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Topological classification of unknotted ring defects

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Abstract

Unknotted ring defects in ordered media are classified in terms of the homotopy theory. It is also investigated what type of point defects will appear when a radius of the ring defect tends to zero.

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§ 1. Introduction

The homotopy theory of defects in ordered media has been developed in the field of condensed matter physics (see [1-3] for review). Topologically stable defects can be classified by means of the homotopy groups of a topological space X which represents internal order of a medium (order parameter space). In the theory, the configurations with defects which can be transformed into each other by continuous deformation are regarded as the same. For example, the topological types of line defects are characterized by conjugacy classes of the first homotopy group (fundamental group), and those of point defects by automorphism classes of the second homotopy group by the action of the first homotopy group. In this paper we develop the mathematical foundation to classify defects of circular shape, or unknotted ring defects which are not penetrated by line defects (Fig.1). We call the defects of this shape the ring defects for short in this paper.

In physical situations, ring defects appear when the two line defects which are characterized by mutually inverse elements of the first homotopy group combine. If all the parts of the two line defects approach each other evenly, they disappear at the same time when they combine, but if they approach unevenly and it happens that some of their parts touch each other and disappear while the others have not, there appear ring defects (Fig.2).

Although the local structure of the ring defect is nothing but that of a line defect, a ring defect is a defect localized in finite volume and does not extend to infinity like a line defect, in that sense it can be categorized as a type of point defects.

In fact, some of ring defects are found to become point defects when their radii tend to zero.

The problem on the classification of ring defects has been explored by Garel[4], who studied the problem by means of the homotopy sets of mappings from tori to order parameter spaces. But these sets do not have group structure and are difficult to analyze. Besides all the elements of the sets do not necessarily correspond to ring defects themselves but some of them represent rings which are penetrated by line defects.

In the present work, we investigate the topological types of order parameter configurations with ring defects by using the homotopy theory. The order parameter configurations with a single ring defect are represented by continuous mappings, $R^3-\Sigma \rightarrow X$, where R^3 denotes the three dimensional Euclidean space and Σ denotes an unknotted ring, e.g. $\Sigma = \{ (x,y,z) \in R^3;\ x^2+y^2=1,\ z=0 \}$. Since $R^3-\Sigma$ is easily shown to be homotopically equivalent to the topological space W which is defined as the spherical surface with the two points N and S (e.g. the north and south poles) shrunk into a point

$$W \equiv S^2/\{ N, S \} \simeq R^3 - \Sigma, \qquad N, S \in S^2, N \neq S \qquad (1)$$

(Fig.3), the homotopy set of the continuous mappings from $R^3-\Sigma$ to X, which is denoted by $[R^3-\Sigma ; X]$, is equivalent to $[W;X]$ as a set. Therefore we can study the isolated ring defects by studying the continuous mappings from the space W to the order parameter space X. Note that W is also homeomorphic to the surface obtained by rotating the circle $\{ (x,y,z) \in R^3;\ (y-1)^2+z^2=1,\ x=0 \}$ around the z-axis, which encloses Σ (Fig.3(c)). If a base point $* \in X$ is fixed and we consider only mappings which send the base point $*_W \in W$ (to which N and S shrink) to $*$, a group structure can be introduced into the set and we call the set the second torus homotopy group $\tau_2(X,*)$ after Fox[5]. Note that this is not the homotopy set of mappings from the torus to the order parameter space and is different from the one Garel has studied[4].

The paper is organized as follows. In Sec.2 the definition of the second torus homotopy group and its relation with the classical homotopy group are given and it is shown that the second torus homotopy group is a semidirect product of the first and second homotopy groups

$$\tau_2(X,*) \cong \pi_1(X,*) \ltimes \pi_2(X,*) ,$$

whose multiplication rule between two elements of $\tau_2(X,*)$ is given using the usual action of $\pi_1(X,*)$ on $\pi_2(X,*)$. In Sec.3, the mapping $\tau_2(X,*) \to \pi_2(X,*)$ which maps a ring defect to the point defect that the ring turns into when it shrinks, is studied. In Sec.4, the correspondence between the elements of $\tau_2(X,*)$ and the physical types of isolated ring defects is examined and it is also studied what would happen if another ring or point defect goes through the ring defect. Some physical examples are presented in Sec.5 and the summary of the paper and discussions are given in Sec.6.

§ 2. Second torus homotopy group

Definitions: Let $(X,*)$ be an arcwise connected topological space with a base point $* \in X$. We define a family of mappings $F_{\tau 2}(X,*)$ as the totality of continuous mappings $f: I^2 \to X$ which satisfy the condition

$$f(u, 0) = f(u, 1) \ , \quad f(0, v) = f(1, v) = * \ ,$$

where $u, v \in I \equiv [0,1]$. The second torus homotopy group $\tau_2(X,*)$ is defined as the homotopy set of $F_{\tau 2}(X,*)$, namely,

$$\tau_2(X,*) \equiv \pi_0(F_{\tau 2}(X,*)) \ . \qquad (2)$$

Equivalently, $\tau_2(X,*)$ is the quotient space of $F_{\tau 2}(X,*)$ under the equivalence relation $\sim_*$: $f \sim_* f'$ means we can connect them by a homotopy h_t ($t \in I$, $h_0 = f$, $h_1 = f'$) with $h_t \in F_{\tau 2}(X,*)$. It is also shown that $\tau_2(X,*) = [W, *_W; X, *]$, i.e. the homotopy set of mappings $W \to X$ which map $*_W \mapsto *$. The multiplication in $\tau_2(X,*)$ is introduced by the multiplication in $F_{\tau 2}(X,*)$, which is defined by

