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GEODESIC FLOWS OF NEGATIVELY CURVED MANIFOLDS WITH SMOOTH STABLE AND UNSTABLE FOLIATIONS

Dedicated to Professor Morio Obata on his 60th birthday

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Abstract. We are concerned with closed C^∞ riemannian manifolds of negative curvature whose geodesic flows have C^∞ stable and unstable foliations. In particular, we show that the geodesic flow of such a manifold is congruent to that of a certain closed riemannian manifold of constant negative curvature if the dimension of the manifold is greater than three and if the sectional curvature lies between $-9/4$ and -1 strictly.

Introduction

The geodesic flows of negatively curved manifolds have been investigated for a long time as a main subject in dynamics and ergodic theory. In particular, in 1960's, Anosov [**2**] introduced the notion of so-called the Anosov flows by abstracting the hyperbolic behavior of the geodesic flows of negatively curved manifolds, and showed that they possess a lot of beautiful properties such as ergodicity, the structural stability and the existence of periodic orbits. By definition, a smooth flow φ_t on a closed riemannian manifold V is called an Anosov flow, if there exists a φ_t-invariant linear splitting $TV = E^- + E^0 + E^+$ of the tangent bundle of V that satisfies the following conditions:

(i) E^0 is the 1-dimensional subbundle of TV spanned by the vector field on V that generates the flow φ_t;

(ii) The subbundles E^- and E^+ of TV are characterized by the inequalities

$$|d\varphi_t \xi^-| \le c_1 \cdot e^{-c_2 t} |\xi^-| \qquad \text{and} \qquad |d\varphi_{-t} \xi^+| \le c_1 \cdot e^{-c_2 t} |\xi^+|$$

for $\xi^- \in E^-$, $\xi^+ \in E^+$ and $t > 0$, where c_1 and c_2 are positive constants.

The splitting $TV = E^- + E^0 + E^+$, which we call the *Anosov splitting* associated with the Anosov flow φ_t, is uniquely determined by φ_t, and is continuous on V. Furthermore it is known that there are foliations $\mathcal{E}^-$ and $\mathcal{E}^+$ of V, called the (strongly) stable and unstable foliations of φ_t, which integrate the subbundles E^- and E^+ of TV respectively.

For a closed riemannian manifold M of negative curvature, it is easy to see that its geodesic flow φ_t defined on the unit tangent bundle $V_M = \{v \in TM : |v| = 1\}$ of M is an Anosov flow. The Anosov splitting $TV_M = E^- + E^0 + E^+$ associated with the geodesic flow φ_t of M is sometimes called the Anosov splitting of M in brief. As was already mentioned, the Anosov splitting of a closed riemannian manifold M of negative curvature is continuous on V_M. In addition, Hirsch–Pugh **[10]**, **[11]** and L. W. Green **[8]** proved independently that the Anosov splitting of M is of class C^1 if the sectional curvature K of M satisfies the pinching condition $-4 < K \leq -1$ or if M is of dimension two. However we have no example of negatively curved manifold whose Anosov splitting is of class C^2 other than the locally symmetric spaces, and this leads us to propose

Conjecture 1. *For a closed riemannian manifold M of negative curvature, if its Anosov splitting is of class C^2, then M should be locally symmetric.*

Actually, E. Ghys **[7]** recently proved that the conjecture is true provided $\dim M = 2$. (See also Hurder–Katok **[12]** for related topics in the case of dimension two.) Our purpose in the present paper is to adduce another evidence which supports the plausibility of the conjecture. More precisely we will prove

Theorem. *Let M be a closed C^∞ riemannian manifold of dimension greater than three. Assume that the sectional curvature K of M satisfies the inequalities $-9/4 < K \leq -1$, and that the Anosov splitting of M is of class C^∞. Then the geodesic flow φ_t of M is congruent to the geodesic flow $\hat{\varphi}_t$ of a certain closed riemannian manifold $\hat{M}$ of constant negative curvature in the sense that there is a C^∞ diffeomorphism Φ of V_M onto $V_{\hat{M}}$ such that $\Phi \circ \varphi_t = \hat{\varphi}_t \circ \Phi$ for all $t \in \mathbf{R}$.*

In **[12]** Hurder and Katok especially proved that the C^2-differentiability of the Anosov splitting of a negatively curved surface always implies the C^∞-differentiability, while Ghys has emploied this result in the proof of her theorem mentioned above. It seems that the "regularity theorem" of Hurder–Katok is also the case with higher dimensional negatively curved manifolds, though we are not able to prove it. This is the reason why we have assumed the C^∞-differentiability of the Anosov splitting in our theorem above. Also we do not know what happens in dimension 3.

With regard to the theorem together with Conjecture 1, it seems to be reasonable to put forward

Conjecture 2. *For a closed riemannian manifold M of negative curvature, if the geodesic flow of M is congruent to that of a closed locally symmetric riemannian manifold $\hat{M}$ of negative curvature, then M is isometric to $\hat{M}$.*

It should be noticed that a diffeomorphism $\Phi : V_M \to V_{\hat{M}}$ commuting with the geodesic flows of M and $\hat{M}$ necessarily preserves the Anosov splittings of TV_M and $TV_{\hat{M}}$ and the canonical contact forms of V_M and $V_{\hat{M}}$ (cf. §2.2). Therefore, under the assumption in Conjecture 2, the geodesic flows of M and $\hat{M}$ are completely isomorphic to each other as hamiltonian systems. In particular, the topological entropy $h_{top}(M)$ and the measure-theoretic entropy $h_{meas}(M) = h_{meas}(M;\mu)$ of the geodesic flow of M (with respect to the Liouville measure $\mu = \Theta \wedge (d\Theta)^n$, where $n + 1 = \dim M$ and Θ denotes the canonical contact form of V_M) coincide respectively with those entropies $h_{top}(\hat{M})$ and $h_{meas}(\hat{M})$ of the geodesic flow of $\hat{M}$. In consequence, we have $h_{top}(M) = h_{meas}(M)$ since $h_{top}(\hat{M}) = h_{meas}(\hat{M})$. Thus Conjecture 2 will follow from Mostow's rigidity theorem **[18]** (see also **[16]** in the case of dimension two) and the conjecture of Katok **[13]** which claims that M is to be locally symmetric provided that $h_{top}(M) = h_{meas}(M)$. It is in fact proved by Katok **[13]** that a closed surface M of negative curvature for which $h_{top}(M) = h_{meas}(M)$ holds is of constant curvature, and it follows that Conjecture 2 is valid in the case of dimension two. See also Burns–Katok **[4]** for further informations about related topics and problems.

Now we exhibit here an outline of the proof of the theorem mentioned above. Throughout this paper, all manifolds, maps and so forth are assumed to be differentiable of class C^∞ unless otherwise stated. We will begin the proof of the theorem with the symplectic geometry. In particular we will concern ourselves with a symplectic manifold (P, Ω) equipped with lagrangian foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ transverse to each other: We call such a quadruplet $P = (P, \Omega, \mathcal{F}^-, \mathcal{F}^+)$ a *bipolarized symplectic manifold.* Among the symplectic manifolds, bipolarized ones have the advantage that they have canonically defined affine connections as we will see in §1. In addition, bipolarized symplectic manifolds naturally appear in geometry of negatively curved manifolds in the following way. Suppose that X is a simply connected complete riemannian manifold with sectional curvature $K \leq -1$. Then, as we will see in §2, the unit tangent bundle $V = V_X$ of X is fibered over the space P of the geodesics in X so that each fiber is an orbit of the geodesic flow of X. Furthermore, the exterior derivative $d\Theta$ of the canonical contact form Θ of V, which is invariant by the geodesic flow, is pushed forward to a symplectic form Ω of P by the fibering $V \to P$, and the foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ of P, obtained by projecting the stable and unstable foliations $\mathcal{E}^-$ and $\mathcal{E}^+$ of V associated with the geodesic flow of X, are lagrangian. In consequence, we are in the presence of the bipolarized symplectic manifold $P = (P, \Omega, \mathcal{F}^-, \mathcal{F}^+)$ associated with the negatively curved manifold X. Moreover, in the case when X is the universal covering of a closed riemannian manifold M of negative curvature whose Anosov splitting is of C^∞, the lagrangian foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ of P are smooth, and therefore the canonical connection ∇ of P is well defined. The most crucial part of the proof of the theorem is the fact that P is locally symmetric with respect to ∇ provided that the sectional curvature of M satisfies the pinching condition $-9/4 < K \leq -1$: This fact will be proved in the last paragraph in §2. This observation naturally leads us to the algebraic studies of affine (locally) symmetric spaces of a certain kind. In particular in §3 we will be interested in a real Lie algebra $\mathbf{g}$ equipped with a linear decomposition $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ such that

$$\begin{array}{lll} [\mathbf{h},\mathbf{h}] \subset \mathbf{h}, & [\mathbf{h},\mathbf{p}^-] \subset \mathbf{p}^-, & [\mathbf{h},\mathbf{p}^+] \subset \mathbf{p}^+, \\ [\mathbf{p}^-,\mathbf{p}^-] = 0, & [\mathbf{p}^+,\mathbf{p}^+] = 0, & [\mathbf{p}^-,\mathbf{p}^+] \subset \mathbf{h}, \end{array}$$

and in the corresponding affine symmetric spaces. By T. Nagano and S. Kobayashi [**19**], [**14**], the simple Lie algebras $\mathbf{g} = \mathbf{h}+\mathbf{p}^-+\mathbf{p}^+$ equipped with linear decompositions satisfying the above conditions are completely classified (cf. §3.4). Appealing to their classification together with some preliminary lemmas obtained in §3, we will be able to show in §4 that in the case when $\dim M \geq 4$ the bipolarized symplectic manifold P that is locally symmetric with respect to its canonical connection ∇ is isomorphic to the bipolarized symplectic manifold $\hat{P}$ associated with a certain riemannian manifold $\hat{X}$ homothetic to the hyperbolic space. Moreover it is possible to lift the isomorphism between P and $\hat{P}$ to a diffeomorphism between the unit tangent bundles V of X and $\hat{V} = V_{\hat{X}}$ of $\hat{X}$ which are fibered over P and $\hat{P}$ respectively, so that the resulting diffeomorphism of V onto $\hat{V}$ commutes with the geodesic flows of X and $\hat{X}$. This will prove the theorem.

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1. Canonical connections of bipolarized symplectic manifolds

In this section, we will see that for a bipolarized symplectic manifold we can always introduce an affine connection in a canonical way, and we begin it with making our terminology for symplectic geometry precise (cf. [**24**], [**1**]). First of all, recall that a *symplectic manifold* is an even-dimensional manifold P equipped with a non-degenerate closed 2-form Ω, which is called a *symplectic form* of P. For a $2n$-dimensional symplectic manifold (P, Ω), its n-dimensional submanifold L such that $i^*\Omega = 0$ for the inclusion $i : L \to P$ is called a *lagrangian submanifold* of (P, Ω), and a *lagrangian foliation* of (P, Ω) is an n-dimensional foliation of (P, Ω) all of whose leaves are lagrangian submanifolds. A symplectic manifold endowed with a lagrangian foliation is sometimes called a *polarized* symplectic manifold. By a *bipolarized* symplectic manifold, we mean a quadruplet $P = (P, \Omega, \mathcal{F}^-, \mathcal{F}^+)$ consisting of a symplectic manifold (P, Ω) and of its lagrangian foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ transverse to each other. Obviously the tangent bundle of a bipolarized symplectic manifold $P = (P, \Omega, \mathcal{F}^-, \mathcal{F}^+)$ carries the linear splitting $TP = F^- + F^+$ into the tangent spaces F^- and F^+ of the foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ respectively such that $\Omega|F^- = \Omega|F^+ = 0$.

Another necessity in defining the canonical connections of bipolarized symplectic manifold is the notion of connection *along* a foliation, which is defined as follows. Let P be a manifold, $\mathcal{F}$ a foliation of P with tangent bundle F, and E a vector bundle over P. A connection ∇ of the vector bundle E along the foliation $\mathcal{F}$ assigns a section $\nabla_\xi \eta$ of E to each pair of smooth sections ξ of F and η of E in the following manner, where f denotes an arbitrary smooth function on P:

(i) $\nabla_\xi \eta$ is bilinear in ξ and η;

(ii) $\nabla_{f\xi}\eta = f\nabla_\xi \eta, \qquad \nabla_\xi(f\eta) = (\xi f)\eta + f\nabla_\xi \eta.$

1.1. To begin with, suppose that P is a manifold and $\mathcal{F}$ is a foliation of P: Denote the tangent bundle of $\mathcal{F}$ by F, and the normal bundle of $\mathcal{F}$ by TP/F. We can always find local coordinates $(p, q) = (p_1, \cdots, p_m, q_1, \cdots, q_n)$ of P such that $\mathcal{F} = \{q = const.\}$ locally, i.e., each leaf of $\mathcal{F}$ is locally of the form $\{q_1 = const_1, \cdots, q_n = const_n\}$, and, using these coordinates, we can define a connection ∇ of the dual bundle $(TP/F)^*$ of TP/F along the foliation $\mathcal{F}$, by $\nabla dq_i \equiv 0$ $(i = 1, \cdots, n)$. It is clear that this definition does not depend on the choice of the coordinates (p, q), and therefore, the connection ∇ is globally defined on P.

1.2 (cf. Weinstein [**23**]). Next, let (P, Ω) be a symplectic manifold, and $\mathcal{F}$ be its lagrangian foliation with tangent bundle F. Then the symplectic form Ω yields the isomorphism $F \cong (TP/F)^*$ given by $\xi \in F \mapsto \Omega(\xi, \cdot) \in (TP/F)^*$, and therefore, we obtain a connection ∇ of F along $\mathcal{F}$ from that of $(TP/F)^*$ along $\mathcal{F}$ defined in §1.1. It is easy to see that ∇ is torsion-free and flat in the sense that $\nabla_\xi\eta - \nabla_\eta\xi - [\xi,\eta] = 0$ and $[\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]} = 0$ for any smooth sections ξ and η of F.

