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Asymptotic distributions of digits in integers

by

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For an increasing sequence of positive integers $n_1, n_2, \cdots$, let $A(x)$ be the number of n_i's up to x. Copeland and Erdös [1] proved that if, for any $\varepsilon>0$,

(1) $$A(x) > x^{1-\varepsilon}$$

provided that x is sufficiently large, or what amounts the same thing

$$\log\frac{x}{A(x)} = o(\log x)$$

as $x \to \infty$, then the infinite decimal $0.n_1n_2\cdots$ is normal to base r when each of these integers n_i is replaced by its r-adic expansion. In the present paper we shall refine this results and make some remarks on it.

§1. Statements of Results

Throughout of this paper r is an integer greater than 1. Every positive integer n can be written uniquely as

$$n = a_1a_2\cdots a_k = \sum_{i=1}^{k} a_i r^{k-i}, \tag{2}$$

where each a_i is one of $0,1,\cdots,r-1$, and $a_1\neq 0$, or equivalently $k=[\log_r n]+1$, where $[t]$ is the greatest integer not exceeding a real number t. For any sequence $b_1,b_2,\cdots,b_\ell$ of 0's, 1's, $\cdots$, and $r-1$'s of length ℓ, we denote by $N(n) = N_r(n;b_1b_2\cdots b_\ell)$ the number of indices i's in the expression (2) such that $a_i= b_1, a_{i+1}=b_2,\cdots,a_{i+\ell-1}=b_\ell$. We put $s_r(n) = \sum_{i=1}^{k} a_i$, so that

$$s_r(n) = \sum_{b=1}^{r-1} bN_r(n;b) \tag{3}$$

Theorem 1. Let $n_1, n_2, \cdots$ be any, finite or infinite, increasing sequence of positive integers. Then, for any ℓ integers $b_1, b_2, \cdots, b_\ell$ with $0 \leqq b_i < r$, we have

$$\left| \sum_{n_i \leqq x} N_r(n_i; b_1 b_2 \cdots b_\ell) - \frac{1}{r} A(x) \log_r x \right|$$

$$\leqq c \left(\frac{\log \frac{x}{A(x)} + \log \log x}{\log x} \right)^{1/2} A(x) \log_r x,$$

provided $x \geqq r^{\ell r^2}$, where $c = 2^8 \ell \sqrt{\ell} \log r$.

By the relation (1) we have the following

Corollary 1. (cf. Heppner [2]). For any, finite or infinite, increasing sequence $n_1, n_2, \cdots$ of positive integers, we have

$$\left| \sum_{n_i \leqq x} s_r(n_i) - \frac{r-1}{2} A(x) \log_r x \right|$$

$$\leqq c_1 \left(\frac{\log \frac{x}{A(x)} + \log \log x}{\log x} \right)^{1/2} A(x) \log_r x,$$

provided $x \geqq r^{r^2}$, where $c_1 = 2^7 r(r-1)\sqrt{\log r}$.

Example 1. Let $\varphi(n)$ be the Enler function. Then we obtain

$$\sum_{\substack{m \leqq n \\ (m,n)=1}} s_r(m) = \frac{r-1}{2} \varphi(n) \log_r n \cdot \left(1 + O\left(\left(\frac{\log \log n}{\log n}\right)^{1/2}\right)\right),$$

using the estimates $\liminf_{n \to \infty} \varphi(n)(\log \log n)/n > 0$.

Example 2. (Shiokawa [8], Heppner [2]).

$$\sum_{p \leqq x} s_r(p) = \frac{r-1}{2} \frac{x}{\log r} \left(1 + O\left(\left(\frac{\log \log x}{\log x}\right)^{1/2}\right)\right),$$

where p runs through prime numbers.

Every irrational number θ with $0 \leqq \theta < 1$ can be uniquely expanded to base r as

