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Ergodicity for an infinite particle system in $\mathbf{R}^d$ of jump type with hard core interaction

by

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Introduction

In this paper, we consider a system of infinitely many hard balls with the same diameter r moving discontinuously in $\mathbb{R}^d$. We denote the configuration space of hard balls by

$$\mathfrak{X} = \{ \xi = \{x_i\} : |x_i - x_j| \geqq r, \quad i \neq j\}.$$

The ball of the system moves by random jump respecting the hard core condition. The system is completely specified by the measure $c(x,dy,\xi)$ which gives the rate of the movement of the ball at the position x to the position y when the entire configuration is ξ. We shall consider the case where $c(x,dy,\xi)$ is given as follows by a translation invariant hard core pair potential Φ which is stable and has a finite range,

$$c(x,dy,\xi) = \exp\{ - \sum_{z\in\xi\setminus\{x\}} \Phi(|y - z|) \} p(|x - y|)dy,$$

where $p(\cdot)$ is a non-negaive function on $[0,+\infty)$ such that $\int_{\mathbb{R}^d} p(|x|)dx = 1$ and $p(\cdot) > 0$ on $[0,2h)$ for some $h > 0$, and Φ is a measurable function on $[0,\infty)$ satisfying the following properties,

(P.1) $\Phi(\cdot) \geqq -C$ for some constant $C \geqq 0$,

(P.2) $\Phi(a) = \infty$ if and only if $a \in [0,r)$,

(P.3) $\Phi(\cdot) = 0$ on $[r',\infty)$ for some constat $r' \geqq r$.

In the previous paper [8] we studied the case where $r = r'$.

We construct the Markov process ξ_t which describes our system. This process has the Gibbs state μ associated with the potential Φ as the stationary measure.

The purpose of this paper is to show the ergodicity of the stationary Markov process in the case where the density of balls is sufficiently small. It is important to develop a topological argument on the configuration space to prove the ergodicity. If the configuration space were connected, any configuration would be attained from any other configuration by moving balls continuously. However, due to the hard core potential, our configuration space is not connected and has more than one connected component. We prove that every pair of different connected components Γ_1 and Γ_2 are jointed by the chain of connected components $\Gamma_1 = \Lambda_1, \Lambda_2, \cdots, \Lambda_k = \Gamma_2$ in which Λ_i and Λ_{i+1} are in "h-communication" for all i with $0 \leqq i \leqq k - 1$. This means that any configuration is attained from any other configuration by means of a finite number of jumps of magnitude equal or less than h. This argument constitutes the most crucial part for the proof of ergodicity. The precise definition of "h-communication" will be given in § 2.

In § 1, we construct the Markov process describing our model by using Liggett's theorem [5] and show that a Gibbs state is a reversible measure for the process. In § 2, using the key lemma about the topological property of the configuration space, we prove the ergodicity of the process. The proof of key lemma is given in § 3. In § 4, we study the central limit theorem of the tagged particle of our process. Kipnis-Varadhan [3] proved the central limit theorem for a tagged particle of simple exclusion process on a lattice. Using the ergodicity of the process and the same technique as [3], we can discuss the central limit theorem. But the proof of non-degeneracy of the covariance matrix is not obtained.

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§ 1. Construction of a Markov process

Let $\mathfrak{M}$ be the set of all countable subsets ξ of $\mathbb{R}^d$ satisfying $\#(\xi \cap K) < \infty$ for any compact subset $K \subset \mathbb{R}^d$. We regard $\xi \in \mathfrak{M}$ as a non-negative integer valued Radon measure on $\mathbb{R}^d$: $\xi(\cdot) = \Sigma\, \delta_{x_i}(\cdot)$. $\mathfrak{X}$ is a compact subset of $\mathfrak{M}$ with the vague topology.

For any $\xi \in \mathfrak{M}$ and $y \in \mathbb{R}^d$ we denote $\xi \cup \{y\}$ by $\xi \cdot y$. Also we denote $\xi \setminus \{z\}$ by $\xi \setminus z$ if $z \in \xi$. The restriction of ξ to any subset K will be denote by ξ_K.

Let $C(\mathfrak{X})$ be the space of all real valued continuous functions on $\mathfrak{X}$ with supremum norm $\| \|_\infty$. We denote by $C_0(\mathfrak{X})$ the set of functions of $C(\mathfrak{X})$ each of which depends only on the configurations in some compact set K;

$$C_0(\mathfrak{X}) = \{f \in C(\mathfrak{X}) : f(\xi) = f(\xi_K) \text{ for some compact set } K\}.$$

It is easily seen that $C_0(\mathfrak{X})$ is dense in $C(\mathfrak{X})$. We define σ-fields $\mathcal{B}(\mathfrak{X})$ and $\mathcal{B}_K(\mathfrak{X})$ by

$$\mathcal{B}(\mathfrak{X}) = \sigma(\ N_A : A \in \mathcal{B}(R^d)\ ),$$

and
$$\mathcal{B}_K(\mathfrak{X}) = \sigma(\ N_A : A \in \mathcal{B}(R^d),\ \ A \subset K\ ),$$

where $N_A(\xi)$ is the number of particles of ξ in A. The σ-field $\mathcal{B}(\mathfrak{X})$ coincides with the topological Borel field on $\mathfrak{X}$.

Before defining a linear operator on $C(\mathfrak{X})$ which generates a Markov process, we define the function $\chi(\underline{x}|\xi)$ for $\underline{x} = \{x_1, x_2, \cdots, x_n\}$ and $\xi \in \mathfrak{M}$ by

$$\chi(\underline{x}|\xi) = \exp\{-\frac{1}{2}\sum_{\substack{y,z\in\underline{x}\\ y\neq z}} \Phi(|y-z|) - \sum_{y\in\underline{x},\, z\in\xi} \Phi(|y-z|)\}.$$

Let $p(\cdot)$ be a non-negative function on $[0,\infty)$ satisfying

(1.1) $$\int_{\mathbb{R}^d} dx\, p(|x|) = 1,$$

(1.2) $$\int_{\mathbb{R}^d} dx\, |x|^2 p(|x|) < \infty,$$

(1.3) $p(\cdot) > 0$ on $[0,2h)$ for some $h > 0$.

Now, we shall define a linear operator on $C_0(\mathfrak{X})$ by

$$Lf(\xi) = \sum_{x\in\xi} \int_{R^d} \{f(\xi^{x,y}) - f(\xi)\}\chi(y|\xi\backslash x)p(|x - y|)dy,$$

where

$$\xi^{x,y} = \begin{cases} (\xi\backslash x)\cdot y, & \text{if } x \in \xi,\ y \notin \xi, \\ \xi, & \text{otherwise.} \end{cases}$$

Since L is dissipative and $C_0(\mathfrak{X})$ is dense in $C(\mathfrak{X})$, L has a minimal closed extension $\bar{L}$. Define bounded operators $L_{j,k}$ on $C(\mathfrak{X})$ for $j = (j_1,\cdots,j_d)$, $k = (k_1,\cdots,k_d) \in \mathbb{Z}^d$ by

$$L_{j,k}f(\xi) = \begin{cases} p_{j,k}^{-1}\sum_{x\in\xi\cap I_j}\int_{I_k} \{f(\xi^{x,y}) - f(\xi)\}\chi(y|\xi\backslash x)p(|x - y|)dy, & \text{if } p_{j,k} > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$p_{j,k} = \int_{(-1,1]^d} dx\, p(|j - k - x|),$$

and

$$I_j = \prod_{i=1}^{d} (j_i - \tfrac{1}{2}, j_i + \tfrac{1}{2}].$$

Then, we have

$$Lf(\xi) = \sum_{j,k\in Z^d} p_{j,k} L_{j,k} f(\xi) \qquad \text{for} \quad f \in C_0(\mathfrak{X}).$$

It follows from the Liggett's theorem [5] that $(\bar{L}, \mathfrak{D}(\bar{L}))$ generates a unique strongly continuous Markov semigroup T_t on $C(\mathfrak{X})$.

