Research Report

KSTS/RR-87/010
8 Dec. 1987

Tensorial Ergodicity of geodesic flows

by

Masahiko Kanai

Masahiko Kanai

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku
Yokohama, 223 Japan

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku, Yokohama, 223 Japan

KSTS/RR-87/010
December 8, 1987

TENSORIAL ERGODICITY OF GEODESIC FLOWS

Masahiko KANAI

Department of Mathematics
Keio University
Yokohama 223, Japan

Introduction

In the 1930's, Birkhoff and von Neumann clarified the mathematical meaning of ergodicity. By their definition, a dynamical system with an invariant measure is said to be ergodic if it is metrically transitive, or equivalently if any L^2-integrable invariant function on the phase space is constant almost everywhere. Soon after, following their work, Hopf and Hedlund actually demonstrated the ergodicity of the geodesic flows on closed surfaces of constant negative curvatu. Furthermore their result was generalized by Anosov to the geodesic flows of arbitrary closed riemannian manifolds of variable negative curvature. The significance of the geodesic flows has been recognized through these works, for they are typical examples of ergodic systems. Now, since the geodesic flow is a differentiable dynamical system, it makes sense to consider the action of the flow on the tensor fields defined on the phase space as well as the action on the functions. In particular, it seems to be reasonable to ask whether the geodesic flow possesses the "tensorial ergodicity"; that is, whether every L^2-integrable tensor field on the phase space which is invariant under the action of the geodesic flow is "constant" almost everywhere. The purpose of the present note is to show this phenomenon of the geodesic flows on certain negatively curved manifolds.

To be more precise, suppose that M is a closed riemannian manifold of negative sectional curvature. The geodesic flow φ_t of M is then defined as a smooth flow on the unit tangent bundle $V = \{v \in TM : |v| = 1\}$ of M. We now restrict ourselves to either of the following two cases: (i) M is of dimension tw (ii) M is locally symmetric. In both cases the unit tangent bundle V has a canonically defined affine connection ∇ as we will see later, and in terms of it, we can define "constant", or more precisely, parallel tensor fields on V. In fact, we say that a differentiable tensor field f on V is *parallel* if its covariant derivative ∇f identically vanishes, and a measurable tensor field on V is said to be *parallel almost everywhere* if it coincides with a certain parallel tensor field almost everywhere. Also a measurable tensor field f on V is said to be *L^2-integrable* if its norm $|f|$ with respect to the canonical riemannian metric of V is L^2-integrable over V relative to the Liouville measure of V. Our main result here is

Theorem. *Every φ_t-invariant L^2-integrable tensor field on V is parallel almost everywhere.*

The proof of the theorem will be given in §1 for locally symmetric spaces, and in §2 for surfaces. Furthermore, in the last section, we will give a reformulation of the theorem and consider a related problem concerned with φ_t-invariant differentiable tensor fields on V.

The author wishes to thank Professor S. Kaneyuki whose suggestion made our description of the canonical connection made in §2 simpler.

1. Locally Symmetric Spaces

In this section we prove the tensorial ergodicity for the geodesic flows of locally symmetric spaces, and we begin it with algebraic description of the unit tangent bundles and the geodesic flows of these spaces in order to introduce the canonical affine connections on the unit tangent bundles. Suppose first that M is a noncompact symmetric space of rank one: Namely, by multiplying the riemannian metric of M by a suitable constant, M is isometric to one of the real hyperbolic space $M_{\mathbf{R}}$, the complex hyperbolic space $M_{\mathbf{C}}$, the quaternion hyperbolic space $M_{\mathbf{H}}$ and the Cayley hyperbolic space $M_{\mathbf{O}}$. The symmetric space M is represented as a homogeneous G-space $M = G/K$ with G being a connected simple Lie group acting on M isometrically, and K a maximal compact subgroup of G. Associated with the representation of M as a homogeneous G-space, the Lie algebra $\mathbf{g}$ of G carries the Cartan decomposition

$$\mathbf{g} = \mathbf{k} + \mathbf{m},$$

where $\mathbf{k}$ is the Lie algebra of K, and $\mathbf{m}$ is a linear subspace of $\mathbf{g}$ that is naturally identified with the tangent space of M at the point $o = K$ of $M = G/K$. Now fix an element ι of $\mathbf{m} \cong T_oM$ of unit length. Direct computation shows that the eigenvalues of $\mathrm{ad}(-\iota) : \mathbf{g} \to \mathbf{g}$ are 0, ± 1 in the case where $M = M_{\mathbf{R}}$, and are 0, ± 1, ± 2 in the cases $M = M_{\mathbf{C}}$, $M_{\mathbf{H}}$ and $M_{\mathbf{O}}$. Denoting the eigenspace of each eigenvalue λ of $\mathrm{ad}(-\iota)$ by $\mathbf{g}^{\lambda}$, we obtain the eigenspace decomposition

$$\mathbf{g} = \mathbf{g}^{-2} + \mathbf{g}^{-1} + \mathbf{g}^{0} + \mathbf{g}^{+1} + \mathbf{g}^{+2}. \tag{1.1}$$

Here after we adopt the convention that $\mathbf{g}^{\lambda} = 0$ for $|\lambda| > 1$ provided $M = M_{\mathbf{R}}$, and $\mathbf{g}^{\lambda} = 0$ for $|\lambda| > 2$ otherwise. Then, by Jacobi's identity, we immediately have

$$[\mathbf{g}^{\lambda}, \mathbf{g}^{\mu}] \subset \mathbf{g}^{\lambda+\mu}; \tag{1.2}$$

that is, $\mathbf{g}$ is a graded Lie algebra. Furthermore, if $\mathbf{m}^{\perp}$ denotes the orthogonal complement of ι in $\mathbf{m}$ relative to the Killing form of $\mathbf{g}$, we have $[\mathbf{k}, \iota] = \mathbf{m}^{\perp}$ since M is a symmetric space *of rank one*. This yields the decomposition $\mathbf{g}^0 = (\mathbf{k} \cap \mathbf{g}^0) + \langle \iota \rangle$ of $\mathbf{g}^0$, where $\langle \iota \rangle$ denotes the linear subspace spanned by ι. Thus we obtain a new decomposition

$$\mathbf{g} = \mathbf{k}^0 + \mathbf{v} \tag{1.3}$$

of the Lie algebra $\mathbf{g}$ into $\mathbf{k}^0 = \mathbf{k} \cap \mathbf{g}^0$ and

$$\begin{aligned} \text{(1.4)} \qquad \mathbf{v} &= \mathbf{v}^{-2} + \mathbf{v}^{-1} + \mathbf{v}^0 + \mathbf{v}^{+1} + \mathbf{v}^{+2} \\ &\quad \text{with} \quad \mathbf{v}^0 = \langle \iota \rangle \quad \text{and} \quad \mathbf{v}^\lambda = \mathbf{g}^\lambda \quad (\lambda = \pm 1, \pm 2). \end{aligned}$$