$$\theta = 0.a_1 a_2 \cdots = a_1 r^{-1} + a_2 r^{-2} + \cdots, \tag{4}$$

where each a_i is one of $0, 1, \cdots, r-1$. Then θ is said to be normal to base r, if the relative frequency $t^{-1}N(t;\theta;b_1 b_2 \cdots b_\ell)$ of a given sequence $b_1, b_2, \cdots, b_\ell$ as in Theorem 1 tends to $r^{-\ell}$ as $t \to \infty$, where $N(t;\theta;b_1 b_2 \cdots b_\ell)$ is the number of occurrences of the sequence $b_1 b_2 \cdots b_\ell$ in the first t digits $a_1 a_2 \cdots a_t$. Let $n_1, n_2, \cdots$ be any increasing sequence of positive integers and let $n_j = a_{j1} a_{j2} \cdots a_{jk_j}$ be the expression (2) of n_j. We define by (4) a number $\theta_r = 0.a_{11} a_{12} \cdots a_{1k_1} a_{21} \cdots a_{2k_2} \cdots$, which will be written simply as $\theta_r = 0.n_1 n_2 \cdots$. Then Copeland and Erdös proved that, for any increasing sequence $n_1, n_2 \cdots$ of positive integers satisfying (1), $t^{-1}N(t;\theta_r;b_1 b_2 \cdots b_\ell) = r^{-\ell} + o(1)$ as $t \to \infty$. It seems difficult, in general, to estimate the remainder $o(1)$ explicitly in terms of t. However, choosing an index $i = i(t)$ such that

$$\sum_{h=1}^{i-1} [\log_r n_h + 1] \leqq t < \sum_{h=1}^{i} [\log_r n_h + 1],$$

we have the following

Theorem 2. Let $n_1, n_2, \cdots$ be any increasing sequence of positive integers satisfying (1). Then, for any $b_1, b_2, \cdots, b_\ell$ as in Theorem 1, we have

$$\frac{N(t)}{t} = \frac{1}{r^\ell} + O\left(\frac{\log \frac{n_{i(t)}}{i(t)} + \log\log t}{\log t} \right)^{1/2},$$

where $\log(n_{i(t)}/i(t)) = o(\log t)$ as $t \to \infty$ and the constant implied depends possibly on r and ℓ.

Example 3. Let $\theta_r = 0.23571113\cdots$ be the number defined by the sequence of primes. Then

$$\frac{1}{t}N(t;\theta_r;b_1b_2\cdots b_\ell) = \frac{1}{r^\ell}\left(1 + O\left(\frac{\log\log t}{\log t}\right)^{1/2}\right).$$

The assumption (1) in the theorem of Copeland and Erdös as well as Corollary 2 is indispensable as the following theorem shows.

Theorem. (Shiokawa [7]) For any given $\varepsilon>0$, there is an increasing sequence $n_1,n_2,\cdots$ of positive integers satisfying

$$A(x) > x^{1-\varepsilon},$$

provided that x is sufficiently large, such that $\theta_r=0.n_1n_2\cdots$ is not normal to base r.

The estimate obtained in Theorem 1 is best possible in the sense that $O((\log(x/A(x)) + \log\log x)/\log x)^{1/2}$ cannot be replaced by $o((\log(x/A(x)) + \log\log x)/\log x)^{1/2}$. The same is true for Theorem 2, and when $r=2$ for Corollary 1, since $N_2(n;1)=s_2(n)$. Indeed we have the following

Theorem 3. Let b be one of $0,1,\cdots,r-1$. Then there is an increasing sequence $n_1,n_2,\cdots$ of positive integers satisfying

$$A(x) > \frac{1}{8}\,\frac{x}{\log x}$$

for all sufficiently large x such that

$$N_r(n;b) \geqq \left(1 + \frac{1}{16}\left(\frac{\log\log x}{r\log x}\right)^{1/2}\right)\frac{1}{r}A(x)\log_r x$$

for infinitely many integers x.

For some specified sequences $n_1,n_2,\cdots$ satisfying the condition (1), one may get better results than those which can be deduced from Theorem 1. For instance, we can show the following by a method similar to that used in [3]: Let ℓ and m be integers such that $0\leqq\ell<m$, and

let $b_1, b_2, \cdots, b_\ell$ as in Theorem 1. Then

$$\sum_{\substack{n\leq x \\ n\equiv a(\mathrm{mod}\ m)}} N_r(n;b_1b_2\cdots b_\ell) = \frac{1}{r}\frac{x}{m}\log_r x + O(x)$$

and

$$\sum_{\substack{n\leq x \\ (n,m)=1}} N_r(n;b_1b_2\cdots b_\ell) = \frac{1}{r}\frac{\varphi(m)}{m}x\log_r x + O(d(m)x).$$

where $d(n)$ is the number of divisors of n. By (1) we also have the estimates for the corresponding sums of digits as Corollary 1.

Finally we exhibit a theorem which is of interest in connection with Example 1.