$$h = f \cdot g \quad \Leftrightarrow \quad h(u, v) = \begin{cases} f(2u, v) & (\ 0 \leqq u \leqq 1/2\ ) \\ g(2u-1, v) & (\ 1/2 \leqq u \leqq 1\ ) \ , \end{cases}$$

and we use the notation $f^{-1} \in F_{\tau 2}(X,*)$ to denote

$$f^{-1}(u, v) = f(1-u, v) \ .$$

We also introduce a family of continuous mappings $F_{\pi 2}(X,*)$ as the totality of continuous mappings $f\colon I^2 \to X$ which satisfy the condition

$$f(u, 0) = f(u, 1) = f(0, v) = f(1, v) = * ,$$

and $F_{\pi 1}(X,*)$ as the totality of mappings $f\colon I \to X$ which satisfy the condition

$$f(0) = f(1) = * .$$

Note that $F_{\pi 2}(X,*) \subset F_{\tau 2}(X,*)$ and that the first and second homotopy groups are obtained by

$$\pi_1(X,*) = \pi_0(F_{\pi 1}(X,*)) , \tag{3}$$

$$\pi_2(X,*) = \pi_0(F_{\pi 2}(X,*)) . \tag{4}$$

Hereafter we use $F_{\tau 2}$, τ_2 etc. to denote $F_{\tau 2}(X,*)$, $\tau_2(X,*)$ etc. in case the abbreviations would not cause any confusion.

Now we define a projection $p\colon F_{\tau 2} \to F_{\pi 1}$ as

$$p(f)(u) = f(u, 0) , \qquad f \in F_{\tau 2} ,$$

then there exists a section $s\colon F_{\pi 1} \to F_{\tau 2}$ defined by

$$s(g)(u, v) = g(u) , \qquad g \in F_{\pi 1} , \tag{5}$$

which satisfies

$$p \circ s = 1 . \tag{6}$$

We use the symbols $*_1$ and $*_2$ to denote constant mappings $I \to * \in X$ and $I^2 \to * \in X$ respectively, namely,

$$*_1(u) = * , \quad \text{and} \quad *_2(u,v) = * ,$$

where $u, v \in I$, and they are related by p and s as

$$p(*_2) = *_1 , \quad \text{and} \quad s(*_1) = *_2 .$$

We state the result first.

Theorem. The second torus homotopy group $\tau_2(X,*)$ is isomorphic to the semidirect product of $\pi_1(X,*)$ and $\pi_2(X,*)$:

$$\tau_2(X,*) \cong \pi_1(X,*) \ltimes \pi_2(X,*) , \tag{7}$$

by the usual action of $\pi_1(X,*)$ on $\pi_2(X,*)$. Namely, every element of $\tau_2(X,*)$ can be represented uniquely by a pair of $\gamma \in \pi_1(X,*)$ and $n \in \pi_2(X,*)$ as $(\gamma , n) \in \tau_2(X,*)$, and the multiplication is given by

$$(\gamma_1, n_1) \cdot (\gamma_2, n_2) = (\gamma_1 \cdot \gamma_2, \gamma_2^{-1}(n_1) + n_2) , \tag{8}$$

where $\gamma_2^{-1}(n_1)$ denotes the image of $n_1 \in \pi_2(X,*)$ by the automorphism introduced by $\gamma_2^{-1} \in \pi_1(X,*)$.

We have used the symbol + to denote the multiplication of π_2 since π_2 is Abelian. To prove this theorem, we show the following lemma first.

Lemma. There exists an exact sequence

$$0 \rightarrow \pi_2(X,*) \overset{i_*}{\rightarrow} \tau_2(X,*) \underset{s_*}{\overset{p_*}{\rightleftarrows}} \pi_1(X,*) \rightarrow 0 \ , \qquad (9)$$

where i_*, p_*, and s_* are homomorphisms, i_* and s_* are injective, and

$$p_* \circ s_* = 1 \ , \qquad (10)$$

proof:

$(F_{\tau 2}, p, F_{\pi 1})$ forms a fibre space, where $F_{\tau 2}$ is a total space and $F_{\pi 1}$ is a base space (see Appendix). The projection p has the section s given by (5) and the fibre on $*_1 \in F_{\pi 1}$ is

$$p^{-1}(*_1) = F_{\pi 2} \ .$$

Then we obtain the homotopy exact sequence for this fibre space,

$$\pi_1(F_{\tau 2}, *_2) \underset{s_*}{\overset{p_*}{\rightleftarrows}} \pi_1(F_{\pi 1}, *_1) \overset{\Delta}{\rightarrow} \pi_0(F_{\pi 2})$$

$$\overset{i_*}{\rightarrow} \pi_0(F_{\tau 2}) \underset{s_*}{\overset{p_*}{\rightleftarrows}} \pi_0(F_{\pi 1}) \ ,$$

which becomes

$$\pi_1(F_{\tau_2},*_2) \underset{s_*}{\overset{p_*}{\rightleftarrows}} \pi_2(X,*) \overset{\Delta}{\rightarrow} \pi_2(X,*) \overset{i_*}{\rightarrow} \tau_2(X,*) \underset{s_*}{\overset{p_*}{\rightleftarrows}} \pi_1(X,*) \quad (11)$$

due to (2), (3), (4) and

$$\pi_1(F_{\pi_1}, *_1) = \pi_2(X,*) \,.$$

The mappings p_*, s_*, and Δ are shown to be homomorphisms although they are not necessarily so for homotopy exact sequences since π_0's are not groups in general. The equation (6) leads to (10), which implies p_* is surjective and s_* is injective. From the fact that the sequence (11) is exact, we observe $\mathrm{Im}(\Delta) = 0$ then we get (9) and i_* is found to be injective (Fig.4). ■