Now consider the unit tangent bundle V of M. The isometric action of G on M is naturally lifted to an action on V, and it is transitive since M is a symmetric space *of rank one*. Further the Lie algebra of the isotropy subgroup K^0 of the action of G on V at the point $\iota \in V$ coincides with the Lie algebra $\mathbf{k}^0$ in (1.3). Hence the decomposition (1.3) of $\mathbf{g}$ means that $V = G/K^0$ is a *reductive* homogeneous G-space; i.e., $[\mathbf{k}^0, \mathbf{v}] \subset \mathbf{v}$. In particular, $\mathbf{v}$ is identified with the tangent space of V at ι. Moreover, since the splitting (1.4) of $\mathbf{v} \cong T_\iota V$ is $\mathrm{ad}(\mathbf{k}^0)$-invariant, it extends to a G-invariant splitting of the tangent bundle of V:

$$\text{(1.5)} \qquad TV = E^{-2} + E^{-1} + E^0 + E^{+1} + E^{+2}.$$

(N e here that $E^{\pm 2} = 0$ in the case of $M = M_{\mathbf{R}}$.) Fix a G-invariant riemannian metric of V for which the splitting (1.5) of the tangent bundle of V is orthogonal. The geodesic flow φ_t of M defined on the unit tangent bundle V of M commutes with the the action of G on V, and the orbit of φ_t passing through ι is given by $\varphi_t(\iota) = (\mathrm{Exp}\, t\iota) \cdot \iota$. This specifically implies that E^0 is spanned by the geodesic spray $\dot\varphi = (d/dt)|_{t=0}\varphi_t$, which is, by definition, the vector field on V generating the flow φ_t on V. In addition, it follows that $d\varphi_t\xi = d(\mathrm{Exp}\, t\iota) \circ \mathrm{Ad}(\mathrm{Exp}(-t\iota))(\xi)$ for $\xi \in \mathbf{v} \cong T_\iota V$, while it holds that $\mathrm{Ad}(\mathrm{Exp}(-t\iota))\xi^\lambda = e^{\lambda t}\xi^\lambda$ for $\xi^\lambda \in \mathbf{v}^\lambda$ since $\mathrm{ad}(-\iota)\xi^\lambda = \lambda\xi^\lambda$ by definition. Hence we have

$$\text{(1.6)} \qquad d\varphi_t\xi^\lambda = e^{\lambda t} \cdot d(\mathrm{Exp}\, t\iota)(\xi^\lambda), \qquad \xi^\lambda \in \mathbf{v}^\lambda \subset T_\iota V, \quad \lambda = 0, \pm 1, \pm 2.$$

In consequence, each subbundle E^λ in the splitting (1.5) is $d\varphi_t$-invariant, and satisfies

$$\text{(1.7)} \qquad |d\varphi_t\xi^\lambda| = e^{\lambda t}|\xi^\lambda|; \qquad \xi^\lambda \in E^\lambda, \quad \lambda = 0, \pm 1, \pm 2.$$

We now proceed to the definition of the canonical affine connection ∇ of the unit tangent bundle V. It requires another affine connection D of V which is defined as the canonical connection of the reductive homogeneous space $V = G/K^0$, and is described as follows (cf. [**KN**]). First extend the decomposition (1.3) of $\mathbf{g}$ to a left-invariant linear splitting $TG = C_{\mathbf{k}^0} + C_{\mathbf{v}}$ of the tangent bundle of G. The first component $C_{\mathbf{k}^0}$ is vertical with respect to the fibering of G over $V = G/K^0$, while $C_{\mathbf{v}}$ is horizontal. For each tangent vector ξ of V, let $\xi^* \in C_{\mathbf{v}}$ be the horizontal lift of ξ by the fibering of G over V. On the other hand, let D^* be the affine connection of G defined by $D^*\zeta \equiv 0$ for all left-invariant vector fields ζ on G. Then the canonical connection D of V as a homogeneous G-space is characterized by $(D_\xi\eta)^* = D^*_{\xi^*}\eta^*$ for any vector fields ξ and η on V. The connection D of V possesses the following properties. (1)

D is G-invariant, that is, $dg(D_\xi\eta) = D_{dg(\xi)}dg(\eta)$ for any $g \in G$ and vector fields ξ and η on V. (2) The torsion tensor T and the curvature tensor R of D have the representations $T(\xi,\eta) = -[\xi,\eta]_{\mathbf{v}}$ and $R(\xi,\eta)\zeta = -[[\xi,\eta]_{\mathbf{k}^0},\zeta]$ for $\xi,\eta,\zeta \in T_\iota V \cong \mathbf{v} \subset \mathbf{g}$, where, for $\xi \in \mathbf{g} = \mathbf{k}^0 + \mathbf{v}$, $\xi_{\mathbf{k}^0}$ and $\xi_{\mathbf{v}}$ denote $\mathbf{k}^0$- and $\mathbf{v}$-components of ξ respectively. (3) Each subbundle E^λ in the splitting (1.5) is D-stable in the sense that the covariant derivative $D_\xi\eta$ is a section of E^λ whenever η is a section of E^λ. (4) The canonical contact form θ of V (cf. [**AM**]; see also §2) is G-invariant, and therefore, is parallel with respect to D. (5) D is φ_t-invariant, i.e., $d\varphi_t(D_\xi\eta) = D_{d\varphi_t\xi}d\varphi_t\eta$ for any vector fields ξ and η on V. Now define the ***canonical connection*** ∇ of V by

$$\nabla_\xi\eta = \begin{cases} D_\xi\eta, & \xi \in E^{-2} + E^{-1} + E^{+1} + E^{+2}, \\ \mathcal{L}_{\dot\varphi}\eta, & \xi = \dot\varphi \in E^0, \end{cases} \tag{1.8}$$

for an arbitrary vector field η on V, where $\mathcal{L}_{\dot\varphi}\eta$ denotes the Lie derivative of η by the geodesic spray $\dot\varphi$. The reason why in the above definition we adopted the Lie derivative in the direction of the geodesic spray instead of the covariant derivative by D is that (locally defined) φ_t-invariant vector fields on V are, in general, non-parallel with respect to D; cf. (1.6). Note that (7) each subbundle E^λ of TV is again ∇-stable, that (8) the Liouville measure $\Lambda = \theta \wedge (d\theta)^n$ ($n + 1 = \dim M$) of V is parallel with respect to the canonical connection ∇, and that (9) ∇ is φ_t-invariant.

In summary, we have obtained the following things on the unit tangent bundle of each noncompact symmetric space of rank one: the geodesic flow φ_t, the splitting (1.5) of the tangent bundle, and the canonical connection ∇.

Henceforth assume that M is a closed locally symmetric riemannian manifold of negative curvature. Then the universal covering $\widetilde{M}$ of M is a noncompact symmetric space of rank one. Moreover the unit tangent bundle V of M is covered by the unit tangent bundle $\widetilde{V}$ of $\widetilde{M}$, and the deck transformations of the covering $\widetilde{V}$ of V preserve the above structures on $\widetilde{V}$. Thus they descend downstairs, and we obtain the corresponding structures on V which we denote by the same symbols: Namely, we are in the presence of the geodesic flow φ_t of M defined on V, the splitting (1.5) of the tangent bundle of V, and the canonical connection ∇ of V. Since the action of the geodesic flow φ_t on V is smooth, it is naturally lifted to an action ψ_t on the vector bundle $T^{(\ell,m)} = (\otimes_\ell TV)\otimes(\otimes_m T^*V)$ of (ℓ,m)-tensors of V as follows. First we require that the diagram

$$\begin{array}{ccc} T^{(\ell,m)} & \xrightarrow{\psi_t} & T^{(\ell,m)} \\ \downarrow & & \downarrow \\ V & \xrightarrow[\varphi_t]{} & V \end{array}$$

commutes, and that the restriction of ψ_t to each fiber of $T^{(\ell,m)}$ is a linear isomorphism. Further ψ_t is given by $\psi_t\xi = d\varphi_t\xi$ specifically for $\xi \in TV$, $\psi_t\alpha = \varphi^*_{-t}\alpha$