We denote by (ξ_t, P_μ) the Markov process associated with $\bar{L}$ having initial distribution μ. Since T_t is a Feller semigroup, we can take a version of ξ_t which is right continuous and has left limits.

For any compact subset $K \subset \mathbb{R}^d$, we denote by $\mathfrak{M}(K)$ and $\mathfrak{M}(K,n)$ the set of all finite subsets of K and the set of all subsets of K having n points respectively.

An alternative description of $\mathfrak{M}(K,n)$ is given by

$$(1.4) \qquad \mathfrak{M}(K,n) = \begin{cases} \{\emptyset\}, & \text{if } n = 0, \\ (K^n)'/S_n, & \text{if } n \geqq 1, \end{cases}$$

where $(K^n)' = \{(x_1,\cdots,x_n) \in K^n : x_i \neq x_j \text{ if } i \neq j\}$ and S_n is the symmetric group of degree n. By means of the factorization (1.4) we introduce a measure $\lambda_{K,z}$ on $\mathfrak{M}(K) = \bigcup_{n=0}^{\infty} \mathfrak{M}(K,n)$ (direct sum) such that

$$\lambda_{K,z}(\emptyset) = 1,$$

and

$$\lambda_{K,z}(\Lambda) = \frac{z^n}{n!} \int_{\tilde{\Lambda}} dx_1 dx_2 \cdots dx_n \quad \text{for a Borel set } \Lambda \text{ in } \mathfrak{M}(K,n), \quad n \geq 1,$$

where $z \geq 0$ and $\tilde{\Lambda}$ is a preimage of Λ by factorization (1.4). The integral of a measurable function f on $\mathfrak{M}(K)$ with respect to this measure is denoted by $\int f(\underline{x})\, d^z\underline{x}$.

Now, we are going to define a Gibbs state. We will see that this Gibbs state is a stationary measure for our process ξ_t.

Definition 1.1 ([2]). A probability measure μ on $\mathfrak{X}$ is called a (grand canonical) Gibbs state with activity $z \geq 0$, if for any compact set K, the restriction of μ on $\mathcal{B}_K(\mathfrak{X})$ is absolutely continuous with respect to $d^z\underline{x}$ and the density is given by

$$\sigma_K(\underline{x}) = \int_{\eta(K)=0} \mu(d\eta)\ \chi(\underline{x}|\eta).$$

Denote by $\mathcal{G}(z)$ the set of all Gibbs states with activity $z \geq 0$. This set $\mathcal{G}(z)$ is convex and compact with respect to the topology of weak convergence, so that the element of $\mathcal{G}(z)$ is represented by the extremal points of $\mathcal{G}(z)$. We denote by $\mathrm{ex}\mathcal{G}(z)$ the set of all extremal points of $\mathcal{G}(z)$.

Remark 1.1 ([4]). There exists $z_0 > 0$ such that if $z < z_0$, then $\#\mathcal{G}(z) = 1$.

Remark 1.2 ([7]). Let $z < z_0$. Then, for any $\mu \in \mathcal{G}(z)$ the limit

$$\rho(z) = \lim_{K \uparrow \mathbb{R}^d} \frac{1}{|K|} \int_{\mathfrak{X}} \xi(K)\ \mu(d\xi).$$

We call $\rho(z)$ the particle density of μ. Also the following property holds. For any $\varepsilon > 0$,

$$\mu\left(\left|\frac{\xi(K)}{|K|} - \rho(z)\right| \geqq \varepsilon \right) \to 0 \qquad \text{as} \quad K \uparrow R^d.$$

Lemma 1.2. If μ is a Gibbs state, then μ is a reversible measure for ξ_t, i.e.

$$\langle T_t f, g\rangle_\mu = \langle f, T_t g\rangle_\mu \qquad \text{for any} \quad f,g \in C(\mathfrak{X}),\ t \geqq 0,$$

where $\langle \cdot , \cdot \rangle_\mu$ is an L^2 inner product with respect to μ.

Proof of Lemma 1.2. Let $j,k \in Z^d$ and $f,g \in C_0(\mathfrak{X})$. Since $p_{j,k} = p_{k,j}$, we have

$$(1.5) \qquad p_{j,k}\{\langle (L_{j,k} + L_{k,j})f, g\rangle_\mu - \langle f, (L_{j.k} + L_{k,j})g\rangle_\mu\}$$

$$= \int_{\mathfrak{X}} \mu(d\xi) \sum_{x\in\xi\cap I_j} \int_{I_k} dy\ f(\xi^{x,y})g(\xi)\chi(y|\xi\backslash x)p(|x - y|)$$

$$+ \int_{\mathfrak{X}} \mu(d\xi) \sum_{x\in\xi\cap I_k} \int_{I_j} dy\ f(\xi^{x,y})g(\xi)\chi(y|\xi\backslash x)p(|x - y|)$$

$$- \int_{\mathfrak{X}} \mu(d\xi) \sum_{x\in\xi\cap I_j} \int_{I_k} dy\ f(\xi)g(\xi^{x,y})\chi(y|\xi\backslash x)p(|x - y|)$$

$$- \int_{\mathfrak{X}} \mu(d\xi) \sum_{x\in\xi\cap I_k} \int_{I_j} dy\ f(\xi)g(\xi^{x,y})\chi(y|\xi\backslash x)p(|x - y|).$$

Choose a compact set K satisfying $f(\xi_K) = f(\xi)$, $g(\xi_K) = g(\xi)$ and $B_{r'}(I_j \cup I_k) \subset K$, where $B_\ell(A)$ is an open ℓ-neighborhood of $A \subset \mathbb{R}^d$. Then, by the definition of a Gibbs state we have

$$\int_{\mathfrak{X}} \mu(d\xi) \sum_{x\in\xi\cap I_j} \int_{I_k} dy\ f(\xi^{x,y})g(\xi)\chi(y|\xi\backslash x)p(|x - y|)$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{K^n} dx_1 \cdots dx_n \sigma_K(x_1 \cdots x_n) \sum_{i=1}^{n} 1\!\!1_{I_j}(x_i) \int_{I_k} dy$$

$$f(x_1 \cdots x_n \cdot y \backslash x_i)\, g(x_1 \cdots x_n)\, \chi(y | x_1 \cdots x_n \backslash x_i)\, p(|x_i - y|)$$

$$= \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!} \int_{K^{n-1}} dx_1 \cdots dx_{n-1} \int_{I_j} dx_n \int_{I_k} dy\; \sigma_K(x_1 \cdots x_n)$$

$$f(x_1 \cdots x_{n-1} \cdot y)\, g(x_1 \cdots x_n)\, \chi(y | x_1 \cdots x_{n-1})\, p(|x_n - y|) .$$