Now we give a proof of the theorem.

proof of Theorem: Since s_* and i_* are injective homomorphisms, π_1 and π_2 are isomorphic to the images $\mathrm{Im}(s_*)$ and $\mathrm{Im}(i_*)$, which are denoted by π_1' and π_2' respectively;

$$\pi_1' \equiv \mathrm{Im}(s_*) \cong \pi_1 \,, \qquad \pi_2' \equiv \mathrm{Im}(i_*) \cong \pi_2 \,.$$

The subgroups π_1' and π_2' have the following properties.

1) $\pi_1' \cap \pi_2' = \{e\}$, where e denotes the unit element. In fact, if $a \in \pi_1' \cap \pi_2'$, then there exist $\gamma \in \pi_1$ and $n \in \pi_2$ such that $a = s_*(\gamma) = i_*(n)$. From (10) we obtain $p_*(a) = p_* \circ s_*(\gamma) = \gamma$, while $p_*(a) = p_* \circ i_*(n) = e$ since the sequence (9) is exact. Then $\gamma = e$ and $a = s_*(e) = e$.

2) $\tau_2 = \pi_1'\pi_2' = \{\gamma' \cdot n' \in \tau_2;\ \gamma' \in \pi_1',\ n' \in \pi_2'\}$. In fact,

any element $a \in \tau_2$ can be expressed as $a = \gamma' \cdot n'$ as follows, where $\gamma' \in \pi_1'$, $n' \in \pi_2'$. Let $\gamma = p_*(a) \in \pi_1$ and $b = s_*(\gamma) \in \tau_2$, then $b^{-1} \cdot a \in \pi_2'$ because $p_*(b^{-1} \cdot a) = \gamma^{-1} \cdot \gamma = e$. From the exactness of (9), there exists an element $n \in \pi_2$ such that $b^{-1} \cdot a = i_*(n)$, which leads an expression of a,

$$a = b \cdot i_*(n) = s_*(\gamma) \cdot i_*(n) \equiv \gamma' \cdot n' \qquad (12)$$

From 1), this expression is unique.

3) π_2' is a normal subgroup of τ_2 because it is a kernel of p_*.

1), 2), and 3) imply that τ_2 is isomorphic to a semidirect product of π_1 and π_2.

Since the expression (12) is unique, any element of τ_2 can be represented by γ and n as

$$(\gamma, n) \equiv s_*(\gamma) \cdot i_*(n) . \qquad (13)$$

The multiplication of elements is given by

$$\begin{aligned}(\gamma_1, n_1) \cdot (\gamma_2, n_2) &= s_*(\gamma_1) \cdot i_*(n_1) \cdot s_*(\gamma_2) \cdot i_*(n_2) \\ &= s_*(\gamma_1) \cdot s_*(\gamma_2) \cdot s_*(\gamma_2)^{-1} \cdot i_*(n_1) \cdot s_*(\gamma_2) \cdot i_*(n_2) \\ &= s_*(\gamma_1 \cdot \gamma_2) \cdot I_{s_*(\gamma_2^{-1})}(n_1') \cdot i_*(n_2) ,\end{aligned}$$

where $n_1' = i_*(n_1)$ and $I_a(b)$ denotes an inner automorphism by a; $I_a(b) = a \cdot b \cdot a^{-1}$. The explicit expression of the inner automorphism is given by

$$I_{s_*(\gamma)}(n) = i_*(\ \gamma(n)\ )\ . \qquad (14)$$

In fact, the left hand side and the right hand side of (14) are represented by Fig.5(a) and (c) respectively, and they are clearly homotopic in $F_{\tau 2}$. Then we get,

$$\begin{aligned}(\gamma_1, n_1)\cdot(\gamma_2, n_2) &= s_*(\gamma_1\cdot\gamma_2)\cdot i_*(\gamma_2^{-1}(n_1))\cdot i_*(n_2) \\ &= s_*(\gamma_1\cdot\gamma_2)\cdot i_*(\gamma_2^{-1}(n_1)+n_2) \\ &= (\ \gamma_1\cdot\gamma_2,\ \gamma_2^{-1}(n_1)+n_2\ )\ ,\end{aligned}$$

which is (8). ■

§ 3. Mapping from τ_2 to π_2

To investigate what type of point defect the ring defect becomes when its radius goes to zero, we define a mapping $\tau_2 \to \pi_2$. This mapping should correspond to the operation that separates the points N and S which were shrunk into the point $*_W$ of W (Fig.3).