Theorem 4. For any ℓ integers $b_1, b_2, \cdots, b_\ell$ with $0\leq b_i<r$, we have

$$\sum_{n\leq x}\sum_{\substack{m\leq n \\ (m,n)=1}} N_r(n;b_1b_2\cdots b_\ell) = \frac{1}{r}\frac{3}{\pi^2}x^2\log_r x\cdot\left(1+O\left(\frac{\log\log x}{\log x}\right)\right),$$

as $x\to\infty$, where the constant implied depends possibly on r and ℓ. Especially,

$$\sum_{n\leq x}\sum_{\substack{m\leq n \\ (m,n)=1}} s_r(m) = \frac{r-1}{2}\frac{3}{\pi^2}x^2\log_r x\cdot\left(1+O\left(\frac{\log\log x}{\log x}\right)\right).$$

§2. Proofs

For the proofs of Theorems 1 and 3, we need the following Lemma which is a refinement of those proved in [6] and [7]. We denote by $T(k,\varepsilon;b)$ and $S(k,\varepsilon;b)$ the number of integers n with $0\leq n<r^k$ such that $|N_r(n;b)-k/r| > k\varepsilon$ and $N_r(n;b)-k/r > k\varepsilon$, respectively.

Lemma. Let b be any integer with $0\leq b<r$. Then we have

$$T(k,\varepsilon;b) < r^k\, k\exp\left(-\frac{1}{32}k\varepsilon^2\right)$$

and

$$S(k,\varepsilon;b) > \frac{1}{16}\, r^k\sqrt{k}\, \exp(-8rk\varepsilon^2)$$

for any ε with $0<\varepsilon<1/8$ and any integer k with $k\varepsilon\geqq 4r$.

Proof of the first inequality. Putting $p(mr,\ell)=\binom{mr}{\ell}(r-1)^{mr-\ell}$ for brevity, we have

$$T(mr,\varepsilon/2) \leqq \sum_{|\ell-m|>mr\varepsilon/2} p(mr,\ell)$$

$$= \sum_{|j|>mr\varepsilon/2} p(mr,m+j) < r^{mr}mr\, \exp(-\frac{1}{16}\, mr\varepsilon^2),$$

provided $mr\varepsilon\geqq 2$, using the inequality (see [5]) $p(mr,m+j) < r^{mr}\exp(-j^2/(4mr))$, where j is an integer with $|j|\geqq 2$. Now let $k=mr+d$, $0\leqq d<r$. Then writing $n<r^k$ as (2) and putting $n_1 = \sum_{i=d+1}^{k} a_i r^{k-i}$, we have $|N(n)-k/r| < |N(n_1)-m|+r$, so that if $|N(n)-k/r| > k\varepsilon$, we have $|N(n_1)-m| > mr\varepsilon/2$, provided $k\varepsilon\geqq 4r$. Therefore we obtain

$$T(k,\varepsilon;b) < r^d T(mr,\varepsilon/2) < r^k k\, \exp(-\frac{1}{32}\, k\varepsilon^2).$$

Proof of the second inequality. Assume first that $b\neq 0$. Then

$$S(mr,2\varepsilon;b) = \sum_{\ell-m>2mr\varepsilon} p(mr,\ell) = \sum_{mr\varepsilon<j<mr-m} p(mr,m+j).$$

Here

$$p(mr,m+j)/p(mr,m) = \prod_{i=1}^{j} (1+\frac{i}{m})^{-1}(1+\frac{i-1}{mr-m}) > \exp(-\frac{2j^2}{m}),$$

provided $j<(mr-m)/2$, noticing that $1-x>e^{-2x}$ with $0\leqq x\leqq 1/2$. Thus we have

$$p(mr,m+j) > \frac{r^{mr}}{\sqrt{m}}\, \exp(-4mr^2\varepsilon^2)$$

for all j with $mr\varepsilon<j<(mr-m)/2$, using the inequality $p(mr,m)>r^{mr}/\sqrt{m}$. Hence

$$S(mr,2\varepsilon;b) > \frac{1}{8} r^{mr+1}\sqrt{m}\, \exp(-4mr^2\varepsilon^2),$$

since $(mr-m)/2-mr\varepsilon > mr/8$ provided $\varepsilon<1/8$.