Let us note that $\sigma_K(x_1 \cdots x_n) = \sigma_K(x_1 \cdots x_{n-1}) \chi(x_n | x_1 \cdots x_{n-1})$ for $x_n \in I_j \cup I_k$. Using this relation, we have

$$(1.6) \qquad \int_{\mathfrak{X}} \mu(d\xi) \sum_{x \in \xi \cap I_j} \int_{I_k} dy\; f(\xi^{x,y})\, g(\xi)\, \chi(y | \xi \backslash x)\, p(|x - y|)$$

$$= \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!} \int_{K^{n-1}} dx_1 \cdots dx_{n-1} \int_{I_j} dx_n \int_{I_k} dy\; \sigma_K(x_1 \cdots x_{n-1})$$

$$f(x_1 \cdots x_{n-1} \cdot y)\, g(x_1 \cdots x_n)\, \chi(y | x_1 \cdots x_{n-1})\, \chi(x_n | x_1 \cdots x_{n-1})\, p(|x_n - y|)$$

$$= \int_{\mathfrak{X}} \mu(d\xi) \sum_{x \in \xi \cap I_k} \int_{I_j} dy\; f(\xi)\, g(\xi^{x,y})\, \chi(y | \xi \backslash x)\, p(|x - y|) .$$

Hence, from (1.5) and (1.6) we have

$$\langle (L_{j,k} + L_{k,j}) f,\ g \rangle_\mu = \langle f,\ (L_{j.k} + L_{k,j}) g \rangle_\mu ,$$

$$\text{for} \quad f, g \in C_0(\mathfrak{X}), \quad j, k \in Z^d.$$

Therefore,

$$\langle Lf,\ g \rangle_\mu = \langle f,\ Lg \rangle_\mu , \quad \text{for} \quad f, g \in C_0(\mathfrak{X}) .$$

Since $\bar{L}$ is the generator for T_t, Lemma 1.2 is proved.

§ 2. Ergodicity of (ξ_t, P_μ)

The primary purpose of this section is to prove the following theorem.

<u>Theorem 2.1.</u> If $z > 0$ is sufficiently small, then the Markov process (ξ_t, P_μ) is ergodic for any $\mu \in \mathrm{ex}\mathcal{G}(z)$.

Let $\hat{T}_t$ be the strongly continuous semigroup on $L^2(\mathfrak{X}, \mu)$ associated with ξ_t and $\hat{L}$ be the generator for $\hat{T}_t$. To prove the ergodicity of the process (ξ_t, P_μ) it is enough to prove the following condition (C.1):

(C.1) If $f \in L_2(\mathfrak{X}, \mu)$ satisfies $\hat{T}_t f = f$ for any $t \geqq 0$, then f is constant.

We shall prove that the condition (C.1) holds for (ξ_t, P_μ), $\mu \in \mathrm{ex}\mathcal{G}(z)$, if $z > 0$ is sufficiently small. The following result [1] about Gibbs states is very useful to show the condition (C.1). Let $\mathcal{E}_\infty(\mathfrak{X})$ be the σ-field given by

$$\mathcal{E}_\infty(\mathfrak{X}) = \bigcap_{K:\text{compact}} \sigma(N_K, \mathcal{B}_{K^c}(\mathfrak{X})),$$

where $\sigma(N_K, \mathcal{B}_{K^c}(\mathfrak{X}))$ is the σ-field generated by $N_K(\xi)$ and $\mathcal{B}_{K^c}(\mathfrak{X})$. If $\mu \in \mathrm{ex}\mathcal{G}(z)$ then $\mu(\Lambda) = 0$ or 1 for any $\Lambda \in \mathcal{E}_\infty(\mathfrak{X})$.

The following Lemma 2.1 follows from this result immediately. For a given compact subset $K \subset \mathbb{R}^d$, $n \in \mathbb{N}$ and $\xi \in \mathfrak{X}$, we denote by $\Lambda(K,n,\xi)$ the interior of the configuration space $\{ \underline{x} \in \mathfrak{M}(K,n) : \chi(\underline{x} | \xi_{K^c}) \neq 0 \}$, and for $m \in \mathbb{N}$ we put

$$K_m = \{ x \in \mathbb{R}^d : |x| \leq \sqrt{d}2^m r \}.$$

<u>Lemma 2.1.</u> Let $f \in L^2(\mathfrak{X},\mu)$ and $\mu \in \mathrm{ex}\mathcal{G}(z)$. If there exists a positive number ε and the following equation (2.1) holds for all $m, n \in N$ satisfying

$$\frac{n}{|K_m|} < \rho(z) + \varepsilon,$$

and for almost all ξ,

$$(2.1) \qquad \int_{\Lambda(K_m,n,\xi)} d'\underline{x} \int_{\Lambda(K_m,n,\xi)} d'\underline{y} \; |f(\underline{x}\cdot\xi_{K_m^c}) - f(\underline{y}\cdot\xi_{K_m^c})| = 0,$$

then f is constant.

Since $\rho(z) \downarrow 0$ as $z \downarrow 0$, the condition (C.1) is obtained by Lemma 2.1, if we prove the following condition (C.2):

(C.2) There exists a positive number c and (2.1) holds for almost all $\xi \in \mathfrak{X}$ and all $(m,n) \in \mathbb{N}\times\mathbb{N}$ satisfying

$$\frac{n}{|K_m|} < c.$$

To show the condition (C.2) it is neccessary to develop the topological argument on the configuration space. We introduce some notion about the configuration space which is weaker than the connectedness.

<u>Definition 2.1.</u> i) Two configurations $\xi \in \mathfrak{X}$ and $\eta \in \mathfrak{X}$ are said to be in h-communication (denote by $\xi \leftarrow h \rightarrow \eta$), if there exist $x \in \xi$ and $y \in \eta$ such that $|x-y| \leq h$ and $\xi^{x,y} = \eta$.

ii) Two subsets Γ and Λ of $\mathfrak{X}$ are said to be in h-communication (denote by $\Gamma \leftarrow h \rightarrow \Lambda$), if there exist $\xi \in \Gamma$ and $\eta \in \Lambda$ such that $\xi \leftarrow h \rightarrow \eta$.

iii) A family of subsets $\{\Lambda(j)\}_{j\in J}$ of $\mathfrak{X}$ is said to be in h-communication, if for any $j', j'' \in J$, there exists a sequence $\{j_1, j_2, \cdots, j_q\}$ such that

$$\Lambda(j') \leftarrow h \rightarrow \Lambda(j_1) \leftarrow h \rightarrow \Lambda(j_2) \leftarrow h \rightarrow \cdots \leftarrow h \rightarrow \Lambda(j_q) \leftarrow h \rightarrow \Lambda(j'') .$$

Let $\{\Lambda_j\}_{j\in J}$ be the set of all connected components of $\Lambda(K_m, n, \xi)$. Then, our key lemma is the following.