We define a family of mappings $F_{\eta 2}(X,*)$ as the totality of mappings $f: I^2 \to X$ which satisfy

$$f(u, 0) = f(u, 1)\ ,\quad f(0, v) = *\ ,\quad f(1, v) = f(1, 0)\ .$$

Note that $F_{\eta 2} \supset F_{\tau 2} \supset F_{\pi 2}$, and there exists a homeomorphism

$$\omega : F_{\eta 2} \to F_{\pi 2}\ ,$$

which is defined as follows. Let q be the mapping $I^2 \to I^2$ defined by

$$q: (u, v) \to (u', v') = (1/2 + r\cdot\cos\theta, 1/2 + r\cdot\sin\theta)$$

where

$$r = (1-u)\cdot\lambda(\theta), \quad \theta = 2\pi\cdot(1-v),$$

(Fig.6). $\lambda(\theta)$ is the length of the segment connecting the center ($u'=1/2$, $v'=1/2$) and the circumference of the square in u'-v' plane in the direction of θ. Then ω is defined by

$$\omega(f)(u', v') = f(u, v),$$

where $q(u,v)=(u',v')$. Although q is not injective, this defines the mapping ω for $f \in F_{\eta 2}$. Therefore there exists a bijection

$$\omega_*: \pi_0(F_{\eta 2}) \to \pi_2(X,*),$$

and the mappings to be examined is

$$\Omega_*: \tau_2(X,*) \to \pi_2(X,*)$$

which is induced by the restriction of ω to $F_{\tau 2}$, or $\Omega = \omega|_{F\tau 2}$.

Now we have the explicit expression:

$$\Omega_*(\ (\gamma\ ,n)\ ) = \gamma\ (n)\ , \tag{15}$$

which can be shown as follows. Recalling (13), first we see
1) $\Omega_*(\ (e,n)\ ) = \Omega_*(\ i_*(n)\ ) = n \in \pi_2(X,*)$, since $\Omega(\ i(f)\ )$ is homotopic with f in $F_{\pi 2}$ where f belongs to the homotopy class that n represents (Fig.7). More generally, we have
2) $\Omega_*(\ (\gamma\ ,n)\ ) = \Omega_*(\ s_*(\gamma\ )\cdot i_*(n)\ )$, and it is easily seen to be $\gamma\ (n)$ (Fig.8). Remark that Ω_* is not a homomorphism in general.

The equation (15) means that the ring defect represented by $(\gamma\ ,n) \in \tau_2$ becomes the point defect $\gamma\ (n) \in \pi_2$ when it shrinks to a point.

§ 4. Physical interpretation of τ_2

The second torus homotopy group $\tau_2(X,*)$ is defined as the homotopy set of $F_{\tau 2}(X,*)$ with a fixed base point $* \in X$, while the continuous deformation we consider in the physical situations is not restricted to such a space. This implies that each element of τ_2 does not necessarily correspond with a distinct type of ring defects. The situation is analogous to the situation that occurs when we consider the correspondence between line defects and elements of π_1 and also between point defects and those of π_2.

4.1 Order parameter configurations with a single ring defect

The topological types of the order parameter configurations with a single ring defect correspond to the elements of [W;X]. To investigate the correspondence between the elements of [W;X] and $\tau_2(X,*)=[W,*_W;\ X,*]$, we define a family of mappings $F'_{\tau 2}(X)$ as the totality of mappings f: $I^2 \to X$ which satisfy,

$$f(0,0) = f(0,v) = f(1,v')\ ,\quad f(u,0) = f(u,1)\ ,\quad u,v,v' \in I\ ,$$

then $\pi_0(F'_{\tau 2})=[W;X]$. Since $F_{\tau 2}(X,*) \subset F'_{\tau 2}(X)$, the inclusion mapping k: $F_{\tau 2} \to F'_{\tau 2}$ can be defined.

We show the following proposition.

Proposition 1. The mapping $\tilde{k}$: $\pi_0(F_{\tau 2}) \to \pi_0(F'_{\tau 2})$ which is induced by k is 1) well-defined, 2) surjective, and 3) $\tilde{k}^{-1}([g])$ coincides with one of the automorphism classes of $\tau_2(X,*)$ by an action of $\pi_1(X,*)$ for any $g \in F'_{\tau 2}(X)$, where we have introduced the notation $[g]$ to denote the homotopy class which g belongs to. The action of $\gamma' \in \pi_1(X,*)$ on the element $(\gamma, n) \in \tau_2(X,*)$ is denoted by $\gamma'((\gamma, n))$ and given by

$$\gamma'((\gamma,n)) = (\gamma', 0)\cdot(\gamma, n)\cdot(\gamma', 0)^{-1}\ , \qquad (16)$$

where $\gamma \in \pi_1(X,*)$ and $n \in \pi_2(X,*)$.

proof:

1) This is obvious because "homotopic in $F_{\tau 2}$" implies "homotopic in $F'_{\tau 2}$".

2) We show that for any $g \in F'_{\tau 2}$ there exists $f \in F_{\tau 2}$ which satisfies $\tilde{k}([f])=[k(f)]=[g]$. In fact, by introducing a path c: $I \to X$ with $c(0)=*$ and $c(1)=g(0,0)$, f can be constructed as

$$f(u, v) = \begin{cases} c(3u) & (0 \leqq u \leqq 1/3) \\ g(3u-1, v) & (1/3 \leqq u \leqq 2/3) \\ c(3-3u) & (2/3 \leqq u \leqq 1). \end{cases}$$

