the Christoffel symbols of $\widetilde{M}$, the second fundamental form of the horosphere $\widetilde{N}_t$, and the velocity vector $\tilde{c}_t'$ of the loop $\tilde{c}_t$. Since the loop $\tilde{c}_t$ contracts exponentially as $t \to \infty$, $\tilde{c}_t$ is enclosed in a certain geodesic ball $U = U_t$ in $\widetilde{M}$ of radius 1 whenever t is large enough. Besides, the coefficients of the metric tensor as well as the Christoffel symbols of $\widetilde{M}$ relative to normal coordinates are uniformly bounded on the unit ball U, since $\widetilde{M}$ satisfies $|R|$, $|\nabla R| \le$ const (cf. Aubin [**A**], p.153). On the other hand, by (1), the second fundamental form of the horosphere $\widetilde{N}_t$ is also bounded: This again follows from Rauch's comparison theorem (cf. [**HI**]). Finally, by (3), the velocity vector $\tilde{c}_t'$ satisfies $|\tilde{c}_t'| \le \mathrm{const} \cdot e^{-t}$. In consequence the coefficients of the equation of the parallel translation along $\tilde{c}_t$ expressed in terms of normal coordinates on U are dominated by $\mathrm{const} \cdot e^{-t}$. Thus, by applying a standard inequality for linear ordinary differential equations to the solution $\widetilde{Y}_t(s)$ of the equation of the parallel transformation along $\tilde{c}_t(s)$, we have $|\widetilde{Y}_t(1) - \widetilde{Y}_t(0)| \le \mathrm{const}_1 \cdot \exp(-t) \cdot \exp\{\mathrm{const}_2 \cdot \exp(-t)\} \cdot |\widetilde{Y}_t(0)|$, and this immediately implies (7).

Combining (6), (7) and (3), we obtain

$$|\,X_t(1) - X_t(0)\,| \le \mathrm{const} \cdot e^{-(2-\Lambda)t} \cdot |X|$$

for any sufficiently large t. This proves the desired convergence (5), and the continuous affine connection D on N turns out to be flat in the sense that the D-parallel transformation is determined only by the homotopy class of a path of the transformation.

2.5. To complete the proof of the theorem it is now sufficient to show that $N = N_0$ carries a C^1 riemannian metric h such that $Dh \equiv 0$. Let g_t be the induced riemannian metric of the hypersurface N_t of M. Then the pull-back $\varphi_t^* g_t$ of g_t by the C^2 diffeomorphism $\varphi_t : N \to N_t$ is a C^1 riemannian metric of N. Now fix a point x in N, and normalize $\varphi_t^* g_t$ so that its norm at x relative to the metric g_0 is equal to 1. Denoting the normalized metric by h_t, we obviously have $(\varphi_t^* \nabla_t) h_t \equiv 0$. Hence for any $y \in N$, $h_t(y)$ is the $\varphi_t^*\nabla_t$-parallel translation of $h_t(x)$ along a curve combining x and y. Recall now that $\varphi_t^*\nabla_t$ converges uniformly to D. On the other hand, by the normalization $|h_t(x)|_{g_0} \equiv 1$, we can pick up a diverging sequence $\{t_k\}$ so that $h_t(x)$ tends to a positive-semidefinite symmetric bilinear form $h^{(0)}(x)$ of T_xN as $t = t_k \to \infty$. Hence $h_t(y)$ also converges to $h^{(0)}(y)$ that is the D-parallel translation of $h^{(0)}(x)$. The resulting positive-semidefinite symmetric $(0, 2)$-tensor field $h^{(0)}$ on N is clearly parallel with respect to D.

In case $h^{(0)}$ is nondegenerate, the proof is already finished. Otherwise we can proceed as follows. Let $E^{(1)}$ be the null space of $h^{(0)}$, which is a C^1 subbundle of TN

closed under the covariant derivative by D, and $g_t^{(1)}$ be the induced riemannian metric of the subbundle $E_t^{(1)} = d\varphi_t E^{(1)}$ of TN_t. We can then apply the argument above again, and show that $E^{(1)}$ possesses a D-parallel positive-semidefinite symmetric bilinear form $h^{(1)}$. Repeating this procedure, we obtain sequences of bundles $TN = E^{(0)} \supset E^{(1)} \supset \cdots$, and of D-parallel positive-semidefinite symmetric bilinear forms $h^{(0)}, h^{(1)}, \cdots$ on them such that each $E^{(k)}$ is the null space of $h^{(k-1)}$. Obviously $h = h^{(0)} + h^{(1)} + \cdots$ is a desired riemannian metric of N that is parallel with respect to the torsion-free "flat" C^0 affine connection D.

In consequence, N turns out to be diffeomorphic to a flat space form, and the proof of the theorem is completed.

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