Lemma 2.2. There exists a positive constant $c(r,h)$ such that for all $\xi \in \mathfrak{X}$ and all $m, n \in N$ satisfying $\frac{n}{|K_m|} < c(r,h)$ $\{\Lambda_j\}_{j\in J}$ are in h-communication.

The proof of Lemma 2.2 is given in § 3.

We shall show the condition (C.2) for $c = c(r,h)$ to prove Theorem 2.1.

From the definition of L and Lemma 1.2, for $g \in C_0(\mathfrak{X})$ we have

$$-2\langle Lg, g\rangle_{\mu}$$

$$= \int_{\mathfrak{X}} \mu(d\xi) \sum_{x\in\xi} \int_{\mathbb{R}^d} \{g(\xi^{x,y}) - g(\xi)\}^2 \chi(y|\xi\backslash x) p(|x - y|) dy.$$

Since f is $\hat{T}_t$-invariant for any $t \geqq 0$, we see that $\hat{L}f = 0$. Since $\hat{L}$ is a minimal closed extension of L, we have

$$\int_{\mathfrak{X}} \mu(d\xi) \sum_{x\in\xi} \int_{\mathbb{R}^d} \{f(\xi^{x,y}) - f(\xi)\}^2 \chi(y|\xi\backslash x) p(|x - y|) dy = 0.$$

From the definition of Gibbs state and (1.3), we have

$$(2.2)\qquad \int_{\Lambda(K_m,n,\bar{\xi})} d^1\underline{x} \sum_{x\in\underline{x}} \int_{B_{2h}(x)} dy \, |f(\underline{x}^{x,y}\cdot\xi_{K_m^c}) - f(\underline{x}\cdot\xi_{K_m^c})| \, \mathbb{1}_{\Lambda(K_m,n,\xi)}(\underline{x}^{x,y}) = 0$$

for all $m, n \in \mathbb{N}$ and almost all $\xi \in \mathfrak{X}$. We define a non-negative function H on $\mathfrak{M}(K_m,n) \times \mathfrak{M}(K_m,n)$ by

$$H(\underline{x},\underline{y}) = \sum\sum \prod_{i=1}^{n} \mathbb{1}_{\Lambda(K_m,n,\xi)}(x_1 \cdots x_{i-1}\cdot y_i \cdots y_n)\, \mathbb{1}_{B_{2h}(x_i)}(y_i).$$

The above sums run over all ordered n-tuples $(x_1,\cdots,x_n)$ and $(y_1,\cdots,y_n)$ such that $\{x_1,\cdots,x_n\} = \underline{x}$ and $\{y_1,\cdots,y_n\} = \underline{y}$. Employing this function, from (2.2) we obtain

$$(2.3)\qquad \int_{\Lambda(K_m,n,\xi)} d^1\underline{x} \int_{\Lambda(K_m,n,\xi)} d^1\underline{y} \, |f(\underline{x}\cdot\xi_{K_m^c}) - f(\underline{y}\cdot\xi_{K_m^c})| \, H(\underline{x},\underline{y}) = 0,$$

for all $m, n \in \mathbb{N}$ and almost all $\xi \in \mathfrak{X}$.

Since $\Lambda(K_m,n,\xi)$ is open, for any $\underline{x}^1, \underline{x}^2 \in \Lambda(K_m,n,\xi)$, we can choose $\varepsilon(\underline{x}^1,\underline{x}^2) \in (0,h)$ such that

$$I(\underline{x}^1,\varepsilon(\underline{x}^1,\underline{x}^2)) \cup I(\underline{x}^2,\varepsilon(\underline{x}^1,\underline{x}^2)) \subset \Lambda(\xi,K_m,n),$$

where

$$I(\underline{x},\varepsilon) = \{\{y_1,\cdots,y_n\} \in \mathfrak{M}(K_m,n) : |y_i - x_i| < \varepsilon\}.$$

We abbreviate $I(\underline{x}^1,\varepsilon(\underline{x}^1,\underline{x}^2)) \cup I(\underline{x}^2,\varepsilon(\underline{x}^1,\underline{x}^2))$ to $I(\underline{x}^1,\underline{x}^2)$. If $\underline{x}^1 \leftarrow h \rightarrow \underline{x}^2$, $H \geqq 1$ on $I(\underline{x}^1,\underline{x}^2) \times I(\underline{x}^1,\underline{x}^2)$. From (2.3), we have

$$(2.4) \qquad \int_{I(\underline{x}^1,\underline{x}^2)} d^1\underline{x} \int_{I(\underline{x}^1,\underline{x}^2)} d^1\underline{y}\ |f(\underline{x}\cdot\xi_{K_m^c}) - f(\underline{y}\cdot\xi_{K_m^c})| = 0.$$

For any $\Gamma_1, \Gamma_2 \subset \Lambda(K_m,n,\xi)$ satisfying $|\Gamma_1 \cap \Gamma_2| > 0$ and

$$\int_{\Gamma_i} d^1\underline{x} \int_{\Gamma_i} d^1\underline{z}\ |f(\underline{x}\cdot\xi_{K_m^c}) - f(\underline{z}\cdot\xi_{K_m^c})| = 0 \quad \text{for } i = 1,2,$$

we have

$$\begin{aligned} &|\Gamma_1 \cap \Gamma_2| \int_{\Gamma_1} d^1\underline{x} \int_{\Gamma_2} d^1\underline{y}\ |f(\underline{x}\cdot\xi_{K_m^c}) - f(\underline{y}\cdot\xi_{K_m^c})| \\ &\leqq |\Gamma_2| \int_{\Gamma_1} d^1\underline{x} \int_{\Gamma_1} d^1\underline{z}\ |f(\underline{x}\cdot\xi_{K_m^c}) - f(\underline{z}\cdot\xi_{K_m^c})| \\ &\quad + |\Gamma_1| \int_{\Gamma_2} d^1\underline{z} \int_{\Gamma_2} d^1\underline{y}\ |f(\underline{z}\cdot\xi_{K_m^c}) - f(\underline{y}\cdot\xi_{K_m^c})| = 0, \end{aligned}$$

and so

$$\int_{\Gamma_1\cup\Gamma_2} d^1\underline{x} \int_{\Gamma_1\cup\Gamma_2} d^1\underline{z}\ |f(\underline{x}\cdot\xi_{K_m^c}) - f(\underline{z}\cdot\xi_{K_m^c})| = 0.$$

Repeating this procedure and using (2.4), we have that if $\Lambda_j \leftarrow h \rightarrow \Lambda_{j'}$, then

$$\int_{\Lambda_j} d^1\underline{x} \int_{\Lambda_{j'}} d^1\underline{y}\ |f(\underline{x}\cdot\xi_{K_m^c}) - f(\underline{y}\cdot\xi_{K_m^c})| = 0.$$

Therefore, it follows from Lemma 2.2 that the condition (C.2) holds for $c = c(r,h)$. This completes the proof of Theorem 2.1.