1.3. Now suppose that $P = (P, \Omega, \mathcal{F}^-, \mathcal{F}^+)$ is a bipolarized symplectic manifold: Denote the tangent bundles of $\mathcal{F}^-$ and $\mathcal{F}^+$ by F^- and F^+, respectively. As was indicated in §1.2, we have a connection ∇^{--} of F^- along $\mathcal{F}^-$ in a canonical manner, which also induces a connection ∇^{--*} of the dual bundle F^{-*} of F^- along $\mathcal{F}^-$. On the other hand, the lagrangian splitting $TP = F^- + F^+$ gives rise to the isomorphism $F^+ \cong F^{-*}$, $\xi \in F^+ \mapsto \Omega(\xi,\cdot) \in F^{-*}$. Thus ∇^{--*} induces a connection ∇^{-+} of F^+ along $\mathcal{F}^-$. Similarly, connections ∇^{++} of F^+ along $\mathcal{F}^+$, and ∇^{+-} of F^- along $\mathcal{F}^+$, are defined. Combining these connections linearly, we obtain an affine connection ∇ of P;

$$\nabla = \nabla^{--} + \nabla^{-+} + \nabla^{+-} + \nabla^{++} : \tag{1.1}$$

(This means that, for arbitrary smooth vector fields ξ and η of P, the connection ∇ of P is given by $\nabla_\xi\eta = \nabla^{--}_{\xi^-}\eta^- + \nabla^{-+}_{\xi^-}\eta^+ + \nabla^{+-}_{\xi^+}\eta^- + \nabla^{++}_{\xi^+}\eta^+$, where ξ^- and η^- (resp. ξ^+ and η^+) denote the F^--components (resp. F^+-components) of ξ and η). We call the affine connection ∇ of P defined in (1.1) the *canonical* connection of the bipolarized symplectic manifold P. It is easy to see that the canonical connection ∇ of P is characterized by the following three properties among all the affine connections of P: (i) ∇ is torsion-free; (ii) The symplectic form Ω is parallel with respect to ∇, i.e., $\nabla\Omega = 0$; (iii) If f is a smooth function defined locally on P so that it is constant on each leaf of $\mathcal{F}^-$ (resp. $\mathcal{F}^+$), then $\nabla_\xi df = 0$ for any $\xi \in F^-$ (resp. $\xi \in F^+$). Furthermore the curvature tensor R of the canonical connection ∇ possesses the following properties for any $\xi^-, \eta^- \in F^-$, $\xi^+, \eta^+ \in F^+$ and $\xi, \eta, \varsigma_1, \varsigma_2 \in TP$:

$$\begin{aligned} &R(\xi^-,\eta^-) = R(\xi^+,\eta^+) = 0; \\ &\Omega(R(\xi,\eta)\varsigma_1, \varsigma_2) + \Omega(\varsigma_1, R(\xi,\eta)\varsigma_2) = 0. \end{aligned} \tag{1.2}$$

In particular, each leaf of the foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ are totally geodesic and flat.

2. Symplectic geometry of the space of the geodesics

We now turn to the study of manifolds of negative curvature from the viewpoint of symplectic geometry. Our aims in the present section are to construct a bipolarized symplectic manifold from a negatively curved manifold (§2.2), and to show that it is locally symmetric with respect to its canonical connection under certain conditions (§2.3): This is the most crucial part in the proof of our theorem.

2.1. In studying non-compact spaces, it is often helpful to take their compactifications into consideration. In particular for manifolds of negative curvature their imaginary boundaries are introduced in concrete forms to construct their compactifications, and has been playing significant roles in geometry, topology and dynamics of negatively curved manifolds. In this paragraph, we will briefly review the notion of the imaginary boundary at infinity of a negatively curved manifold (see, for details, Eberlein–O'Neill [6]). Suppose first that X is an $(n+1)$-dimensional simply connected complete riemannian manifold with sectional curvature $K \leq 0$. By the Cartan–Hadamard theorem, the exponential map $\exp : T_x X \to X$ of X at each point $x \in X$ is a diffeomorphism, and in consequence, X is diffeomorphic to the euclidean space $\mathbf{R}^{n+1}$. A *ray* in X is a geodesic of X parametrized by the arc length $t \in [0, \infty)$, and two rays r_1 and r_2 in X are said to be *asymptotic* if $\operatorname{dist}(r_1(t), r_2(t))$ is bounded in $t \geq 0$: The asymptote is evidently an equivalence relation between rays in X. Intuitively speaking, each ray is directed towards a point at infinity, and two rays will reach the same point at infinity if they are asymptotic. Hence the *imaginary boundary B of X at infinity* is defined as the set of the asymptote classes of the rays in X:

$$B = \{\text{the rays in } X\}/(\text{to be asymptotic}).$$

Let V_x be the set of unit tangent vectors of X at $x \in X$, and for each $v \in V_x$, denote by $A_x(v) \in B$ the asymptote class represented by the ray $\exp_x tv$, $t \geq 0$. From the assumption on the curvature, it is derived that the map $A_x : V_x \to B$ is bijective, and that $A_y^{-1} \circ A_x : V_x \to V_y$ is a homeomorphism for any $x, y \in X$. Thus we can give B a topology by identifying B with any V_x $(x \in X)$ through the 1–1 correspondence $A_x : V_x \to B$. With respect to this topology, all of the maps $A_x : V_x \to B$ $(x \in X)$ are homeomorphisms, and in particular B is homeomorphic to the n-sphere. Moreover B is attached to X to form a compactification $X \cup B$ of X homeomorphic to the $(n+1)$-ball so that the map $e_x : V_x \times (0, \infty] \to (X \cup B) \setminus \{x\}$, defined by $e_x(v,t) = \exp_x tv$ for $t < \infty$ and $e_x(v,t) = A_x(v)$ for $t = \infty$, is a homeomorphism. It is easy to see that each isometric transformation of X has a natural extension to a homeomorphism of $X \cup B$ onto itself.

Especially, for the hyperbolic space H^{n+1} (i.e., the simply connected complete riemannian manifold of constant curvature -1), which is realized by so-called the Poincaré model as the open unit disc $\{|x| < 1\}$ in the euclidean space $\mathbf{R}^{n+1}$ equipped with a conformally deformed metric, the boundary sphere $S^n = \{|x| = 1\}$ in $\mathbf{R}^{n+1}$ is naturally identified with the imaginary boundary B of H^{n+1} at infinity. Furthermore, in this case, the imaginary boundary has the following nice structures which can not be expected in general for the imaginary boundaries of manifolds of variable negative curvature. One of them is the differentiable structure. In fact, for the hyperbolic space H^{n+1}, the family of maps $A_x : V_x \to B$ has the property that $A_y^{-1} \circ A_x : V_x \to V_y$ is always C^∞ for any $x, y \in H^{n+1}$, and this gives the imaginary boundary B a natural differentiable structure. It is easy to see that this differentiable structure of B coincides with that of B induced by the identification of B with the sphere S^n in $\mathbf{R}^{n+1}$ in the Poincaré model, and that the extension of every isometric transformation of H^{n+1} to a homeomorphism of $H^{n+1} \cup B$ is actually a diffeomorphism. In addition, in the case of $n+1 \geq 3$, the imaginary boundary $B = S^n \subset \mathbf{R}^{n+1}$ of the hyperbolic space has a canonical conformal structure induced from that of the euclidean space $\mathbf{R}^{n+1}$, and for each isometric transformation of H^{n+1}, its extension to B is a conformal transformation. Furthermore, every conformal transformation of B is obtained in this way; that is, the conformal extension of each isometric transformation of H^{n+1} gives an isomorphism of the isometric transformation group $\mathrm{Iso}(H^{n+1})$ of the hyperbolic space H^{n+1} onto the conformal transformation group $\mathrm{Con}(B)$ of the imaginary boundary B. These facts were extensively utilized in the proof of Mostow's rigidity theorem [**17**], [**18**].

2.2. Assume further that X is of curvature $K \leq -1$. Next we are going to construct a bipolarized symplectic manifold $P = (P, \Omega, \mathcal{F}^-, \mathcal{F}^+)$ from X. Let $V = V_X = \bigcup_{x \in X} V_x$ be the unit tangent bundle of X, and put $P = \{(b^-, b^+) \in B \times B : b^- \neq b^+\}$. Note that the assumption on the curvature implies the "convexity"of the imaginary boundary B of X: More precisely, the curvature assumption guarantees that for any distinct points b^- and b^+ of B there is a geodesic line l of X, unique up to the reparametrization, such that $l(t) \to b^\pm$ in $X \cup B$ as $t \to \pm\infty$. Thus the map $\pi : V \to P$, defined by $\pi(v) = (A_x(-v), A_x(v))$ for $v \in V_x$ and $x \in X$, constitutes an $\mathbf{R}$-fibering of V over P. In other words, P can be considered as the space of the geodesic lines in X. We can also explain this by saying that the additive group $\mathbf{R}$ of the real numbers acts on V as the geodesic flow, and P is identified with the orbit space $\mathbf{R} \setminus V$ with the projection $\pi : V \to P = \mathbf{R} \setminus V$. In particular, P has a unique differentiable structure for which the projection $\pi : V \to P$ is smooth. Moreover we can introduce a symplectic form Ω on P in the following way, so-called the symplectic reduction. Denote by Θ the canonical contact form of V: Θ is the pull-back of a 1-form Θ_0 on TX by the inclusion $V \to TX$, while, in the local coordinates $(x_i, \dot{x}_i)$ of TX induced from local coordinates (x_i) of X in which each tangent vector $v \in TX$ of X is represented by $v = \sum \dot{x}_i (\partial/\partial x_i)$, Θ_0 is given by $\Theta_0 = \sum g_{ij} \dot{x}_i dx_j$, where $g_{ij} = \langle \partial/\partial x_i, \partial/\partial x_j \rangle$ with

$\langle\cdot,\cdot\rangle$ being the riemannian metric of X (cf. [1]). Then Liouville's theorem claims that both Θ and its exterior derivative $d\Theta$ are invariant by the geodesic flow, and therefore, $d\Theta$ is pushed forward to P by the projection $\pi : V \to P$ so that the resulting 2-form Ω on P, which is characterized by $d\Theta = \pi^*\Omega$, is a symplectic form of P.

Now let φ_t be the geodesic flow of X defined on the unit tangent bundle V of X. Although X is non-compact, the Anosov splitting $TV = E^- + E^0 + E^+$ associated with the geodesic flow φ_t is canonically defined in a geometric manner, and it satisfies

$$(2.1)\qquad \begin{aligned} |d\varphi_t(\xi)| &\le const\cdot e^{-t}|\xi|, &&\text{for } \xi\in E^-,\ t>0;\\ |d\varphi_{-t}(\xi)| &\le const\cdot e^{-t}|\xi|, &&\text{for } \xi\in E^+,\ t>0. \end{aligned}$$

Since the subbundles E^- and E^+ of TV are invariant by the geodesic flow φ_t, and are transverse to the orbits of the geodesic flow φ_t, they induce the continuous splitting $TP = F^- + F^+$ of the tangent bundle of P into the n-dimensional subbundles F^- and F^+: The differential $d\pi_v$ of the projection $\pi : V \to P$ at each point $v \in V$ maps $E_v^\pm \subset T_vV$ onto $F^\pm_{\pi(v)} \subset T_{\pi(v)}P$ isomorphically. It is easy to see that this splitting of TP corresponds to the product structure of $P \subset B \times B$. Furthermore, through the projection $\pi : V \to P$, the stable and unstable foliations $\mathcal{E}^-$ and $\mathcal{E}^+$ of V tangent to E^- and E^+ descend respectively to the C^0 foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ of P which consist of the C^1-leaves $(B\times\{b^+\})\cap P$ and $(\{b^-\}\times B)\cap P$ ($b^\pm \in B$) respectively. For these foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ of P, we have

(2.2) **Lemma.** *Both $\mathcal{F}^-$ and $\mathcal{F}^+$ are lagrangian foliations of (P,Ω), and in consequence, $P = (P,\Omega,\mathcal{F}^-,\mathcal{F}^+)$ is a bipolarized symplectic manifold.*

Proof. To see that $\mathcal{F}^-$ is lagrangian, it is sufficient to prove $d\Theta|E^- = 0$, where, as before, Θ denotes the canonical contact form of V. By Liouville's theorem and the first inequality in (2.1), we immediately have

$$\begin{aligned} |d\Theta(\xi_1,\xi_2)| &= |(\varphi_t^* d\Theta)(\xi_1,\xi_2)| = |d\Theta(d\varphi_t\xi_1, d\varphi_t\xi_2)|\\ &\le const\cdot|d\Theta|\,|d\varphi_t\xi_1|\,|d\varphi_t\xi_2| \le const\cdot|d\Theta|\,|\xi_1|\,|\xi_2|\cdot e^{-2t} \end{aligned}$$

for $\xi_1,\xi_2 \in E^-$ and $t>0$. As $|d\Theta|$ is bounded on V, we obtain $d\Theta(\xi_1,\xi_2) = 0$ by letting $t\to\infty$. ∎

One should notice that the argument employed in the proof of the lemma also implies that $\Theta|(E^- + E^+) = 0$ for the canonical contact form Θ of V. On the other hand, for the geodesic spray $\varphi' = (\partial/\partial t)|_{t=0}\varphi_t$ on V, which is by definition the vector field on V generating the flow φ_t and spanning the subbundle E^0 of TV, it is clear that $\Theta(\varphi') = 1$. Thus the canonical contact form Θ of V is completely determined by the Anosov splitting $TV = E^- + E^0 + E^+$ and hence only by the geodesic flow φ_t.