Now let $k=mr+d$, $0\leqq d<r$. Then $N(n_1)-m > 2mr\varepsilon$ implies $N(n)-k/r > N(n_1)-k/r > k\varepsilon$, provided $k\varepsilon\geqq 4r$. Therefore we obtain

$$S(k,\varepsilon;b) > r^d S(mr,2\varepsilon;b) > \frac{1}{8} r^k\sqrt{k}\exp(-8rk\varepsilon^2).$$

For the remaining case of $b=0$, we get

$$S(k,\varepsilon;0) \geqq \sum_{\substack{r^{k-1}\leqq n<r^k \\ N_r(n;0)-k/r>k\varepsilon}} 1 = \frac{r-1}{r}S(k,\varepsilon;r-1)$$

$$> \frac{1}{16}r^k\sqrt{k}\exp(-8rk\varepsilon^2)$$

as required, and the proof of Lemma is completed.

Proof of Theorem 1. Assume first that $\ell=1$. Putting $k=[\log_r x + 1]$, we have

$$\Big|\sum_{n_i\leqq x} N(n_i) - \frac{A(x)}{r}\log_r x\Big| \leqq \Big|\sum_{n_i\leqq x} N(n_i) - \frac{A(x)}{r}k\Big| + \Big|\frac{A(x)}{r}k - \frac{A(x)}{r}\log_r x\Big|$$

$$\leqq \Big(\sum_{\substack{n_i\leqq x \\ |n(n_i)-k/r|\leqq k\varepsilon}} + \sum_{\substack{n_i\leqq x \\ |n(n_i)-k/r|>k\varepsilon}}\Big)\Big|N(n_i)-\frac{k}{r}\Big| + \frac{A(x)}{r}$$

$$\leqq A(x)k\varepsilon + \frac{A(x)}{r} + kT(k,\varepsilon;b),$$

where $\varepsilon=\varepsilon(x)$ is defined by

$$\varepsilon^2 = \frac{2^5}{k}\Big(\log\frac{r^k}{A(x)} + 2\log k\Big),$$

so that, by the assumption $x\geqq r^{r^2}$, we find $k\varepsilon\geqq 4r$ and

$$\varepsilon^2 < \frac{2^8}{\log_r x}\log\frac{x}{A(x)} + \log(\log_r x) . \tag{5}$$

Thus we may use the first inequality of Lemma and obatin $T(k,\varepsilon;b) < A(x)/k$.

Therefore

$$\left| \sum_{n_i \leqq x} N(n_i) - r^{-1}A(x)\log_r x \right| < A(x)k\varepsilon + A(x)/r + A(x)$$

$$< 4r\varepsilon\, A(x)\, \log_r x,$$

noticing that $\varepsilon \geqq 4r/k > 1/\log_r x$, which together with (5) leads to

$$\leqq 2^6\, r \left(\frac{\log(x/A(x)) + \log(\log_r x)}{\log_r x} \right)^{1/2} A(x)\log_r x. \qquad (6)$$

Now let $\ell \geqq 2$. It follows from the definition that

$$\left| N_r(n;b_1b_2\cdots b_\ell) - \sum_{j=0}^{\ell-1} N_{r^\ell}([nr^{-i}];B) \right| \leqq \ell,$$

where $B = \sum_{i=1}^{\ell} b_i r^{\ell-1}$, so that

$$\left| \sum_{n_i \leqq x} N_{r^\ell}(n_i;b_1b_2\cdots b_\ell) - r^{-\ell}A(x)\log_r x \right|$$

$$\leqq \sum_{j=0}^{\ell-1} \left| \sum_{n_i \leqq x} N_{r^\ell}([n_i r^{-j}];B0 - r^{-\ell}A(x)\log_{r^\ell}(xr^{-j}) \right| + 2\ell A(x),$$

and hence using (6) with r^ℓ and xr^{-j} in place of r and x,

$$\leqq \ell\, 2^7 \left(\frac{\log(x/A(x)) + \log(\log_{r^\ell} x)}{\log_{r^\ell} x} \right)^{1/2} A(x)\log_{r^\ell} x + 2\ell A(x),$$

provided $x \geqq r^{2\ell}$, which leads to Theorem 1.

Proof of Theorem 2. We put for brevity

$N(t)=N(t;\theta_r;b_1b_2\cdots b_\ell)$, $N(m)=N_r(m;b_1b_2\cdots b_\ell)$, $K(m)=[\log_r m]+1$, and $n=n(t)$.

Then

$$t = \sum_{n_i \leqq n} K(n_i) + O(\log n) \qquad (7)$$

and

$$N(t) = \sum_{n_i \leq n} N(n_i) + O(\log n) + O(A(n)),$$

so that

$$\frac{N(t)}{t} = \frac{\sum_{n_i \leq n} N(n_i)}{\sum_{n_i \leq n} K(n_i)} + O(\frac{\log n}{t}) , \tag{8}$$

noticing that $A(n) = O(\log n)$ by (1).