§ 3. Proof of Lemma 2.2

First of all, we introduce some notions about configurations. Let B be a convex subset of K_m and k be a non-negative integer. A configuration $\{x_i\}_{i=1}^{k}$ in B is called standard in B if $d(B^c, \{x_i\}_{i=1}^{k}) > 2r$ and $|x_i - x_j| > 4r$, for $1 \leqq i < j \leqq k$, where $d(A_1, A_2) = \inf\{|x - y| : x \in A_1,\ y \in A_2\}$, for $A_1, A_2 \subset \mathbb{R}^d$. Note that the standard configuration in B_1 is also standard in B_2 if $B_1 \subset B_2$.

Also a configuration $\underline{x} \in \Lambda(K_m, n, \xi)$ is called B-standard if $\underline{x} \cap B$ is standard in B. We abbreviate a K_m-standard configuration to a standard configuration.

If the number of balls in B is sufficiently large, then there is no B-standard configuration. For each convex set B we assign the number $\mathcal{N}(B)$ as the largest number of balls in B for B-standard configurations, i.e. ,

$$\mathcal{N}(B) = \max\{k \geqq 0 : \text{there exists a B-standard configuration } \underline{x} \text{ with } n_B(\underline{x}) = k\},$$

where $n_B(\underline{x})$ is number of balls of $\underline{x}$ in B. If $k \leqq \mathcal{N}(B)$, there exists a standard configuration of k balls. Since $\mathcal{N}(B_\ell(b))$ is determined by the radius of $B_\ell(b)$, we abbreviate $\mathcal{N}(B_\ell(b))$ to $\mathcal{N}(\ell)$. It is easily seen that

$$\mathcal{N}(\ell) \geqq \left(\frac{\ell - 2r}{2\sqrt{d}\, r}\right)^d. \tag{3.1}$$

For $\underline{x} \in \Lambda(K_m, n, \xi)$, let $\Lambda(\underline{x})$ be the connected component of $\Lambda(K_m, n, \xi)$ containing $\underline{x}$.

Lemma 3.1. Any standard configuration in B can be attained from other standard configuration in B by moving balls continuously within B preserving the hard core condition and without interference the boundary condition, more precisely, if $\underline{y}$ and $\underline{z}$ are standard configurations in B such that $n_B(\underline{y}) = n_B(\underline{z})$, then

$$\Lambda(\underline{y}\cdot\underline{w}) = \Lambda(\underline{z}\cdot\underline{w}),$$

for any configuration $\underline{w}$ in $K_m \backslash B$.

This lemma will be proved later.

From Lemma 3.1 all standard configurations are contained in one connected component Λ^s of $\Lambda(K_m,n,\xi)$. For the connected components Γ_1 and Γ_2 of $\Lambda(K_m,n,\xi)$ we write $\Gamma_1 \leftrightarrow \Gamma_2$, if there exists a sequence of connected comoponents $\Lambda_1, \Lambda_2, \cdots, \Lambda_k$ of $\Lambda(K_m,n,\xi)$ such that

$$\Gamma_1 \leftarrow h \rightarrow \Lambda_1 \leftarrow h \rightarrow \Lambda_2 \leftarrow h \rightarrow \cdots \leftarrow h \rightarrow \Lambda_k = \Gamma_2.$$

We introduce several numbers j_0, ℓ_0 and $c(r,h)$ which will be used in the proof of Lemma 2.2. Let j_0 be an integer such that $2^{j_0-1} \leqq r/h \vee 8d < 2^{j_0}$. Put $\ell_0 = 2^{j_0}r$ and $c(r,h) = |B_{\sqrt{d}\ell_0}(0)|^{-1}$. Then, $j_0 \geqq 4$ and $c(r,h)|K_m| = 2^{d(m-j_0)}$.

We also denote the convex hull of $A \subset R^d$ by $[A]$. For a given ordered sequence $B_1, B_2, \cdots, B_k$ of open balls, we define $B[i]$, $2 \leqq i \leqq k$ by $B[i] = [B_1 \cup B_2] \cup [B_2 \cup B_3] \cup \cdots [B_{i-1} \cup B_i]$. See, e.g. Fig.1.

Fig.1.

To prove Lemma 2.2, it is enough to prove the following lemma.

Lemma 2.2'. Let $\xi \in \mathfrak{X}$ and m be an integer with $m \geqq j_0$. If $n < 2^{d(m-j_0)}$, then for any $\underline{x} \in \Lambda(K_m, n, \xi)$ there exist a sequence of open balls $B_1, B_2, \cdots, B_k$ and a sequence of configurations $\underline{y}^1, \underline{y}^2, \cdots, \underline{y}^k \in \Lambda(K_m, n, \xi)$ such that for $1 \leqq i \leqq k$

(i) $\underline{y}^i$ is B_i-standard,

(ii) $\underline{y}^i{}_{B[i]^c} = \underline{x}^i{}_{B[i]^c}$,

(iii) $n_{B[i]\setminus B_i}(\underline{y}^i) = 0$ i.e. $n_{B[i]}(\underline{y}^i) = n_{B_i}(\underline{y}^i)$,

(iv) $\Lambda(\underline{x}) \leftrightarrow \Lambda(\underline{y}^1) \leftrightarrow \Lambda(\underline{y}^2) \leftrightarrow \cdots \leftrightarrow \Lambda(\underline{y}^k) = \Lambda^s$.

We shall prove this lemma by induction. The following lemma play an important role for the proof.

Lemma 3.2. Let $B_1 = B_{\ell_1}(b_1)$, $B_2 = B_{\ell_2}(b_2) \subset K_m$ for some $\ell_2 \geqq \ell_1 \geqq \ell_0$, $b_1, b_2 \in R^d$. Take a B_1-standard configuration $\underline{y} \in \Lambda(K_m, n, \xi)$. If $n_{[B_1 \cup B_2]}(\underline{y}) \leqq N(\ell_1 - 5r)$, then a B_2-standard configuration is obtained from $\underline{y}$ by moving balls in $[B_1 \cup B_2]$ into B_2 by means of finite many jumps of magnitude equal or less than h. More precisely, there exists a B_2-standard configuration $\underline{z}$ such that

(i) $\Lambda(\underline{z}) \leftrightarrow \Lambda(\underline{y})$,

(ii) $n_{B_2}(\underline{z}) = n_{[B_1 \cup B_2]}(\underline{z}) = n_{[B_1 \cup B_2]}(\underline{y})$,

(iii) $\underline{z}_{B_2{}^c} = \underline{z}_{[B_1 \cup B_2]^c} = \underline{y}_{[B_1 \cup B_2]^c}$.

Proof of Lemma 2.2'. We construct an increasing sequence $E_1, E_2, \cdots, E_{m-j_0}$ of cubes in K_m satisfying the condition (3.2) as follows,

$$(3.2) \qquad n_{E_j}(\underline{x}) < 2^{dj}, \qquad \text{for } 1 \leqq j \leqq m - j_0.$$

First, we put

$$E_{m-j_0} = \{(a_1, \cdots, a_d) \in R^d : -2^m r < a_i \leqq 2^m r, \ 1 \leqq i \leqq d\}.$$

From the assumption of Lemma 2.2' we have $n_{E_{m-j_0}}(\underline{x}) \leqq n < 2^{dm}$. We decompose E_m into the disjoint union of congruent 2^d cubes with edge length $2^m r$. Pick up one of the cubes having the smallest number of balls of $\underline{x}$ and denote it E_{m-1}. Then, $n_{E_{m-j_0-1}}(\underline{x}) \leqq 2^{d(m-1-j_0)}$. Repeating this procedure, we can construct a sequence $E_1 \subset E_2 \subset \cdots \subset E_{m-j_0}$ satisfying the condition (3.2).