for $\alpha \in T^*V$ and $\psi_t(\xi_1 \otimes \cdots \xi_\ell \otimes \alpha_1 \otimes \cdots \otimes \alpha_m) = (\psi_t\xi_1) \otimes \cdots \otimes (\psi_t\xi_\ell) \otimes (\psi_t\alpha_1) \otimes \cdots \otimes (\psi_t\alpha_m)$ for $\xi_j \in TV$ and $\alpha_k \in T^*V$. Now the splitting (1.5) of the tangent bundle of V yields the splitting of $T^{(\ell,m)}$ into the subbundles

$$E^{(\lambda_1,\cdots,\lambda_\ell;\mu_1,\cdots,\mu_m)} = E^{\lambda_1} \otimes \cdots \otimes E^{\lambda_\ell} \otimes E^{\mu_1 *} \otimes \cdots \otimes E^{\mu_m *}$$

with $\lambda_1, \cdots, \lambda_\ell, \mu_1, \cdots, \mu_m = 0, \pm 1, \pm 2$. Each of these bundles is ψ_t-invariant, and satisfies

$$(1.9) \qquad |\psi_t\tau| = \exp\left(\sum_{j=1}^{\ell} \lambda_j - \sum_{k=1}^{m} \mu_k\right) t \cdot |\tau|, \qquad \tau \in E^{(\lambda_1,\cdots,\lambda_\ell;\mu_1,\cdots,\mu_m)},$$

by (1.6) and (1.7). (Remind here that we have chosen a G-invariant riemannian metric of V so that the splitting (1.5) of the tangent bundle of V is orthogonal). For a section f of $E^{(\lambda_1,\cdots,\lambda_\ell;\mu_1,\cdots,\mu_m)}$, define a new section $\psi_t f$ of $E^{(\lambda_1,\cdots,\lambda_\ell;\mu_1,\cdots,\mu_m)}$ by

$$(\psi_t f)(v) = \psi_t \circ f \circ \varphi_{-t}(v), \qquad v \in V.$$

We say that a section f of $E^{(\lambda_1,\cdots,\lambda_\ell;\mu_1,\cdots,\mu_m)}$ is *φ_t-invariant* if $\psi_t f = f$: In other words, f is ψ_t-invariant if and only if its "graph" $f(V)$ is a ψ_t-invariant subset of $E^{(\lambda_1,\cdots,\lambda_\ell;\mu_1,\cdots,\mu_m)}$. Note also that the canonical connection ∇ of V naturally induces a connection of $T^{(\ell,m)}$, which we denote by the same symbol ∇, and each subbundle $E^{(\lambda_1,\cdots,\lambda_\ell;\mu_1,\cdots,\mu_m)}$ of $T^{(\ell,m)}$ is ∇-stable. Every differentiable section f of $E^{(\lambda_1,\cdots,\lambda_\ell;\mu_1,\cdots,\mu_m)}$ should satisfy

$$(1.10) \qquad \nabla(\psi_t f) = \psi_t(\nabla f),$$

since the canonical connection ∇ of V is φ_t-invariant. To prove the tensorial ergodicity, it is now sufficient to show

(1.11) **Proposition.** *An L^2-integrable section f of $E^{(\lambda_1,\cdots,\lambda_\ell;\mu_1,\cdots,\mu_m)}$ is parallel almost everywhere with respect to ∇ whenever f is ψ_t-invariant.*

The proof of the proposition is divided into two cases according to whether $\sum_{j=1}^{\ell} \lambda_j - \sum_{k=1}^{m} \mu_k = 0$ or not. We first consider the easier one.

Case 1. Suppose that $\Sigma\lambda_j - \Sigma\mu_k \neq 0$. For a measurable ψ_t-invariant section f of $E^{(\lambda_1,\cdots,\lambda_\ell;\mu_1,\cdots,\mu_m)}$, we obtain

$$(1.12) \qquad |f(v)| = |\psi_{-t} \circ f \circ \varphi_t(v)| = e^{-(\Sigma\lambda_j - \Sigma\mu_k)t} |f \circ \varphi_t(v)|$$

from (1.9). Now choose a constant $c > 0$ so that $\Lambda(W) > 0$ for $W = \{v \in V : |f(v)| \leq c\}$, where Λ denotes the Liouville measure of the unit tangent bundle V. Since the Liouville measure is invariant under the the geodesic flow, Birkhoff's individual ergodic theorem (see e.g. [P]), together with the ergodicity of the geodesic flow, implies that $t^{-1}\int_0^t \chi_W \circ \varphi_s(v)\, ds$ converges to $\Lambda(W)/\Lambda(V) > 0$ as t goes to infinity for almost all $v \in V$, where χ_W denotes the characteristic function of W. In particular, for almost all $v \in V$ there exists a diverging sequence $\{t_k\}$ such that $\varphi_{t_k}(v) \in W$, i.e., $|f|(\varphi_{t_k}(v)) \le c$. Thus, in the case where $\Sigma\lambda_j - \Sigma\mu_k > 0$, we can obtain $f \equiv 0$ almost everywhere by letting $t = t_k \to \infty$ in (1.12). The case of $\Sigma\lambda_j - \Sigma\mu_k < 0$ can be treated in a similar way.

Case 2. Next we consider the remaining case $\Sigma\lambda_j - \Sigma\mu_k = 0$, which requires two preliminary notions — the covariant derivative in distribution sense, and an averaging operator. We have first to explain the former one. For a riemannian vector bundle F over V, denote, by $L^2(F)$, the Hilbert space of L^2 sections of F equipped with L^2-norm $\|\cdot\|_{L^2}$, and by $C^k(F)$, the Banach space of C^k sections of F equipped with the C^k-norm $\|\cdot\|_{C^k}$. Define the contraction of L^2 sections of F and of its dual bundle F^* by

$$(f\,|\,\sigma) = \int_V \langle f(v), \sigma(v)\rangle\, d\Lambda(v) \qquad \text{for} \quad f \in L^2(F), \quad \sigma \in L^2(F^*),$$

ere the integrand $\langle f(v), \sigma(v)\rangle$ denotes the contraction of $f(v) \in F_v$ and $\sigma(v) \in F_v^*$. It satisfies $|(f\,|\,\sigma)| \le \|f\|_{L^2} \cdot \|\sigma\|_{L^2}$, and the correspondence $f \in L^2(F) \mapsto (f\,|\,\cdot) \in L^2(F^*)^*$ yields an isomorphism $L^2(F) \cong L^2(F^*)^*$. In the rest of this section, put $F = E^{(\lambda_1,\cdots,\lambda_l;\mu_1,\cdots,\mu_m)}$ for simplicity. The φ_t-invariance of the Liouville measure Λ implies

$$(1.13) \qquad (\psi_t f\,|\,\psi_t\sigma) = (f, \sigma) \qquad \text{for} \quad f \in L^2(F), \quad \sigma \in L^2(F^*).$$