3) It is easily seen that $[k(s(d)\cdot f\cdot s(d^{-1}))]=[g]$ for any $d\in F_{\pi 1}$, where $[k(f)]=[g]$. We have used the notation $d^{-1}\in F_{\pi 1}$ to denote $d^{-1}(u)=d(1-u)$ $(u\in I)$. We have to show that for any f and $f'\in F_{\tau 2}$ with $[k(f)]=[k(f')]=[g]$, there exists $d\in F_{\pi 1}$ which satisfies $[s(d)\cdot f'\cdot s(d^{-1})]=[f]$. There exist the homotopies g_t and g'_t $(t\in I,\ g_0=g'_0=g,\ g_1=k(f),\ g'_1=k(f'))$ with g_t, $g'_t\in F'_{\tau 2}$. If $d\in F_{\pi 1}$ is defined by

$$d(u) = \begin{cases} g_{1-2u}(0, 0) & (0 \leqq u \leqq 1/2) \\ g'_{2u-1}(0, 0) & (1/2 \leqq u \leqq 1), \end{cases}$$

it can be shown $[s(d)\cdot f'\cdot s(d^{-1})]=[f]$. ■

From the proposition 1, it is seen that two configurations with a ring defect can be transformed into each other if and only if they are characterized by the same automorphism class of $\tau_2(X,*)$ by the action of $\pi_1(X,*)$ given by (16).

Point defects are also represented by elements (e, n) of τ_2, which form a normal subgroup π_2'. This is expected because the closed surface W can also enclose point defects. The fact that the ring defects are classified by the automorphism classes of τ_2 by π_1 is consistent with the fact that the point defects are classified by the automorphism classes of π_2 by π_1.

4.2 Ring defects in the presence of other ring or point defects

The ring defects has the feature which either the line or point defects do not have, i.e. other ring or point defects can go through the ring defect. This transformation is continuous but cannot be treated as a continuous path in the space $F'_{\tau 2}(X)$ because this continuous transformation can be considered only on the surface of the torus which encloses the ring defect but not on the space W. Therefore we introduce a family of mappings $F_{\theta 2}(X)$ which is the totality of continuous mappings $f: I^2 \rightarrow X$ that satisfy

$$f(0, v) = f(1, v), \quad f(u, 0) = f(u, 1)$$

with $u, v \in I$ and define an equivalence relation $\sim$ in $F_{\tau 2}$ as follows. For the elements f and f' of $F_{\tau 2}(X,*)$, $f \sim f'$ means that they can be connected by a homotopy h_t ($t \in I$, $h_0 = f$, $h_1 = f'$) with $h_t \in F_{\theta 2}(X)$. Note that $f \sim f'$ if f and f' belong to the same homotopy class of $F_{\tau 2}$, i.e. [f]=[f'], but the reverse is not necessarily true.

Proposition 2. The quotient space of $F_{\tau 2}(X,*)$ under the above equivalence relation $F_{\tau 2}(X,*)/\sim$ is isomorphic to the set of conjugacy classes of $\tau_2(X,*)$ as a set.

proof: If [f] and [f'] of τ_2 where $f, f' \in F_{\tau 2}$ belong to the same conjugacy class, there exists $g \in F_{\tau 2}$ such that $[f'] = [g \cdot f \cdot g^{-1}]$. The homotopy h_t can be defined by

$$h_t(u, v) = \begin{cases} g(1-t+3u, v) & (0 \leqq u \leqq t/3) \\ f((3u-t)/(3-2t), v) & (t/3 \leqq u \leqq 1-t/3) \\ g(4-3u-t, v) & (1-t/3 \leqq u \leqq 1), \end{cases}$$

which satisfies $h_t \in F_{\theta\,2}$, $h_0=f$ and $[h_1]=[f']$ therefore $f \sim h_1 \sim f'$.

On the other hand if $f \sim f'$, then there exists the homotopy h_t with $h_0=f$, $h_1=f'$, and $h_t \in F_{\theta\,2}$. The element g of $F_{\tau\,2}$ which satisfies $[g \cdot f' \cdot g^{-1}]=[f]$ can be defined as $g(u, v)=h_u(0, v)$. ∎

The proposition 2 shows that two types of ring defect can be transformed into each other by continuous deformation allowing another ring or point defect to go through the ring if and only if they are characterized by the same conjugacy class of $\tau_2(X,*)$.

In Fig.9, the two pairs of filled and open circles denote cross section of the ring defects denoted by 1 and 2 respectively, and the heavy and light lines (the dotted lines) denote the cross sections of closed surfaces W which enclose the ring defect 1 (the ring defect 2). The arrows indicate the positive direction of u-axis. α, β, and γ denote the elements of τ_2 which represent the configurations on the closed surfaces indicated by heavy, light, and dotted lines, respectively. From Fig.9(b), it is obvious that

$$\alpha = \gamma \cdot \beta \cdot \gamma^{-1}, \qquad (17)$$

and it can be seen that there are two elements α and β, which characterize the ring defect 1. They may belong to different automorphism classes and $\Omega_*(\alpha)$ may not equal to $\Omega_*(\beta)$. If the ring 1 moves upwards and shrinks (Fig.9(a)), then it will become the point defect characterized by $\Omega_*(\alpha) \in \pi_2$. On the other hand, if the ring 1 moves downwards and shrinks (Fig.9(c)), then it will become the point defect characterized by $\Omega_*(\beta)$. In this sense, it might be roughly said that the ring defect 1, which is originally characterized by α is transformed into the ring defect characterized by β by means of being penetrated by the ring defect γ (Fig.9 (a)→(b)→(c)).