Let B_i be the inscribed open ball in E_i for $0 \leqq i \leqq m - j_0$ and B_{m+1-j_0} be the interior of K_m. The radius of B_i is $2^i \ell_0$ for $0 \leqq i \leqq m - j_0$. Taking (3.1) in account and using (3.2) we have

$$(3.3) \qquad n_{B[i]}(\underline{x}) \leqq n_{E_i}(\underline{x}) < 2^{id} \leqq \mathcal{N}(2^{i-1}\ell_0 - 5r),$$

$$\text{for } 0 \leqq i \leqq m + 1 - j_0.$$

From (3.3) $\underline{x}$ is a B_0 -standard configuration satisfying $n_{[B_0 \cup B_1]}(\underline{x}) \leqq \mathcal{N}(\ell_0 - 5r)$. Then, we can apply Lemma 3.2 and have that there exsits a B_1-standard configuration $\underline{y}^1$ satisfying

$$\Lambda(\underline{y}^1) \leftrightarrow \Lambda(\underline{x}),$$

(3.4) $$n_{B_1}(\underline{y}^1) = n_{[B_0 \cup B_1]}(\underline{y}^1) = n_{[B_1 \cup B_0]}(\underline{x}),$$

(3.5) $$\underline{y}^1{}_{B_1^c} = \underline{y}^1{}_{[B_0 \cup B_1]^c} = \underline{x}_{[B_0 \cup B_1]^c}.$$

From (3.4) and (3.5) we have

$$\begin{aligned} n_{[B_1 \cup B_2]}(\underline{y}^1) &= n_{B_1}(\underline{y}^1) + n_{[B_1 \cup B_2] \setminus B_1}(\underline{y}^1) \\ &= n_{[B_0 \cup B_1]}(\underline{x}) + n_{[B_1 \cup B_2] \setminus [B_0 \cup B_1]}(\underline{x}) \\ &= n_{B[2]}(\underline{x}) \leq \mathcal{N}(2\ell_0 - 5r). \end{aligned}$$

Then, we can apply Lemma 3.2 and we obtain a B_2-standard configuration $\underline{y}^2$ satisfying

$$\Lambda(\underline{y}^2) \leftrightarrow \Lambda(\underline{y}^1),$$

$$n_{B_2}(\underline{y}^2) = n_{[B_1 \cup B_2]}(\underline{y}^2) = n_{[B_1 \cup B_2]}(\underline{y}^1),$$

$$\underline{y}^2{}_{B_2^c} = \underline{y}^2{}_{[B_1 \cup B_2]^c} = \underline{y}^1{}_{[B_1 \cup B_2]^c}.$$

Repeating this procedure, we construct the sequence of configurations $\underline{y}^1, \underline{y}^2, \cdots, \underline{y}^{m+1-j_0}$. Since $\Lambda(K_m, n, \xi)$ is open, there is no ball in the boundary ∂K_m of K_m, therefore, $\underline{y}^{m+1-j_0}$ is a standard configuration. Now, we have a sequence $\underline{y}^1, \underline{y}^2, \cdots, \underline{y}^{m+1-j_0}$ of configurations satisfying the conditions (i) ~ (iv) in Lemma 2.2'.

Proof of Lemma 3.1. This lemma is trivial when $\underline{y} = \underline{z}$ or $n_B(\underline{y}) = 1$. So, we assume that $\underline{y} \neq \underline{z}$ and $n_B(\underline{y}) \geqq 2$. Suppose that $y \in \underline{y}$ and $z \in \underline{z}$ satisfy $|y - z| \leqq 2r$, then we have

$$|z - y'| \geqq |y - y'| - |y - z|$$

$$> 4r - 2r = 2r, \qquad \text{for } y' \in \underline{y}\backslash y.$$

In the same way, we have

$$|y - z'| > 2r \qquad \text{for } z' \in \underline{z}\backslash z.$$

Then, there is at most one ball $z \in \underline{z}$ with $|y - z| \leqq 2r$, for any $y \in \underline{y}$ and there is at most one ball $y \in \underline{y}$ with $|y - z| \leqq 2r$, for any $z \in \underline{z}$. So, we can take pairings (y_i, z_i), $1 \leqq i \leqq q$ such that

$$\{ y \in \underline{y} : |y - z_i| \leqq 2r\} = \{y_i\},$$

$$\{ z \in \underline{z} : |z - y_i| \leqq 2r\} = \{z_i\},$$

$$\#\{ y \in \underline{y} : d(y,\underline{z}) \leqq 2r\} = q.$$

Put $\underline{x}\backslash\{x_1, \cdots, x_q\} = \{x_{q+1}, \cdots, x_k\}$, $\underline{y}\backslash\{y_1, \cdots, y_q\} = \{x_{q+1}, \cdots, x_k\}$. For $1 \leqq i \leqq q$, $\alpha \in [0,1]$ we put

$$z_i(\alpha) = (1 - \alpha) z_i + \alpha y_i .$$

Then, for $1 \leqq i \leqq q$

$$|z_i(\alpha) - z_i'| = |(z_i - z') - \alpha(z_i - y_i)|$$

$$\geqq |z_i - z'| - \alpha|z_i - y_i|$$

$$> 2r, \qquad \text{for} \quad z' \in \underline{z}\backslash z_i, \quad \alpha \in [0,1],$$

and

$$d(B^c, z_i(\alpha)) \geqq d(B^c, z_i) \wedge d(B^c, y_i) > 2r, \quad \text{for} \quad \alpha \in [0,1].$$

Hence, for $1 \leqq i \leqq q$ we can move the ball z_i to the position y_i without interference from any ball of $(\underline{z}\backslash z_i)\cdot\underline{w}$. Repeating this procedure, we obtain the configuration $\{y_1, \cdots y_q, z_{q+1}, \cdots, z_k\}$ from $\underline{z}$ by moving balls continuously within B preserving the hard core condition and without interference from the boundary condition, more precisely,

$$(3.6) \qquad \Lambda(y_1 \cdots y_q \cdot z_{q+1} \cdots z_k \cdot \underline{w}) = \Lambda(\underline{z}\cdot\underline{w}).$$

Since $d(\underline{y}, \{z_{q+1}, \cdots, z_k\}) > 2r$, for any i with $q+1 \leqq i \leqq k$, we have

$$|z' - z''| > 2r, \quad z', z'' \in \{y_1, \cdots, y_i, z_i, \cdots, z_k\}, \quad z' \neq z''.$$

Hence, we can obtain the configuration $\{y_1, \cdots, y_{i+1}, z_{i+2}, \cdots, z_k\}$ from $\{y_1, \cdots, y_i, z_{i+1}, \cdots, z_k\}$ by moving the ball z_{i+1} to y_{i+1} continuously within B and without interference from the boundary condition, more precisely,

$$(3.7) \qquad \Lambda(y_1 \cdots y_{i+1} \cdot z_{i+2} \cdots z_k \cdot \underline{w}) = \Lambda(y_1 \cdots y_i \cdot z_{i+1} \cdots z_k \cdot \underline{w}).$$

for $q \leqq i \leqq k-2$ and

(3.8) $$\Lambda(\underline{y}\cdot\underline{w}) = \Lambda(y_1 \cdots y_{k-1}\cdot z_k \cdot \underline{w}).$$

Combining (3.6), (3.7) and (3.8) we complete the proof.