Now putting $k=K(n)$, we get

$$A(n)k - \sum_{n_i \leq n} K(n_i) = \sum_{j=1}^{k-1} A(r^j-1), \tag{9}$$

and so

$$\frac{\sum_{n_i \leq n} N(n_i)}{\sum_{n_i \leq n} K(n_i)} = \frac{\sum_{n_i \leq n} N(n_i)}{A(n)k} (1 + \frac{\sum_{j=1}^{k-1} A(r^j-1)}{\sum_{n_i \leq n} K(n_i)}). \tag{10}$$

Here

$$\sum_{j=1}^{k-1} A(r^j) \leq \sum_{1 \leq j \leq [\log_r A(r^{k-1})]} r^j + (k-1-[\log_r A(r^{k-1})])A(r^{k-1})$$

$$\leq (3 + \log_r \frac{r^{k-1}}{A(r^{k-1})}) A(r^{k-1}),$$

so that by (1) and (9)

$$\sum_{n_i \leq n} K(n_i) = A(n)k(1 + o(1)), \tag{11}$$

and consequently

$$\frac{\sum_{j=1}^{k-1} A(r^j-1)}{\sum_{n_i \leq n} K(n_i)} = O(\frac{1}{k}) + O\left(\frac{A(r^{k-1})}{kA(n)}(\log\frac{n}{A(n)} + \log\frac{A(n)}{A(r^{k-1})} + \log\frac{r^{k-1}}{n})\right)$$

$$= O(\frac{1}{\log n}) + O(\frac{1}{\log n}\log\frac{n}{A(n)}).$$

Therefore we obtain by (1) and Theorem 1

$$\frac{\sum_{n_i \leqq n} N(n_i)}{\sum_{n_i \leqq n} K(n_i)} = \frac{1}{r^{\ell}} + O\left(\frac{\log \frac{n}{A(n)} + \log \log n}{\log n} \right)^{1/2} ,$$

which together with (8) yealds Theorem 2, since by (7) and (11) $\log t = \log n + o(\log n)$, and so $\log(n/A(n)) = o(\log t)$.

Proof of Theorem 3. Let k_o be a sufficiently large integer. We define

$$\varepsilon_k = \left(\frac{3 \log k}{16rk}\right)^{1/2},$$

so that

$$r^k \sqrt{k} \exp(-8rk\varepsilon_k^2) = r^k/k$$

and $k\varepsilon_k \geqq 4r$ for $k \geqq k_o$. Then for each integer $k \geqq k_o$ we can choose, by the second inequality of Lemma, $[r^k/(16k)+1]$ integers n's with $r^{k-1} \leqq n < r^k$ such that $N_r(n;b) - k/r > k\varepsilon_k$. We denote by $n_1, n_2, \cdots$ the increasing sequence consisting of all these integer for all $k \geqq k_o$.

By the partial summation formula, we have

$$\sum_{j=k_o}^{k} \frac{r^j}{j} \leqq \frac{r^{k+1}}{(r-1)k} + \frac{4r^k}{(r-1)k^2} ,$$

so that

$$A(r^k) \leqq \frac{1}{16} \sum_{j=k_o}^{k} \frac{r^j}{j} + k \leqq \frac{r^{k+1}}{8k} , \tag{12}$$

$$\frac{A(x)}{x} \geqq \frac{A(r^{[\log_r x]})}{r^{[\log_r x]+1}} > \frac{1}{8 \log_r x} , \tag{13}$$

and

$$\sum_{j=k_o}^{k} (k-j) \frac{r^i}{j} \leqq \frac{8r^k}{k} . \tag{14}$$

Using (12) and (14), we obtain

$$\sum_{n_i \leq r^k} N_r(n;b) - \frac{1}{r} A(r^k)k$$

$$= \sum_{j=k_o}^{k} \sum_{r^{j-1} \leq n_i < r^j} N(_r(n;b) - \frac{k}{r}) - \frac{1}{r} \sum_{j=k_o}^{k} (k-j)[\frac{r^j}{16j} + 1]$$

$$> k\varepsilon_k \frac{r^k}{16k} - \frac{r^{k-1}}{2k} - \frac{k^2}{24} > \frac{\varepsilon_k}{4} \frac{A(r^k)}{r} k$$

for all $k \geq k_o$; which as well as (13) implies that the sequence $n_1, n_2, \cdots$ satisfies the properties mentioned in the theorem.