§ 5. Physical examples

The list of systems of physical interest, their order parameter spaces, and the first, second, and second torus homotopy groups is given in the table. Since the second torus homotopy group τ_2 is a semidirect product of π_1 and π_2, τ_2 is isomorphic to π_1 (π_2) if $\pi_2=0$ ($\pi_1=0$). Noting $\tau_2(X\times Y, (*_X,*_Y)) = \tau_2(X,*_X) \times \tau_2(Y,*_Y)$, there are only two systems with non trivial τ_2 in the table; the nematic phase of liquid crystal and the dipole free A-phase of superfluid ^3He. We give τ_2 for these systems as examples. We use the same notation for groups with that of ref.1.

5.1 Nematic liquid crystal

The order parameter space X, its π_1, and π_2 are given by

$$X = RP^2, \quad \pi_1 = Z_2 = \{0,1\}, \quad \pi_2 = Z .$$

Since the automorphisms of π_2 by π_1 are

$$0(n) = n, \quad 1(n) = -n, \qquad n \in \pi_2 ,$$

the multiplication law of τ_2 will be

$$(\gamma_1, n_1)\cdot(\gamma_2, n_2) = \begin{cases} (\ \gamma_1 + \gamma_2 (\mathrm{mod}\ 2),\ n_1 + n_2\ ) & \text{for } \gamma_2 = 0 \\ (\ \gamma_1 + \gamma_2 (\mathrm{mod}\ 2), -n_1 + n_2\ ) & \text{for } \gamma_2 = 1 \end{cases}$$

and inverse elements are obtained by

$$(0,\ n)^{-1} = (0,\ -n)\ , \quad (1,\ n)^{-1} = (1,\ n)\ .$$

The expression $\gamma_1 + \gamma_2 (\mathrm{mod}\ r)$ means the sum is taken in the mod r sense. The automorphism classes by π_1 are

$$A_{(m,n)} = \{\ (m,n),\ (m,-n)\ \}, \quad m = 0,1; \quad n = 0,1,2,\cdots \qquad (18)$$

and the conjugacy classes are

$$C_{(0,n)} = A_{(0,n)}\ , \qquad n = 0,1,2,3,\cdots$$
$$C_{(1,0)} = \bigcup_{n=0}^{\infty} A_{(1,2n)} \qquad (19)$$
$$C_{(1,1)} = \bigcup_{n=0}^{\infty} A_{(1,2n+1)}\ .$$

The class $A_{(0,n)}$ gives the classification of point defects.

5.2 Dipole free A-phase of ^3He

For this system

$$X = (SO(3) \times S^2)/Z_2 \ ,$$

$$\pi_1 = Z_4 = \{ 0, 1, 2, 3 \} \ , \quad \pi_2 = Z \ .$$

and the automorphisms of π_2 by π_1 are

$$0(n) = 2(n) = n \ , \qquad 1(n) = 3(n) = -n \ .$$

The multiplications and inverse elements are given by

$$(\gamma_1, n_1)\cdot(\gamma_2, n_2) = \begin{cases} (\gamma_1 + \gamma_2 (\mathrm{mod}\ 4),\ n_1 + n_2) & \text{for } \gamma_2 = 0,\ 2 \\ (\gamma_1 + \gamma_2 (\mathrm{mod}\ 4), -n_1 + n_2) & \text{for } \gamma_2 = 1,\ 3 \end{cases}$$

and

$$(0,\ n)^{-1} = (0,\ -n), \quad (1,\ n)^{-1} = (3,\ n),$$

$$(2,\ n)^{-1} = (2,\ -n), \quad (3,\ n)^{-1} = (1,\ n).$$

The automorphism classes by π_1 are

$$A_{(m,n)} = \{ (m,\ n),\ (m,\ -n) \}, \quad m = 0,1,2,3 \ ; \quad n = 0,1,2,3,\cdots \qquad (20)$$

and the conjugacy classes are

$$C_{(0,n)} = A_{(0,n)}\ ,\quad n = 0,1,2,3,\cdots$$
$$C_{(1,0)} = \bigcup_{n=0}^{\infty} A_{(1,2n)}$$
$$C_{(1,1)} = \bigcup_{n=0}^{\infty} A_{(1,2n+1)}$$
$$C_{(2,n)} = A_{(2,n)}\ ,\quad n = 0,1,2,3,\cdots \tag{21}$$
$$C_{(3,0)} = \bigcup_{n=0}^{\infty} A_{(3,2n)}$$
$$C_{(3,1)} = \bigcup_{n=0}^{\infty} A_{(3,2n+1)}\ .$$

The class $A_{(0,n)}$ gives the classification of point defects.

§ 6. Summary and discussions

We have investigated the topological classification of ring defects by analyzing the homotopy set τ_2 of the mappings from the space W, which is homotopically equivalent to $R^3-\Sigma$, to the order parameter space. It has been shown that τ_2 is a semidirect product of π_1 and π_2, which is the manifestation of the duality of the ring defects, namely, the local structure of the singularity is identical with that of the line defects, while the defect is confined in finite volume like a point defect. The topological types of isolated ring defects correspond to the automorphism classes of τ_2 by the π_1-action, whereas one type of the ring defects can be transformed into another which is characterized by the conjugate element of the first one in the presence of other ring or point defects if they are allowed to go through the ring defect.

The multiplication of two automorphism classes of τ_2 gives the type of ring defect which is obtained when the two ring defects merge in the way shown in Fig.10(a). When the multiplication of two automorphism classes does not result in a single automorphism class, the result of the defect combination will depend on the path along which they merge in the presence of other line defects (Fig.10(b)).