Now the covariant derivative by the canonical connection ∇ is regarded as a differential operator $\nabla : C^1(F) \to C^0(T^*V \otimes F)$, and the adjoint operator $\nabla^* : C^1(TV \otimes F^*) \to C^0(F^*)$ of ∇ is again realized as a first order differentiable operator. In fact, for $\tau \in C^1(TV \otimes F^*)$, define $\nabla^*\tau \in C^0(F^*)$ by contracting T^*V- and TV-components of $-\nabla\tau \in C^0(T^*V \otimes TV \otimes F^*)$. Then we actually have

$$(1.14) \qquad (\nabla f\,|\,\tau) = (f\,|\,\nabla^*\tau), \qquad \text{for} \quad f \in C^1(F), \quad \tau \in C^1(TV \otimes F^*),$$

nce the Liouville measure Λ (regarded as a volume form of V) is parallel with respect to the canonical connection ∇. With regard to (1.14), it is possible to introduce a ***covariant derivative in distribution sense*** as an operator $\nabla : L^2(F) \to C^1(TV \otimes F^*)^*$ by

$$(\nabla f\,|\,\tau) = (f\,|\,\nabla^*\tau) \qquad \text{for} \quad f \in L^2(F), \quad \tau \in C^1(TV \otimes F^*).$$

In particular, we say that $\nabla f = g$ ***in distribution sense*** for $f, g \in L^2(F)$, if $(\nabla f\,|\,\tau) = (g\,|\,\tau)$ for all $\tau \in C^1(TV \otimes F^*)$. It is not hard to see that for $f \in L^2(F)$, $\nabla f = 0$ in distribution sense if and only if f is parallel almost everywhere with respect to the canonical connection ∇ (cf. [S;§II.6]). Hence, to prove the proposition, it suffices to show $\nabla f = 0$ in distribution sense for all ψ_t-invariant L^2 sections f of F. Now recall that the tangent bundle of V has the splitting $TV = E^- + E^0 + E^+$ with $E^- = E^{-1} + E^{-2}$ and $E^+ = E^{+1} + E^{+2}$. This also gives rise to the decomposition of the covariant derivative in distribution sense into the "restricted derivatives" $\nabla^\nu : L^2(F) \to C^1(E^\nu \otimes F^*)^*$ ($\nu = 0, \pm$). In particular, we can easily prove that $\nabla^0 f = 0$ in distribution sense if $f \in L^2(F)$ is ψ_t-invariant. This can be seen as follows. Let $f \in L^2(F)$ be ψ_t-invariant. We have to prove $(\nabla^0 f \,|\, \tau) = (f \,|\, \nabla^{0*}\tau) = 0$ for all $\tau \in C^1(E^0 \otimes F^*)$, where $\nabla^{0*} : C^1(E^0 \otimes F^*) \to C^0(F^*)$ denotes the adjoint of ∇^0. Without loss of generality, we may assume that τ is of the form $\tau = \dot\varphi \otimes \sigma$, $\sigma \in C^1(F^*)$, where $\dot\varphi$ denotes the geodesic spray as before. Then, by (1.8), we have $\nabla^{0*}\tau = \nabla^{0*}(\dot\varphi \otimes \sigma) = \mathcal{L}_{\dot\varphi}\sigma = (d/dt)|_{t=0}\psi_t\sigma$, and therefore, (1.13) and ψ_t-invariance of f imply

$$(\nabla^0 f \,|\, \tau) = (f \,|\, \frac{d}{dt}\Big|_{t=0} \psi_t\sigma) = \frac{d}{dt}\Big|_{t=0} (f \,|\, \psi_t\sigma) = \frac{d}{dt}\Big|_{t=0} (f \,|\, \sigma) = 0.$$

In consequence, to prove the proposition, it is enough to show $\nabla^\pm f = 0$ in distribution sense for all ψ_t-invariant $f \in L^2(F)$.

We now turn to the definition of an *averaging operator* $A^+ : L^2(F) \to L^2(F)$. First let $\mathcal{B}$ be the Banach space of bounded linear operators on $L^2(F)$ whose norm is denoted by $\|\cdot\|_{\mathcal{B}}$, and define the *weak topology* of $\mathcal{B}$ so that for B, $B_k \in \mathcal{B}$ ($k = 1, 2, \cdots$),

$$\begin{gathered} B_k \to B \text{ in the weak topology if and only if} \\ (B_k f \,|\, \sigma) \to (Bf \,|\, \sigma) \text{ for any } f \in L^2(F),\ \sigma \in L^2(F^*). \end{gathered}$$

Then the unit ball $\mathcal{B}_1 = \{B \in \mathcal{B} : \|B\|_{\mathcal{B}} \le 1\}$ in $\mathcal{B}$ is sequentially compact relative to the weak topology (cf. [**R**;§10.6]). Now for each real number t, define $A_t \in \mathcal{B}$ by

$$(A_t f)(v) = \frac{1}{t}\int_0^t (\psi_s f)(v)\,ds, \qquad f \in L^2(F), \quad v \in V.$$

By (1.9), $\psi_t : L^2(F) \to L^2(F)$ is an isometry (recall that we are assuming $\Sigma\lambda_j - \Sigma\mu_k = 0$), and therefore we have $\|A_t\|_{\mathcal{B}} \le 1$. Hence, there is a sequence $\{t_k\}$ with $t_k \to \infty$ for which A_{t_k} converges to some $A^+ \in \mathcal{B}_1$ in the weak topology.

(1.15) **Lemma.** *For any $f \in L^2(F)$, $A^+ f$ is ψ_t-invariant.*

Proof. For $f \in L^2(F)$ and $T \in \mathbf{R}$, it follows that

$$\begin{aligned} \|\psi_T A_t f - A_t f\|_{L^2} &= \frac{1}{t}\left\| \int_0^t \psi_{s+T} f\,ds - \int_0^t \psi_s f\,ds \right\|_{L^2} \\ &\le \frac{1}{t}\left\{ \left\| \int_t^{t+T} \psi_s f\,ds \right\|_{L^2} + \left\| \int_0^T \psi_s f\,ds \right\|_{L^2} \right\} \le \frac{2T}{t}\|f\|_{L^2}. \end{aligned}$$

On the other hand, by letting $t = t_k \to \infty$, $(\psi_T A_t f \,|\, \sigma) = (A_t f \,|\, \psi_T^* \sigma)$ converges to $(A^+f \,|\, \psi_T^*\sigma) = (\psi_T A^+ f \,|\, \sigma)$, and $(A_t f \,|\, \sigma)$ to $(A^+ f \,|\, \sigma)$ for any $\sigma \in L^2(F^*)$. Hence we can conclude that $(\psi_T A^+ f \,|\, \sigma) = (A^+ f \,|\, \sigma)$ for all $f \in L^2(F)$ and $\sigma \in L^2(F^*)$. This proves the lemma. ∎

Suppose now that $f \in L^2(F)$ is ψ_t-invariant. Then we can always take $g \in C^1(F)$ so that $\|f-g\|_{L^2}$ is sufficiently small. Since $\|A^+\|_B \leq 1$ and $A^+f = f$, we obtain $\|f - A^+g\|_{L^2} = \|A^+(f-g)\|_{L^2} \leq \|f-g\|_{L^2}$. Furthermore it holds that

$$\left|(\nabla^+(f - A^+g) \,|\, \tau)\right| = \left|(f - A^+g \,|\, \nabla^{+*}\tau)\right| \leq \|f - A^+g\|_{L^2} \cdot \|\tau\|_{C^1}$$

for any $\tau \in C^1(E^+ \otimes F^*)$, where $\nabla^{+*} : C^1(E^+ \otimes F^*) \to C^0(F^*)$ denotes the adjoint of ∇^+. Thus, to see that $\nabla^+ f = 0$ in distribution sense, it is sufficient to prove

(1.16) **Lemma.** $\nabla^+ A^+ g = 0$ *in distribution sense for any* $g \in C^1(F)$.