By virtue of Lemma (2.2), the canonical connection of the bipolarized symplectic manifold P is defined provided that the Anosov splitting $TV = E^- + E^0 + E^+$ of X is of class C^∞. Further in this case the foliations $\mathcal{E}^\pm$ of V and $\mathcal{F}^\pm$ of P are C^∞, and the imaginary boundary B of X at infinity has a C^∞-differentiable structure so that the projections of $P = B \times B \setminus$ (the diagonal set) onto B are smooth.

2.3. Now suppose moreover that X appears as the universal covering of a certain closed riemannian manifold M of negative curvature: Of course we may assume, without a loss of generality, that the sectional curvature K of M satisfies the inequalities

$$-\lambda^2 \le K \le -1 \qquad (\lambda \ge 1). \tag{2.3}$$

The fundamental group Γ of M acts on the universal covering X of M by the isometric deck transformations, and it induces smooth actions of Γ on V and on P in a canonical way: Note that this is possible because the action of Γ on X preserves *all* of the structures of X. In particular, the action of Γ on P preserves the symplectic form Ω and the transverse lagrangian foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ introduced in §2.2: In other words, Γ acts on the bipolarized symplectic manifold $P = (P, \Omega, \mathcal{F}^-, \mathcal{F}^+)$ by its automorphisms. In this situation, we have

(2.4) **Proposition.** *Assume that the Anosov splitting of M is of C^∞ and that $\lambda < 3/2$. Then the canonical connection of the bipolarized symplectic manifold P is locally symmetric; that is, $\nabla R = 0$ for the canonical connection ∇ of P and its curvature tensor R.*

Proof. First, note that the pinching condition (2.3) yields the following estimates for the hyperbolicity of the geodesic flow φ_t of X which improve the previous ones (2.1):

$$\begin{aligned} &const^{-1} \cdot e^{-\lambda t}|\xi| \le |d\varphi_t(\xi)| \le const \cdot e^{-t}|\xi|, \quad \text{for } \xi \in E^-,\ t > 0; \\ &const^{-1} \cdot e^{-\lambda t}|\xi| \le |d\varphi_{-t}(\xi)| \le const \cdot e^{-t}|\xi|, \quad \text{for } \xi \in E^+,\ t > 0. \end{aligned} \tag{2.5}$$

Furthermore, since the splitting $TV = E^- + E^0 + E^+$ is φ_t-invariant, (2.5) implies

$$\begin{aligned} &const^{-1} \cdot e^{t}|\xi| \le |d\varphi_{-t}(\xi)| \le const \cdot e^{\lambda t}|\xi|, \quad \text{for } \xi \in E^-,\ t > 0, \\ &const^{-1} \cdot e^{t}|\xi| \le |d\varphi_t(\xi)| \le const \cdot e^{\lambda t}|\xi|, \quad \text{for } \xi \in E^+,\ t > 0. \end{aligned} \tag{2.6}$$

Now define a $(0,4)$-tensor field $\check{R}$ on P by $\check{R}(\xi_1, \xi_2, \xi_3, \xi_4) = \Omega(R(\xi_1, \xi_2)\xi_3, \xi_4)$ for $\xi_1, \xi_2, \xi_3, \xi_4 \in TP$: Then $\check{R}$ and its covariant derivative $\nabla\check{R}$ are tensor fields on P which are invariant under the action of Γ, since Γ acts on $P = (P, \Omega, \mathcal{F}^-, \mathcal{F}^+)$ by its automorphisms, and to prove the proposition it suffices to show that $\nabla\check{R} = 0$ since $\nabla\Omega = 0$. Let $S = \pi^*(\nabla\check{R})$ be the pull-back of $\nabla\check{R}$ by the projection $\pi : V \to P$, which is a $(0,5)$-tensor field on V. Suppose that $\xi_1^-, \xi_2^-, \xi_3^- \in E^-$ and $\eta_1^+, \eta_2^+ \in E^+$. Since S is φ_t-invariant, (2.5) and (2.6) imply

$$\begin{aligned} |S(\xi_1^-, \xi_2^-, \xi_3^-, \eta_1^+, \eta_2^+)| &= |(\varphi_t^* S)(\xi_1^-, \xi_2^-, \xi_3^-, \eta_1^+, \eta_2^+)| \\ &= |S(d\varphi_t\xi_1^-, d\varphi_t\xi_2^-, d\varphi_t\xi_3^-, d\varphi_t\eta_1^+, d\varphi_t\eta_2^+)| \\ &\le const \cdot |S| \cdot |d\varphi_t\xi_1^-|\,|d\varphi_t\xi_2^-|\,|d\varphi_t\xi_3^-|\,|d\varphi_t\eta_1^+|\,|d\varphi_t\eta_2^+| \\ &\le const \cdot |S| \cdot |\xi_1^-|\,|\xi_2^-|\,|\xi_3^-|\,|\eta_1^+|\,|\eta_2^+| \cdot e^{(2\lambda-3)t} \end{aligned}$$

for $t > 0$. Since S is smooth on V, and is invariant under the action of Γ on V for which the quotient $\Gamma \setminus V$ is compact, $|S|$ is bounded on V. Thus by letting $t \to \infty$ in the above inequalities we have $S(\xi_1^-, \xi_2^-, \xi_3^-, \eta_1^+, \eta_2^+) = 0$ in the case of $\xi_1^-, \xi_2^-, \xi_3^- \in E^-$ and $\eta_1^+, \eta_2^+ \in E^+$ (recall that $\lambda < 3/2$). The other cases can be treated in the same way, and we obtain $S = 0$, which implies $\nabla \check{R} = 0$. ∎

Note that in the case of $\dim M = 2$ the proposition is valid without the pinching condition on the curvature.

3. Algebraic studies of bipolarized symmetric spaces

Examining the symplectic geometry of a negatively curved manifold in the previous section, we found a bipolarized symplectic manifold which is locally symmetric with respect to its canonical connection, and what we have to do in the rest of the present paper is to determine its structure to conclude the theorem mentioned in the introduction. We will begin the proof of the theorem with arguments which are not require the symplectic form of the locally symmetric bipolarized symplectic manifold under consideration, and this leads us to the following definition. Suppose that $P = (P, \nabla, \mathcal{F}^-, \mathcal{F}^+)$ is a quadruplet consisting of a manifold P, a torsion-free affine connection ∇ of P, and two foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ of P transverse to each other with $\dim P = \dim \mathcal{F}^- + \dim \mathcal{F}^+$. Then P is called a *bipolarized symmetric* (resp. *locally symmetric*) *space* if the following three conditions are satisfied:

(i) (P, ∇) is an affine symmetric (resp. locally symmetric) space;

(ii) The tangent bundles F^- and F^+ of the foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ are closed with respect to the covariant derivative by ∇, that is, for any vector field ξ of P and a section $\eta^\pm$ of $F^\pm$, $\nabla_\xi \eta^\pm$ is again a section of $F^\pm$;

(iii) The curvature tensor R of ∇ satisfies $R(\xi^-, \eta^-) = R(\xi^+, \eta^+) = 0$ for $\xi^-, \eta^- \in F^-$ and $\xi^+, \eta^+ \in F^+$.

A bipolarized symplectic manifold which is symmetric (resp. locally symmetric) with respect to its canonical connection is obviously a bipolarized symmetric (resp. locally symmetric) space because of (1.1) and (1.2). As we will see later, local geometry of a bipolarized symmetric space is described by a real Lie algebra $\mathbf{g}$ equipped with a linear decomposition $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ which satisfies the following conditions:

(i) The splitting $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ obeys the bracket rules

$$[\mathbf{h},\mathbf{h}] \subset \mathbf{h}, \qquad [\mathbf{h},\mathbf{p}^-] \subset \mathbf{p}^-, \qquad [\mathbf{h},\mathbf{p}^+] \subset \mathbf{p}^+,$$
$$[\mathbf{p}^-,\mathbf{p}^-] = 0, \qquad [\mathbf{p}^+,\mathbf{p}^+] = 0, \qquad [\mathbf{p}^-,\mathbf{p}^+] \subset \mathbf{h};$$

(ii) The adjoint representation of the subalgebra $\mathbf{h}$ of $\mathbf{g}$ on $\mathbf{p} = \mathbf{p}^- + \mathbf{p}^+$ is faithful; that is, for $\xi \in \mathbf{h}$, $[\xi, \mathbf{p}] = 0$ implies $\xi = 0$;

(iii) There is an element $\delta \in \mathbf{h}$ such that $[\delta, \xi] = 0$, $-\xi$ or ξ according to whether $\xi \in \mathbf{h}$, $\mathbf{p}^-$ or $\mathbf{p}^+$, respectively.

We call a real Lie algebra $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ equipped with a linear decomposition satisfying the above conditions (i)–(iii) a *bipolarized symmetric Lie algebra*, and the purpose of this section is to study bipolarized (locally) symmetric spaces in terms of bipolarized symmetric Lie algebras as preliminaries for the proof of the theorem. It should be remarked that the latter conditions (ii) and (iii) are rather subordinate, and are imposed to the bipolarized symmetric Lie algebras in order that we can treat the non-semisimple case as well. In fact, they will be emploied only in the proofs of Lemmas (3.5) and (3.6) in which we treat non-semisimple bipolarized symmetric Lie algebras, while in the case when $\mathbf{g}$ is semisimple the condition (iii) is always derived from (i), and more strongly in the case when $\mathbf{g}$ is simple the condition (i) implies both of (ii) and (iii). Concerned with (semisimple) Lie algebras $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ endowed with linear splittings satisfying the first condition (i), we should refer to the works by Berger [**3**], Nagano [**19**], Kobayashi–Nagano [**14**], Tanaka [**21**] and by others, which had been done in other contexts, but some of them are still helpful in our discussions.

3.1. In this paragraph, we will reveal the fundamental relation between bipolarized symmetric spaces and bipolarized symmetric Lie algebras. First suppose that $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ is a bipolarized symmetric Lie algebra, from which we are going to construct a bipolarized symmetric space. Let σ be the involutive automorphism of $\mathbf{g}$ defined by $\sigma\xi = \xi$ for $\xi \in \mathbf{h}$ and $\sigma\xi = -\xi$ for $\xi \in \mathbf{p} = \mathbf{p}^- + \mathbf{p}^+$, and take an analytic group G with Lie algebra $\mathbf{g}$ so that the automorphism σ is integrated to an involutive automorphism Σ of G. Furthermore let H be the analytic subgroup of G corresponding to the subalgebra $\mathbf{h}$ of $\mathbf{g}$: H coincides with a connected component of the fixed point set of Σ, and consequently H is closed in G. Thus we can form an affine symmetric space $(P = G/H, \nabla)$ (cf. [**15**]). In addition, the abelian subalgebras $\mathbf{p}^-$ and $\mathbf{p}^+$ of $\mathbf{g}$ induce a G-invariant splitting $TP = F^- + F^+$ of the tangent bundle of P. Both F^- and F^+ are integrable, and yield G-invariant foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ of P tangent to F^- and F^+ respectively. It is easy to see that $P = (P, \nabla, \mathcal{F}^-, \mathcal{F}^+)$ is a bipolarized symmetric space.

Next we conversely construct bipolarized symmetric Lie algebra from a bipolarized locally symmetric space. Let $P = (P, \nabla, \mathcal{F}^-, \mathcal{F}^+)$ be a bipolarized locally symmetric space, and p a point of P. Denote by $\mathbf{p}^\pm$ the tangent space of the foliation $\mathcal{F}^\pm$ at p, and by $\mathbf{h}$ the space of linear automorphisms α of $\mathbf{p} = \mathbf{p}^- + \mathbf{p}^+ = T_pP$ satisfying the following two conditions:

(3.1.1) $\quad \alpha\mathbf{p}^- \subset \mathbf{p}^-, \quad \alpha\mathbf{p}^+ \subset \mathbf{p}^+;$

(3.1.2) $\quad \alpha \circ R(\xi,\eta) - R(\xi,\eta) \circ \alpha = R(\alpha\xi,\eta) + R(\xi,\alpha\eta) \quad \text{for } \xi,\eta \in \mathbf{p}.$

In the latter condition, R denotes the curvature tensor of P at p: Note that $R(\xi,\eta) \in \mathbf{h}$ for all $\xi,\eta \in \mathbf{p}$. Now put $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$, and define a Lie bracket operation $[\,\cdot\,,\,\cdot\,]$ on $\mathbf{g}$ by $[\alpha,\beta] = \alpha\circ\beta - \beta\circ\alpha$ for $\alpha,\beta \in \mathbf{h}$; $[\alpha,\xi] = \alpha\xi$ for $\alpha \in \mathbf{h}$, $\xi \in \mathbf{p}$; and $[\xi,\eta] = -R(\xi,\eta)$ for $\xi,\eta \in \mathbf{p}$. It is obvious that $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ is a bipolarized symmetric Lie algebra. Moreover if $\hat{P} = (\hat{P} = G/H, \hat{\nabla}, \hat{\mathcal{F}}^-, \hat{\mathcal{F}}^+)$ denotes a bipolarized symmetric space constructed from $\mathbf{g}$ as above, then P is locally isomorphic to $\hat{P}$; that is, P is covered by partially defined diffeomorphisms of P into $\hat{P}$ which send the affine connection and the foliations of P to those of $\hat{P}$.