Proof of Theorem 4. We note first that for any interger b with $0 \leq b < r$

$$\sum_{n \leq x} N_r(n;b) = \frac{x}{r} \log_r x + O(x),$$

(which can be proved for instance by using the same idea as in [4],) and consequently, the same argument as in the proof of Theorem 1 will yield

$$\sum_{n \leq x} N_r(n;b_1 b_2 \cdots b_\ell) = \frac{x}{r^\ell} \log_r x + O(x). \tag{15}$$

Now we put

$$\lambda = [\log_r(\log n)] \quad \text{and} \quad J = [nr^{-\lambda}]$$

so that $Jr^\lambda \leq n < (J+1)r^\lambda$. We note that for any integer $j \geq 0$,

$$N(j) \leq N(m) \leq N(j) + \lambda$$

if m is an integer with $jr^\lambda \leq m < (j+1)r^\lambda$, where $N(n) = N_r(n;b_1 b_2 \cdots b_\ell)$.
Then, using the equality (see [5])

$$\sum_{\substack{m \leq q \\ (m,n)=1}} 1 = \frac{(n)}{n} q - \sum_{d|n} \mu(d)\{\frac{q}{d}\} ,$$

where n and q are positive integers and $\{x\}=x-[x]$, we get

$$\sum_{\substack{m\leq n\\(m,n)=1}} N(m) = \sum_{j=1}^{j} \sum_{\substack{(j-1)r^{\lambda}<m\leq jr^{\lambda}\\(m,n)=1}} N(m) + \sum_{Jr^{\lambda}<m\leq n} N(m)$$

$$= \sum_{j=1}^{J} N(j) \sum_{\substack{(j-1)r^{\lambda}<m\leq jr^{\lambda}\\(m,n)=1}} 1 + O(\lambda\varphi(n)) + O(r^{\lambda}\log n)$$

$$= \frac{\varphi(n)}{n} r^{\ell} \sum_{j=1}^{J} N(j) - \sum_{j=1}^{J} N(j) \sum_{d|n} \mu(d)(\{\frac{jr^{\ell}}{d}\} - \{\frac{(j-1)r^{\ell}}{d}\})$$

$$+ O(\varphi(n) \log\log n). \qquad (16)$$

Here it follows from (15) that

$$\frac{\varphi(n)}{n} r^{\ell} \sum_{j=1}^{J} N(j) = \frac{\varphi(n)}{r} \log n + O(\varphi(n) \log\log n). \qquad (17)$$

On the other hand, we have

$$\sum_{j=1}^{J} N(j) \sum_{d|n} \mu(d)(\{\frac{ir^{\ell}}{d}\} - \{\frac{(j-1)r^{\ell}}{d}\})$$

$$= N(J) \sum_{d|n} \mu(d)\{\frac{Jr^{\ell}}{d}\} - \sum_{j=2}^{J} (N(j) - N(j-1) \sum_{d|n} \mu(d)\{\frac{(j-1)r^{\ell}}{d}\}$$

$$= O(N(J)d(n)) + O(d(n) \sum_{j=1}^{J} |N(j)-N(j-1)|)$$

$$= O(d(n)n/\log n), \qquad (18)$$

where $d(n)$ is the number of divisors of n, since

$$\sum_{j=1}^{J} |N(j)-N(j-1)| \leq \sum_{k\geq 0} \sum_{J=1}^{J} \sum_{r^{k}||j} (k+1) = O(J).$$

Combining (16), (17), and (18), we get

$$\sum_{\substack{m \leq n \\ (m,n)=1}} N(m) = \frac{\varphi(n)}{r} \log_r n + O(\varphi(n) \log \log n) + O\left(\frac{d(n)n}{\log n}\right).$$

Therefore we obtain

$$\sum_{n \leq x} \sum_{\substack{m \leq n \\ (m,n)=1}} N(m) = \frac{1}{r} \sum_{n \leq x} \varphi(n) \log_r n$$

$$+ O\left(\sum_{n \leq x} \varphi(n) \log \log n\right) + O\left(\sum_{n \leq x} d(n) \frac{n}{\log n}\right)$$

$$= \frac{1}{r} \frac{3}{\pi^2} x^2 \log_r x + O(x^2 \log \log x),$$

using the inequalities

$$\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x)$$

and

$$\sum_{n \leq x} d(n) = x \log x + O(x),$$

and the proof of Theorem 3 is completed.

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