Proof. Recall first that the action of ψ_t on F is isometric on each fiber (1.9), while $|\psi_t \alpha| \leq e^{-t}|\alpha|$ for $\alpha \in E^{+*}$ and $t \geq 0$ by (1.7). Thus, for $\nabla^+ g \in C^0(E^{+*} \otimes F)$, we obtain $\|\psi_t \nabla^+ g\|_{C^0} \leq e^{-t} \cdot \|g\|_{C^1}$, $t \geq 0$. Hence it follows from (1.10) that

$$\|\nabla^+ A_t g\|_{C^0} \leq \frac{1}{t} \int_0^t \|\nabla^+ \psi_s g\|_{C^0} ds \leq \frac{1 - e^{-t}}{t} \|g\|_{C^1}.$$

Consequently we obtain $(A_t g \,|\, \nabla^{+*}\tau) = (\nabla^+ A_t g \,|\, \tau) \to 0$ as $t \to \infty$ for $\tau \in C^1(E^+ \otimes F^*)$. Meanwhile, by the definition of A^+, $(A_t g \,|\, \nabla^{+*}\tau)$ converges to $(A^+ g \,|\, \nabla^{+*}\tau) = (\nabla^+ A^+ g \,|\, \tau)$ as $t = t_k$ goes to ∞. This proves $(\nabla^+ A^+ g \,|\, \tau) = 0$ for all $\tau \in C^1(E^+ \otimes F^*)$, and concludes the lemma. ∎

In consequence, we have $\nabla^+ f = 0$ in distribution sense for any ψ_t-invariant $f \in L^2(F)$. Of course, it is also possible to show that $\nabla^- f = 0$ in distribution sense for all ψ_t-invariant $f \in L^2(F)$, and this completes the proof of Proposition (1.11) in the case of $\Sigma\lambda_j - \Sigma\mu_k = 0$.

2. Negatively Curved Surfaces

We now proceed to the demonstration of the tensorial ergodicity for the geodesic flows on negatively curved surfaces. We first introduce a canonical connection on the unit tangent bundle of such a surface, and then show the tensorial ergodicity by modifying the arguments in the preceding section slightly. The method emploied here to construct the canonical connection is basically the same with Kanai [K] except the point that a suggestion given by S. Kaneyuki made our definition much simpler than [K] (cf. [KK], [KW]). Anyway the canonical connection is roughly speaking defined by combining contact geometry of the unit tangent bundle together with consideration on the dynamics of the geodesic flow.

First of all, we briefly review contact geometry of the unit tangent bundles of riemannian manifolds. Let M be a riemannian manifold whose unit tangent bundle is denoted by V. For a local coordinate system $\{x_i\}$ of M, every tangent vector of M is represented as $v = \sum \dot{x}_i \partial/\partial x_i$, and in consequence, we obtain the local coordinate system $\{x_i, \dot{x}_i\}$ of the tangent bundle TM. In terms of these coordinates, it is possible to define a 1-form θ_0 on TM which is expressed locally as $\theta_0 = \sum g_{ij}\dot{x}_i dx_j$, where g_{ij}'s denote the coefficients of the metric tensor of M in the coordinate system $\{x_i\}$. Now pull back θ_0 by the inclusion of the unit tangent bundle V into TM. The resulting 1-form θ of V, called the *canonical contact form*, relates to the geodesic flow φ_t of M in the following manner. (1) The canonical contact form θ and its exterior derivative $d\theta$ are φ_t-invariant (Liouville's theorem). (2) $\theta(\dot{\varphi}) = 1$ for the geodesic spray $\dot{\varphi} = (d/dt)|_{t=0}\varphi_t$. (3) $d\theta(\dot{\varphi}, \cdot) = 0$. In addition, we can easily show that (4) the 2-form $d\theta$ is nondegenerate on the subbundle $E = \{\xi \in TV : \theta(\xi) = 0\}$.

Next suppose that M is a closed riemannian manifold of negative sectional curvature. Then the geodesic flow φ_t of M is an Anosov flow; that is, the ngent bundle of V carries a unique φ_t-invariant continuous splitting $TV = E^- + E^0 + E^+$ into linear subbundles satisfying the following two conditions: (1) E^0 is spanned by the geodesic spray $\dot{\varphi}$; (2) For each $\xi^\pm \in E^\pm$, $d\varphi_t\xi^\pm$ contracts exponentially as $t \to \mp\infty$. We call the splitting $TV = E^- + E^0 + E^+$ the *Anosov splitting* of M. Fundamental relations between the Anosov splitting and the canonical contact form θ can be summarized in

(2.1) **Lemma.** *(1)* $\theta(\xi) = 0$ *whenever* $\xi \in E^- + E^+$.
(2) For any $\xi^\pm, \eta^\pm \in E^\pm$, $d\theta(\xi^-, \eta^-) = d\theta(\xi^+, \eta^+) = 0$.

Proof. Suppose that $\xi^- \in E^-$. Then Liouville's theorem implies that $|\theta(\xi^-)| = |(\varphi_t^*\theta)(\xi^-)| = |\theta(d\varphi_t\xi^-)| \leq \|\theta\| \cdot |d\varphi_t\xi^-|$. Since $|d\varphi_t\xi^-|$ tends to zero as t goes to ∞, we have $\theta(\xi^-) = 0$. Similarly we can show that $\theta(\xi^+) = 0$ for $\xi^+ \in E^+$, and this proves (1). The second assertion (2) can be proved in a similar way. ∎

The first assertion in the lemma implies that the 2-form $d\theta$ is nondegenerate when it is restricted to $E \times E$, where $E = \{\xi \in TV : \theta(\xi) = 0\} = E^- + E^+$. In other words, $d\theta$ is a symplectic structure of the vector bundle E. Then the second assertion claims that $E = E^- + E^+$ is a lagrangian splitting of E with respect to the symplectic structure $d\theta$. By virtue of these observations, we can define a continuous pseudo riemannian metric g of V in the following way. First let I be the continuous involution of E characterized by $I|E^\pm = \pm\mathrm{id}$. Then $g_0(\xi, \eta) = d\theta(\xi, I\eta)$ $(\xi, \eta \in E)$ is a pseudo riemannian structure of E, and $g = g_0 + \theta \otimes \theta$ is the desired pseudo riemannian metric of V.

To introduce the canonical connection of V, assume especially that the Anosov splitting of M is C^1-*differentiable.* This assumption is fulfilled especially if M is a surface, or if the sectional curvature of M satisfies the pinching condition $-4 < K_M \leq -1$ (see Hirsch–Pugh [$\mathbf{HP}_1$], [$\mathbf{HP}_2$]). Under this condition, the pseudo riemannian metric g of V we have just defined is C^1-differentiable, and has the continuous Levi-Civita connection ∇. It is not hard to see that the canonical connection ∇ possesses the following properties. (1) The subbundles E^0, $E^{\pm}$ of TV are ∇-stable. (2) $\nabla\theta = 0$, $\nabla d\theta = 0$ and $\nabla\Lambda = 0$, where $\Lambda = \theta \wedge (d\theta)^n$ $(n+1 = \dim M)$ denotes the Liouville measure. (3) ∇ is invariant under the geodesic flow φ_t. (4) $\nabla_{\dot\varphi}\xi = \mathcal{L}_{\dot\varphi}\xi$ for any vector field ξ on V.

Notice here that the Anosov splitting of a closed locally symmetric riemannian manifold of negative curvature is C^∞-differentiable. Thus its unit tangent bundle has the canonical connection as above. However it does not coincide with the canonical connection introduced in the preceding section unless M is of constant negative curvature. In fact, for the complex, quaternion and Cayley hyperbolic spaces, the canonical connections of their unit tangent bundles given in §1 have torsion, while the canonical connection defined here is torsion-free.