3.2. Now we are going to consider the space of the leaves of the foliation $\mathcal{F}^-$ or $\mathcal{F}^+$ of a bipolarized symmetric space $P = (P, \nabla, \mathcal{F}^-, \mathcal{F}^+)$. To begin with, let $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ be a bipolarized symmetric Lie algebra, and denote by δ the element of $\mathbf{h}$ such that $\mathrm{ad}(\delta)|\mathbf{h} = 0$ and $ad(\delta)|\mathbf{p}^\pm = \pm 1$. Associated with it, there is a 1-parameter group of the inner automorphisms $\varphi_t = \mathrm{Exp}\, t\, \mathrm{ad}(\delta)$ $(t \in \mathbf{R})$ of $\mathbf{g}$ such that $\varphi_t\xi = \xi$, $e^{-t}\xi$ or $e^t\xi$ according to whether $\xi \in \mathbf{g}$, $\mathbf{p}^-$ or $\mathbf{p}^+$, respectively. For every ideal $\mathbf{a}$ of $\mathbf{g}$, these automorphisms φ_t give rise to the splitting

(3.2) $$\mathbf{a} = (\mathbf{h} \cap \mathbf{a}) + (\mathbf{p}^- \cap \mathbf{a}) + (\mathbf{p}^+ \cap \mathbf{a}).$$

In fact, $\mathbf{a}$ is invariant by φ_t, and for $\xi = \xi^0 + \xi^- + \xi^+ \in \mathbf{a}$ $(\xi^0 \in \mathbf{h},\ \xi^\pm \in \mathbf{p}^\pm)$ it follows that $e^{-t}\varphi_t\xi = e^{-t}\xi^0 + e^{-2t}\xi^- + \xi^+ \in \mathbf{a}$. By letting $t \to \infty$ we obtain $\xi^+ \in \mathbf{a}$, and in a similar way we have $\xi^- \in \mathbf{a}$, and in consequence $\xi^0 \in \mathbf{a}$. This shows (3.2). Now take an analytic group G and its subgroup H corresponding to $\mathbf{g}$ and $\mathbf{h}$ as in §3.1. From them we can form a bipolarized symmetric space $P = (P = G/H, \nabla, \mathcal{F}^-, \mathcal{F}^+)$ as we have already seen. Furthermore let $H^{\pm}$ be the analytic subgroup of G corresponding to the subalgebra $\mathbf{h}^{\pm} = \mathbf{h} + \mathbf{p}^{\pm}$ of $\mathbf{g}$. Then the coset space $B^{\pm} = G/H^{\pm}$ is naturally considered as the space of the leaves of the foliation $\mathcal{F}^{\pm}$. It is the topological structure of $B^{\pm}$ that we intend to study in this paragraph, and it will be done by dividing the arguments into two cases according to whether $\mathbf{g}$ is semisimple or not.

Semisimple case. First suppose that $\mathbf{g}$ is semisimple. In this case we can immediately show that both H^{-} and H^{+} are closed in G: In fact $H^{\pm}$ coincides with a connected component of the normalizer $\{g \in G : \mathrm{Ad}(g)\mathbf{p}^{\pm} \subset \mathbf{p}^{\pm}\}$ of $\mathbf{p}^{\pm}$ in G (cf. Tanaka [**21**]). Thus $B^{\pm} = G/H^{\pm}$ is a homogeneous manifold. Furthermore we have

(3.3) **Lemma.** *The universal coverings of B^{-} and B^{+} are diffeomorphic to a certain compact riemannian symmetric space B_0. Furthermore B_0 is not irreducible unless $\mathbf{g}$ is simple.*

Proof. As was indicated by Kobayashi–Nagano [**14**], it follows from the assumption of semisimplicity that there exists a Cartan decomposition $\mathbf{g} = \mathbf{k} + \mathbf{l}$ of $\mathbf{g}$ with $\mathbf{k}$ being a maximal compact subalgebra of $\mathbf{g}$, such that

$$\mathbf{h} = (\mathbf{h} \cap \mathbf{k}) + (\mathbf{h} \cap \mathbf{l}), \qquad \mathbf{p} = (\mathbf{p} \cap \mathbf{k}) + (\mathbf{p} \cap \mathbf{l}), \tag{3.4}$$

where $\mathbf{p} = \mathbf{p}^{-} + \mathbf{p}^{+}$ as before. Moreover it is not hard to see that for any Cartan decompositions $\mathbf{g} = \mathbf{k}_1 + \mathbf{l}_1$ and $\mathbf{g} = \mathbf{k}_2 + \mathbf{l}_2$ of $\mathbf{g}$ satisfying the condition (3.4) there is an automorphism of $\mathbf{g}$ which sends $\mathbf{k}_1$ and $\mathbf{l}_1$ onto $\mathbf{k}_2$ and $\mathbf{l}_2$ respectively and keeps $\mathbf{h}$ and $\mathbf{p}$ invariant. Now let K be the compact analytic subgroup of G with Lie algebra $\mathbf{k}$, and as in §3.1 denote by σ the involutive automorphism of the Lie algebra $\mathbf{g}$ characterized by $\sigma|\mathbf{h} = 1$ and $\sigma|\mathbf{p} = -1$. Then by (3.4), $\mathbf{k}$ is invariant by σ, and therefore the automorphism Σ of G that integrates σ keeps K invariants and fixes $K \cap H$. Thus we obtain a compact riemannian symmetric space $K/K \cap H$. Furthermore (3.4) implies that all of the projections $\mathbf{p} \cap \mathbf{k} \to \mathbf{p}^{\pm}$, $\mathbf{p} \cap \mathbf{l} \to \mathbf{p}^{\pm}$ into $\mathbf{p}^{-}$ and $\mathbf{p}^{+}$ are bijective (cf. [**14**] again), and, from this fact it follows that the restricted action of $K(\subset G)$ on the homogeneous space $B^{\pm} = G/H^{\pm}$ of G is transitive, and that $\mathbf{k} \cap \mathbf{h} = \mathbf{k} \cap \mathbf{h}^{\pm}$, i.e., the Lie subgroup $K \cap H$ of K and the isotropy subgroup $K \cap H^{\pm}$ of the action of K on $B^{\pm}$ have the same Lie algebra. Thus $B^{\pm}$ is diffeomorphic to the homogeneous space $K/K \cap H^{\pm}$ which is finitely covered by the compact riemannian symmetric space $K/K \cap H$, and in consequence the universal covering of $B^{\pm}$ is diffeomorphic to the universal covering B_0 of $K/K \cap H$. To prove the second assertion in the lemma, assume further that $\mathbf{g}$ is not simple: Let $\mathbf{g} = \mathbf{g}_1 + \cdots + \mathbf{g}_\nu$ $(\nu > 0)$ be a decomposition of $\mathbf{g}$ into prime ideals. Recall that $\mathbf{g}_\mu = (\mathbf{h} \cap \mathbf{g}_\mu) + (\mathbf{p}^{-} \cap \mathbf{g}_\mu) + (\mathbf{p}^{+} \cap \mathbf{g}_\mu)$ by (3.2): This means that $\mathbf{g}_\mu$ is a bipolarized symmetric Lie algebra in itself. Thus the previous argument shows that each $\mathbf{g}_\mu$ carries a Cartan decomposition $\mathbf{g}_\mu = \mathbf{k}_\mu + \mathbf{l}_\mu$ with $\mathbf{h} \cap \mathbf{g}_\mu = (\mathbf{h} \cap \mathbf{k}_\mu) + (\mathbf{h} \cap \mathbf{l}_\mu)$ and $\mathbf{p} \cap \mathbf{g}_\mu = (\mathbf{p} \cap \mathbf{k}_\mu) + (\mathbf{p} \cap \mathbf{l}_\mu)$. Now put $\mathbf{k} = \mathbf{k}_1 + \cdots + \mathbf{k}_\nu$ and $\mathbf{l} = \mathbf{l}_1 + \cdots + \mathbf{l}_\nu$. Then $\mathbf{g} = \mathbf{k} + \mathbf{l}$ is a Cartan decomposition of $\mathbf{g}$ that satisfies the condition (3.4). In particular (3.4) implies $\mathbf{k} = (\mathbf{k} \cap \mathbf{h}) + (\mathbf{k} \cap \mathbf{p})$. Furthermore $\mathbf{k} \cap \mathbf{p}$ admits the ad$(\mathbf{k} \cap \mathbf{h})$-invariant splitting $\mathbf{k} \cap \mathbf{p} = (\mathbf{k}_1 \cap \mathbf{p}) + \cdots + (\mathbf{k}_\nu \cap \mathbf{p})$, and in consequence the adjoint representation of $\mathbf{k} \cap \mathbf{h}$ on $\mathbf{k} \cap \mathbf{p}$ is not irreducible. Hence the riemannian symmetric space $K/K \cap H$ and its universal covering B_0 is not irreducible. ∎

Non-semisimple case. Next we consider the case when $\mathbf{g}$ is not semisimple. In this case we should assume that G is *simply connected.* Under this assumption the automorphisms φ_t of $\mathbf{g}$ induce a 1-parameter group of the automorphisms Φ_t of G with $(d\Phi_t)_1 = \varphi_t$. The fact we have to prove first is

(3.5) **Lemma.** *Both H^- and H^+ are closed in G.*

Proof. To see this, we first show that $L_0^- \cap L_0^+ = \{o\}$, where $o = H$ denotes the origin of the symmetric space $P = G/H$, and $L_0^\pm$ the leaf of the foliation $\mathcal{F}^\pm$ passing through o. Since the automorphisms Φ_t of G fix H, they descend to a 1-parameter group of automorphisms Ψ_t of P which fix the origin o, keep L_0^- and L_0^+ invariant, and contract L_0^- and expand L_0^+ provided $t > 0$. Thus, if there were a point $p \neq o$ in P lying in L_0^- and L_0^+ simultaneously, then Ψ_t's would move p along L_0^- since $p \in L_0^-$ and along L_0^+ since $p \in L_0^+$, in contradiction. Hence we have $L_0^- \cap L_0^+ = \{o\}$. Note that this implies also that for any leaves L^- of $\mathcal{F}^-$ and L^+ of $\mathcal{F}^+$ their intersection $L^- \cap L^+$ contains at most one point. Now we turn to the proof of the lemma. For this purpose it suffices to show that the leaves of the foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ are closed in P. Assume in contrary that a leaf L^- of $\mathcal{F}^-$ is not closed in P. Take an accumulating point p of L^- that does not belong to L^-, and let U be a "flow box" of the foliation pair $(\mathcal{F}^-, \mathcal{F}^+)$ around p: By definition, U is a neighborhood of p where local coordinates $p_1^-, \cdots, p_n^-, p_1^+, \cdots, p_n^+$ of P are defined so that each leaf of $\mathcal{F}^\pm$ is of the form $\{p_i^\pm = const_i\}$ in U. Then L^- should intersect a leaf $\{p_i^+ = const_i\}$ of $\mathcal{F}^+$ infinitely many times, but this contradicts the previous assertion. Thus the leaves of the foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ are closed in P, and consequently, the subgroups H^- and H^+ are closed in G. ∎

The above lemma guarantees that $B^\pm = G/H^\pm$ is a homogeneous manifold, and concerned with the topology of $B^\pm$ we have

(3.6) **Lemma.** *Either B^- or B^+ is non-compact unless $\mathbf{g}$ is semisimple.*

Proof. Since $\mathbf{g}$ is not semisimple, $\mathbf{g}$ has an abelian ideal $\mathbf{a} \neq 0$. Recall (3.2): $\mathbf{a} = (\mathbf{h} \cap \mathbf{a}) + (\mathbf{p}^- \cap \mathbf{a}) + (\mathbf{p}^+ \cap \mathbf{a})$. First we prove that $\mathbf{p}^- \cap \mathbf{a} \neq 0$ or $\mathbf{p}^+ \cap \mathbf{a} \neq 0$. Assume in contrary that $\mathbf{p}^- \cap \mathbf{a} = \mathbf{p}^+ \cap \mathbf{a} = 0$. In this case, we have $[\mathbf{h} \cap \mathbf{a}, \mathbf{p}^-] \subset \mathbf{p}^- \cap \mathbf{a} = 0$ and similarly $[\mathbf{h} \cap \mathbf{a}, \mathbf{p}^+] = 0$, which imply that $\mathbf{h} \cap \mathbf{a} = 0$ (recall that the adjoint representation of $\mathbf{h}$ on $\mathbf{p} = \mathbf{p}^- + \mathbf{p}^+$ is to be faithful); but this contradicts $\mathbf{a} \neq 0$. Thus we have either $\mathbf{p}^- \cap \mathbf{a} \neq 0$ or $\mathbf{p}^+ \cap \mathbf{a} \neq 0$. For simplicity, assume $\mathbf{p}^+ \cap \mathbf{a} \neq 0$. Now let A be the analytic subgroup of G corresponding to the abelian ideal $\mathbf{a}$ of $\mathbf{g}$. Then G/A is simply connected as we have been assuming that G is simply connected, and its Lie algebra splits as $\mathbf{g}/\mathbf{a} = (\mathbf{h}/\mathbf{h} \cap \mathbf{a}) + (\mathbf{p}^-/\mathbf{p}^- \cap \mathbf{a}) + (\mathbf{p}^+/\mathbf{p}^+ \cap \mathbf{a})$ and satisfies the first and the third conditions in the definition of the bipolarized symmetric Lie algebras. In particular, the arguments in the proof of Lemma (3.5) fully work, and imply that the analytic subgroup $H^-/H^- \cap A$ of G/A with Lie algebra $\mathbf{h}^-/\mathbf{h}^- \cap \mathbf{a} = (\mathbf{h}/\mathbf{h} \cap \mathbf{a}) + (\mathbf{p}^-/\mathbf{p}^- \cap \mathbf{a})$ is closed in G/A. Thus we can form a homogeneous space $(G/A)/(H^-/H^- \cap A)$, over which $B^- = G/H^-$ is naturally fibered with fibers being homeomorphic to $A/A \cap H^-$. Hence, to show the non-compactness of B^-, it suffices to prove that $A/A \cap H^-$ is non-compact. Recall that $\mathbf{a} = (\mathbf{h} \cap \mathbf{a}) + (\mathbf{p}^- \cap \mathbf{a}) + (\mathbf{p}^+ \cap \mathbf{a})$ is abelian, and therefore A is also abelian. Furthermore A is simply connected since so is G. Thus A is isomorphic to a linear space, and the analytic subgroup $A \cap H^-$ of A with Lie algebra $\mathbf{a} \cap \mathbf{h}^- = (\mathbf{h} \cap \mathbf{a}) + (\mathbf{p}^- \cap \mathbf{a})$ is isomorphic to a linear subspace. Consequently $A/A \cap H^- \cong \mathbf{R}^m$ with $m = \dim(\mathbf{p}^+ \cap \mathbf{a}) > 0$, and $A/A \cap H^-$ is non-compact. This proves the lemma. ∎