Some of the ring defects leave point defects when they shrink. The theory predicts that the elements which belong to the same conjugacy class can result in different point defects (see (15), (19), and (21)). However, this is not a contradiction and similar situation occurs for the point defects in the presence of the line defects, i.e. the point defects can be created or annihilated by being brought around the line defect. Similarly in the presence of another ring defect or point defect, the ring defect results in different types of point defects depending on whether it shrinks directly or after the ring or point defects pass through the ring defect.

In conclusion, we have found that the ring defects have richer topological structure than either point defects or line defects.

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Appendix

In the appendix we show that ($F_{\tau 2}$, p, $F_{\pi 1}$) is a fibre space. It is obvious that p is continuous and surjective. We have to demonstrate that it has the covering homotopy property for any topological space Y, i.e. for any homotopy $f_t\colon Y \to F_{\pi 1}$ where $t \in I$, and for any lift of f_0, $g\colon Y \to F_{\tau 2}$, which satisfies $p \circ g = f_0$, there exists a lift of f_t, $g_t\colon Y \to F_{\tau 2}$ which satisfies $g_0 = g$ (and $p \circ g_t = f_t$). Actually if we define g_t as

$$g_t(y)(u,v) = \begin{cases} f_{t-3v}(y)(u) & (0 \leqq v \leqq t/3) \\ g(y)(u,\ (3v-t)/(3-2t)) & (t/3 \leqq v \leqq 1-t/3) \\ f_{3(v-1)+t}(y)(u) & (1-t/3 \leqq v \leqq 1)\ , \end{cases}$$

where

$$y \in Y, \quad t, u, v \in I,$$
$$f_t\colon Y \to F_{\pi 1}, \qquad g\colon Y \to F_{\tau 2},$$

then g_t is a lift of f_t, and satisfies $g_0 = g$. ■

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Table

Physical systems	Topological space	π_2	τ_2	π_1
Planer spin	S^1	0	Z	Z
Heisenberg spin	S^2	Z	Z	0
Nematics	RP^2	Z	$Z_2 \ltimes Z$	Z_2
Biaxial nematics	$SO(3)/D_2$	0	Q	Q
Dipole free A-phase(^3He)	$(SO(3)\times S^2)/Z_2$	Z	$Z_4 \ltimes Z$	Z_4
Dipole locked A-phase(^3He)	SO(3)	0	Z_2	Z_2
Dipole free B-phase(^3He)	$SO(3)\times S^1$	0	$Z\times Z_2$	$Z\times Z_2$
Dipole locked B-phase(^3He)	$S^2\times S^1$	Z	$Z\times Z$	Z

Table caption

The first, second, and second torus homotopy groups of the topological spaces for the systems with physical interest. The notations for groups are the same with ref.1.

Figure captions

Fig.1 An unknotted ring defect.

Fig.2 (a) Two line defects characterized by γ and $\gamma^{-1} \in \pi_1$ approach unevenly. (b) Some of their parts combine to disappear (the dotted lines) and a ring defect is left.

Fig.3 (a) The spherical surface S^2 and (b) the closed surface W with the north pole N and south pole S of S^2 shrunk into a point. (c) The surface obtained by rotating the circle $\{(x,y,z) \in R^3; (y-1)^2+z^2=1,\ x=0\}$ around the z-axis.

Fig.4 The homotopy exact sequence of the fibre space $(F_{\tau_2}, p, F_{\pi_1})$.

Fig.5 The diagrams which illustrate that (a) $s_*(\gamma)\cdot n\cdot s_*(\gamma^{-1})$ and (c) $\gamma(n)$ represent the same element of τ_2. The heavy lines denote the regions of I^2 which are mapped into the base point $*$ of X.

Fig.6 The mapping q from I^2 of the u-v plane to I^2 of the u'-v' plane. The left side of the square in the u-v plane (the heavy line) is mapped into the circumference of the square in the u'-v' plane (the heavy line) and the right side of the square in the u-v plane (the dotted line) is mapped into the center of the square in the u'-v' plane.

Fig.7 The diagrams which illustrate that (a) f of $F_{\pi 2}$ and (b) Ω (i(f)) are homotopic. The heavy lines indicate the regions which are taken into the base point $*$ of X.

Fig.8 The diagrams which illustrate $\Omega_*((\gamma, n)) = \gamma(n)$. (a), (b), and (c) represent (γ, n), $\Omega_*((\gamma, n))$, and $\gamma(n)$, respectively.

Fig.9 The ring defect 2 whose cross section is denoted by the pair of open circles goes through the ring defect 1 denoted by the pair of filled circles; (a)→(b)→(c). The heavy and light lines (the dotted line) denote the cross section of the closed surfaces W, which enclose the ring defect 1 (the ring defect 2). α, β, and γ denote the elements of τ_2 which represent the order parameter configurations on the surfaces denoted by the heavy, light, and dotted lines, respectively. The arrows indicate the direction of u-axis.

Fig.10 (a) Two ring defects combine into one. (b) Two different paths along which two ring defects combine in the presence of a line defect.