We are now in the position to prove the tensorial ergodicity for surfaces. Let M be a 2-dimensional closed riemannian manifold of negative curvature. Then the subbundles $E^{\pm}$ appearing in the Anosov splitting of M are both 1-mensional, and therefore we obtain functions $h^{\pm}(v,t)$ $(v \in V,\ t \in \mathbf{R})$ such that

$$(2.2) \qquad |d\varphi_t\xi^{\pm}| = e^{\pm h^{\pm}(v,t)}|\xi^{\pm}| \qquad \text{for} \quad \xi^{\pm} \in E_v^{\pm}, \quad v \in V,$$

where $E_v^{\pm}$ denotes the fiber of $E^{\pm}$ over $v \in V$. These functions should satisfy

$$(2.3) \qquad c_1^{-1}t \le h^{\pm}(v,t) \le c_1 t \qquad \text{and} \qquad |h^+(v,t) - h^-(v,t)| \le c_2,$$

where c_1 and c_2 are positive constants. The first assertion actually follows from the assumption on the curvature of M together with the compactness of M. Meanwhile, the last inequality is a consequence of the following three facts: (1) The symplectic structure $d\theta$ of $E = E^- + E^+$ is preserved by the geodesic flow; (2) $E = E^- + E^+$ is a lagrangian splitting of the symplectic vector bundle $(E, d\theta)$; (3) The angles between the lines E_v^- and E_v^+ in T_vV are uniformly bounded in $v \in V$. Here set $E^{\pm 1} = E^{\pm}$, $h^{\pm 1}(v,t) = h^{\pm}(v,t)$ and $h^0(v,t) \equiv 0$. Then, by (2.2), the natural lift ψ_t of the geodesic flow φ_t to the vector bundle

$$E^{(\lambda_1,\cdots,\lambda_\ell;\mu_1,\cdots,\mu_m)} = E^{\lambda_1} \otimes \cdots \otimes E^{\lambda_\ell} \otimes E^{\mu_1 *} \otimes \cdots \otimes E^{\mu_m *}$$

with $\lambda_1, \cdots, \lambda_\ell, \mu_1, \cdots, \mu_m = 0, \pm 1$ satisfies

$$(2.4) \qquad |\psi_t\tau| \le c_3 \cdot \exp\left\{\sum_{j=1}^{\ell} \lambda_j h^{\lambda_j}(v,t) - \sum_{k=1}^{m} \mu_k h^{\mu_k}(v,t)\right\} \cdot |\tau|$$

for $\tau \in E^{(\lambda_1,\cdots,\lambda_\ell;\mu_1,\cdots,\mu_m)}$. Thus, in the case where $\Sigma\lambda_j - \Sigma\mu_k \ne 0$, $\psi_t\tau$, $\tau \in E^{(\lambda_1,\cdots,\lambda_\ell;\mu_1,\cdots,\mu_m)}$, contracts exponentially as t diverges to ∞ or $-\infty$ by (2.3). This proves as in §1 that each measurable ψ_t-invariant section of $E^{(\lambda_1,\cdots,\lambda_\ell;\mu_1,\cdots,\mu_m)}$ with $\Sigma\lambda_j - \Sigma\mu_k \ne 0$ vanishes almost everywhere.

Next we consider the case of $\Sigma\lambda_j - \Sigma\mu_k = 0$. Also in this case, the proof goes as in the preceding section. Notice here that (2.4) together with the last inequality in (2.3) guarantees that ψ_t is an "almost isometry" on each fiber of $E^{(\lambda_1,\cdots,\lambda_\ell;\mu_1,\cdots,\mu_m)}$; that is, there is a constant c_4 such that

$$|\psi_t\tau| \leq c_4 \cdot |\tau| \quad \text{for} \quad \tau \in E^{(\lambda_1,\cdots,\lambda_\ell;\mu_1,\cdots,\mu_m)}, \quad t \in \mathbf{R}.$$

This again makes it possible for us to follow the arguments in §1 with suitable slight modification, and we can conclude the tensorial ergodicity of the geodesic flows of negatively curved surfaces.

3. Differentiable Tensor Fields

As we have observed in the previous sections, the geodesic flows of negatively curved surfaces and locally symmetric spaces possess the tensorial ergodicity. In particular, in the former case, the tensorial ergodicity is stated in terms of the canonical connection of the unit tangent bundle, while the defi- ,ion of the canonical connection requires only the C^1-differentiability of the Anosov splitting. On the other hand, we know that there are actually a number of negatively curved manifolds other than surfaces which have C^1-differentiable Anosov splittings: For example, closed riemannian manifolds whose sectional curvatures are bounded between -4 and -1 strictly have this property (remind Hirsch–Pugh [$\mathbf{HP}_1$], [$\mathbf{HP}_2$]). Thus the unit tangent bundles of these manifolds carry canonical connections, and it makes sense to ask whether the geodesic flows of these manifolds possess the tensorial ergodicity. Unfortunately we have no answer to the this question. However, if we restrict ourselves to considering only *differentiable* tensor fields, we can obtain a partial result, and that is the purpose of the present section.

To mention it clearly, it is convenient to reformulate our problem slightly. Suppose that M is a closed riemannian manifold of negative curvature, and let $\widetilde{M}$ be the universal covering of M. Then the unit tangent bundle $\widetilde{V}$ of $\widetilde{M}$ has two group actions: One of them is the action of $\mathbf{R}$ on $\widetilde{V}$ as the geodesic flow ˆ $\widetilde{M}$, and the other is the action of the fundamental group $\Gamma = \pi_1(M)$ of M that is obtained by lifting the covering action of Γ on $\widetilde{M}$ to $\widetilde{V}$. Since those actions on $\widetilde{V}$ commute with each other, the orbit space of one of these actions again has an group action induced by the other. In fact, the induced action of $\mathbf{R}$ on the the orbit space $\widetilde{V}/\Gamma$, which is naturally identified with the unit tangent bundle V of M, is nothing but the geodesic flow of M. Meanwhile, Γ acts on the orbit space of $P = \widetilde{V}/\mathbf{R}$ of the geodesic flow of $\widetilde{M}$. It is easy to see that the orbit space P naturally becomes a smooth manifold, and the action of Γ on P is differentiable. These two actions, the geodesic flow of M and the action of Γ on P, are essentially equivalent in the viewpoint of ergodic theory. For instance, the ergodicity of the geodesic flow of M can be explained by the ergodicity of the Γ-action on P, and *vice versa*. Furthermore, in some cases, such as the problem we are now involved in, the description is much simpler for the Γ-action on P than for the geodesic flow of M. So, in the rest, we will discuss on the Γ-action on P instead of the geodesic flow of M.