3.3. Here we give two examples of bipolarized symmetric Lie algebras $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ and the associated bipolarized symmetric spaces $(P, \nabla, \mathcal{F}^-, \mathcal{F}^+)$ which will play important roles in the proof of our theorem. In particular, we are interested in "transversal geometry" of the leaves of the foliation $\mathcal{F}^+$, and to explain it more precisely, we have to introduce a notion in the theory of foliations. Suppose generally that P is a manifold foliated by two transverse foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ with $\dim P = \dim \mathcal{F}^- + \dim \mathcal{F}^+$, and let p_1 and p_2 be arbitrary points of P lying on the same leaf L^- of $\mathcal{F}^-$: Denote by L_i^+ $(i = 1, 2)$ the leaf of $\mathcal{F}^+$ passing through p_i. Then, (corresponding to each homotopy class of curves in L^- combining p_1 and p_2,) there is a diffeomorphism φ of a neighborhood U_1 of p_1 in L_1^+ onto a neighborhood U_2 of p_2 in L_2^+ such that $\varphi(p_1) = p_2$ and that $\varphi(q_1) = q_2$ for $q_i \in U_i$ $(i = 1, 2)$ if and only if q_1 and q_2 lie on the same leaf of $\mathcal{F}^-$. We call such a partially defined diffeomorphism φ between leaves of $\mathcal{F}^+$ a *canonical transformation of* $\mathcal{F}^+$ *along* $\mathcal{F}^-$. In exhibiting examples of bipolarized symmetric spaces $(P, \nabla, \mathcal{F}^-, \mathcal{F}^+)$ below, we will try to make clear what geometry of the leaves of $\mathcal{F}^+$ is preserved by the canonical transformations of $\mathcal{F}^+$ along $\mathcal{F}^-$. Of course, as is expected from the definition of canonical transformations itself, this is closely related to the G-invariant geometry of the space $B^- = G/H^-$ of the leaves of the foliation $\mathcal{F}^-$ introduced in the previous paragraph.

Example 1 (Projective geometry). First we give an example concerned with the projective geometry. To begin with, let

$$\mathbf{g} = \mathbf{sl}(n+1;\mathbf{R}) = \{\alpha \in \mathbf{gl}(n+1;\mathbf{R}) : \operatorname{trace}\alpha = 0\}$$

be the real unimodular Lie algebra, where, for a positive integer n, $\mathbf{gl}(n;\mathbf{R})$ denotes the Lie algebra of $n \times n$ matrices with real entries, and consider its linear decomposition $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ given by

$$\mathbf{h} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \lambda \end{pmatrix} : \alpha \in \mathbf{gl}(n;\mathbf{R}),\ \lambda \in \mathbf{R},\ \operatorname{trace}\alpha + \lambda = 0 \right\},$$

$$\mathbf{p}^- = \left\{ \begin{pmatrix} 0 & 0 \\ \xi^- & 0 \end{pmatrix} : \xi^- = (\xi_1^- \cdots \xi_n^-) \in \mathbf{R}^n \right\},$$

$$\mathbf{p}^+ = \left\{ \begin{pmatrix} 0 & \xi^+ \\ 0 & 0 \end{pmatrix} : \xi^+ = \begin{pmatrix} \xi_1^+ \\ \vdots \\ \xi_n^+ \end{pmatrix} \in \mathbf{R}^n \right\} :$$

It is easy to check that $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ is a bipolarized symmetric Lie algebra. Now let $P = (P = G/H, \nabla, \mathcal{F}^-, \mathcal{F}^+)$ be the associated bipolarized symmetric space, where $G = SL(n+1;\mathbf{R})$ and H is the closed analytic subgroup of G with Lie algebra $\mathbf{h}$ which is isomorphic to the identity component $GL_+(n;\mathbf{R})$ of $GL(n;\mathbf{R})$. In this case, we can show that the corresponding homogeneous space $B^- = G/H^-$ introduced in §3.2 is diffeomorphic to the n-sphere. To see this, first recall that $G = SL(n+1;\mathbf{R})$ acts linearly on the euclidean space $\mathbf{R}^{n+1}$, while the standard sphere S^n is considered to be embedded in $\mathbf{R}^{n+1}$ by $S^n = \{|x| = 1\}$. In addition, S^n can be identified with the space of rays in $\mathbf{R}^{n+1}$ starting from the origin, and this identification gives rise to an action of G on S^n which is easily seen to be projective. Moreover we can immediately show that the isotropy subgroup of the action of G on S^n at the point ${}^t(0,\cdots,0,1) \in S^n$ coincides with the subgroup H^- of G. Thus S^n can be considered as the homogeneous space $B^- = G/H^-$. These observations imply further that for the projection π of $P = G/H$ onto $B^- = G/H^- = S^n$ its restriction to a leaf L^+ of $\mathcal{F}^+$, which is a totally geodesic flat submanifold of P, is a projective diffeomorphism of L^+ (with the induced affine connection) onto B^- (identified with the standard sphere). Thus the projective geometry of the leaves of $\mathcal{F}^+$ is preserved by the canonical transformations of

$\mathcal{F}^+$ along $\mathcal{F}^-$ (note that, for two leaves L_1^+ and L_2^+ of $\mathcal{F}^+$, the canonical transformation $\varphi : L_1^+ \to L_2^+$ between them is given by $\varphi = (\pi|L_2^+)^{-1} \circ (\pi|L_1^+)$).

Example 2 (Conformal geometry). The second example we are going to exhibit is related to the conformal geometry and arises as the geometry at infinity of the space of constant negative curvature. Consider the real simple Lie algebra

$$\mathbf{g} = \mathbf{so}(n+1,1) = \{\alpha \in \mathbf{gl}(n+2;\mathbf{R}) : \ {}^t\alpha\epsilon + \epsilon\alpha = 0\},$$

where

$$\epsilon = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \in \mathbf{gl}(n+2;\mathbf{R}),$$

which carries the linear decomposition $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ with

$$\mathbf{h} = \left\{ \begin{pmatrix} 0 & 0 & \lambda \\ 0 & \alpha & 0 \\ \lambda & 0 & 0 \end{pmatrix} : \ {}^t\alpha + \alpha = 0, \ \lambda \in \mathbf{R} \right\},$$

$$\mathbf{p}^- = \left\{ \begin{pmatrix} 0 & \xi^- & 0 \\ -{}^t\xi^- & 0 & {}^t\xi^- \\ 0 & \xi^- & 0 \end{pmatrix} : \xi^- = (\xi_1^- \cdots \xi_n^-) \in \mathbf{R}^n \right\},$$

$$\mathbf{p}^+ = \left\{ \begin{pmatrix} 0 & -{}^t\xi^+ & 0 \\ \xi^+ & 0 & \xi^+ \\ 0 & {}^t\xi^+ & 0 \end{pmatrix} : \xi^+ = \begin{pmatrix} \xi_1^+ \\ \vdots \\ \xi_n^+ \end{pmatrix} \in \mathbf{R}^n \right\}.$$

Again it is easy to see that $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ is a bipolarized symmetric Lie algebra. Now put $G = SO_0(n+1,1)$, the identity component of the lorentzian orthogonal group $O(n+1,1) = \{g \in GL(n+2;\mathbf{R}) : \ {}^tg\epsilon g = \epsilon\}$, and let H be the analytic subgroup of G with Lie algebra $\mathbf{h}$ which is isomorphic to the Lie group $CO(n) = \{\lambda g : g \in SO(n), \ \lambda > 0\}$. Then the conformal structure of $\mathbf{p}^+$ defined by the inner product $\langle \xi^+, \eta^+ \rangle = \sum_{k=1}^n \xi_k^+ \eta_k^+$ ($\xi^+, \eta^+ \in \mathbf{p}^+$) is $\mathrm{Ad}(H)$-invariant; that is, for every $h \in H$, there is a constant $\kappa > 0$ such that $\langle \mathrm{Ad}(h)\xi^+, \mathrm{Ad}(h)\eta^+ \rangle = \kappa \langle \xi^+, \eta^+ \rangle$ for all $\xi^+, \eta^+ \in \mathbf{p}^+$. Moreover it is easy to see that this is the ***unique*** $\mathrm{Ad}(H)$-invariant conformal structure of $\mathbf{p}^+$. Hence, for the bipolarized symmetric space $P = (P = G/H, \nabla, \mathcal{F}^-, \mathcal{F}^+)$ associated with $\mathbf{g}$, the tangent bundle F^+ of the foliation $\mathcal{F}^+$ has a unique G-invariant conformal structure, which automatically gives the leaves of $\mathcal{F}^+$ conformal structures. Now we show that these conformal structures of the leaves of $\mathcal{F}^+$ are preserved by the canonical transformations of $\mathcal{F}^+$ along $\mathcal{F}^-$. First recall the Minkowski space $\mathbf{R}^{n+1,1}$ which is , by definition, the $(n+2)$-dimensional euclidean space endowed with the lorentzian metric $ds^2 = dx_0^2 + \cdots + dx_n^2 - dx_{n+1}^2$, where $x = (x_0, \cdots, x_{n+1})$ denotes the canonical coordinate system. We regard the standard sphere S^n as being embedded in the Minkowski space $\mathbf{R}^{n+1,1}$ in the way that $S^n = \{x_0^2 + \cdots + x_n^2 - x_{n+1}^2 = 0\} \cap \{x_{n+1} = 1\}$. Each point of S^n corresponds to the light-like line in $\mathbf{R}^{n+1,1}$ passing through both the origin and itself. On the other hand, the group $G = SO_0(n+1,1)$ acts on $\mathbf{R}^{n+1,1}$ as linear isometries, and the action keeps the light cone $\{x_0^2 + \cdots + x_n^2 - x_{n+1}^2 = 0\}$ invariant. Thus G acts also on the sphere S^n, and the action preserves the induced conformal structure of $S^n \subset \mathbf{R}^{n+1,1}$ and is transitive on S^n. Moreover the isotropy subgroup of the action at ${}^t(1, 0, \cdots, 0, 1) \in S^n$ coincides with the analytic subgroup H^- of G with Lie algebra $\mathbf{h} = \mathbf{h} + \mathbf{p}^-$, and therefore the homogeneous space $B^- = G/H^-$, which is identified with the space of the leaves of the foliation $\mathcal{F}^-$, is diffeomorphic to the n-sphere. Now it is not hard to see that the restriction of the projection π of $P = G/H$ onto $B^- = G/H^-$ to a leaf L^+ of $\mathcal{F}^+$ is a conformal diffeomorphism of the leaf L^+ (equipped with the conformal structure described earlier) onto B^- (which is conformally equivalent to the standard sphere S^n). This shows that the canonical transformation of $\mathcal{F}^+$ along $\mathcal{F}^-$ preserve the conformal structures of the leaves of $\mathcal{F}^+$.

3.4. The proof of our theorem, which will be done in the following section, requires the classification of the simple bipolarized symmetric Lie algebras, while in [**19**] and [**14**] Nagano and Kobayashi gave the complete classification of them. In particular we will make use of their classification in the following form:

(3.7) **Proposition** (see [**19**], [**14**]). *Suppose that* $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ *is a simple bipolarized symmetric Lie algebra such that the associated homogeneous space* $B^\pm = G/H^\pm$ *introduced in §3.3 is covered by the n-sphere. Then* $\mathbf{g}$ *is isomorphic to either of the bipolarized symmetric Lie algebras given in Examples 1 and 2 of §3.3.*

In deriving the proposition from the classification of Kobayashi–Nagano, recall that $B^\pm$ is covered by a compact riemannian symmetric space B_0 (Lemma (3.2)), for, in the classification table of Kobayashi–Nagano, they describe the compact riemannian symmetric spaces B_0 associated with the Lie algebras in the same time: By picking up from the classification table the bipolarized symmetric Lie algebras for which the associated riemannian symmetric spaces are homeomorphic to the n-sphere, we conclude the proposition.

4. Construction of a congruence between geodesic flows

In the preceding sections we have prepared for the proof of the theorem below that is our main subject in the present paper, and we are now in the position where the proof is to be completed.