Before proving Lemma 3.2, we prepare the following two lemmas which will be used for the proof. From now on, we put $B = B_\ell(b)$ for $\ell \geqq \ell_0$, for simplicity.

Lemma A-1. For $x \in \partial B$ put $x(\alpha) = x - \alpha(x-b)/\ell$, $\alpha \in [0,\ell]$. Then,

$$d(B^c \cap B_r(x)^c, x(\alpha)) > r, \qquad \text{if } \alpha > r^2/\ell.$$

Fig.2.

Lemma A-1 implies that even if balls are arranged closely on ∂B as in Fig. 2, we can move the ball at x to $x(\alpha)$ by means of a jump of range α preserving the hard core condition. The proof of this lemma is easy, so we omit the proof.

Lemma A-2. Let $\underline{y}$ and $\underline{z}$ be standard configurations in B with $n_B(\underline{y}) = n_B(\underline{z}) - 1$ and $\underline{w}$ be a configuration in $K_m \backslash B$ with $\underline{w} \cap \partial B \neq \phi$ and $\underline{y}\cdot\underline{w} \in \Lambda(K_m, n, \xi)$. If $n_B(\underline{y}) \leqq \mathcal{N}(\ell-5r)$,

$$\Lambda(\underline{y}\cdot\underline{w}) \leftarrow h \rightarrow \Lambda(\underline{z}\cdot\underline{w}\backslash x), \qquad \text{for any } x \in \underline{w} \cap \partial B.$$

Proof of Lemma A-2. From Lemma 3.1 we can assume $\underline{y}$ is a standard configuration in $B_{\ell-5r}(b)$.

For $x \in \underline{w} \cap \partial B$ put $x(\alpha) = x - \alpha(x-b)/\ell$, $\alpha \in [0,\ell]$. Since, $\ell_0 > r^2/h \vee 8dr$, we have $h \wedge r > r^2/\ell$. Then, it follows from Lemma A-1 that

$$d(\underline{w}\backslash x, x(h\wedge r)) > d(B^c \cap B_r(x)^c, x(h\wedge r)) > r,$$

so that

$$\underline{y}\cdot\underline{w}^{x,x(h\wedge r)} \in \Lambda(K_m, n, \xi).$$

Since $|x - x(h\wedge r)| \leq h$, we have

$$\underline{y}\cdot\underline{w} \leftarrow h \rightarrow \underline{y}\cdot\underline{w}^{x,x(h\wedge r)}. \tag{3.9}$$

Since $\underline{y}$ is the standard configuration in $B_{\ell-5r}(b)$, there is no ball of $\underline{y}$ outside of $B_{\ell-7r}(b)$. Then, we have

$$d(y\cdot\underline{w}\backslash x, x(\alpha)) > r\vee\alpha, \quad \text{for} \quad \alpha \in [h\wedge r, 3r], \tag{3.10}$$

and

$$d(\underline{y}, x(3r)) > 4r. \tag{3.11}$$

The condition (3.10) implies that we can move the ball at $x(h\wedge r)$ to the position $x(3r)$ continuously along the line segment connecting these two points, so that

$$\Lambda(\underline{y}\cdot\underline{w}^{x,x(h\wedge r)}) = \Lambda(\underline{y}\cdot\underline{w}^{x,x(3r)}). \tag{3.12}$$

Also the condition (3.11) implies the that configuration $\underline{y}\cdot x(3r)$ is standard in B, so that we have the following relation (3.13) from Lemma 3.1.

$$(3.13)\qquad \Lambda(\underline{y}\cdot\underline{w}^{x,x(3r)}) = \Lambda(\underline{z}\cdot\underline{w}\backslash x).$$

Combining (3.9), (3.12) and (3.13) we complete the proof.

Proof of Lemma 3.2. For $\alpha\in[0,1]$ put

$$B(\alpha) = B_{\ell(\alpha)}(b(\alpha)),$$

where $b(\alpha) = \alpha b_2 + (1-\alpha)b_1$ and $\ell(\alpha) = \alpha\ell_2 + (1-\alpha)\ell_1$. Since $[B_1 \cup B_2]\backslash B_1 \subset \bigcup_{\alpha\in[0,1]} \partial B(\alpha)$ there exists $\alpha\in[0,1)$ such that $x\in\partial B(\alpha)$, for any $x\in\underline{y}\cap([B_1\cup B_2]\backslash B_1)$. First, we put

$$\alpha_1 = \min\{\alpha\geqq 0 : d(\underline{y}_{[B_1\cup B_2]\backslash B_1}, \partial B(\alpha)) = 0\},$$

and pick up one of the balls of $\underline{y}_{[B_1\cup B_2]\backslash B_1}$ in $\partial B(\alpha_1)$ and denote it by x_1. Next, we put

$$\alpha_2 = \min\{\alpha\geqq \alpha_1 : d(\underline{y}_{[B_1\cup B_2]\backslash B_1}\backslash\{x_1\}, \partial B(\alpha)) = 0\},$$

and pick up one of the balls of $\underline{y}_{[B_1\cup B_2]\backslash B_1}\backslash\{x_1\}$ in $\partial B(\alpha_2)$ and denote it by x_2. Repeating this procedure, we can take $x_1, x_2, \cdots, x_k$ and $0 = \alpha_0 \leqq \alpha_1 \leqq \cdots \leqq \alpha_k \leqq \alpha_{k+1} = 1$ such that

$$\{x_1, x_2, \cdots, x_k\} = \underline{y}_{[B_1\cup B_2]\backslash B_1},$$

$$\{x_i, x_{i+1}, \cdots, x_k\} \subset [B_1 \cup B(\alpha_i)]^c,\quad 1\leqq i\leqq k,$$

$$x_i \in \partial B(\alpha_i), \qquad 1 \leqq i \leqq k.$$

Let $\underline{y}^i$, $0 \leqq i \leqq k$ and $\underline{z}^i$, $1 \leqq i \leqq k+1$ be $B(\alpha_i)$-standard configurations such that

$$\underline{y}^i{}_{B(\alpha_i)^c} = \underline{y}^i{}_{[B_1 \cup B(\alpha_i)]^c} = \{x_{i+1}, \cdots, x_k\} \cdot \underline{y}_{[B_1 \cup B_2]^c},$$

$$\underline{z}^i{}_{B(\alpha_i)^c} = \underline{z}^i{}_{[B_1 \cup B(\alpha_i)]^c} = \{x_i, \cdots, x_k\} \cdot \underline{y}_{[B_1 \cup B_2]^c}.$$

It is easily seen that $\underline{y} = \underline{y}^0$ and $\underline{z}^{k+1}$ is the B_2-standard configuration satisfying (ii) and (iii) in Lemma 3.2. Therefore it is enough to show the following two relations (3.14) and (3.15) to finish the proof of Lemma 3.2.