First we restate the results we have obtained in the earlier sections in the new framework. Throughout the following discussions, let $\pi : \widetilde{V} \to P$ be the projection of $\widetilde{V}$ onto the orbit space $P = \widetilde{V}/\mathbf{R}$. Then, for the lift $T\widetilde{V} = \widetilde{E}^- + \widetilde{E}^0 + \widetilde{E}^+$ of the Anosov splitting $TV = E^- + E^0 + E^+$ of M, the subbundle $\widetilde{E} = \widetilde{E}^- + \widetilde{E}^+$ of $T\widetilde{V}$ is horizontal relative to the $\mathbf{R}$-fibering $\pi : \widetilde{V} \to P$. For each tangent vector ξ of P, let $\xi^* \in \widetilde{E}$ be the horizontal lift of ξ. On the other hand, each 1-form α of P is pulled back to the 1-form $\alpha^* = \pi^*\alpha$ of $\widetilde{V}$. Hence it is also possible to define the horizontal lift of any tensor of P: For a tensor τ of P, denote the tensor of $\widetilde{V}$ obtained by lifting τ horizontally to $\widetilde{V}$ by τ^*. A measurable tensor field τ on P is said to be *locally L^2-integrable*,or simply, L^2_{loc} if the horizontal lift τ^* is a locally L^2-integrable tensor field on $\widetilde{V}$; i.e., $\int_K |\tau^*|^2 < \infty$ for any compact subset K of $\widetilde{V}$. Notice further that for a invariant tensor field τ on P, its lift τ^* to $\widetilde{V}$ is invariant by the geodesic flow of $\widetilde{M}$ as well as the Γ-action on $\widetilde{V}$. Therefore τ^* induces a tensor field $\hat{\tau}$ on the unit tangent bundle $V = \widetilde{V}/\Gamma$ of M which is invariant under the geodesic flow of M. Of course, $\hat{\tau}$ is L^2-integrable over V provided τ is L^2_{loc}.

A. Locally Symmetric Spaces. Suppose specifically that M is locally symmetric. Then the unit tangent bundle $\widetilde{V}$ of $\widetilde{M}$ has the canonical connection ∇ as in §1, and both the geodesic flow of $\widetilde{M}$ and the Γ-action on $\widetilde{V}$ preserve the connection ∇. Thus we can define a Γ-invariant connection D on P by $(D_\xi\tau)^* = \nabla_{\xi^*}\tau^*$ for any vector field ξ and tensor field τ on P. The connection D can be also described as follows. Since the identity component G of the isometric transformation group of the symmetric space $\widetilde{M}$ acts on $\widetilde{V}$ transitively, G acts also on P transitively so that the projection $\pi : \widetilde{V} \to P$ is G-equivariant. In fact, if G^0 denotes the connected Lie subgroup of G with the Lie algebra $\mathbf{g}^0$ in the gradation (1.1) of the Lie algebra $\mathbf{g}$ of G, then P is represented as the homogeneous space G/G^0. Further, (1.2) guarantees that $P = G/G^0$ is a ductive homogeneous G-space, and D is just the canonical connection of P as a homogeneous G-space (cf. [**KN**]). It is easy to show that a tensor field on P is parallel with respect to D if and only if it is invariant under the action of G on P. Thus the tensorial ergodicity of the geodesic flow of M we proved in §1 implies

(3.1) **Corollary.** *Every Γ-invariant L^2_{loc} tensor field on P is G-invariant almost everywhere.*

B. Negatively Curved Surfaces. Next consider a 2-dimensional closed riemannian manifold M of negative curvature. The canonical connection ∇ of the unit tangent bundle V of M which we constructed in §2 is lifted to a connection of $\widetilde{V}$, and the lifted connection on $\widetilde{V}$, which we denote by the same symbol ∇, is invariant under both the Γ-action on $\widetilde{V}$ and the geodesic flow of $\widetilde{M}$. Thus we can again define a continuous Γ-invariant connection D of P by $(D_\xi \tau)^* = \nabla_{\xi^*} \tau^*$ for any vector field ξ and tensor field τ on P. The following assertion is an immediate consequence of the tensorial ergodicity for the geodesic flows of surfaces.

(3.2) **Corollary.** *A Γ-invariant L^2_{loc} tensor field on P is parallel almost everywhere with respect to D.*

C. Manifolds with Pinched Negative Curvature. We now proceed to the study of continuous or differentiable tensor fields on P which are invariant under the action of Γ on P. Suppose that M is a closed riemannian manifold whose sectional curvature satisfies the pinching condition

$$-\left(\frac{c+1}{c}\right)^2 < K_M \leq -1 \tag{3.3}$$

with a nonnegative integer c. Then we first have

(3.4) **Proposition.** *Under the condition (3.3), every Γ-invariant continuous tensor field on P of odd degree $\leq 2c+1$ vanishes.*

Proof. Recall here that the exterior derivative $d\tilde{\theta}$ of the canonical contact form $\tilde{\theta}$ of $\widetilde{V}$ is invariant under the geodesic flow of $\widetilde{M}$ (Liouville's theorem). Thus it induces a 2-form ω on P which is characterized by the condition $\pi^*\omega = d\tilde{\theta}$, and it is easily seen to be a Γ-invariant smooth symplectic form of P. Especially it yields the Γ-equivariant isomorphism $TP \cong T^*P$ defined by $\xi \mapsto \omega(\xi, \cdot)$. This guarantees that it is enough to consider only covariant tensor fields. Let g be a continuous covariant tensor field on P of degree $2d+1 \leq 2c+1$ which is invariant under the action of Γ on P. Then $f = \hat{g}$ is a continuous covariant tensor field on the unit tangent bundle V of M which is invariant under the geodesic flow φ_t of M, and to see that $g \equiv 0$, it is sufficient to prove $f|E^{\lambda_1} \times \cdots \times E^{\lambda_{2d+1}} \equiv 0$ for $\lambda_1, \cdots, \lambda_{2d+1} = \pm$, where $TV = E^- + E^0 + E^+$ denotes the Anosov splitting of M. Let N^- be the number of $-$'s appearing in $\lambda_1, \cdots, \lambda_{2d+1}$, and N^+ the number of $+$'s. Note that we have only two possibilities $N^- > N^+$ and $N^- < N^+$ since $N^- + N^+ = 2d+1$ is odd. We here give the proof in the former case, the other is treated in the same way. By the compactness of M, we can take a constant Λ so that $-\{(c+1)/c\}^2 < -\Lambda^2 \leq K_M \leq -1$ $(1 \leq \Lambda < (c+1)/c)$, and the standard comparison theorem implies the following estimates for the hyperbolicity of the geodesic flow φ_t:

$$e^t|\xi^-| \leq |d\varphi_{-t}\xi^-| \leq e^{\Lambda t}|\xi^-|, \qquad e^{-\Lambda t}|\xi^-| \leq |d\varphi_t\xi^-| \leq e^{-t}|\xi^-|,$$
$$e^t|\xi^+| \leq |d\varphi_t\xi^+| \leq e^{\Lambda t}|\xi^+|, \qquad e^{-\Lambda t}|\xi^+| \leq |d\varphi_{-t}\xi^+| \leq e^{-t}|\xi^+|,$$

where $\xi^\pm \in E^\pm$ and $t \geq 0$. Thus we have the following for any $\xi^{\lambda_j} \in E^{\lambda_j}$ and $t > 0$ since f is φ_t-invariant:

$$\left|f(\xi^{\lambda_1}, \cdots, \xi^{\lambda_{2d+1}})\right| = \left|(\varphi_t^* f)(\xi^{\lambda_1}, \cdots, \xi^{\lambda_{2d+1}})\right| \tag{3.5}$$

$$
\begin{aligned}
&= \left|f(d\varphi_t\xi^{\lambda_1}, \cdots, d\varphi_t\xi^{\lambda_{2d+1}})\right| \\
&\le \|f\|\,|d\varphi_t\xi^{\lambda_1}|\cdots|d\varphi_t\xi^{\lambda_{2d+1}}| \\
&\le \|f\|\,|\xi^{\lambda_1}|\cdots|\xi^{\lambda_{2d+1}}|\cdot e^{(\Lambda N^+ - N^-)t}.
\end{aligned}
$$