(4.1) **Theorem.** *For a closed riemannian manifold M of dimension greater than three, if its sectional curvature satisfies the pinching condition $-9/4 < K \leq -1$, and if the Anosov splitting of M is of class C^∞, then the geodesic flow φ_t of M is congruent to the geodesic flow $\hat{\varphi}_t$ of a certain closed riemannian manifold $\hat{M}$ of constant negative curvature; that is, there exists a diffeomorphism Φ of V_M onto $V_{\hat{M}}$ such that $\Phi \circ \varphi_t = \hat{\varphi}_t \circ \Phi$ for all $t \in \mathbf{R}$.*

Throughout this section suppose that M is a closed riemannian manifold of dimension $n+1$ which satisfies the conditions in the theorem, and let X be the universal covering of M on which the fundamental group $\Gamma = \pi_1(M)$ of M acts by the deck transformations. Denote by $P = (P, \Omega, \mathcal{F}^-, \mathcal{F}^+)$ the bipolarized symplectic manifold constructed from X. Note that the foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ have smooth tangent bundles F^- and F^+ since the Anosov splitting of M is smooth: In other words, the imaginary boundary B of X has a natural differentiable structure for which B is diffeomorphic to the standard sphere S^n, and the foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ of P correspond to the product structure of $P = B \times B \setminus$ (the diagonal set). Further recall that P is locally symmetric with respect to its canonical connection ∇ as was indicated in Proposition (2.4).

4.1. In §§4.1–4.2, assume that $n+1 = \dim M \geq 3$. Starting from the bipolarized locally symmetric space $P = (P, \nabla, \mathcal{F}^-, \mathcal{F}^+)$, it is possible to construct a bipolarized symmetric Lie algebra $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ from P, and secondarily a bipolarized symmetric space $\hat{P} = (\hat{P} = G/H, \hat{\nabla}, \hat{\mathcal{F}}^-, \hat{\mathcal{F}}^+)$ from $\mathbf{g}$ (cf. §3.1): In the case when $\mathbf{g}$ is not semisimple, take the 1-connected Lie group G in order that we can apply Lemmas (3.5) and (3.6). Then we can find a developing $\Psi : P \to \hat{P}$ which preserves the affine connections and the foliations of P and $\hat{P}$. Note that this is possible because P is locally modeled on $\hat{P}$, and is simply connected as we have been assuming that $n+1 = \dim M \geq 3$, while in the case of $\dim M = 2$, P is not simply connected: This is the reason why we should assume that $\dim M \geq 3$. Now P is fibered over B with the projection $(b^-, b^+) \in P = B \times B \setminus$ (diagonal) $\mapsto b^\mp \in B$ so that each fiber is a leaf of the foliation $\mathcal{F}^\pm$, while $\hat{P} = G/H$ is a fiber bundle over the homogeneous space $\hat{B}^\pm = G/H^\pm$ whose fibers are the leaves of the foliation $\hat{\mathcal{F}}^\pm$ (cf. §3.2). Thus we obtain an induced developing $\Psi^\pm : B \to \hat{B}^\pm$, which is actually a covering since B is compact. This especially excludes the case that $\mathbf{g}$ is not semisimple: In fact, if $\mathbf{g}$ were not semisimple, then either $\hat{B}^-$ or $\hat{B}^+$ would be non-compact by Lemma (3.6), in contradiction to the fact that $\Psi^\pm : B \to \hat{B}^\pm$ is a covering. Hence $\mathbf{g}$ should be semisimple. Furthermore $\mathbf{g}$ is to be simple, for $\hat{B}^\pm$ is covered by a compact riemannian symmetric space $\hat{B}_0$, which is not irreducible in the case when $\mathbf{g}$ is not simple (Lemma (3.3)), while $\hat{B}^\pm$ is covered by B which is diffeomorphic to the sphere. Thus we can apply Proposition (3.7) to $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$, and conclude that we have only two possibilities $\mathbf{g} \cong \mathbf{sl}(n+1;\mathbf{R})$ and $\mathbf{g} \cong \mathbf{so}(n+1,1)$.

Now we show that $\mathbf{g}$ is never isomorphic to $\mathbf{sl}(n+1;\mathbf{R})$. Assume in contrary that $\mathbf{g} \cong \mathbf{sl}(n+1;\mathbf{R})$. In this case we can choose $SL(n+1;\mathbf{R})/GL_+(n;\mathbf{R})$ as $\hat{P}$. The canonical transformations of $\hat{\mathcal{F}}^+$ along $\hat{\mathcal{F}}^-$ preserve the projective structures of the leaves of $\hat{\mathcal{F}}^+$ (recall Example 1 of §3.3). Note that this is also the case with $P = (P, \nabla, \mathcal{F}^-, \mathcal{F}^+)$ since P is locally isomorphic to $\hat{P}$. Thus the projective structures of the leaves of $\mathcal{F}^+$ can be pushed forward to B by the fibering $P \to B$, $(b^-, b^+) \mapsto b^+$, so that B is projectively equivalent to the standard n-sphere and that the action of the fundamental group Γ of M on B preserves the projective structure. However, this is a contradiction for the following reason. The action of each element $\gamma \in \Gamma \setminus \{1\}$ on the imaginary boundary B at infinity has the characteristic property that there are distinct two fixed points $b^-, b^+ \in B$ of γ such that $\gamma^k b \to b^\pm$ as $k \to \pm\infty$ for every $b \in B \setminus \{b^-, b^+\}$: In fact, for each $\gamma \in \Gamma \setminus \{1\}$ there is a unique geodesic line l in X that is invariant under the isometric action of γ on X, since X is closed (cf. [**6**]), and the points $b^\pm = l(\pm\infty)$ of B satisfy the above condition. On the other hand, we can find a closed geodesic c in B that does not pass through b^-; this is possible because B is projectively equivalent to the standard sphere. Then, by taking $k > 0$ large enough, $\gamma^k c$ is a closed geodesic of B enclosed in an arbitrarily small neighborhood of b^+: This is of course impossible, and consequently, we can exclude the case of $\mathbf{g} \cong \mathbf{sl}(n+1;\mathbf{R})$.

4.2. The argument above shows that $\mathbf{g} \cong \mathbf{so}(n+1,1)$, and therefore we may set $\hat{P} = SO_0(n+1,1)/CO(n)$ as in Example 2 of §3.3. Our next purpose is to show that the developing $\Psi : P \to \hat{P}$ is in fact bijective. First note that the model space $\hat{P}$ can be constructed from the hyperbolic space H^{n+1} as in the manner of §2.2, and especially is of the form $\hat{P} = S^n \times S^n \setminus$ (diagonal), where the n-sphere S^n is considered as the imaginary boundary of the hyperbolic space H^{n+1} at infinity. It is obvious that the developing Ψ of P into $\hat{P}$ is given by $\Psi(b^-, b^+) = (\Psi^- b^-, \Psi^+ b^+)$ for $(b^-, b^+) \in P = B \times B \setminus$ (diagonal), where Ψ^- and Ψ^+ are diffeomorphisms of B onto S^n. In particular, it is clear that Ψ is injective since so are Ψ^- and Ψ^+. We are now going to prove that $\Psi^- = \Psi^+$ in order to show that Ψ is surjective. For each element

γ of the fundamental group Γ of M that can be simultaneously considered as an automorphism of $P = (P, \nabla, \mathcal{F}^-, \mathcal{F}^+)$, $\Psi \circ \gamma \circ \Psi^{-1}$ is a partially defined automorphism of $\hat{P} = (\hat{P}, \hat{\nabla}, \hat{\mathcal{F}}^-, \hat{\mathcal{F}}^+)$, which is uniquely extended to a globally defined automorphism $\iota(\gamma)$ of $\hat{P}$ (the extension is possible because $\hat{P}$ is simply connected). Thus we have a faithful discrete representation ι of Γ into the automorphism group $\mathrm{Aut}(\hat{P})$ of $\hat{P}$. Put $\hat{\Gamma} = \iota(\Gamma) \subset \mathrm{Aut}(\hat{P})$.

On the other hand, it is easy to see that there are canonical isomorphisms among the group $\mathrm{Iso}(H^{n+1})$ of the isometric transformations of the hyperbolic space H^{n+1}, the group $\mathrm{Con}(S^n)$ of the conformal transformations of the imaginary boundary S^n of H^{n+1} equipped with the standard conformal structure, and the automorphism group $\mathrm{Aut}(\hat{P})$ of $\hat{P}$;

$$\mathrm{Iso}(H^{n+1}) \cong \mathrm{Con}(S^n) \cong \mathrm{Aut}(\hat{P}). \tag{4.2}$$

The first isomorphism is introduced in accordance with the fact that each isometric transformation g of H^{n+1} is naturally extended to a conformal transformation of the imaginary boundary S^n of H^{n+1} at infinity (cf. §2.1), while the second isomorphism assigns each conformal transformation h of S^n the automorphism $\tilde{h}$ of $\hat{P}$ defined by $\tilde{h}(b^-, b^+) = (hb^-, hb^+)$ for $(b^-, b^+) \in \hat{P}$. By means of these isomorphisms, the group $\hat{\Gamma}$ can be considered as a discrete subgroup of each of those transformation groups appearing in (4.2).

Now put $f = \Psi^+ \circ (\Psi^-)^{-1}$ which is a diffeomorphism of the imaginary boundary S^n onto itself. Then we can show that

$$f \circ \hat{\gamma} = \hat{\gamma} \circ f \qquad \text{for all} \quad \hat{\gamma} \in \hat{\Gamma} \subset \mathrm{Con}(S^n). \tag{4.3}$$

In fact, for $\hat{\gamma} = \iota(\gamma) \in \hat{\Gamma}$ $(\gamma \in \Gamma)$, it holds that

$$\begin{aligned}(\hat{\gamma}b^-, \hat{\gamma}b^+) &= \hat{\gamma}(b^-, b^+) = \Psi \circ \gamma \circ \Psi^{-1}(b^-, b^+) \\ &= (\Psi^- \circ \gamma \circ (\Psi^-)^{-1} b^-, \Psi^+ \circ \gamma \circ (\Psi^+)^{-1} b^+)\end{aligned}$$

for $(b^+, b^-) \in \hat{P}$, where $\hat{\gamma} \in \hat{\Gamma}$ is considered as an element of $\mathrm{Con}(S^n)$ in the first term, and as an element of $\mathrm{Aut}(\hat{P})$ in the second, while $\gamma \in \Gamma$ is considered as an automorphism of P in the third term, and as a diffeomorphism of B in the last: This implies that

$$\hat{\gamma} = \Psi^- \circ \gamma \circ (\Psi^-)^{-1} = \Psi^+ \circ \gamma \circ (\Psi^+)^{-1}$$

as an element of $\mathrm{Con}(S^n)$, and in consequence, we have (4.3).

Next consider $\hat{\Gamma}$ as a discrete subgroup of $\mathrm{Iso}(H^{n+1})$: $\hat{\Gamma}$ is torsion-free since so is Γ, and therefore we can form a complete riemannian manifold $\hat{M} = \hat{\Gamma} \setminus H^{n+1}$ of constant curvature -1. Furthermore the manifolds $\hat{M}$ and M, which are both aspherical, have the isomorphic fundamental groups $\hat{\Gamma}$ and Γ. Thus $\hat{M}$ is homotopy equivalent to M, and we have $H_{n+1}(\hat{M};\mathbf{Z}_2) \cong H_{n+1}(M;\mathbf{Z}_2) \cong \mathbf{Z}_2$, which means that $\hat{M}$ is closed. Hence we have proved that $\hat{\Gamma}$ is a uniform lattice of the Lie group $\mathrm{Iso}(H^{n+1})$ with $n+1 \geq 3$.

In this situation, Mostow's rigidity theorem claims that the diffeomorphism f of S^n satisfying the condition (4.3) with the uniform lattice $\hat{\Gamma}$ should be a conformal transformation of S^n ([**17**], [**18**]; see also [**22**], [**20**]). Thus f induces an isometric transformation $\bar{f}$ of H^{n+1} such that $\bar{f} \circ \hat{\gamma} = \hat{\gamma} \circ \bar{f}$ for all $\hat{\gamma} \in \hat{\Gamma} \subset \mathrm{Iso}(H^{n+1})$. On the other hand, it is well known that there is no such isometric transformation of H^{n+1} other than the identity map. Thus we have $\bar{f} = id$, and $f = \Psi^+ \circ (\Psi^-)^{-1} = id$. This shows that $\Psi^- = \Psi^+$, which immediately implies the surjectivity of Ψ.

In a summary we have

(4.4) **Lemma.** *Under the preceding assumptions, there exists an isomorphism Ψ of $(P, \nabla, \mathcal{F}^-, \mathcal{F}^+)$ onto $(\hat{P}, \hat{\nabla}, \hat{\mathcal{F}}^-, \hat{\mathcal{F}}^+)$.*

4.3. Recall that the affine connection $\hat{\nabla}$ of the bipolarized symmetric space $(\hat{P}, \hat{\nabla}, \hat{\mathcal{F}}^-, \hat{\mathcal{F}}^+)$ is the canonical connection of the bipolarized symplectic manifold $(\hat{P}, \hat{\Omega}, \hat{\mathcal{F}}^-, \hat{\mathcal{F}}^+)$ that can be constructed from the hyperbolic space H^{n+1}, while the bipolarized symplectic manifold $(P, \Omega, \mathcal{F}^-, \mathcal{F}^+)$ was constructed from the universal covering X of M. Here we show that the isomorphism Ψ of $(P, \nabla, \mathcal{F}^-, \mathcal{F}^+)$ onto $(\hat{P}, \hat{\nabla}, \hat{\mathcal{F}}^-, \hat{\mathcal{F}}^+)$ also preserves the symplectic forms Ω of P and $\hat{\Omega}$ of $\hat{P}$ up to multiplication by a constant provided that $n+1 = \dim M \geq 4$. We begin it with the following algebraic lemma.