(3.14) $\quad \Lambda(\underline{y}^i) = \Lambda(\underline{z}^{i+1}), \qquad$ for $0 \leqq i \leqq k$,

(3.15) $\quad \Lambda(\underline{z}^i) \leftarrow h \rightarrow \Lambda(\underline{y}^i), \qquad$ for $1 \leqq i \leqq k$.

Since $\underline{y}^i$ and z^{i+1} are $[B_1 \cup B(\alpha_i)]$-standard, (3.14) follows from Lemma 3.1, and (3.15) follows from Lemma 3.2 directly.

§ 4. Asymptotics for a tagged particle

In this section, we study the behavior of one of the balls in our process. We call this ball the tagged particle. In order to follow the motion of the tagged particle it is convenient to regard the process ξ_t as a Markov process (x_t, η_t) on the locally compact space $\mathbb{R}^d \times \mathfrak{X}_0$, where

$$\mathfrak{X}_0 = \{ \eta \in \mathfrak{X} : \eta \cap B_r(0) = \phi \}.$$

x_t is the position of the tagged particle and η_t is the entire configuration seen from the tagged particle. We can see that η_t is a Markov process whose generator $\bar{\mathfrak{L}}$ is the closure of the operator given by

$$\mathfrak{L}f(\eta) = \int_{\mathbb{R}^d} \{f(\tau_{-u}\eta) - f(\eta)\}\chi(u|\eta)p(|u|)du$$
$$+ \sum_{z\in\eta} \int_{\mathbb{R}^d \setminus B_r(0)} \{f(\eta^{z,y}) - f(\eta)\}\chi(y|\eta\setminus z)p(|z - y|)dy,$$

where

$$\tau_u\eta = \{x_i + u\}, \qquad \text{if } \eta = \{x_i\}.$$

We denote by S_t the semigroup for $\bar{\mathfrak{L}}$ and (η_t, P^0_ν) the Markov process generated by $\bar{\mathfrak{L}}$ with initial distribution ν.

From Remark 1.2, $\#\mathcal{G}(z) = 1$ for sufficiently small $z > 0$ and so $\mu \in \mathcal{G}(z)$ is shift invariant. For any shift invariant $\mu \in \mathrm{ex}\mathcal{G}(z)$ we define

$$\mu_0(d\eta) = \frac{1}{\mu(\mathfrak{X}_0)} \chi(0|\eta)\mu(d\eta).$$

Using the same argument as Lemma 1.2 and Theorem 2.1, we have the following lemma.

Lemma 4.1. If $z > 0$ is sufficiently small, then $(\eta_t, P^o_{\mu_0})$ is an ergodic reversible Markov process for any $\mu \in \mathrm{ex}\mathcal{G}(z)$.

The process x_t is driven by the process η_t in the following way. We introduce measurable sets

$$\Delta = \{(\eta,\zeta) \in \mathfrak{X}_0 \times \mathfrak{X}_0 : \eta = \zeta\},$$

$$\Gamma_A = \{(\eta,\zeta) \in \mathfrak{X}_0 \times \mathfrak{X}_0 \setminus \Delta : \zeta = \tau_{-u}\eta \quad \text{for some} \quad u \in A\},$$

$$\text{for} \quad A \in \mathcal{B}(\mathbb{R}^d),$$

and define the family of σ-fields $\{\mathfrak{F}_t\}$ by

$$\mathfrak{F}_t = \sigma(\eta_s : s \in (-\infty, t]).$$

Then,

$$N((t_1,t_2] \times A) = \sum_{s \in (t_1,t_2]} 1_{\Gamma_A}(\eta_{s-}, \eta_s) \quad \text{for } 0 < t_1 < t_2,$$

is an $\mathfrak{F}_t$-adapted σ-finite random measure and we have

$$x_t = x_0 + \int_{(0,t]} \int_{R^d} u\, N(dsdu).$$

Using the same argument as Theorem 2.4 of [3], we have the following result.

<u>Theorem 4.1.</u> For sufficiently small $z > 0$ if $\mu \in \mathrm{ex}\mathcal{G}(z)$ then

$$\lambda x_{t/\lambda^2} \to D \cdot B_t \qquad \text{as } \lambda \to 0$$

in the sense of distribution in the Skorohod space, where B_t is d-dimensional Brownian motion and D is $d \times d$ matrix such that

$$(D^2)_{ij} = \int_{\mathbb{R}^d} du \int_{\mathfrak{X}_0} d\mu_0 \; u_i u_j \chi(u|\cdot) p(|u|)$$

$$- 2 \int_{[0,\infty)} dt \; \langle S_t F_i, F_j \rangle_{\mu_0},$$

$$F(\eta) = \int_{\mathbb{R}^d} du \; u \; p(|u|) \chi(u|\eta).$$

Unfortunately, we haven't proved the non-degeneracy of D.

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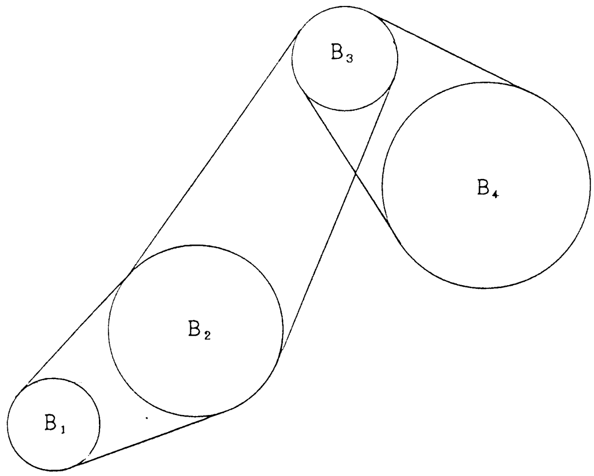

Fig.1. Example of B[4]

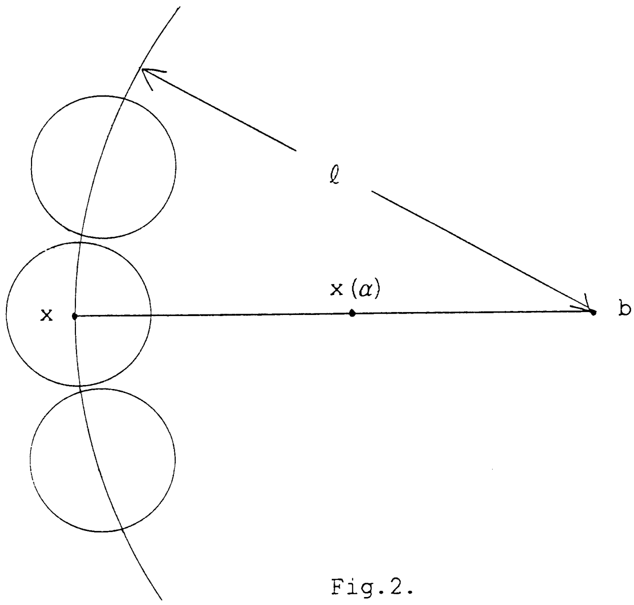

Fig.2.