Notice here that $N^- \ge d+1$ and $N^+ \le d$ since $N^- > N^+$ with $N^- + N^+ = 2d+1$. On the other hand, we have been assuming $\Lambda < (c+1)/c \le (d+1)/d$. Thus we obtain $\Lambda N^+ - N^- < 0$, and, in consequence, $f(\xi^{\lambda_1}, \cdots, \xi^{\lambda_{2d+1}}) = 0$ by letting $t \to \infty$ in (3.5). ∎

Now assume especially that $c \ge 1$ in (3.3), that is, the curvature of M lies between -4 and -1 strictly. Then, by the theorem of Hirsch–Pugh [**HP**$_1$], [**HP**$_2$], the Anosov splitting of M is C^1-differentiable, and therefore the unit tangent bundle V of M has a canonical connection ∇ (recall §2), which again gives rise to a Γ-invariant continuous connection D on P as in the previous case. The connection D of P has another simpler description. As we have already mentioned in the proof of Proposition (3.4), P has a canonically defined Γ-invariant symplectic form ω. In addition, the Anosov splitting $T\widetilde{V} = \widetilde{E}^- + \widetilde{E}^0 + \widetilde{E}^+$ of $\widetilde{M}$ is also invariant under the geodesic flow of $\widetilde{M}$, and the subbundles $\widetilde{E}^\pm$ are transverse to the fibers of the $\mathbf{R}$-fibering of $\widetilde{V}$ over P. Thus they induce the C^1 subbundles $F^\pm$ of TP such that $TP = F^- + F^+$ is a lagrangian splitting of the tangent bundle of P; i.e., $\dim F^\pm = \dim P/2$ and $\omega(\xi^-, \eta^-) = \omega(\xi^+, \eta^+) = 0$ for any $\xi^\pm, \eta^\pm \in F^\pm$ (cf. Lemma (2.1)). Furthermore, the subbundles $F^\pm$ are integrable since so are $E^\pm$, and, in consequence, we obtain Γ-invariant lagrangian foliations $\mathcal{F}^\pm$ of the symplectic manifold (P, ω) which integrate $F^\pm$. Now define a Γ-invariant C^1 pseudo riemannian metric g of P by $g(\xi, \eta) = \omega(\xi, I\eta)$ for $\xi, \eta \in TP$, where I denotes the involution of TP given by $I|_{F^\pm} = \pm\mathrm{id}$. Then D coincides with the Levi–Civita connection of the pseudo riemannian metric g of P. It is obvious that for any Γ-invariant differentiable tensor field f on P, the covariant derivative Df is again Γ-invariant. Thus Proposition (3.4) has

(3.6) **Corollary.** *Under the assumption (3.3) with $c \ge 1$, we always have $Df \equiv 0$ for any Γ-invariant C^1 tensor field f on P of even degree $\le 2c$.*

An application of Corollary (3.6) was already made by author [**K**] concerned with the smoothness problem of the Anosov splitting. It proceeds as follows. Suppose that M is a closed riemannian manifold of negative curvature whose Anosov splitting is C^∞. Then the canonical connection ∇ on the unit tangent bundle V of M is C^∞, and therefore so is the connection D of P. In particular the curvature tensor R of D is defined as a Γ-invariant smooth tensor field on P of degree four, and we can apply Corollary (3.6) to the curvature tensor R with $c = 2$:

$$DR \equiv 0 \tag{3.7}$$

provided $-9/4 < K_M \le -1$. This equation means that the affine connection D on P is locally symmetric, and has a quite strong implication. In fact, we proved in [**K**] that under (3.7) the geodesic flow φ_t of M is completely isomorphic to the geodesic flow of a certain closed riemannian manifold of *constant* negative curvature. Hence we obtain

(3.8) **Theorem** ([**K**]). *Suppose that M is a closed riemannian manifold with sectional curvature $-9/4 < K_M \leq -1$. If the Anosov splitting of M is C^∞, then the geodesic flow of M is isomorphic to the geodesic flow of a certain closed riemannian manifold of constant negative curvature.*

In the case of dimension two, a much stronger result had been obtained by Ghys [**G**]. He proved that a 2-dimensional closed riemannian manifold M of negative curvature should be of constant curvature provided that the Anosov splitting of M is C^2-differentiable (cf. Hurder–Katok [**HK**]). Thus Theorem (3.8) is a partial generalization of the theorem of Ghys. Furthermore Katok and Feres recently improved our arguments extensively, and showed that in Theorem (3.8) we can replace the pinching condition $-9/4 < K_M \leq -1$ by $K_M < 0$ in the case of $\dim M = 3$ (Katok [**Kt**]), and by $-4 < K_M \leq -1$ in the case of $\dim M = 4$ (Feres [**F**]). Actually they proved the identity (3.7) under their conditions by the aid of the multiplicative ergodic theorem of Oseledec. This may suggest the possibility that the multiplicative ergodic theorem would sometimes strengthen our results (3.4) and (3.6).

References

[**AM**] R. Abraham and J. E. Marsden, "Foundations of Mechanics," 2nd ed., Benjamin, Reading, 1978.

[**A**] D. V. Anosov, *Geodesic flows on closed riemannian manifolds with negative curvature* (English translation), Proc. Steklov Inst. Math. **90** (1969).

[**AS**] D. V. Anosov and Ya. G. Sinai, *Some smooth ergodic systems* (English translation), Russian Math. Surveys **22** (1967), 103–167.

[**F**] R. Feres, *Rigidity of geodesic flows on negatively curved 4-manifolds*, to appear.

[**G**] E. Ghys, *Flots d'Anosov dont les feuilletages stables sont différentiables*, Ann. scient. Éc. Norm. Sup. **20** (1987), 251–270.

[**H**] S. Helgason, "Differential Geometry, Lie Groups, and Symmetric Spaces," Academic Press, New York, 1978.

[**HP$_1$**] M. W. Hirsch and C. C. Pugh, *Stable manifolds and hyperbolic sets*, in "Proc. Sympos. Pure Math. vol.14," Amer. Math. Soc., Providence, 1970, pp. 133–163.

[**HP$_2$**] M. W. Hirsch and C. C. Pugh, *Smoothness of horocycle foliations*, J. Diff. Geom. **10** (1975), 225–238.

[**HK**] S. Hurder and A. Katok, *Differentiability, rigidity and Godbillon–Vey classes for Anosov flows*, to appear.

[**K**] M. Kanai, *Geodesic flows of negatively curved manifolds with smooth stable and unstable foliations*, to appear in Ergod. Th. & Dynam. Sys..

[**KK**] S. Kaneyuki and M. Kozai, *Paracomplex structures and affine symmetric spaces*, Tokyo J. Math. **8** (1985), 81–98.

[KW] S. Kaneyuki and F. L. Williams, *Almost paracontact and parahodge structures on manifolds*, Nagoya Math. J. 99 (1985), 173–187.

[Kt] A. Katok, *Rigidity of geodesic flows on negatively curved 3-manifolds*, to appear.

[KN] S. Kobayashi and K. Nomizu, "Foundations of Differential Geometry, Vol. II," Interscience, New York, 1969.

[P] W. Parry, "Topics in Ergodic Theory," Cambridge Univ. Press, Cambridge, 1981.

[R] H. L. Royden, "Real Analysis," 2nd ed., MacMillan, New York, 1968.

[S] L. Schwartz, "Théorie des Distributions," 3rd ed., Hermann, Paris, 1966.