(4.5) **Lemma.** *Let $\mathbf{g} = \mathbf{so}(n+1,1) = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ $(n \geq 1)$ be the bipolarized symmetric Lie algebra exhibited in Example 2 of §3.3. If $n \neq 2$, then the non-degenerate 2-forms ω on $\mathbf{p} = \mathbf{p}^- + \mathbf{p}^+$ which satisfy the following two conditions are unique up to the multiplication by constants:*

- (i) *The splitting $\mathbf{p} = \mathbf{p}^- + \mathbf{p}^+$ is lagrangian with respect to ω; that is, $\omega|\mathbf{p}^- = \omega|\mathbf{p}^+ = 0$;*
- (ii) *ω is $\mathrm{Ad}(H)$-invariant, where $H \cong CO(n)$ denotes the analytic subgroup of $G = SO_0(n+1,1)$ with Lie algebra $\mathbf{h}$.*

In the case of $n = 2$, the uniqueness of this kind does not hold, and this is the reason why we should assume that $\dim M \neq 3$ in Theorem (4.1). The above lemma follows from quite elementary computations, and the proof is omitted.

In the rest of the present section, assume that $n+1 = \dim M \geq 4$. Further, in constructing the bipolarized symmetric Lie algebra $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ from the bipolarized locally symmetric space $(P, \nabla, \mathcal{F}^-, \mathcal{F}^+)$ (see §3.1), put the following condition together with the conditions (3.1.1) and (3.1.2):

$$\omega(\alpha\xi, \eta) + \omega(\xi, \alpha\eta) = 0 \qquad \text{for all } \xi, \eta \in \mathbf{p};$$

where ω denotes the restriction of the symplectic form Ω of P to $\mathbf{p} = T_pP$ ($p \in P$). Under this additional condition, the resulting Lie algebra $\mathbf{g} = \mathbf{h} + \mathbf{p}^- + \mathbf{p}^+$ is still a bipolarized symmetric Lie algebra, and the arguments in §3 and §4.1–4.2 work fully without any change to yield the same conclusion, Lemma (4.4). Moreover, by this modification, it follows that the symplectic form ω on $\mathbf{p}$ satisfies the conditions in Lemma (4.5). Meanwhile, the symplectic form $\hat{\Omega}$ of the model space $\hat{P}$ is invariant under the action of $G = SO_0(n+1,1) = \mathrm{Iso}(H^{n+1})$ on $\hat{P} = G/H$, and, in particular, the restriction $\hat{\omega}$ of $\hat{\Omega}$ to the tangent space at the origin of $\hat{P}$ is $\mathrm{Ad}(H)$-invariant. Thus, by Lemma (4.5) together with the fact that the symplectic form of any bipolarized symplectic manifold is parallel with respect to its canonical connection (cf. §1.3), the pull-back $\Psi^*\hat{\Omega}$ of the symplectic form $\hat{\Omega}$ of $\hat{P}$ by Ψ should be a constant multiple of the symplectic form Ω of P. On the other hand, the bipolarized symplectic manifold $\hat{P} = (\hat{P}, \hat{\Omega}, \hat{\mathcal{F}}^-, \hat{\mathcal{F}}^+)$ is constructed from the hyperbolic space H^{n+1}. Thus, a suitable homothetic change of the metric of H^{n+1}, which also has the effect of the change of the symplectic form $\hat{\Omega}$ by multiplication by a constant, ensures that Ψ is an isomorphism between the bipolarized symplectic manifolds $P = (P, \Omega, \mathcal{F}^-, \mathcal{F}^+)$ and $\hat{P} = (\hat{P}, \hat{\Omega}, \hat{\mathcal{F}}^-, \hat{\mathcal{F}}^+)$. Let $\hat{X}$ be the riemannian manifold homothetic to H^{n+1} whose associated bipolarized symplectic manifold $\hat{P}$ is isomorphic to P under Ψ.

4.4. Let $V = V_X = \{v \in TX : |v| = 1\}$ be the unit tangent bundle of the universal covering X of M, which is fibered over P so that each fiber is an orbit of the geodesic flow φ_t of X. Also let $\hat{V} = V_{\hat{X}}$ be the unit tangent bundle of the riemannian manifold $\hat{X}$. We are now going to lift the isomorphism $\Psi : (P, \Omega, \mathcal{F}^-, \mathcal{F}^+) \to (\hat{P}, \hat{\Omega}, \hat{\mathcal{F}}^-, \hat{\mathcal{F}}^+)$ to a diffeomorphism $\Phi : V \to \hat{V}$ which commutes with the geodesic flows φ_t of X and $\hat{\varphi}_t$ of $\hat{X}$. For a while, identify $P = (P, \Omega, \mathcal{F}^-, \mathcal{F}^+)$ with $\hat{P} = (\hat{P}, \hat{\Omega}, \hat{\mathcal{F}}^-, \hat{\mathcal{F}}^+)$ under the isomorphism Ψ.

To begin with, recall that the connected component $G = \mathrm{Iso}_0(\hat{X}) \cong SO_0(n+1,1)$ of the isometric transformation group $\mathrm{Iso}(\hat{X})$ of $\hat{X}$ acts on $\hat{V}$ and on $\hat{P} = P$ naturally. These actions give rise to natural embeddings of the Lie algebra $\mathbf{g} \cong \mathbf{so}(n+1,1)$

of G into the Lie algebra $\mathcal{X}(\hat{V})$ of the vector fields on $\hat{V}$, and into the Lie algebra $\mathcal{X}(\hat{P}) = \mathcal{X}(P)$ of the vector fields on $\hat{P} = P$: For each $\xi \in \mathbf{g}$, denote by $\tilde{\xi} \in \mathcal{X}(\hat{V})$ and $\bar{\xi} \in \mathcal{X}(\hat{P}) = \mathcal{X}(P)$ the corresponding elements. On the other hand, let $T\hat{V} = \hat{E}^- + \hat{E}^0 + \hat{E}^+$ be the Anosov splitting associated with the geodesic flow $\hat{\varphi}_t$ of $\hat{X}$. We introduce a function $\alpha : \mathbf{g} \times \hat{V} \to \mathbf{R}$ which is characterized by the following condition: For each $\xi \in \mathbf{g}$ and $\hat{v} \in \hat{V}$, $\alpha(\xi, \hat{v})\hat{\varphi}'(\hat{v})$ is the $\hat{E}^0_{\hat{v}}$-component of $\tilde{\xi}(\hat{v}) \in T_{\hat{v}}\hat{V} = \hat{E}^-_{\hat{v}} + \hat{E}^0_{\hat{v}} + \hat{E}^+_{\hat{v}}$, where $\hat{\varphi}' = (\partial/\partial t)|_{t=0}\hat{\varphi}_t$ denotes the geodesic spray on $\hat{V}$ which spans the 1-dimensional subbundle $\hat{E}^0$ of $T\hat{V}$. It is easy to see that for each $\xi \in \mathbf{g}$ the function $\alpha(\xi, \cdot)$ on $\hat{V}$ is constant along each orbit of the geodesic flow $\hat{\varphi}_t$ of $\hat{X}$, and therefore α is reduced to a function on $\mathbf{g} \times \hat{P} = \mathbf{g} \times P$, which is denoted by the same symbol α. For each $\xi \in \mathbf{g}$, let α_ξ be the function on $P = \hat{P}$ defined by $\alpha_\xi(p) = \alpha(\xi, p)$ for $p \in P$.

Now let $TV = E^- + E^0 + E^+$ be the Anosov splitting associated with the geodesic flow φ_t of X: Note that with respect to the $\mathbf{R}$-fibering $\pi : V \to P$, E^0 is the vertical subbundle of TV that is spanned by the geodesic spray $\varphi' = (\partial/\partial t)|_{t=0}\varphi_t$, while $E = E^- + E^+$ is horizontal. For each $\varsigma \in TP = T\hat{P}$, denote by $\varsigma^* \in E$ the horizontal lift of ς by $\pi : V \to P$.

(4.6) **Lemma.** *The mapping* $\mathbf{g} \to \mathcal{X}(V)$, $\xi \mapsto \check{\xi} = \bar{\xi}^* + (\alpha_\xi \circ \pi)\varphi'$ *of the Lie algebra* $\mathbf{g}$ *of* G *into the Lie algebra* $\mathcal{X}(V)$ *of the vector fields on* V *is a Lie algebra monomorphism.*

Proof. Recall that the symplectic form Ω on P is the push-forward of the exterior derivative $d\Theta$ of the canonical contact form Θ of V by the projection $\pi : V \to P$. Moreover, as we have seen in §2.2, $\Theta(\varphi') = 1$ holds for the geodesic spray φ', and the subbundle $E = E^- + E^+$ of TV is characterized by $\Theta|E = 0$. Thus a standard formula for the $\mathbf{R}$-fibering $V \to P$ yields

$$(4.7) \qquad [\varsigma_1^*, \varsigma_2^*] = [\varsigma_1, \varsigma_2]^* - \Omega(\varsigma_1, \varsigma_2)\varphi' \qquad \text{for } \varsigma_1, \varsigma_2 \in \mathcal{X}(P) = \mathcal{X}(\hat{P}).$$

Furthermore the equation of the same form holds also for the fibering $\hat{V} \to \hat{P} = P$, and it implies

$$(4.8) \qquad d\alpha_\eta(\bar{\xi}) - d\alpha_\xi(\bar{\eta}) - \alpha_{[\xi,\eta]} = \Omega(\bar{\xi}, \bar{\eta}) \qquad \text{for } \xi, \eta \in \mathbf{g},$$

since the mapping $\mathbf{g} \to \mathcal{X}(\hat{V})$, $\xi \to \tilde{\xi}$ is a Lie algebra homomorphism. Now it follows easily from (4.7) and (4.8) that the mapping $\mathbf{g} \to \mathcal{X}(V)$ defined by $\xi \mapsto \check{\xi} = \bar{\xi}^* + (\alpha_\xi \circ \pi)\varphi'$ is actually a Lie algebra homomorphism. ∎

One should notice that it has been possible to prove the lemma because the isomorphism Ψ of P onto $\hat{P}$ preserves the symplectic forms of P and $\hat{P}$.

It is also easy to see that for each $\xi \in \mathbf{g}$ the corresponding vector field $\check{\xi}$ on V is complete. Thus the embedding $\mathbf{g} \to \mathcal{X}(V)$ of the Lie algebra $\mathbf{g}$ obtained in Lemma (4.6) induces an action of the Lie group G on V, and the action satisfies the following conditions:

(4.9.1) The action of G on V is transitive;

(4.9.2) The action commutes with the geodesic flow φ_t of X; that is, $g \circ \varphi_t = \varphi_t \circ g$ for all $g \in G$ and $t \in \mathbf{R}$;

(4.9.3) The Anosov splitting $TV = E^- + E^0 + E^+$ is invariant under the action.

Of course the original action of G on $\hat{V}$ satisfies the corresponding conditions, and especially by (4.9.1) we obtain a diffeomorphism Φ of V onto $\hat{V}$ which possesses the following properties:

(4.10.1) Φ is a lift of Ψ; i.e., $\hat{\pi} \circ \Phi = \Psi \circ \pi$ for the projection $\pi : V \to P$ and $\hat{\pi} : \hat{V} \to \hat{P}$.

(4.10.2) Φ is G-equivariant; i.e., $\Phi \circ g = g \circ \Phi$ for all $g \in G$;

(4.10.3) Φ commutes with the geodesic flows of X and $\hat{X}$; i.e., $\Phi \circ \varphi_t = \hat{\varphi}_t \circ \Phi$;

(4.10.4) Φ maps the Anosov splitting $TV = E^- + E^0 + E^+$ of X to the Anosov splitting $T\hat{V} = \hat{E}^- + \hat{E}^0 + \hat{E}^+$ of $\hat{X}$.

4.5. By means of the isomorphism Ψ of $P = (P, \Omega, \mathcal{F}^-, \mathcal{F}^+)$ onto $\hat{P} = (\hat{P}, \hat{\Omega}, \hat{\mathcal{F}}^-, \hat{\mathcal{F}}^+)$, the fundamental group Γ of M, which acts on P by automorphisms, has a faithful representation ι into the automorphism group $\mathrm{Aut}(\hat{P})$ of $\hat{P}$ defined by $\iota(\gamma) = \Psi \circ \gamma \circ \Psi^{-1} \in \mathrm{Aut}(\hat{P})$ for $\gamma \in \Gamma \subset \mathrm{Aut}(P)$: Put $\hat{\Gamma} = \iota(\Gamma)$. The isomorphism $\Psi : P \to \hat{P}$ is clearly ι-equivariant in the sense that $\Psi \circ \gamma = \iota(\gamma) \circ \Psi$ for all $\gamma \in \Gamma$. Furthermore, it is not hard to see that the automorphism group $\mathrm{Aut}(\hat{P}) = \mathrm{Aut}(\hat{P}, \hat{\Omega}, \hat{\mathcal{F}}^-, \hat{\mathcal{F}}^+)$ of $\hat{P}$ is canonically isomorphic to the isometry group $\mathrm{Iso}(\hat{X})$ of $\hat{X}$, where $\hat{X}$ denotes, as in §§4.3–4.4, the riemannian manifold homothetic to the hyperbolic space H^{n+1} for which $\hat{P}$ is the associated bipolarized symplectic manifold (cf. (4.2)). Thus $\hat{\Gamma}$ can be simultaneously considered as a discrete subgroup of $\mathrm{Iso}(\hat{X})$, and therefore acts on the unit tangent bundle $\hat{V} = V_{\hat{X}}$ of $\hat{X}$ in the canonical way. On the other hand, the group Γ, which acts on X by the isometric deck transformations, has a canonical action on the unit tangent bundle $V = V_X$ of X. Our purpose here is to deform the lift $\Phi : V \to \hat{V}$ of Ψ constructed in the previous paragraph to an ι-equivariant diffeomorphism of V onto $\hat{V}$ so that it still commutes with the geodesic flows of X and $\hat{X}$.