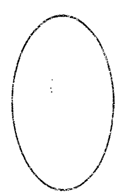


Fig. 1

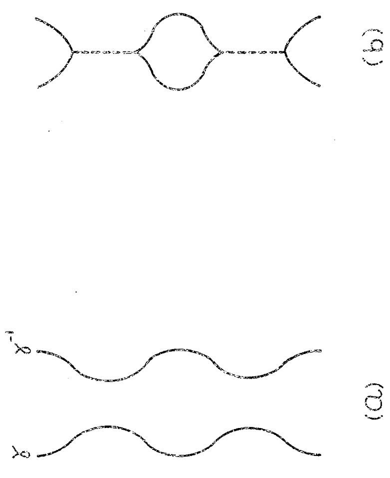


Fig. 2

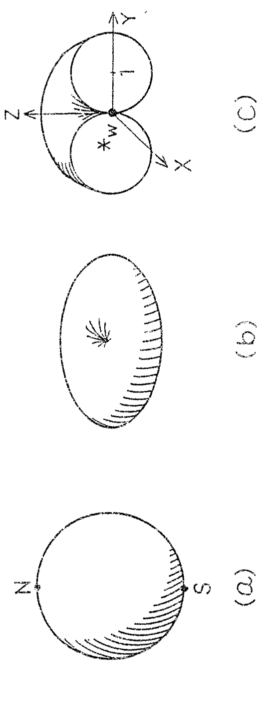
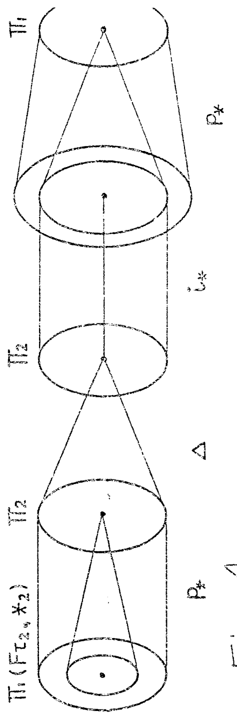


Fig. 3



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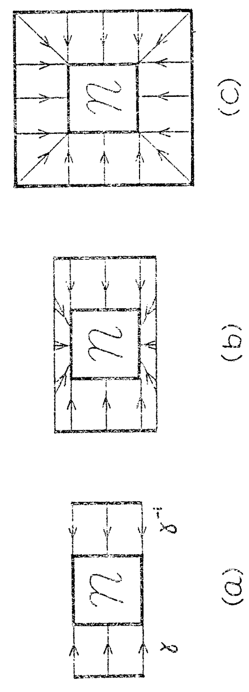

Fig. 5

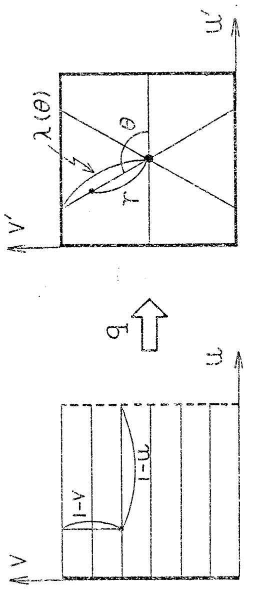

Fig. 6

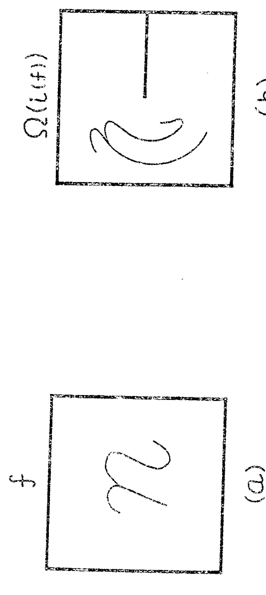

Fig. 7

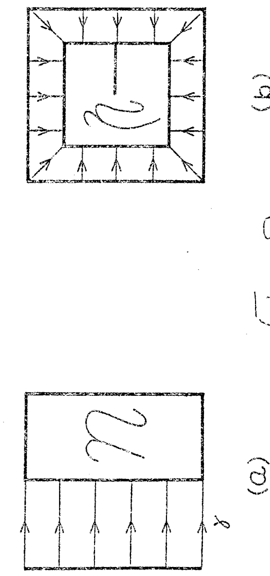

Fig. 8

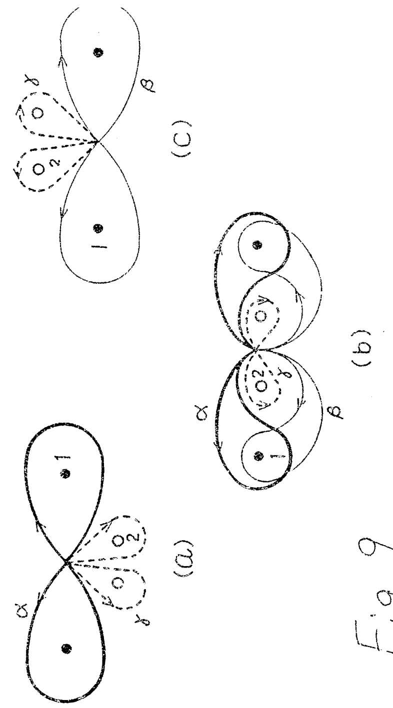

Fig. 9

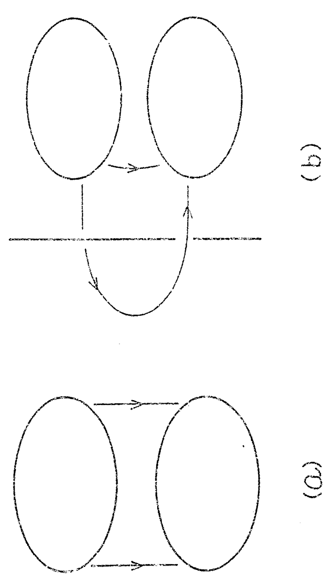

Fig. 10