Let I be the involution of V defined by $Iv = -v$ for $v \in V$, and let $\hat{I}$ be the corresponding involution of $\hat{V}$. It is clear that

$$(4.11) \qquad I \circ \varphi_{-t} \circ I = \varphi_t \quad \text{and} \quad \hat{I} \circ \hat{\varphi}_{-t} \circ \hat{I} = \hat{\varphi}_t \quad \text{for } t \in \mathbf{R},$$

where φ_t and $\hat{\varphi}_t$ denote the geodesic flows of X and $\hat{X}$ respectively. First we are going to deform the diffeomorphism $\Phi : V \to \hat{V}$ so that

$$(4.12) \qquad \hat{\varphi}_t \circ \Phi = \Phi \circ \varphi_t \quad \text{and} \quad \hat{I} \circ \Phi = \Phi \circ I.$$

Note that I is a lift of the involution J of $P = B \times B \backslash$(diagonal) defined by $J(b^-, b^+) = (b^+, b^-)$ for $b^-, b^+ \in B$ with $b^- \neq b^+$. Similarly $\hat{I}$ is a lift of the involution $\hat{J}$ of $\hat{P} = S^n \times S^n \backslash$(diagonal) defined in the same way, where S^n denotes the imaginary boundary of $\hat{X}$ at infinity. Furthermore, the diffeomorphism Ψ of P onto $\hat{P}$ is the "product" of a diffeomorphism of B onto S^n by itself (see §4.2 again), and in consequence, Ψ commutes with the involutions of P and $\hat{P}$; i.e., $\hat{J} \circ \Psi = \Psi \circ J$. Since Φ is a lift of Ψ, there is a function $f : V \to \mathbf{R}$ such that

$$(4.13) \qquad \hat{I} \circ \Phi(v) = \hat{\varphi}_{f(v)} \circ \Phi \circ I(v) \quad \text{for } v \in V.$$

Now together with (4.10.3), (4.11) and (4.13) we have

$$\begin{aligned} \hat{\varphi}_{f(v)-t} \circ \Phi \circ I(v) &= \hat{\varphi}_{-t} \circ \hat{\varphi}_{f(v)} \circ \Phi \circ I(v) = \hat{\varphi}_{-t} \circ \hat{I} \circ \Phi(v) \\ &= \hat{I} \circ \hat{\varphi}_t \circ \Phi(v) = \hat{I} \circ \Phi \circ \varphi_t(v) = \hat{\varphi}_{f \circ \varphi_t(v)} \circ \Phi \circ I \circ \varphi_t(v) \\ &= \hat{\varphi}_{f \circ \varphi_t(v)} \circ \Phi \circ \varphi_{-t} \circ I(v) = \hat{\varphi}_{f \circ \varphi_t(v)} \circ \hat{\varphi}_{-t} \circ \Phi \circ I(v) \\ &= \hat{\varphi}_{f \circ \varphi_t(v)-t} \circ \Phi \circ I(v), \end{aligned}$$

and therefore we obtain $f \circ \varphi_t(v) = f(v)$ for all $v \in V$ and $t \in \mathbf{R}$; that is, f is constant on each orbit of the geodesic flow φ_t. Secondarily we will show that f is constant on each leaf of the stable and unstable foliations $\mathcal{E}^-$ and $\mathcal{E}^+$ of V which integrate the subbundles E^- and E^+ of TV respectively. Fix $v \in V$ and put $a = f(v)$. For the diffeomorphisms $\hat{I} \circ \Phi$ and $\hat{\varphi}_a \circ \Phi \circ I$ of V onto $\hat{V}$, we have

$$d(\hat{I} \circ \Phi)(\xi) = d(\hat{\varphi}_a \circ \Phi \circ I)(\xi) + df(\xi)\hat{\varphi}' \qquad \text{for } \xi \in T_v V$$

by (4.13), where $\hat{\varphi}'$ denotes the geodesic spray on $\hat{V}$. Further, by (4.10.4), both of these diffeomorphisms $\hat{I} \circ \Phi$ and $\hat{\varphi}_a \circ \Phi \circ I$ of V onto $\hat{V}$ send the subbundles E^- and E^+ of TV to the subbundles $\hat{E}^+$ and $\hat{E}^-$ of $T\hat{V}$ respectively; that is, $d(\hat{I} \circ \Phi)(\xi), d(\hat{\varphi}_a \circ \Phi \circ I)(\xi) \in \hat{E}^{\mp}$ whenever $\xi \in E_v^{\pm}$, and therefore $df(\xi) = 0$ for $\xi \in E_v^{\pm}$. This means that f is constant along the leaves of the foliations $\mathcal{E}^-$ and $\mathcal{E}^+$, and in consequence, f is constant on V. Set $f \equiv T$ on V, and replace Φ by $\hat{\varphi}_{T/2} \circ \Phi$. Then it is obvious that this Φ satisfies the condition (4.12).

Next we prove that the diffeomorphism Φ of V onto $\hat{V}$ which satisfies (4.12) is

ι-equivariant. Since Φ is a lift of the ι-equivariant diffeomorphism Ψ of P onto $\hat{P}$, there is a function $f : \Gamma \times V \to \mathbf{R}$ such that $\iota(\gamma) \circ \Phi(v) = \hat{\varphi}_{f(\gamma,v)} \circ \Phi \circ \gamma(v)$ for $\gamma \in \Gamma$ and $v \in V$. However, for each $\gamma \in \Gamma$, we can prove, by using (4.12) and (4.10.4) as before, that the function $f(\gamma, \cdot)$ on V is constant. Put $f(\gamma) = f(\gamma, \cdot)$, and consider f as a function on Γ. It is obvious that

$$\iota(\gamma) \circ \Phi = \hat{\varphi}_{f(\gamma)} \circ \Phi \circ \gamma \qquad \text{for } \gamma \in \Gamma, \tag{4.14}$$

and our purpose here is to show that $f \equiv 0$. Take $\gamma \in \Gamma \setminus \{1\}$. Then there is a vector $v \in V$ and $T > 0$ such that

$$\gamma v = \varphi_T v, \qquad \gamma(-v) = \varphi_{-T}(-v), \tag{4.15}$$

since $M = \Gamma \setminus X$ is a closed manifold: In fact, it suffices to take a unit vector v tangent to the geodesic line in X that is invariant under the isometric action of γ on X. The action of γ on P fixes the point $\pi(v)$ of P, where $\pi : V \to P$ denotes the projection, and therefore the point $\Psi \circ \pi(v)$ of $\hat{P}$ is fixed by the action $\iota(\gamma)$ on $\hat{P}$. Therefore, for $\hat{\gamma} = \iota(\gamma)$ and for $\hat{v} = \Phi(v)$, there is a constant $\hat{T}$ such that

$$\hat{\gamma}\hat{v} = \hat{\varphi}_{\hat{T}}\hat{v}, \qquad \hat{\gamma}(-\hat{v}) = \hat{\varphi}_{-\hat{T}}(-\hat{v}). \tag{4.16}$$

From (4.14)–(4.16) and (4.12), we have

$$\begin{aligned}\hat{\varphi}_{\hat{T}}(\hat{v}) = \hat{\gamma}(\hat{v}) = \iota(\gamma) \circ \Phi(v) &= \hat{\varphi}_{f(\gamma)} \circ \Phi \circ \gamma(v) \\ &= \hat{\varphi}_{f(\gamma)} \circ \Phi \circ \varphi_T(v) = \hat{\varphi}_{f(\gamma)} \circ \hat{\varphi}_T \circ \Phi(v) = \hat{\varphi}_{f(\gamma)+T}(\hat{v}),\end{aligned}$$

and in consequence it follows that

$$\hat{T} = f(\gamma) + T. \tag{4.17}$$

Similarly (4.12) and (4.14)–(4.16) imply that

$$\begin{aligned}\hat{\varphi}_{-\hat{T}}(-\hat{v}) = \hat{\gamma}(-\hat{v}) = \iota(\gamma) \circ \hat{I} \circ \Phi(v) &= \iota(\gamma) \circ \Phi \circ I(v) \\ &= \hat{\varphi}_{f(\gamma)} \circ \Phi \circ \gamma(-v) = \hat{\varphi}_{f(\gamma)} \circ \Phi \circ \varphi_{-T}(-v) \\ &= \hat{\varphi}_{f(\gamma)} \circ \hat{\varphi}_{-T} \circ \Phi(-v) = \hat{\varphi}_{f(\gamma)-T}(-\hat{v}),\end{aligned}$$

and we have

$$-\hat{T} = f(\gamma) - T. \tag{4.18}$$

It is an immediate consequence of (4.17) and (4.18) that $f(\gamma) = 0$ for $\gamma \in \Gamma$, and this shows that Φ is ι-equivariant.

Now Theorem (4.1) follows immediately. In fact, consider $\hat{\Gamma}$ as a discrete subgroup of the isometric transformation group $\mathrm{Iso}(\hat{X})$ of $\hat{X}$, and put $\hat{M} = \hat{\Gamma} \setminus \hat{X}$: $\hat{M}$ is a closed riemannian manifold of constant negative curvature (cf. §4.2), and $\hat{\Gamma} \setminus \hat{V}$ coincides with the unit tangent bundle $V_{\hat{M}}$ of $\hat{M}$, while $V_M = \Gamma \setminus V$ obviously. Thus the ι-equivariant diffeomorphism Φ of V onto $\hat{V}$ descends to a diffeomorphism of V_M onto $V_{\hat{M}}$ that commutes with the geodesic flows of M and $\hat{M}$. This proves Theorem (4.1).

4.6. Here is a digression on another topic on the geodesic flows of negatively curved manifolds. First of all, recall that the geodesic flow φ_t of a riemannian manifold M is said to be *smoothly conjugate* (resp. *topologically conjugate*) to the geodesic flow $\hat{\varphi}_t$ of another riemannian manifold $\hat{M}$ if there is a diffeomorphism (resp. homeomorphism) Φ of V_M onto $V_{\hat{M}}$ such that each orbit of φ_t is mapped to an orbit of $\hat{\varphi}_t$ by Φ with orientations preserved. In spite of the fact that smooth conjugacy is in general a weaker notion than congruence between geodesic flows in the sense of Theorem (4.1), the previous arguments in §§4.3–4.5 yield

(4.19) **Proposition.** *For a closed riemannian manifold M of dimension $n+1 \neq 3$ with negative curvature, if the geodesic flow φ_t of M is smoothly conjugate to the geodesic flow $\hat{\varphi}_t$ of a certain closed riemannian manifold $\hat{M}$ of constant curvature -1, then φ_t is congruent to $\hat{\varphi}_t$ provided that the metric of $\hat{M}$ is changed suitably homothetically.*

Note that, modulo Conjecture 2 we posed in the introduction, M is to be homothetic to $\hat{M}$ under the assumption in the proposition. In contrast, M. Gromov [9] proved that the fundamental group of a closed riemannian manifold determines its geodesic flow up to topological conjugacy: More precisely, his result claims that, for two closed riemannian manifolds M and $\hat{M}$ of negative curvature, their geodesic flows are topologically (but not necessarily smoothly) conjugate to each other whenever the fundamental group of M is isomorphic to that of $\hat{M}$. Thus, in Proposition (4.19), the assumption of the smoothness of the conjugacy is never removed.

Proof. First note that the fundamental groups of M and $\hat{M}$ are isomorphic to each other. Put $\Gamma = \pi_1(M)$, $\hat{\Gamma} = \pi_1(\hat{M})$, and let $\iota : \Gamma \to \hat{\Gamma}$ be an isomorphism. The smooth conjugacy between V_M and $V_{\hat{M}}$ is lifted to an ι-equivariant smooth conjugacy from the unit tangent bundle V of the universal covering X of M onto the unit tangent bundle $\hat{V}$ of the hyperbolic space H^{n+1} which covers $\hat{M}$ isometrically. Thus, by denoting by $P = (P, \Omega, \mathcal{F}^-, \mathcal{F}^+)$ and $\hat{P} = (\hat{P}, \hat{\Omega}, \hat{\mathcal{F}}^-, \hat{\mathcal{F}}^+)$ the bipolarized symplectic manifolds associated with X and H^{n+1} respectively, we have an ι-equivariant diffeomorphism Ψ of P onto $\hat{P}$ which maps the foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ of P to the foliations $\hat{\mathcal{F}}^-$ and $\hat{\mathcal{F}}^+$ of $\hat{P}$ respectively. Now let ∇ and $\hat{\nabla}$ be the canonical connections of P and $\hat{P}$, and let $h = (\Psi^{-1})^*\nabla - \hat{\nabla}$ be the second fundamental form of the diffeomorphism $\Psi^{-1} : \hat{P} \to P$: h is a $\hat{\Gamma}$-invariant $(1,2)$-tensor field on $\hat{P}$, and therefore an argument employed in the proof of Proposition (2.4) implies that h should vanish on $\hat{P}$. Thus Ψ

is an ι-equivariant isomorphism of $(P, \nabla, \mathcal{F}^-, \mathcal{F}^+)$ onto $(\hat{P}, \hat{\nabla}, \hat{\mathcal{F}}^-, \hat{\mathcal{F}}^+)$. Then it follows from the arguments in §§4.3–4.5 that the geodesic flow of M is congruent to that of $\hat{M}$ provided that the metric of $\hat{M}$ is changed homothetically so that $P = (P, \Omega, \mathcal{F}^-, \mathcal{F}^+)$ and $\hat{P} = (\hat{P}, \hat{\Omega}, \hat{\mathcal{F}}^-, \hat{\mathcal{F}}^+)$ are isomorphic to each other under the diffeomorphism Ψ. ∎

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